RESTRICTIONS OF $\Omega_m(q)$-MODULES TO ALTERNATING GROUPS

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We consider the restriction of an irreducible $F\Omega_m(q)$-module $M$ to a subgroup $H$ where $F^*(H) \cong A_n$ and where $F$ is algebraically closed with $(\text{char}(F), q) \neq 1$. Given certain restrictions on the highest weight of $M$, we show that if $m > n^6$, then $M \downarrow H$ is reducible.

1. Introduction.

In the study of the maximal subgroups of classical groups, the following question arises: Given an absolutely irreducible module $M$ for $K$ and a subgroup $H$, when does $M \downarrow H$ remain absolutely irreducible? In this article $K \cong \Omega_m(q)$ is the commutator subgroup of an $m$-dimensional orthogonal group over $F_q$, and $F^*(H) \cong A_n$ is the alternating group of degree $n$. We treat the case that the field of definition of $M$ has characteristic dividing $q$.

Let $F$ be an algebraically closed field containing $F_q$, the field with $q$ elements, such that char($F$) > 3. Then $K < K$ where $K \cong \Omega_m(F)$ and we may assume that $M$ is a $FK$-module. By [6, Theorem 43], every absolutely irreducible $FK$-module is the restriction of an irreducible $FK$-module of the same weight. So we may assume that $M = M(\lambda)$ is an irreducible $FK$-module with highest weight $\lambda$. Let $\ell = \lfloor m/2 \rfloor$ be the Lie rank of $K$ and let $\{\lambda_i\}$ be the fundamental dominant weights of $K$. The labeling of these weights corresponds to the labeling of the Dynkin diagrams for $K$ as given in [3].

Hypothesis 1.1. Assume the following are true:

1. If $m$ is even, then $\lambda = \left( \sum_{i=1}^{\ell-2} a_i \lambda_i \right) + a_{\ell-1}(\lambda_{\ell-1} + \lambda_\ell); a_i \in \mathbb{Z}, a_i \geq 0$.

2. If $m$ is odd, then $\lambda = \left( \sum_{i=1}^{\ell-1} a_i \lambda_i \right) + 2a_\ell \lambda_\ell; a_i \in \mathbb{Z}, a_i \geq 0$.

3. If $\mu_i = \sum_{j=i}^{\ell-1} a_j$, $m$ even or if $\mu_i = \sum_{j=i}^{\ell} a_j$, $m$ odd then
   (a) $\mu_1 < p = \text{char}(F_q)$;
   (b) $1 < \sum \mu_i = k < \ell$.
Conditions (1) and (2) imply that $M$ is not a faithful module for any proper covering group of $\overline{K}$. We now state our main result:

**Theorem 1.2.** Assume that $H, K$ and $M = M(\lambda)$ are as above with $n, m \geq 10$ and $(q, 6) = 1$. Suppose further that $\lambda$ satisfies Hypothesis 1.1. If $m > n^6$, then $M|_H$ is reducible.

Our strategy is to produce a small subspace in $M$ with a large stabilizer in $H$ and then, using Frobenius reciprocity, produce an upper bound for $\dim(M)$. We produce a lower bound for $\dim(M)$ as an $FK$-module using the length of the Weyl group orbit of a subdominant weight in $M$. The result then follows by comparing these two bounds.

### 2. A construction of $\overline{W}(\lambda)$.

In this section we construct the Weyl module $\overline{W}(\lambda)$ of $\overline{K}$ with highest weight $\lambda$. Then $M$ is a homomorphic image of this module. Our construction proceeds by first constructing the Weyl module $W(\lambda)$ for a complex Lie group $G$ of the same type and rank as $K$, then we use Kostant’s $Z$-form to produce $\overline{W}(\lambda)$. For notational convenience we assume that $\{\lambda_i\}$ are the fundamental dominant weights for $G$ as well as for $K$, and accordingly, assume that $\lambda$ is a dominant weight of $G$.

Let $V$ be a complex, $m$-dimensional vector space possessing a non-degenerate orthogonal form $f(.,.)$ and let $\mathcal{B}$ be a basis for $V$ so that

$$\mathcal{B} = \begin{cases} \{e_i, f_i \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is even} \\ \{e_i, f_i, x \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is odd} \end{cases}$$

with $f(e_i, e_j) = f(f_i, f_j) = f(x, e_i) = f(x, f_i) = 0$, $f(e_i, f_j) = \delta_{i,j}$ and $f(x,x) = 2$. We then define $G = \Omega(V)$ and let $T$ be the maximal torus of $G$ with respect to $\mathcal{B}$. Set $V_e = \langle e_i \mid 1 \leq i \leq \ell \rangle$ and $V_f = \langle f_i \mid 1 \leq i \leq \ell \rangle$.

