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Cotilting modules and bimodules over arbitrary associative rings are studied. On the one hand we find a connection between reflexive modules with respect to a cotilting (bi)module U and a notion of U-torsionless linear compactness. On the other hand we provide concrete examples of cotilting bimodules over linearly compact noetherian serial rings.

Cotilting theory is a generalization of Morita duality in a sense that is analogous to that in which tilting theory is a generalization of Morita equivalence. Indeed, cotilting modules first appeared as vector space duals of tilting modules over finite dimensional algebras (see, e.g., [**H**, IV, 7.8]), just as injective cogenerators are such duals of progenerators. Later, R.R. Colby [**Cb1**] studied finitely generated cotilting bimodules over noetherian rings, proving that they induce finitistic generalized Morita dualities, similar to the finite dimensional algebra case. More recently, in [**Cb2**] he investigated a more general class of representable dualities, namely (nonfinitistic) generalized Morita dualities. He proved that the existence of such a duality implies the existence of a second pair of functors between classes that complement the reflexive ones, obtaining a result which is close to a dual form of the celebrated Tilting Theorem [**BrBu**], [**HaRi**].

For arbitrary rings R and S, a Morita duality between left S-modules and right R-modules is given by the contravariant Hom functors induced by a so called *Morita bimodule* ${}_SW_R$, namely, one such that (i) the classes of W-reflexive modules contain R_R, W_R, ${}_SS$ and ${}_SW$, and are closed under submodules, factor modules and extensions; or, equivalently, (ii) ${}_SW_R$ is balanced, and W_R and ${}_SW$ are injective cogenerators. Colby's generalized Morita dualities in [**Cb2**] are those induced by a bimodule ${}_SU_R$ such that a natural weakening of (i) holds (just closure under factor modules is left out). Generalizing the notion of injective cogenerator, the authors of [**CpDeTo**] and [**CpToTr**] defined a *cotilting module* U_R over a ring R as one such that $\mathrm{Cogen}(U_R) = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, U_R)$. In [**CpDeTo**, Proposition 1.7] it is shown that this notion is dual to that of tilting module by means of the following

Proposition. *A module U_R is a cotilting module if and only if it satisfies the conditions*

(1) $\mathrm{inj}\dim(U_R) \le 1$,

(2) $\operatorname{Ext}^1_R(U^\alpha_R, U_R) = 0$ *for any cardinal* α,
(3) $\operatorname{Ker}\operatorname{Hom}_R(-, U_R) \cap \operatorname{Ker}\operatorname{Ext}^1_R(-, U_R) = 0$.

To obtain a homological generalization of (ii), as in [**Cp**], we say that a balanced bimodule ${}_SU_R$ in which both U_R and ${}_SU$ are cotilting modules is a *cotilting bimodule*.

In this paper we continue the study of cotilting (bi)modules over arbitrary rings that was begun in [**Cp**]. There it was shown that any cotilting bimodule ${}_SU_R$ induces a pair of dualities between quite large subcategories of torsion-free and torsion modules in Mod-R and S-Mod, respectively. This result naturally generalizes Morita dualities to torsion theories, and it is still dual to the Tilting Theorem.

A third major component of Morita duality theory is B. Müller's theorem [**X**, Corollary 4.2] that the reflexive modules relative to a Morita bimodule are precisely the linearly compact modules. In Section 1 we investigate the related notion of torsionless linear compactness and its connection to the reflexivity of modules. This allows us to find a bridge between Colby's generalized Morita duality and cotilting bimodules by showing that a cotilting bimodule U induces a generalized Morita duality if and only if the classes of the U-reflexive modules coincide with those of the U-torsionless linearly compact modules. This is accomplished, in part, by answering a question posed in [**Cp**].

Perhaps the most accessible collection of examples of tilting modules over non-artinian rings are those over hereditary noetherian serial rings. They and their endomorphism rings were classified in [**CbFu**]. In Section 2 we show that the Morita dual of a tilting module possesses most of the properties of a cotilting bimodule. Then in Section 3 we employ these results and Warfield's theorems on noetherian serial rings in [**Wa**] to show that the dual of any tilting module over a noetherian serial ring with selfduality is a cotilting bimodule. Thus we obtain a class of concrete examples of cotilting bimodules that are not, in general, finitely generated.

We denote by R and S two arbitrary associative rings with unit, and by Mod-R and S-Mod the category of all unitary right R- and left S-modules, respectively. All the classes of modules that we introduce are to be considered as full subcategories of modules closed under isomorphisms. Given a module U, we denote by $\operatorname{add}(U)$ the class of all direct summands of any finite direct sum of copies of U, and by $\operatorname{Cogen}(U)$ the class of all modules *cogenerated* by U, that is all the modules M such that there exists an exact sequence $0 \to M \to U^\alpha$, for some cardinal α. We denote by $\operatorname{Rej}_U(-)$ the *reject* radical, defined by $\operatorname{Rej}_U(M) = \cap\{\operatorname{Ker}(f) \mid f \in \operatorname{Hom}_R(M, U)\}$, i.e., the least submodule M_0 of M such that M/M_0 belongs to $\operatorname{Cogen}(U)$. Given a bimodule ${}_SU_R$, we denote by Δ both the functors $\operatorname{Hom}_R(-, U)$ and $\operatorname{Hom}_S(-, U)$, and by Γ both the functors $\operatorname{Ext}^1_R(-, U)$ and $\operatorname{Ext}^1_S(-, U)$. For any module M, $\delta_M\ M \to \Delta^2(M)$ denotes the evaluation morphism. M is called Δ-*reflexive* if δ_M is an isomorphism. Note that if U_R is a cotilting module, then $(\operatorname{Ker}\Delta, \operatorname{Ker}\Gamma)$ is a torsion theory in Mod-R, associated to the idempotent radical $\operatorname{Rej}_U(-) = \operatorname{Ker}(\delta_-)$. For further notation, we refer to [**AF**], [**S**] and [**CE**].

1. Reflexivity and torsionless linear compactness.

We start this section pointing out some facts on Δ-reflexivity of modules, with respect to a cotilting module U_R, which generalize part of [**Cp**, Lemma 4 and Proposition 5]:

Lemma 1.1. *Let U_R be a cotilting module, and let $S = \operatorname{End}(U_R)$. Then:*

(a) *U_R and ${}_SS$ are Δ-reflexive.*
(b) *If ${}_SL \in S$-Mod is a factor of any Δ-reflexive module (in particular, if ${}_SL$ is finitely generated), then δ_L is an epimorphism.*
(c) *If ${}_SL \in \operatorname{Cogen}({}_SU)$ is a factor of any Δ-reflexive module, then L is Δ-reflexive.*
(d) *For any $M_R \in$ Mod-R, we have $\operatorname{Coker}(\delta_M) \in \operatorname{Ker}\Gamma$.*
(e) *Let $M_R \in$ Mod-R. Then M_R is Δ-reflexive if and only if $M_R \in \operatorname{Ker}\Gamma$ and $\Delta(M)$ is Δ-reflexive.*
(f) *If $M_R \in \operatorname{Ker}\Gamma$ and $\Delta(M_R)$ is a factor of any Δ-reflexive module (in particular, if $\Delta(M_R)$ is finitely generated), then M_R is Δ-reflexive.*
(g) *If $L_R \leq M_R$, M_R is Δ-reflexive and $M/L \in \operatorname{Ker}\Gamma$, then L_R is Δ-reflexive.*

Proof. (a) $\Delta^2(U_R) \cong \Delta({}_SS) \cong U_R$ and $\Delta^2({}_SS) \cong \Delta(U_R) \cong {}_SS$ canonically.