Suppose that $\lambda$ satisfies hypothesis 1.1 and $d = \max\{i \mid \mu_i \neq 0\}$ so that $\mu = (\mu_1, \ldots, \mu_d)$ is a proper partition of $k$. Let $T$ be the tableau of shape $\mu$ with entries $t_{i,j} = j + \sum_{s<i} \mu_s$. Define the following subgroups of the symmetric group $\mathcal{S}_k$:

$$\mathcal{R}_\mu = \{\sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same row as } t_{i,j} \text{ for all } i, j\}$$

$$\mathcal{C}_\mu = \{\sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same column as } t_{i,j} \text{ for all } i, j\}$$

and elements of $CS_k$:

$$r_\mu = \sum_{\sigma \in \mathcal{R}_\mu} \sigma \quad \text{and} \quad c_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \text{sgn}(\sigma)\sigma$$
Define \( \kappa_{i,j} : V^\otimes k \to V^\otimes (k-2) \) by \( \kappa_{i,j}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = f(v_{i_i}, v_{i_j})(v_{i_1} \otimes \cdots \otimes v_{i_i} \otimes \cdots \otimes v_{i_j} \otimes \cdots \otimes v_{i_k}) \) for \( 1 \leq i < j \leq k \) and set
\[
\mathcal{K} = \bigcap_{i,j} \ker(\kappa_{i,j}).
\]

\( S_k \) acts on \( V^\otimes k \) by place permutation, specifically:
\[
\sigma(v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{i_{\sigma^{-1}(i_1)}} \otimes \cdots \otimes v_{i_{\sigma^{-1}(k)}}.
\]
This action commutes with the diagonal action of \( G \) on \( V^\otimes k \).

Given \( v \in V^\otimes k \), we define one additional element \( r^v_\mu \) of the group algebra \( CS_k \) as follows: Let \( R^v_\mu = \{ \sigma \in R_\mu \mid \sigma(v) = v \} \) and let \( \{ s_i \} \) be a left transversal for \( R^v_\mu \) in \( R_\mu \). Define \( r^v_\mu = \sum_i s_i \). Notice that \( r^v_\mu(v) = |R^v_\mu| r^v_\mu(v) \).

By [2, Theorem 19.22], \( W(\lambda) = c_\mu r^\mu_\mu(V^\otimes k) \cap \mathcal{K} \) is the Weyl module for \( G \) with highest weight \( \lambda \). Since \( V \) is a complex vector space, \( c_\mu r^\mu_\mu(V^\otimes k) = \langle c_\mu r^\mu_\mu(v) \mid v \in V^\otimes k \rangle \).

Define \( V_Z = Z[\mathcal{B}] \) and let \( \overline{V} = V_Z \otimes Z \mathbb{F} \). Then \( \overline{f}(\ , \ ) = f(\ , \ ) \otimes 1_F \) is a non-degenerate orthogonal form on \( \overline{V} \). Without loss of generality, we may assume that \( \overline{K} = \Omega(\overline{V}) \). Moreover if \( \overline{e}_i = e_i \otimes 1_F, \overline{f}_i = f_i \otimes 1_F \) and \( \overline{x} = x \otimes 1_F \), then
\[
\overline{B} = \begin{cases} 
\{ \overline{e}_i, \overline{f}_i \mid 1 \leq i \leq \ell \} & \text{if } m \text{ is even} \\
\{ \overline{e}_i, \overline{f}_i, \overline{x} \mid 1 \leq i \leq \ell \} & \text{if } m \text{ is odd}
\end{cases}
\]
is a standard basis for \( \overline{V} \) with respect to \( \overline{f}(\ , \ ) \). We identify \( r_\mu \) and \( c_\mu \) with the elements \( r_\mu \otimes 1_F \) and \( c_\mu \otimes 1_F \) of \( FS_k \).

Suppose that \( L \subset \text{End}(V) \) is the adjoint module for \( G \) so that \( L \) is a complex Lie algebra of type \( D_\ell \) or \( B_\ell \). Let \( \Delta = \{ r_1, \ldots, r_\ell \} \) be the set of simple roots corresponding to the torus \( T \) and let \( \Phi \) be the root system generated by \( \Delta \). Set \( \Delta_0 = \{ r_1, \ldots, r_{\ell-1} \} \) and let \( \Phi_0 \subset \Phi \) be the subset generated by \( \Delta_0 \). Using the setup of [1, §11.2], \( \{ \epsilon_r, h_{r_i} \mid r \in \Phi, 1 \leq i \leq \ell \} \) is a Chevalley basis for \( L \) and \( \{ \epsilon_r, h_{r_i} \mid r \in \Phi_0, 1 \leq i \leq \ell - 1 \} \) is a Chevalley basis for \( L_0 \subset L \) where \( L_0 \) is a Lie algebra of type \( A_{\ell-1} \). Let \( G_0 < N_G(V_\ell \oplus V_f) \) such that \( G_0 \cong SL_\ell(C) \). Then, by [1, Theorem 11.3.2], \( G = \langle \exp(\zeta e_r) \mid r \in \Phi, \zeta \in C \rangle \) and \( G_0 = \langle \exp(\zeta e_r) \mid r \in \Phi_0, \zeta \in C \rangle \). Note that neither \( G \) nor \( G_0 \) is the adjoint group for \( L \) or \( L_0 \), respectively. We may consider \( V_e \) to be the natural module for \( G_0 \). Under this identification, \( V_f \) is the dual of \( V_e \).

Assume that \( U(L) \) is the universal enveloping algebra of \( L \). From [3, §26], Kostant's \( Z \)-form \( U_Z(L) \) is the \( Z \)-span of \( \{ \epsilon_r^m/m! \mid r \in \Phi, m \in \mathbb{Z}^+ \} \). Given any vector \( v \) of weight \( \lambda \) in \( W(\lambda) \), \( U_Z(L) v \otimes_Z \mathbb{F} = \overline{W}(\lambda) \) where \( \overline{W}(\lambda) \) is the Weyl module for \( \overline{K} \) with highest weight \( \lambda \). By the previous remarks, \( U_Z(L_0) \subset U_Z(L) \), which implies that \( U_Z(L_0) v \otimes_Z \mathbb{F} \subset \overline{W}(\lambda) \).

Define \( v_{\mu_i} = \otimes_{j=1}^{\mu_i} e_i \) and \( v_{\mu} = \otimes_{i=1}^{d} v_{\mu_i} \).
Lemma 2.1. We have:

1. \( c_\mu(v_\mu) \) is a vector of weight \( \lambda \) in \( W(\lambda) \);
2. \( \mathcal{U}_Z(L_0)c_\mu(v_\mu) = c_\mu r_\mu(V_e^{\otimes k}) \cap \mathbb{Z}[e_1, \ldots, e_d]^{\otimes k} \).

Proof. First note that \( \mathcal{R}_\mu^{v_\mu} = \mathcal{R}_\mu \) so that \( r_\mu^{v_\mu}(v_\mu) = v_\mu \) and that \( c_\mu(v_\mu) \neq 0 \). This implies that \( c_\mu(v_\mu) \in c_\mu r_\mu(V_e^{\otimes k}) \). It is clear that \( c_\mu(v_\mu) \in \mathcal{K} \) so we have \( c_\mu(v_\mu) \in W(\lambda) \). Let \( t \in T \) and write \( t = \text{diag}(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1}) \) or \( t = \text{diag}(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1}, t') \) depending on the parity of \( m \). Then

\[
t v = c_\mu(tc_\mu(v_\mu)) = c_\mu \left( \bigotimes_{i=1}^{d} \mu_i v_{\mu i} \right) = \left( \prod_{i=1}^{d} \mu_i \right) c_\mu(v_\mu).
\]

From the definition of \( \mu \) it follows that \( c_\mu(v_\mu) \) is a vector of weight \( \lambda \) and so (1) follows. With the identification of \( V_e \) with the natural module of \( G_0 \), we see by [2, Theorem 15.15] that \( c_\mu r_\mu(V_e^{\otimes k}) \) is the Weyl module for \( G_0 \) corresponding to the partition \( \mu \) of \( k \) via the Schur functor. The argument above restricted to \( t \in T \cap G_0 \) shows that \( c_\mu(v_\mu) \) is a highest weight vector in \( c_\mu r_\mu(V_e^{\otimes k}) \). In particular \( \mathcal{U}(L_0)c_\mu(v_\mu) = c_\mu r_\mu(V_e^{\otimes k}) \). Using the proof of [4, Theorem 8.3.1], we have

\[
\mathcal{U}_Z(L_0)c_\mu(v_\mu) = c_\mu r_\mu(V_e^{\otimes k}) \cap \mathbb{Z}[e_1, \ldots, e_d]^{\otimes k}
\]

which completes our proof.

Lemma 2.2. Suppose \( \tau = \tau_{i_1} \otimes \cdots \otimes \tau_{i_k} \) where \( \{\tau_i\} \) is a collection of mutually orthogonal, linearly independent singular vectors. Then:

1. If \( \text{sgn}(\sigma_\tau)\sigma_\tau(\tau) \neq \tau \) for all \( \sigma_\tau \notin C_\mu, \sigma_\tau \in \mathcal{R}_\mu \), then \( c_\mu r_\mu^\tau(\tau) \neq 0 \);
2. \( c_\mu r_\mu^\tau(\tau) \in \overline{W}(\lambda) \).