(b) Let $K \to M \to L \to 0$ be an exact sequence in S-Mod, with M Δ-reflexive. Then we have the exact sequence $0 \to \Delta(L) \to \Delta(M) \to I \to 0$, where I embeds into $\Delta(K)$, so that $\Gamma(I) = 0$. Therefore we get the commutative exact diagram

$$\begin{array}{ccccc} M & \longrightarrow & L & \longrightarrow & 0 \\ \cong\downarrow \delta_M & & \downarrow \delta_L & & \\ \Delta^2(M) & \longrightarrow & \Delta^2(L) & \longrightarrow & 0 \end{array}$$

which shows that δ_L is epic.

(c) Clearly ${}_SL \in \operatorname{Cogen}({}_SU)$ if and only if δ_L is monic. We can conclude by (b).

(d) By adjunction, we get $\Delta(\delta_M) \circ \delta_{\Delta(M)} = \mathrm{id}_{\Delta(M)}$, so that $\Delta(\delta_M)$ is epic. Therefore, from the exact sequence $0 \to M/\operatorname{Rej}_U(M) \to \Delta^2(M) \to \operatorname{Coker}(\delta_M) \to 0$ we see that $\Gamma(\operatorname{Coker}(\delta_M)) \hookrightarrow \Gamma\Delta^2(M) = 0$.

(e) Again from the identity $\Delta(\delta_M) \circ \delta_{\Delta(M)} = \mathrm{id}_{\Delta(M)}$, we see that if δ_M is an isomorphism, then $\delta_{\Delta(M)}$ is too, and of course $M \in \operatorname{Cogen}(U_R) = \operatorname{Ker}\Gamma$.

Conversely, if $\delta_{\Delta(M)}$ is an isomorphism, then $\Delta(\delta_M)$ must be monic, i.e., $\operatorname{Coker}(\delta_M) \in \operatorname{Ker}\Delta$. Moreover $\operatorname{Coker}(\delta_M) \in \operatorname{Ker}\Gamma$ because of (d). Since $(\operatorname{Ker}\Delta, \operatorname{Ker}\Gamma)$ is a torsion theory, we conclude that $\operatorname{Coker}(\delta_M) = 0$, i.e., δ_M is epic. Under the further assumption that $M \in \operatorname{Ker}\Gamma$, we conclude that δ_M is an isomorphism.

(f) Since $\Delta(M) \in \operatorname{Cogen}({}_SU)$, (c) applies, giving $\Delta(M)$ reflexive. We conclude by (e).

(g) From the exact sequence $0 \to L \to M \to M/L \to 0$ in Mod-R, by assumption we get the exact sequence $0 \to \Delta(M/L) \to \Delta(M) \to \Delta(L) \to 0$. We conclude using (e) and (f). □

It is well known that linear compactness plays a fundamental role in the study of duality. Here we introduce a concept of linear compactness with respect to a torsion theory, drawing inspiration from [**GpGaWi**, §3]:

Definition 1.2. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in Mod-R. Then a right R-module M is called *$\mathcal{F}$-linearly compact* if $M \in \mathcal{F}$ and for any inverse system of morphisms $\{p_\lambda : M \to M_\lambda\}$ with $M_\lambda \in \mathcal{F}$ and $\operatorname{Coker}(p_\lambda) \in \mathcal{T}$, for all λ's, it happens that $\operatorname{Coker}(\varprojlim p_\lambda) \in \mathcal{T}$.

If U_R is a cotilting module, a module $M \in$ Mod-R is called *U-torsionless linearly compact* (U-tl.l.c., for short) if M is $\operatorname{Ker}\Gamma$-linearly compact.

Note that $M \in$ Mod-R is linearly compact iff M is Mod-R-linearly compact, i.e., it is linearly compact with respect to the trivial torsion theory $(\{0\}, \text{Mod-}R)$. In particular if U_R is a cotilting module, then the U-torsionless linear compactness coincides with the usual linear compactness iff U_R is an injective cogenerator.

Torsionfree linear compactness is inherited by any inverse limit of the type in Definition 1.2, as the following result due to A. Tonolo shows:

Proposition 1.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in* Mod-R*, and let $M \in$* Mod-R *be $\mathcal{F}$-linearly compact. Then for any inverse system $\{p_\lambda : M \to M_\lambda\}$ with $M_\lambda \in \mathcal{F}$ and* $\operatorname{Coker}(p_\lambda) \in \mathcal{T}$*, the module $\varprojlim M_\lambda$ is $\mathcal{F}$-linearly compact too.*

Proof. First of all, let us note that $\varprojlim M_\lambda \in \mathcal{F}$, because $\mathcal{F}$ is closed under inverse limits. Next, let $\{p'_\mu : \varprojlim M_\lambda \to M'_\mu : \mu \in \Lambda'\}$ be any inverse system with $M'_\mu \in \mathcal{F}$ and $\operatorname{Coker}(p'_\mu) \in \mathcal{T}$ for all μ's. Let us prove that $\operatorname{Coker}(\varprojlim p'_\mu) \in \mathcal{T}$. Note that the cokernel of each map $p'_\mu \circ \varprojlim p_\lambda$, for all μ's, is in $\mathcal{T}$, because it is an extension of a factor of $\operatorname{Coker}(\varprojlim p_\lambda)$, which is in $\mathcal{T}$, by the torsion module $\operatorname{Coker}(p'_\mu)$. Hence, by assumption, we get that the morphism $\varprojlim(p'_\mu \circ \varprojlim p_\lambda) \cong \varprojlim p'_\mu \circ \varprojlim p_\lambda$ has a torsion cokernel. This implies that $\operatorname{Coker}(\varprojlim p'_\mu) \in \mathcal{T}$. □

In [**Cp**, Proposition 10] it was proved that if ${}_SU_R$ is a cotilting bimodule, then any U-tl.l.c. module is Δ-reflexive; and the question of whether the converse is true was posed. To give a partial answer, we start with a theorem which generalizes a well known result, substantially due to Müller [**Mu**] (see also [**X**, Theorem 4.1]):

Theorem 1.4. *Let U_R be a cotilting module, and let $S = \mathrm{End}(U_R)$. Then the following are equivalent for any $M \in$ Mod-R:*

(1) *M is U-tl.l.c.*

(2) *M is Δ-reflexive, and for all $L \overset{i}{\hookrightarrow} \Delta(M)$ we have $\mathrm{Coker}\,\Delta(i) \in \mathrm{Ker}\,\Delta$.*