Proof. Since \( \tau \) is a summand of \( c_\mu r_\mu^\tau(\tau) \) and all other summands of \( c_\mu r_\mu^\tau(\tau) \) have the form \( \text{sgn}(\sigma_\tau)\sigma_\tau(\tau) \), part (1) must hold. There is \( g \in \overline{K} \) such that \( g(\tau_{i_j}) = \alpha_{i_j} \tau_{i_j} \) such that \( \alpha_{i_j} \neq 0 \) for all \( 1 \leq i \leq k \). If \( w = e_{i_1} \otimes \cdots \otimes e_{i_k} \), then \( r_\mu^w = r_\mu^\tau \). As

\[
c_\mu r_\mu^w(w) \in c_\mu r_\mu(V_e^{\otimes k}) \cap \mathbb{Z}[e_1, \ldots, e_d]^{\otimes k},
\]

Lemma 1 implies that \( c_\mu r_\mu^w(w) \in \mathcal{U}_Z(L)v \). Writing \( \overline{\tau} = \alpha_{i_1} \tau_{i_1} \otimes \cdots \otimes \alpha_{i_k} \tau_{i_k} \), we then have

\[
c_\mu r_\mu^w(\overline{\tau}) \in \mathcal{U}_Z(L) v \otimes_{\mathbb{Z}} F = \overline{W}(\lambda).
\]

Finally, as \( \overline{W}(\lambda) \) is a \( \mathbb{F}\overline{K} \)-module, \( g^{-1} c_\mu r_\mu^w(\overline{\tau}) = c_\mu r_\mu^w(\overline{\tau}) \in \overline{W}(\lambda) \). 

Though \( W(\lambda) \) is a irreducible module for \( G \), \( \overline{W}(\lambda) \) may not be an irreducible module for \( \overline{K} \); however, it does possess a unique maximal submodule by [6, Lemma 80] which we denote by \( \text{Rad}(\overline{W}(\lambda)) \). Moreover, \( M \cong \overline{W}(\lambda)/\text{Rad}(\overline{W}(\lambda)) \).
We now discuss the orthogonal forms on \( V^{\otimes k} \) and \( W(\lambda) \). Suppose \( v, w \in V^{\otimes k} \) where \( v = v_1 \otimes \cdots \otimes v_k \) and \( w = w_1 \otimes \cdots \otimes w_k \). We define \( f^k(\,,\,) \) by
\[
f^k(v, w) = \prod_{i=1}^k f(v_i, w_i).
\]

\( f^k(\,,\,) \) is a non-degenerate, \( G \)-invariant orthogonal form on \( V^{\otimes k} \). This form is also invariant under the action of \( S_k \). Note that
\[
f^k[c_{\mu}(v), c_{\mu}(w)] = \sum_{\sigma \in C_{\mu}} \text{sgn}(\sigma) f^k[\sigma(v), c_{\mu}(w)]
= \sum_{\sigma \in C_{\mu}} \text{sgn}(\sigma) f^k[v, \sigma^{-1}c_{\mu}(w)]
= \sum_{\sigma \in C_{\mu}} f^k[v, c_{\mu}(w)]
= |C_{\mu}| f^k[v, c_{\mu}(w)].
\]

We define \( f^k_{\mu}(\,,\,) \) on \( c_{\mu}(V^{\otimes k}) \) by
\[
f^k_{\mu}[c_{\mu}(v), c_{\mu}(w)] = f^k[v, c_{\mu}(w)].
\]

By a similar argument as above, we see that \( f^k[v, c_{\mu}(w)] = f^k[w, c_{\mu}(v)] \), so this form is symmetric. Since \( f^k(\,,\,) \) is bilinear and \( G \)-invariant, \( f^k_{\mu}(\,,\,) \) is also bilinear and \( G \)-invariant. Therefore \( f^k_{\mu}(\,,\,) \) is a \( G \)-invariant orthogonal form on \( W(\lambda) \subset c_{\mu}(V^{\otimes k}) \). As before, \( \overline{f}^k(\,,\,) = f^k(\,,\,) \otimes 1_F \) is a \( \overline{K} \)-invariant orthogonal form on \( \overline{V}^{\otimes k} \) and \( \overline{f}^k_{\mu}(\,,\,) = f^k_{\mu}(\,,\,) \otimes 1_F \) is a \( \overline{K} \)-invariant orthogonal form on \( \overline{W}(\lambda) \). This form is possibly degenerate. We denote the radical of this form as \( \overline{W}(\lambda) \). The following lemma is generally known, although we present a proof:

**Lemma 2.3.** \( \text{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp \).

**Proof.** Define \( \overline{\sigma}_{-\mu} = \bigotimes_{j=1}^{\mu_i} \overline{1}_i \) and \( \overline{\sigma}_{-\mu} = \bigotimes_{i=1}^d \overline{\sigma}_{-\mu_i} \). Noting that \( r_{\overline{\mu}}^{\overline{\sigma}_{-\mu}} = 1 \), \( c_{\mu}(v_{-\mu}) \neq 0 \in \overline{W}(\lambda) \) by Lemma 2.2. A similar argument as in the proof of Lemma 2.1 shows that \( c_{\mu}(v_{-\mu}) \) is a vector of weight \(-\lambda\). Hypothesis 1.1 implies that \( d < \ell \). In particular, there is an element \( \omega_0 \) of the Weyl group of \( \overline{K} \) such that \( \omega_0[c_{\mu}(v_{-\mu})] = c_{\mu}(v_{\mu}) \). This means that \( M = M(\lambda) \) must be self-dual. Clearly we have that \( \overline{W}(\lambda)^\perp \subset \text{Rad}(\overline{W}(\lambda)) \) and that \( \overline{W}(\lambda)/\overline{W}(\lambda)^\perp \) is non-degenerate, so this latter module is also self-dual. Since \( M \) is self-dual and is a homomorphic image of \( \overline{W}(\lambda)/\overline{W}(\lambda)^\perp \), \( \overline{W}(\lambda)/\overline{W}(\lambda)^\perp \) must possess a submodule isomorphic to \( M \). Since \( M \cong \overline{W}(\lambda)/\text{Rad}(\overline{W}(\lambda)) \) and \( \text{Rad}(\overline{W}(\lambda)) \) does not possess a constituent which is isomorphic to \( M \), we must have \( \text{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp \) and our result follows. \( \square \)
Lemma 2.4. Let \( \{v_i, w_i \mid 1 \leq i \leq k\} \) be a hyperbolic basis for some 2\( k \)-dimensional subspace of \( \overline{V} \). Set \( v = v_1 \otimes \cdots \otimes v_k \) and \( w = w_1 \otimes \cdots \otimes w_k \).

Then:

1. \( c_\mu r_\mu(v) \neq 0, c_\mu r_\mu(w) \neq 0 \);
2. \( c_\mu r_\mu(v), c_\mu r_\mu(w) \in \overline{W}(\lambda) \);
3. \( \bar{F}_\mu[c_\mu r_\mu(v), c_\mu r_\mu(w)] \neq 0 \).

Proof. Parts (1) and (2) follow from Lemma 2.2 since \( r_v = r_w = r_\mu \) and the \( v_i \) are distinct, similarly for \( w_i \). If \( \sigma_1, \sigma_2 \in S_k \), then

\[
\bar{F}_\mu[k(\sigma_1(v), \sigma_2(w))] = \prod_{i=1}^{k} \bar{F}[v_{\sigma_1^{-1}(i)}, w_{\sigma_2^{-1}(i)}] = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 \\ 0 & \text{otherwise} \end{cases}
\]

Recall that \( \mathcal{R}_\mu \cap C_\mu = 1 \). Then we have

\[
\bar{F}_\mu[c_\mu r_\mu(v), c_\mu r_\mu(w)] = \bar{F}_\mu[r_\mu(v), r_\mu(w)] = \sum_{\sigma \in \mathcal{R}_\mu} \bar{F}_\mu[\sigma(v), r_\mu(w)] = \sum_{\sigma \in \mathcal{R}_\mu} \bar{F}_\mu[\sigma(v), \sigma(w)] = |\mathcal{R}_\mu|.
\]

Part (3) then follows as \( |\mathcal{R}_\mu| = \prod_{i=1}^{d} \mu_i! \) and \( \mu_i < \text{char}(F_q) \) for all \( i \). \( \square \)

Lemma 2.5. \( M \) possesses a vector of weight \( \lambda_k \).