Proof. (1) $\Rightarrow$ (2). Let M_R be U-tl.l.c., let L be a submodule of $\Delta(M)$ and let $\{L_\lambda : \lambda \in \Lambda\}$ be the upward directed family of the finitely generated submodules of L. Thus, if we denote by $i_\lambda : L_\lambda \hookrightarrow L \overset{i}{\hookrightarrow} \Delta(M)$ the canonical inclusions, we get $\varinjlim L_\lambda = L$ and $\varinjlim i_\lambda = i$. Let now $p_\lambda = \Delta(i_\lambda) \circ \delta_M : M \to \Delta(L_\lambda)$. Then $\{p_\lambda : \lambda \in \Lambda\}$ is an inverse system of morphisms in $\mathrm{Ker}\,\Gamma$. In order to show that $\mathrm{Coker}(p_\lambda) \in \mathrm{Ker}\,\Delta$ for any λ, let us consider the commutative diagram in Mod-R

$$\begin{array}{ccccc} L_\lambda & \xrightarrow{i_\lambda} & \Delta(M) & = & \Delta(M) \\ \cong \downarrow \delta_{L_\lambda} & & \downarrow \delta_{\Delta(M)} & & \| \\ \Delta^2(L_\lambda) & \xrightarrow{\Delta^2(i_\lambda)} & \Delta^3(M) & \xrightarrow{\Delta(\delta_M)} & \Delta(M) \end{array}$$

where δ_{L_λ} is an isomorphism because of Lemma 1.1(c), which proves that $\Delta(p_\lambda) = \Delta(\delta_M) \circ \Delta^2(i_\lambda)$ is monic, i.e., $\mathrm{Coker}(p_\lambda) \in \mathrm{Ker}\,\Delta$.

Thus the hypothesis (1) applies, giving $\mathrm{Coker}(\varprojlim p_\lambda) \in \mathrm{Ker}\,\Delta$. Moreover

$$(*) \qquad \varprojlim p_\lambda = \varprojlim \Delta(i_\lambda) \circ \delta_M \cong \Delta(\varinjlim i_\lambda) \circ \delta_M = \Delta(i) \circ \delta_M.$$

First, if we choose $L = \Delta(M)$ we clearly get $\mathrm{Coker}(\delta_M) \cong \mathrm{Coker}(\varprojlim p_\lambda) \in \mathrm{Ker}\,\Delta$. On the other hand, since $M \in \mathrm{Ker}\,\Gamma$, δ_M is injective and $\mathrm{Coker}(\delta_M) \in \mathrm{Ker}\,\Gamma$ because of Lemma 1.1(d). Therefore $\mathrm{Coker}(\delta_M) = 0$, i.e., M is Δ-reflexive.

Finally, in the case L is arbitrary, since δ_M is an isomorphism, from (*) we get $\mathrm{Coker}(\Delta(i)) \cong \mathrm{Coker}(\varprojlim p_\lambda) \in \mathrm{Ker}\,\Delta$.

(2) $\Rightarrow$ (1). Let $\{p_\lambda : M \to M_\lambda\}$ be an inverse system of morphisms in Mod-R, with $M, M_\lambda \in \mathrm{Ker}\,\Gamma$ and $\mathrm{Coker}(p_\lambda) \in \mathrm{Ker}\,\Delta$ for all λ's.

In the sequel, we will refer to the following exact sequences

$$\text{(ex1)} \qquad 0 \longrightarrow K_\lambda \longrightarrow M \xrightarrow{\alpha_\lambda} I_\lambda \longrightarrow 0$$

$$\text{(ex2)} \qquad 0 \longrightarrow I_\lambda \xrightarrow{\beta_\lambda} M_\lambda \longrightarrow C_\lambda \longrightarrow 0$$

with $K_\lambda = \mathrm{Ker}(p_\lambda)$, $I_\lambda = \mathrm{Im}(p_\lambda)$, $C_\lambda = \mathrm{Coker}(p_\lambda)$ and $\beta_\lambda \circ \alpha_\lambda = p_\lambda$.

First, let us prove that all the K_λ, I_λ, M_λ are Δ-reflexive. Note that the sequence (ex1) is in $\mathrm{Ker}\,\Gamma$, and M is Δ-reflexive by assumption, so that from Lemma 1.1(g) we obtain that K_λ is Δ-reflexive too. Moreover, looking at the embedding $\Delta(\alpha_\lambda) : \Delta(I_\lambda) \hookrightarrow \Delta(M)$, by hypothesis we have that $\operatorname{Coker}(\Delta^2(\alpha_\lambda)) \in \operatorname{Ker}\Delta$. Thus we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\lambda & \longrightarrow & M & \xrightarrow{\alpha_\lambda} & I_\lambda & \longrightarrow & 0 & & \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\delta_{I_\lambda} & & & & \\ 0 & \longrightarrow & \Delta^2(K_\lambda) & \longrightarrow & \Delta^2(M) & \xrightarrow{\Delta^2(\alpha_\lambda)} & \Delta^2(I_\lambda) & \longrightarrow & \operatorname{Coker}(\Delta^2(\alpha_\lambda)) & \longrightarrow & 0 \end{array}$$

from which we get (thanks to Lemma 1.1(d)) $\operatorname{Coker}(\Delta^2(\alpha_\lambda)) \cong \operatorname{Coker}(\delta_{I_\lambda}) \in \operatorname{Ker}\Gamma$. Thus $\operatorname{Coker}(\delta_{I_\lambda}) = 0$, i.e., I_λ is Δ-reflexive. Next, from (ex2) we get the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\lambda & \xrightarrow{\beta_\lambda} & M_\lambda & \longrightarrow & C_\lambda & \longrightarrow & 0 \\ & & \cong\downarrow\delta_{I_\lambda} & & \downarrow\delta_{M_\lambda} & & & & \\ & & \Delta^2(I_\lambda) & \xrightarrow{\Delta^2(\beta_\lambda)} & \Delta^2(M_\lambda) & \longrightarrow & C'_\lambda & \longrightarrow & 0 \end{array}$$

where $C_\lambda \in \operatorname{Ker}\Delta$ by assumption. Let us prove that $C'_\lambda \in \operatorname{Ker}\Delta$ too. From the embedding

(ex3) $$0 = \Delta(C_\lambda) \longrightarrow \Delta(M_\lambda) \xrightarrow{\Delta(p_\lambda)} \Delta(M)$$

we get, by hypothesis, that $\operatorname{Coker}(\Delta^2(p_\lambda)) \in \operatorname{Ker}\Delta$. From $\Delta^2(p_\lambda) = \Delta^2(\beta_\lambda) \circ \Delta^2(\alpha_\lambda)$ we see that $C'_\lambda = \operatorname{Coker}(\Delta^2(\beta_\lambda)) \in \operatorname{Ker}\Delta$ too. Therefore, applying the functor Δ to the previous diagram we obtain the commutative exact diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \Delta(M_\lambda) & \xrightarrow{\Delta(\beta_\lambda)} & \Delta(I_\lambda) \\ & & \downarrow\Delta(\delta_{M_\lambda}) & & \downarrow\cong \\ 0 & \longrightarrow & \Delta^3(M_\lambda) & \xrightarrow{\Delta^3(\beta_\lambda)} & \Delta^3(I_\lambda) \end{array}$$

which shows that $\Delta(\delta_{M_\lambda})$ is monic. Since $\Delta(\delta_{M_\lambda}) \circ \delta_{\Delta(M_\lambda)} = \mathrm{id}_{\Delta(M_\lambda)}$, we conclude that $\delta_{\Delta(M_\lambda)}$ is an isomorphism, so that M_λ is Δ-reflexive, because of Lemma 1.1(e).

Finally, from (ex3), we derive the embedding $\varinjlim \Delta(p_\lambda) : \varinjlim \Delta(M_\lambda) \hookrightarrow \Delta(M)$, so that $\operatorname{Coker}(\Delta(\varinjlim \Delta(p_\lambda))) \in \operatorname{Ker}\Delta$ by assumption. Therefore we get the commutative exact diagram

$$\begin{array}{ccccccc} \Delta^2(M) & \xrightarrow{\Delta(\varinjlim \Delta(p_\lambda))} & \Delta(\varinjlim \Delta(M_\lambda)) \cong \varprojlim \Delta^2(M_\lambda) & \longrightarrow & \operatorname{Coker}(\Delta(\varinjlim \Delta(p_\lambda))) & \longrightarrow & 0 \\ \cong\uparrow\delta_M & & \cong\uparrow\varprojlim \delta_{M_\lambda} & & & & \\ M & \xrightarrow{\varprojlim p_\lambda} & \varprojlim M_\lambda & \longrightarrow & \operatorname{Coker}(\varprojlim p_\lambda) & \longrightarrow & 0 \end{array}$$

which shows that $\operatorname{Coker}(\varprojlim p_\lambda) \cong \operatorname{Coker}(\Delta(\varinjlim \Delta(p_\lambda))) \in \operatorname{Ker}\Delta$. □

The next result points out some good properties of U-tl.l.c. modules.

Corollary 1.5. *Let U_R be a cotilting module.*

(a) *If* $M \to M' \to T \to 0$ *is exact in* Mod-R, *and* M *is* U*-tl.l.c.*, $M' \in \operatorname{Ker}\Gamma$ *and* $T \in \operatorname{Ker}\Delta$, *then* M' *is* U*-tl.l.c. too.*

(b) *If* $M \in$ Mod-R *is a factor of any* U*-tl.l.c. module, then* δ_M *is surjective and* $M/\operatorname{Rej}_U(M)$ *is* U*-tl.l.c. too.*

Proof. (a) is an immediate consequence of Proposition 1.3. In order to prove (b), let us consider an epimorphism $L \xrightarrow{\varphi} M \to 0$, with L U-tl.l.c. From Theorem 1.4 we get that L is Δ-reflexive and, considering the embedding $0 \longrightarrow \Delta(M) \xrightarrow{\Delta(\varphi)} \Delta(L)$, also that $\operatorname{Coker}(\Delta^2(\varphi)) \in \operatorname{Ker}\Delta$. On the other hand, from the commutative exact diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\varphi} & M & \longrightarrow & 0 & & \\ \cong\big\downarrow \delta_L & & \big\downarrow \delta_M & & & & \\ \Delta^2(L) & \xrightarrow{\Delta^2(\varphi)} & \Delta^2(M) & \longrightarrow & \operatorname{Coker}(\Delta^2(\varphi)) & \longrightarrow & 0 \end{array}$$

we see that $\operatorname{Coker}(\Delta^2(\varphi)) \cong \operatorname{Coker}(\delta_M) \in \operatorname{Ker}\Gamma$, because of Lemma 1.1(d). Hence $\operatorname{Coker}(\Delta^2(\varphi)) \cong \operatorname{Coker}(\delta_M) = 0$, so that δ_M is surjective and $M/\operatorname{Rej}_U(M) \cong \Delta^2(M)$ is U-tl.l.c. because of (a). □

Proposition 1.6. *Let* U_R *be a cotilting module and let* $S = \operatorname{End}(U_R)$. *Then* U_R *is* U*-tl.l.c. if and only if* $\Delta\Gamma(S/I) = 0$ *for every left ideal* I *of* S.

Proof. The module U_R is Δ-reflexive because of Lemma 1.1(a). Therefore, by Theorem 1.4, U_R is U-tl.l.c. if and only if for any exact sequence of the form $0 \longrightarrow I \xrightarrow{i} \Delta(U_R) \cong {}_SS \longrightarrow S/I \longrightarrow 0$ it happens that $\operatorname{Coker}(\Delta(i)) \in \operatorname{Ker}\Delta$. Finally, from the previous sequence we get the exact sequence $0 \longrightarrow \Delta(S/I) \longrightarrow \Delta(S) \xrightarrow{\Delta(i)} \Delta(I) \longrightarrow \Gamma(S/I) \longrightarrow 0$, which shows that $\operatorname{Coker}(\Delta(i)) \cong \Gamma(S/I)$. □

We switch now to the case of a cotilting bimodule.

Corollary 1.7. *Let* ${}_SU_R$ *be a cotilting bimodule and let* ${}_SS$ (R_R, *respectively*) *be noetherian. Then* U_R (${}_SU$, *respectively*) *is* U*-tl.l.c.*

Proof. By assumption, for any left ideal I of S the cyclic module S/I is finitely presented, and so it belongs to the class $\mathcal{C}$, as proved in [**Cp**, Proposition 5 d)]. Moreover, from [**Cp**, Theorem 6 a)], we get $\Gamma(\mathcal{C}) \subseteq \operatorname{Ker}\Delta$, so that $\Delta\Gamma(S/I) = 0$. We finish the proof applying Proposition 1.6. □

We are now ready to answer the question posed in [**Cp**, Remark 11].

Theorem 1.8. *Let* ${}_SU_R$ *be a cotilting bimodule. The following conditions are equivalent for any module* $M_R \in \operatorname{Ker}\Gamma$:

(1) M_R *is* U*-tl.l.c.*

(2) M_R *is* Δ*-reflexive and for all* ${}_SL \leq \Delta(M)$ *we have* $\Delta\Gamma(\Delta(M)/L) = 0$.

(3) *Any S-submodule of $\Delta(M)$ is Δ-reflexive.*

Proof. (1) $\Leftrightarrow$ (2). Since $\Gamma\Delta(M) = 0$, for any embedding $i : {}_SL \hookrightarrow \Delta(M)$ we get $\mathrm{Coker}(\Delta(i)) = \Gamma(\Delta(M)/L)$. Now apply Theorem 1.4.

(2) $\Rightarrow$ (3). For any ${}_SL \leq \Delta(M)$ we get the exact sequence $\Delta^2(M) \to \Delta(L) \to \Gamma(\Delta(M)/L) \to 0$ where, by assumption, since (1) $\Leftrightarrow$ (2), $\Delta^2(M)$ is U-tl.l.c. and $\Gamma(\Delta(M)/L) \in \mathrm{Ker}\,\Delta$. So Corollary 1.5(a) applies, giving $\Delta(L)$ reflexive. Since L is clearly in $\mathrm{Ker}\,\Gamma$, from Lemma 1.1(e) we obtain that L is Δ-reflexive.