Proof. Let \( \{\overline{e}_i, \overline{f}_i \mid 1 \leq i \leq k\} \) be a subset of our standard basis \( \overline{B} \) for \( \overline{V} \). By part (2) of Lemma 2.4, \( c_\mu r_\mu(\overline{e}_1 \otimes \cdots \otimes \overline{e}_k) \in \overline{W}(\lambda) \). An argument similar to that used in Lemma 2.1 shows that \( c_\mu r_\mu(\overline{e}_1 \otimes \cdots \otimes \overline{e}_k) \) is a vector of weight \( \lambda_k \). Hence \( \lambda_k \) is a subdominant weight of \( \lambda \). Condition (3) of Hypothesis 1.1 insures that \( \lambda \) is \( p \)-restricted. Therefore using the results of \( [5] \), \( M \) possesses a vector of weight \( \lambda_k \). \( \square \)

3. Elementary abelian 3-subgroup \( E_k \).

Assume that \( k \leq n/3 - 2 \) and recall that \( F^*(H) \) possesses a subgroup \( H_0 \) isomorphic to \( S_{n-2} \). Let

\[
E_k \cong \langle (123), (456), \ldots, (3k-2, 3k-1, 3k) \rangle < A_n
\]

be a subgroup of \( H_0 \) generated by commuting 3-cycles in \( F^*(H) \) so that \( E_k \) is an elementary abelian 3-group of rank \( k \). Then

\[
N_k = N_{H_0}(E_k) \cong S_3 \wr S_k \times S_{n-3k-2}
\]

\[
C_k = C_{H_0}(E_k) \cong E_k \times S_{n-3k-2}
\]
and let $H_k < C_k$ so that $H_k \cong S_{n-3k-2}$. Note that $C_{N_k}(H_k) \cong S_3 \wr S_k$ and this subgroup controls fusion in $E_k$. Let $\sigma \neq 1 \in E_k$ and assume that $\sigma$ is the product of $k_1$ disjoint 3-cycles. Then $C_{N_k}(\sigma) \cong \mathbb{Z}_3 \wr S_{k_1} \times S_{3} \wr S_{k-k_1} \times S_{n-3k-2}$ which implies $|\sigma^{N_k}| = 2k^2(k_1)k_2$.

Let $\varphi \in E^*_k = \text{Hom}(E_k, \mathbb{F}^*)$. The group $N_k$ acts on this group by $\varphi^g : \sigma \mapsto \varphi(g^{-1}\sigma g)$ for $g \in N_k$, $\sigma \in E_k$. We abuse notation slightly and define $\varphi^{-1}$ by $\varphi^{-1} : \sigma \mapsto \varphi(\sigma^{-1})$ for all $\sigma \in E_k$. Recall that $I_{N_k}(\varphi) = \{g \in N_k|\varphi^g = \varphi\}$ is the inertia group of $\varphi$ in $N_k$ and note that $H_k \subseteq I_{N_k}(\varphi)$.

If $\varphi \in E^*_k$ is non-trivial, then the previous remarks concerning the action of $N_k$ on $E_k$ imply that $[N_k : I_{N_k}(\varphi)] = 2k^2(k_1)$ for some $k_1$, $1 \leq k_1 \leq k$ and that $\varphi^{-1} \in \varphi^{N_k}$. Since $(k_1) \geq k$ unless $k = k_1$, in which case $2k^1 \geq 2k$, we have $[N_k : I_{N_k}(\varphi)] \geq 2k$.

4. Decomposition of $\mathbb{V}|_{E_k}$ and $C_k$-invariant subspace of $\mathbb{W}(\lambda)$.

We continue to assume that $k \leq n/3 - 2$ and we now consider the restriction of $\mathbb{V}$ to $E_k$. Since $\text{char}(\mathbb{F}) \neq 3$, we have $\mathbb{V}|_{E_k} \cong \bigoplus_{\varphi \in E^*_k} \mathbb{V}_\varphi$ where $\mathbb{V}_\varphi$ is the homogeneous component of $\varphi$. Let $\overline{v}_1 \in \mathbb{V}_{\varphi_1}$ and $\overline{v}_2 \in \mathbb{V}_{\varphi_2}$. Then $(g\overline{v}_1, g\overline{v}_2) = \varphi(g\varphi_2(g)\overline{v}_1, \overline{v}_2)$ for all $g \in E_k$. If $\varphi_1^{-1} \neq \varphi_2$ then $\langle \overline{v}_1, \overline{v}_2 \rangle = 0$ which implies $\mathbb{V}_{\varphi_1} \perp \mathbb{V}_{\varphi_2}$ when $\varphi_1^{-1} \neq \varphi_2$. Since $\mathbb{V}$ is non-degenerate, $\text{dim}(\mathbb{V}_{\varphi}) = \text{dim}(\mathbb{V}) - \text{dim}(\mathbb{V}_{\varphi_1})$ and it follows that $\mathbb{V}_\varphi \oplus \mathbb{V}_{\varphi^{-1}}$ must be non-degenerate and therefore possesses a hyperbolic basis.

Pick $\varphi \neq 1$ so that $\mathbb{V}_\varphi \neq 0$. Since $g\mathbb{V}_\varphi = \mathbb{V}_{\varphi^g}$ for $g \in N_k$, we may consider $\mathbb{V}_\varphi$ to be an $N_k(\varphi)$-module. Let $E^*_{k-1}$ be a maximal subgroup of $E^*_k$ which does not contain $\varphi$. Define $\mathcal{O}_+ = \varphi E^*_{k-1} \cap \varphi^{N_k}$ and $\mathcal{O}_- = \varphi^{-1}E^*_{k-1} \cap \varphi^{N_k}$ so that $\mathcal{O}_+ \cup \mathcal{O}_- = \varphi^{N_k}$ and $|\mathcal{O}_+| = |\mathcal{O}_-| \geq k$. Moreover $\varphi_i \in \mathcal{O}_+$ if and only if $\varphi_i^{-1} \in \mathcal{O}_-$. We assume that $\mathcal{O}_+ = \{\varphi_1\}$ and that $\mathcal{O}_- = \{\varphi_i^{-1}\}$.

Then $\left(\bigoplus_{\varphi_i \in \mathcal{O}_+} \mathbb{V}_{\varphi_i}\right) \oplus \left(\bigoplus_{\varphi_i^{-1} \in \mathcal{O}_-} \mathbb{V}_{\varphi_i^{-1}}\right)$ is an $N_k$-submodule of $\mathbb{V}_{N_k}$. If $\varphi' \in \varphi^{N_k}$ then, as $C_{N_k}(H_k)$ also controls fusion in $E^*_k$, there is a $g \in C_{N_k}(H_k)$ such that $g\mathbb{V}_{\varphi'} = \mathbb{V}_{\varphi'}$. In particular $\mathbb{V}_{\varphi} \cong \mathbb{V}_{\varphi'}$ as $H_k$-modules. Define $D = \text{dim}(\mathbb{V}_{\varphi})$ so that $D = \text{dim}(\mathbb{V}_{\varphi_i})$ for all $i$.

Given the above decomposition, we form the following:

$$\mathbb{V}_+ = \bigotimes_{i=1}^{k} \mathbb{V}_{\varphi_i} \quad \text{and} \quad \mathbb{V}_- = \bigotimes_{i=1}^{k} \mathbb{V}_{\varphi_i^{-1}}.$$

Recall that $D = \text{dim}(\mathbb{V}_{\varphi_i})$ and assume that $\{\overline{v}_{i,j} : 1 \leq j_i \leq D\}$ is a hyperbolic basis for $\mathbb{V}_{\varphi_i} \oplus \mathbb{V}_{\varphi_i^{-1}}$. Define $\overline{v}^{j_1 \cdots j_k} = \bigotimes_{i=1}^{k} \overline{v}_{i,j_i}$. Then $\{\overline{v}^{j_1 \cdots j_k} : 1 \leq j_i \leq D\}$ forms a hyperbolic basis for $\mathbb{V}_+ \oplus \mathbb{V}_-$. If $\sigma \in S_k$, then $\sigma(\overline{v}^{j_1 \cdots j_k}) = \overline{v}^{j_1 \cdots j_k}$ if and only if $\sigma = 1$.
since the $V_{\varphi}$ are distinct. Moreover, $r^{{\pi}_{1}^{j_1} \cdots \cdot j_k}_{\mu} = r_{\mu}$ for all $\pi_{1}^{j_1} \cdots \cdot j_k \in V_+$. Similarly for $\pi_{1}^{j_1} \cdots \cdot j_k \in V_-$. 

By parts (1) and (2) of Lemma 2.4, and as $V_\pm$ are both totally singular, $c_{\mu}r_{\mu}(V_\pm) \subset \overline{W}(\lambda)$. By part (3) of Lemma 2.4, $\mathcal{R}_\mu[c_{\mu}r_{\mu}(\pi_{1}^{j_1} \cdots \cdot j_k), c_{\mu}r_{\mu}(\pi_{1}^{j_1} \cdots \cdot j_k)] \neq 0$. Whenever $(j_1, \ldots, j_k) \neq (j_1', \ldots, j_k')$, we have that $\mathcal{R}_\mu[c_{\mu}r_{\mu}(\pi_{1}^{j_1} \cdots \cdot j_k), c_{\mu}r_{\mu}(\pi_{1}^{j_1'} \cdots \cdot j_k')] = 0$. Therefore $\{c_{\mu}r_{\mu}(\pi_{1}^{j_1} \cdots \cdot j_k), c_{\mu}r_{\mu}(\pi_{1}^{j_1} \cdots \cdot j_k) | 1 \leq j_i \leq D\}$ is a hyperbolic basis for $c_{\mu}r_{\mu}(V_+) \bigoplus c_{\mu}r_{\mu}(V_-)$.