(3) $\Rightarrow$ (2). By assumption $\Delta(M)$ is Δ-reflexive, and so M is Δ-reflexive too, because of Lemma 1.1(e). Next, for any ${}_SL \leq \Delta(M)$ we get the canonical exact sequence $0 \to L \to \Delta(M) \to \Delta(M)/L \to 0$, with both L and $\Delta(M)$ Δ-reflexive. Then $\Delta\Gamma(\Delta(M)/L) = 0$ because of [**Cp**, Lemma 4 d)]. □

Corollary 1.9. *Let ${}_SU_R$ be a cotilting bimodule. The following conditions are equivalent:*

(1) *every Δ-reflexive right R-module is U-tl.l.c.,*
(2) *the class of all the Δ-reflexive left S-modules is closed under submodules.*

Proof. Apply (1) $\Leftrightarrow$ (3) of Theorem 1.8. □

We now have the following connection between cotilting bimodules and those bimodules ${}_SU_R$ that induce Colby's *generalized Morita dualities* [**Cb2**] in the sense that the classes of Δ-reflexive modules are closed under extensions and submodules, and contain ${}_SS$ and R_R, respectively.

Corollary 1.10. *Let ${}_SU_R$ be a cotilting bimodule. Then ${}_SU_R$ induces a generalized Morita duality if and only if the class of the Δ-reflexive modules coincides with the class of the U-torsionless linearly compact modules, both in S-*Mod *and in* Mod-R.

Proof. For any cotilting bimodule ${}_SU_R$, the regular modules ${}_SS$ and R_R are Δ-reflexive, because of Lemma 1.1(a), and, similarly, any extension of two Δ-reflexive modules is Δ-reflexive too, because of [**Cp**, Proposition 5 a)]. Now apply Corollary 1.9. □

2. Morita duals of tilting bimodules.

Originally cotilting bimodules arose as k-duals of tilting bimodules. Namely, consider two finite dimensional k-algebras R and S, and denote by $D(-)$ the vector space k-duality. In this context a cotilting bimodule is just the dual $D({}_RV_S)$ of a finite dimensional tilting bimodule ${}_RV_S$, so cotilting theory for finite dimensional algebras is just a perfect dual of tilting theory. Moreover, since $D(R_R)$ is an injective cogenerator in R-Mod and adjunction induces a natural isomorphism of left S-modules $D(V_S) \cong \mathrm{Hom}_R({}_RV_S, D(R_R))$, it follows that $D(V_S)$ is a cotilting left S-module in our sense (see the proof of 2.4 below). Arguing in the same way for $D({}_RV)$, we obtain that ${}_SU_R = D({}_RV_S)$ is a cotilting bimodule in our sense.

Nevertheless, if we do not restrict our attention to finitely generated modules, cotilting theory is as far from tilting theory as Morita duality is from Morita equivalence. Even in this classical case, the theory seems to be quite hidden: We do not know, for instance, if the equivalent conditions of Corollary 1.9 hold true.

Obviously, a natural way to generalize this construction is to consider Morita duals of tilting bimodules. In this pursuit we are fortunate that standard methods yield the following extensions of the adjointness of the functors $\mathrm{Hom}_A(V,-)$ and $V\otimes_S-$ and of the contravariant functors $\mathrm{Hom}_A(-,W)$ and $\mathrm{Hom}_R(-,W)$ induced by bimodules ${}_AV_S$ and ${}_AW_R$ (see [**AF**, §20]):

Lemma 2.1. *Let ${}_SN$ and M_R be modules and ${}_AV_S$ and ${}_AW_R$ be bimodules.*

(a) *If ${}_AW$ is injective, then there are natural isomorphisms*

$$\mathrm{Hom}_A(\mathrm{Tor}_n^S(V,N),W) \cong \mathrm{Ext}_S^n(N,\mathrm{Hom}_A(V,W))$$

for $n = 1,2,\dots$.

(b) *If ${}_AW$ and W_R are both injective, then there are natural isomorphisms*

$$\mathrm{Ext}_A^n(V,\mathrm{Hom}_R(M,W)) \cong \mathrm{Ext}_R^n(M,\mathrm{Hom}_A(V,W))$$

for $n = 1,2,\dots$.

Proof. (a) This is [**CE**, page 120, Proposition 5.1].

(b) Being unable to find a reference for this part, we shall sketch a proof. Let

$$\cdots \to P_2 \to P_1 \to P_0 \to {}_AV \to 0$$

be a projective resolution of ${}_AV$, and note that the conditions on W yield an injective resolution

$$0 \to \mathrm{Hom}_A(V,W) \to \mathrm{Hom}_A(P_0,W) \to \mathrm{Hom}_A(P_1,W) \to \mathrm{Hom}_A(P_2,W) \to \cdots$$

of $\mathrm{Hom}_A(V,W)_R$. Then (see [**R**, Chapter 7]) one obtains the desired isomorphisms from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_A(P_0,\mathrm{Hom}_R(M,W)) & \longrightarrow & \mathrm{Hom}_A(P_1,\mathrm{Hom}_R(M,W)) & \longrightarrow & \cdots \\
\cong\downarrow & & \cong\downarrow & & \\
\mathrm{Hom}_R(M,\mathrm{Hom}_A(P_0,W)) & \longrightarrow & \mathrm{Hom}_R(M,\mathrm{Hom}_A(P_1,W)) & \longrightarrow & \cdots
\end{array}$$

□

For the remainder of this section A and R are supposed to be Morita dual rings via faithfully balanced bimodule ${}_AW_R$ that is a (linearly compact) injective cogenerator on both sides. Moreover we assume that ${}_AV$ is a (linearly compact) tilting module with endomorphism ring $S = \mathrm{End}({}_AV)$, and we let

$$ {}_SU_R = \mathrm{Hom}_A(V, W). $$

We futher assume that ${}_AV$ is not projective (equivalently, not a (pro)generator), so that the bimodule ${}_SU_R$ is not just another Morita bimodule.

For convenience sake, given any bimodule ${}_AM_B$ we shall denote by Δ_M the two contravariant functors $\mathrm{Hom}_?(-, {}_AM_B)$ and by Γ_M their first derived functors $\mathrm{Ext}^1_?(-, {}_AM_B)$, where $? = A$ or B. Also we put $H_M = \mathrm{Hom}_A(M, -)$, $T_M = M \otimes_B -$, $H'_M = \mathrm{Ext}^1_A(M, -)$ and $T'_M = \mathrm{Tor}^B_1(M, -)$.

Thus by adjointness we have

$$ \Delta_U \cong H_V \Delta_W : \text{Mod-}R \to S\text{-Mod} $$

and

$$ \Delta_U \cong \Delta_W T_V : S\text{-Mod} \to \text{Mod-}R $$

and by Lemma 2.1

$$ \Gamma_U \cong H'_V \Delta_W : \text{Mod-}R \to S\text{-Mod} $$

and

$$ \Gamma_U \cong \Delta_W T'_V : S\text{-Mod} \to \text{Mod-}R. $$

Also there are natural transformations

$$ \delta : \mathrm{id}_{\text{Mod-}R} \to \Delta_U \Delta_U \qquad \text{and} \qquad \delta : \mathrm{id}_{S\text{-Mod}} \to \Delta_U \Delta_U $$

and

$$ \gamma : \Gamma_U \Gamma_U \to \mathrm{id}_{\text{Mod-}R} \qquad \text{and} \qquad \gamma : \Gamma_U \Gamma_U \to \mathrm{id}_{S\text{-Mod}} $$

with the δ's via the usual evaluation maps, and the γ's derived from the natural transformations of the Tilting Theorem [**CbFu**, 1.4] and the Δ_W's. Thus we obtain

Duality 2.2. *There are dualities*

$$ \Delta_U : \mathcal{Y}_R \rightleftarrows {}_S\mathcal{Y} : \Delta_U $$
$$ \Gamma_U : \mathcal{X}_R \rightleftarrows {}_S\mathcal{X} : \Gamma_U $$

where the $\mathcal{Y}$'s and $\mathcal{X}$'s are the full subcategories on whose objects the δ's and the γ's, respectively, are isomorphisms.

Let us denote by ${}_A\mathcal{C}$ and $\mathcal{C}_R$ the classes of all linearly compact left A- and right R-modules, respectively. Moreover, $({}_A\mathcal{T}, {}_A\mathcal{F})$ denotes the torsion theory generated by the tilting module ${}_AV$, and $({}_S\mathcal{T}, {}_S\mathcal{F})$ the torsion theory cogenerated by the cotilting module ${}_SU = \mathrm{Hom}_A({}_AV_S, {}_AW)$ (see the proof of 2.4 below).

By assumption, the bimodule ${}_AW_R$ induces a duality of the form

$$ \Delta_W : \mathcal{C}_R \rightleftarrows {}_A\mathcal{C} : \Delta_W $$

and the tilting bimodule ${}_AV_S$ induces the two equivalences

$$H_V : {}_A\mathcal{T} \rightleftarrows {}_S\mathcal{F} : T_V \quad \text{and} \quad H'_V : {}_A\mathcal{F} \rightleftarrows {}_S\mathcal{T} : T'_V.$$

Therefore, letting

$${}_A\mathcal{X} = {}_A\mathcal{C} \cap {}_A\mathcal{T} \quad \text{and} \quad {}_A\mathcal{Y} = {}_A\mathcal{C} \cap {}_A\mathcal{F}$$

we see that

$$\begin{aligned} \mathcal{X}_R &\supseteq \Delta_W({}_A\mathcal{Y}), & \mathcal{Y}_R &\supseteq \Delta_W({}_A\mathcal{X}), \\ {}_S\mathcal{X} &\supseteq H'_V({}_A\mathcal{Y}), & {}_S\mathcal{Y} &\supseteq H_V({}_A\mathcal{X}). \end{aligned}$$

Since ${}_AV$ is a tilting module, ${}_AV$ and ${}_AW$ belong to ${}_A\mathcal{X}$. Thus

$$\begin{aligned} U_R &= \Delta_W({}_AV) \in \mathcal{Y}_R, & R_R &= \Delta_W({}_AW) \in \mathcal{Y}_R, \\ {}_SU &= H_V({}_AW) \in {}_S\mathcal{Y}, & {}_SS &= H_V({}_AV) \in {}_S\mathcal{Y}, \end{aligned}$$

so, in particular, we have:

2.3. Balance. The bimodule ${}_SU_R$ is faithfully balanced.

Since ${}_AV$ is a $*$-module, ${}_SU = \mathrm{Hom}_A({}_AV_S, {}_AW)$ and ${}_AW$ is an injective cogenerator, as in [**CpToTr**, 2.3 3)], we obtain:

2.4. Properties of ${}_SU$. ${}_SU$ is a cotilting module.

One would hope that U_R is one too. Perhaps not in general, but we do have the following:

2.5. Properties of U_R.

(a) There is an exact sequence $0 \to U_R \to W' \to W'' \to 0$, where $W', W'' \in \mathrm{add}(W_R)$. In particular U_R is finitely cogenerated and $\mathrm{inj\,dim}(U_R) \le 1$.
(b) There is an exact sequence $0 \to U' \to U'' \to W_R \to 0$, where $U', U'' \in \mathrm{add}(U_R)$. In particular $\mathrm{Ker}\,\Delta_U \cap \mathrm{Ker}\,\Gamma_U = 0$.
(c) $\Delta_W({}_A\mathcal{X}) \subseteq \mathrm{Ker}\,\Gamma_U$. In particular $\mathrm{Ext}^1_R(M, U) = 0$ for all $M_R \hookrightarrow U^n_R$ (n finite).

Proof. (a) Since ${}_AV$ is a tilting module, there is an exact sequence of the form $0 \to A' \to A'' \to {}_AV \to 0$, with $A', A'' \in \mathrm{add}({}_AA)$. Now apply Δ_W.

(b) Similarly to the previous case, applying Δ_W to the exact sequence $0 \to {}_AA \to V' \to V'' \to 0$, where $V', V'' \in \mathrm{add}({}_AV)$, we obtain the required exact sequence. Finally, applying $\mathrm{Hom}_R(M, -)$ to that, we see that $\mathrm{Hom}_R(M, U) = 0 = \mathrm{Ext}^1_R(M, U)$ implies $\mathrm{Hom}_R(M, W) = 0$, and so $M = 0$.

(c) For any $M \in \Delta_W({}_A\mathcal{X})$ we clearly have $\Delta_W(M) \in {}_A\mathcal{X} \subseteq {}_A\mathcal{T} = \mathrm{Ker}\,\mathrm{Ext}^1_A(V, -)$. Therefore, we see by Lemma 2.1(b) that $\mathrm{Ext}^1_R(M, U) \cong \mathrm{Ext}^1_A(V, \Delta_W(M)) = 0$. □

From 2.3, 2.4 and 2.5 we immediately have:

Proposition 2.6. *The bimodule ${}_SU_R$ is a cotilting bimodule if and only if $\mathrm{Ext}^1_R(U^\alpha, U) = 0$ for any cardinal α.*

3. Cotilting bimodules over noetherian serial rings.

In [**CbFu**] Colby and Fuller determined all the tilting bimodules ${}_RV_S$ over a noetherian serial ring R. In this concluding section we shall see that if R has self-duality induced by ${}_RW_R$ then ${}_SU_R = \mathrm{Hom}_R(V, W)$ is a cotilting bimodule. Thus we obtain a large collection of cotilting bimodules (that are not even finitely generated) in addition to the classical ones over finite dimensional algebras.

According to [**Wa**, Theorem 5.11], a noetherian serial ring is a finite direct sum of indecomposable artinian serial rings and prime noetherian serial rings. Warfield proved that every finitely generated module and every injective module over such a ring is a direct sum of uniserial modules. The structure of artinian serial rings is well known (see [**AF**, §32]).

Let R be a prime noetherian serial ring with right Kupisch series

$$e_1R, \ldots, e_nR$$

so that, setting $J = \mathrm{J}(R)$

$$e_1J \cong e_2R, \ldots, e_{n-1}J \cong e_nR \quad \text{and} \quad e_nJ \cong e_1R$$

(see [**CbFu**, §3]). According to Warfield [**Wa**]

$$e_iR > e_iJ > e_iJ^2 > \ldots \qquad \text{and} \qquad Re_i > Je_i > J^2e_i > \ldots$$

are complete lists of the submodules of e_iR and Re_i, for $i = 1, \ldots, n$. Thus, setting $S_i = e_iR/e_iJ$, the composition factors of e_iR are, from the top down,

$$S_i, S_{i+1}, \ldots, S_n, S_1, S_2, \ldots, S_n, \ldots .$$

On the other hand, as Warfield showed, every finitely generated indecomposable R-module is uniserial. It follows that the indecomposable injective R-modules are also uniserial. There are just $n+1$ indecomposable injective right R-modules

$$E_1 = \mathrm{E}(S_1), \ldots, E_n = \mathrm{E}(S_n) \quad \text{and} \quad E_0$$

with $\mathrm{Soc}(E_0) = 0$, each E_i is artinian, and for any $i = 1, \ldots, n$ the submodules of E_i are

$$0 < \mathrm{Soc}(E_i) < \mathrm{Soc}^2(E_i) < \ldots$$

where $\mathrm{Soc}^k(M) = \mathrm{Ann}_M(J^k)$. And the composition factors of E_i, from the bottom up, are

$$S_i, S_{i-1}, \ldots, S_1, S_n, S_{n-1}, \ldots, S_1, \ldots$$

while the composition factor of E_0 are

$$\ldots S_n, \ldots, S_i, S_{i-1}, \ldots S_1, S_n, S_{n-1}, \ldots, S_i, S_{i-1}, \ldots S_1, \ldots .$$

In particular any proper factor of an indecomposable injective module is the injective envelope of its socle, and every proper submodule of E_0 is isomorphic to an indecomposable projective module.

Lemma 3.1. *Let R be a noetherian serial ring. If X_R is an indecomposable R-module of finite length, then for any cardinal α there is a cardinal γ such that $X^\alpha \cong X^{(\gamma)}$.*

Proof. Let $Q = R/\operatorname{Ann}_R(X)$. Then X_Q is a faithful indecomposable module over the artinian QF-3 ring Q. Thus X is the unique indecomposable injective projective right Q-module (see [**AF**, §31 and §32]). But X^α is both injective and, since Q is artinian, projective. Moreover X_Q^α is a direct sum of indecomposable modules, since Q is artinian. □

Lemma 3.2. *Let R be a prime noetherian serial ring with indecomposable injective modules $E_1, \dots, E_n$ and E_0 as above. Then for any cardinal α there are cardinals β, γ such that $E_i^\alpha \cong E_i^{(\beta)} \oplus E_0^{(\gamma)}$.*

Proof. Since R is semiperfect and J is finitely generated, we see that

$$\operatorname{Soc}(E_i^\alpha) = \operatorname{Ann}_{E_i^\alpha}(J) = \operatorname{Ann}_{E_i}(J)^\alpha = \operatorname{Soc}(E_i)^\alpha.$$

But if $i \neq j$ then $\operatorname{Soc}(E_i)e_j = 0$. Thus $\operatorname{Soc}(E_i^\alpha) = S_i^{(\beta)}$. So we see that $E_i^\alpha \cong E_i^{(\beta)} \oplus E$, with $\operatorname{Soc}(E) = 0$. But the only indecomposable injective with zero socle is E_0, so $E \cong E_0^{(\gamma)}$. □

Proposition 3.3. *If U is a finitely cogenerated module over a noetherian serial ring R such that $\operatorname{Ext}_R^1(U, U) = 0$, then $\operatorname{Ext}_R^1(U^\alpha, U) = 0$ for any cardinal α.*

Proof. Since U is finitely cogenerated, we have

$$U = E_{i_1} \oplus \cdots \oplus E_{i_k} \oplus X_1 \oplus \cdots \oplus X_l$$

where $E_{i_j} = \mathrm{E}(S_{i_j})$, $j = 1, \dots, k$, and X_i, $i = 1, \dots, l$, are uniserial modules of finite length. Thus by Lemmas 3.1 and 3.2 we have

$$U^\alpha = E_{i_1}^{(\beta_1)} \oplus \cdots \oplus E_{i_k}^{(\beta_k)} \oplus E_0^{(\gamma)} \oplus X_1^{(\delta_1)} \oplus \cdots \oplus X_l^{(\delta_l)}.$$

Now, since $\operatorname{Ext}_R^1(-, X_i)$ converts direct sums to direct products, we need only check that $\operatorname{Ext}_R^1(E_0, X_i) = 0$ for all $i = 1, \dots, l$. To this end, consider the minimal injective resolution

$$0 \longrightarrow X_i \longrightarrow E_i \xrightarrow{\nu} E_j \longrightarrow 0.$$

Here we need to show that

$$\operatorname{Hom}_R(E_0, E_i) \xrightarrow{\operatorname{Hom}_R(E_0,\nu)} \operatorname{Hom}_R(E_0, E_j) \longrightarrow 0$$

is exact. So let $0 \neq \beta \in \operatorname{Hom}_R(E_0, E_j)$ with $K = \operatorname{Ker}(\beta)$. Then there is $m \in \mathbb{N}$ such that $E_0/KJ^m \cong E_{i_1}$. But $\operatorname{Ext}_R^1(E_{i_1}, X_i) = 0$, being a direct summand of $\operatorname{Ext}_R^1(U, U)$, so

$$\operatorname{Hom}_R(E_0/KJ^m, E_i) \xrightarrow{\operatorname{Hom}_R(E_0/KJ^m,\nu)} \operatorname{Hom}_R(E_0/KJ^m, E_j) \longrightarrow 0$$

is exact. Thus, setting $\eta : E_0 \to E_0/KJ^m$, we have a commutative diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\beta} & E_j \\ \eta\big\downarrow & & \big\| \\ E_0/KJ^m & \xrightarrow{\overline{\beta}} & E_j \\ \gamma\big\downarrow & & \big\| \\ E_i & \xrightarrow{\nu} & E_j \end{array}$$

which shows that $\mathrm{Hom}_R(E_0, \nu)(\gamma \circ \eta) = \nu \circ \gamma \circ \eta = \overline{\beta} \circ \eta = \beta$. □

Theorem 3.4. *Let R be a noetherian serial ring with self-duality induced by a bimodule ${}_RW_R$. If ${}_RV$ is a tilting module and $S = \mathrm{End}({}_RV)$, then ${}_SU_R = \mathrm{Hom}_R(V, W)$ is a cotilting bimodule.*

Proof. According to Proposition 2.6, it only remains to observe that $\mathrm{Ext}^1_R(U^\alpha, U) = 0$ for any α. And this is true thanks to Proposition 3.3, since $\mathrm{Ext}^1_R(U, U) = 0$ and U_R is finitely cogenerated because of 2.5(c) and (a). □

Let us pause to point out a couple of facts about self-duality for noetherian serial rings.

Proposition 3.5. *If R is a left linearly compact indecomposable prime noetherian serial ring, then R has a self-duality.*

Proof. Assume, as we may, that R is basic. Let $E = E_1 \oplus \cdots \oplus E_n$ be the minimal cogenerator. Then E is artinian, hence linearly compact. Thus, setting $S = \mathrm{End}({}_RE)$, S_S is linearly compact and the bimodule ${}_RE_S$ defines a Morita duality. Now it is easy to see that $A_k = \mathrm{Ann}_E(J^k)$ is the minimal cogenerator over R/J^k, and that the bimodule ${}_{R/J^k}A_{kS/\,\mathrm{Ann}_S(A_k)}$ defines a Morita duality. But R/J^k is a basic QF-ring (see [**AF**, §32.6]) and hence ${}_{R/J^k}A_k \cong {}_{R/J^k}R/J^k$. But then

$$S/\,\mathrm{Ann}_S(A_k) \cong \mathrm{End}({}_{R/J^k}A_k) \cong R/J^k$$

as rings. Now both $\{J^k \mid k \geq 1\}$ and $\{\mathrm{Ann}_S(A_k) \mid k \geq 1\}$ are downward directed sets of ideals with $\cap_k J^k = 0$ [**Wa**, Theorem 5.11] and so $\cap_k \mathrm{Ann}_S(A_k) = 0$. Therefore, since ${}_RR$ and S_S are both linearly compact, we have

$$R \cong \varprojlim R/J^k \cong \varprojlim S/\,\mathrm{Ann}_S(A_k) \cong S.$$

□

As Warfield [**Wa**] showed, a prime noetherian serial ring R is isomorphic to the $n \times n$ $(D : M)$-upper triangular matrix ring $\mathrm{UTM}_n(D : M)$, consisting of those matrices over a local noetherian serial ring D whose entries below the main diagonal all come from the unique maximal ideal M of D. It follows from Proposition 3.5 and [**X**, Theorem 4.3, Lemma 4.9 and Proposition 3.3] that R has self-duality if and only if D is linearly compact. According to [**Wb**] and [**DiMl**], any artinian serial ring has self-duality. Thus from Proposition 3.5 and [**Mu**] (see again [**X**, Theorem 4.3]) we have:

Proposition 3.6. *A noetherian serial ring has a self-duality if and only if it is left* (*equivalently right*) *linearly compact.*

Finally, we note that any tilting module ${}_RV$ over a hereditary noetherian ring (which was shown to be a finitistic cotilting module in [**CbFu**]) satisfies at least two of the three conditions needed to be a cotilting module in our sense whenever R has selfduality.

Proposition 3.7. *Let R be a hereditary linearly compact noetherian serial ring and let ${}_RV$ be a tilting module. Then ${}_RV$ is a finitistic cotilting module with* $\operatorname{Ext}^1_R(V^\alpha, V) = 0$ *for all cardinal numbers α.*

Proof. According to [**CbFu**, Proposition 2.1], ${}_RV$ is a finitistic cotilting module, and since it is finitely generated

$${}_RV = P \oplus T,$$

with P finitely generated projective and $T = T_1 \oplus \cdots \oplus T_l$, with all the T_i's uniserial modules of finite length. Since $\operatorname{Ext}^1_R(V, V) = 0$, and since, by Lemma 3.1,

$$V^\alpha = P^\alpha \oplus T_1^{(\delta_1)} \oplus \cdots \oplus T_l^{(\delta_l)},$$

it only remains to show that $\operatorname{Ext}^1_R(P^\alpha, P) = 0$ and $\operatorname{Ext}^1_R(P^\alpha, T_i) = 0$ for $i = 1, \ldots, l$.

Let ${}_RW_R$ induce a self-duality and observe that the canonical right R-isomorphism

$$R \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(R_R, W), W) \longrightarrow \operatorname{Hom}_R({}_RW_R, {}_RW_R)$$

is also a left R-map. Now P^α is flat by Chase's Theorem [**AF**, 19.20], since R is noetherian, and so by Lemma 2.1(a)

$$\begin{aligned}\operatorname{Ext}^1_R(P^\alpha, {}_RR) &\cong \operatorname{Ext}^1_R(P^\alpha, \operatorname{Hom}_R({}_RW_R, {}_RW)) \\ &\cong \operatorname{Hom}_R(\operatorname{Tor}^R_1(W, P^\alpha), W) = 0.\end{aligned}$$

Thus, assuming, as we may, that P is a direct summand of ${}_RR$, we do have

$$\operatorname{Ext}^1_R(P^\alpha, P) = 0.$$

On the other hand, if T_i has length m, and $A = R/J^m$, then ${}_AT_i$ is injective [**AF**, Theorem 32.6] and

$${}_RT_i \cong \operatorname{Hom}_A({}_AA_R, {}_AT_i),$$

so that

$$\begin{aligned}\mathrm{Ext}^1_R(P^\alpha, {}_RT_i) &\cong \mathrm{Ext}^1_R(P^\alpha, \mathrm{Hom}_A({}_AA_R, {}_AT_i)) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(A_R, P^\alpha), {}_AT_i) = 0.\end{aligned}$$

□

Remark 3.8. (1) Krause and Saorín [**KrSa**, Proposition 3.8] have recently shown that if M_R is a finitely generated module, then every M^α is isomorphic to a direct summand of some $M^{(\delta)}$ if and only if $S = \mathrm{End}(M_R)$ is left coherent and right perfect and ${}_SM$ is finitely presented. Thus we see that if R is right artinian and (hence) S is left artinian in a cotilting triple (S, U, R) in the sense of [**Cb1**, §2], then $\mathrm{Ext}^1_R(U^\alpha, U) = 0 = \mathrm{Ext}^1_S(U^\alpha, U)$ for any cardinal α.

(2) Over rings of finite representation type, cotilting triples yield more examples of cotilting modules. Indeed, in a cotiling triple (S, U, R), if it happens that R is a ring of finite representation type (so that every R-module is a direct sum of finitely generated modules), then since U_R is a finitistic cotilting module [**Cb1**, Theorem 3.3], we also have $\mathrm{Ker}\,\mathrm{Hom}_R(-, U_R) \cap \mathrm{Ker}\,\mathrm{Ext}^1_R(-, U_R) = 0$, so that U_R is a cotilting module in the present sense. If in addition S has finite representation type (in particular, if R is hereditary [**CbFu**, Proposition 2.2]), then ${}_SU_R$ is a cotilting bimodule.

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Università di Padova
Via Belzoni 7, 35131 Padova
Italy
E-mail address: colpi@math.unipd.it

University of Iowa
Iowa City, IA 52242
E-mail address: kfuller@math.uiowa.edu