Lemma 4.1. We have:

(1) $V_\pm \cong c_{\mu}r_{\mu}(V_\pm)$ as $\mathbb{F}C_k$-modules;  

(2) If $k$ is even, then $C_k$ stabilizes a 1-dimensional subspace of $M$;  

(3) If $k$ is odd, then $C_k$ stabilizes a $D$-dimensional subspace of $M$.

Proof. Given the hyperbolic basis $\{\pi_{1}^{j_1} \cdots \cdot j_k, \pi_{1}^{j_1} \cdots \cdot j_k | 1 \leq j_i \leq D\}$ for $V_+ \oplus V_-$, it is clear that the map $\pi_{1}^{j_1} \cdots \cdot j_k \mapsto c_{\mu}r_{\mu}(\pi_{1}^{j_1} \cdots \cdot j_k)$ is a $C_k$-invariant bijection. Therefore $V_+ \cong c_{\mu}r_{\mu}(V_+)$ as $\mathbb{F}C_k$-modules. The case for $V_-$ follows by a similar argument, proving part (1). Suppose that $k$ is even and recall that $V_{\varphi_i} \cong V_{\varphi_j}$ and $V_{\varphi_i^{-1}} \cong V_{\varphi_j^{-1}}$ as $\mathbb{F}H_k$-modules. As $H_k \cong S_{n-2k-2}$ and all irreducible $\mathbb{F}S_{n-2k-2}$ are self-dual, $H_k$ stabilizes a 1-dimensional subspace of $V_{\varphi_i} \otimes V_{\varphi_j}$. It follows by induction that $H_k$ stabilizes a 1-dimensional subspace of $V_+$. If $k$ is odd, then the same argument leads to a $D$-dimensional subspace being stabilized by $H_k$. As $E_k$ acts as scalars on $V_\pm$, these spaces are, in fact, stabilized by $C_k$. Using part (1), $C_k$ stabilizes a subspace $\mathcal{W}_0$ of one of these dimensions in $\overline{W}(\lambda)$. Since $c_{\mu}r_{\mu}(V_+) \oplus c_{\mu}r_{\mu}(V_-)$ possesses a hyperbolic basis, $\mathcal{W}_0 \cap \overline{W}(\lambda) = 0$. If we let 

$$M_0 = \left( \mathcal{W}_0 + \overline{W}(\lambda) \right) / \overline{W}(\lambda)$$

then Lemma 2.3 implies that $M_0 \subset \overline{W}(\lambda) / \overline{W}(\lambda) \cong M$, hence (2) and (3).

5. Proof of Theorem 1.2.

We are now in a position to prove Theorem 1.2:

Since $M$ possesses a vector $\pi_{\lambda_k}$ of weight $\lambda_k$ by Lemma 2.5, we can produce a lower bound for $\dim(M)$ as follows: Let Weyl($K$) be the Weyl group of $K$ and recall that $\ell$ is the Lie rank of $K$. We compute $C_{\text{Weyl}(K)}(\lambda_k)$ using [3, §13.1], and compute $|\lambda_k^{\text{Weyl}(K)}|$, whence

$$\dim(M) \geq |\lambda_k^{\text{Weyl}(K)}| = 2^k \binom{\ell}{k}.$$
Case 1. First suppose that $k \geq n/3 - 1$. We assume that $\dim(\mathcal{V}) \geq 2n^4$, so $\ell \geq n^4$. Since $\dim(M) \leq \sqrt{|H|} \leq \sqrt{n!}$, Eq. (1) implies that $2^k \binom{\ell}{k} \leq \sqrt{n!}$. Trivially, $2^{n^4/2} > \sqrt{n!}$ for all $n \geq 1$, so that $k < n^4/2 \leq \ell/2$. Using the fact that $\binom{\ell}{k_1} < \binom{\ell}{k_2}$ if $k_1 < k_2 < \ell/2$, $\binom{\ell}{k}$ will be minimal when $k = n/3 - 1$ and $\ell = n^4$. Note also that $\binom{\ell}{k} = \prod_{i=1}^{k} \frac{\ell - i + 1}{k - i + 1} \geq \frac{(\ell - k + 1)^k}{k!}$. We have:

$$2^{n/3 - 1} \left( \frac{n^4}{n/3 - 1} \right) < \sqrt{n!},$$

$$2^{n/3 - 1} \frac{(n^4 - n/3 + 2)n^{3/1}}{(n/3)^{n/3 - 1}} < (n^{1/2})^{n - 1},$$

$$2^{n/3 - 1} (n^3 - 1)^{n/3 - 1} < n^{(n - 1)/2},$$

$$n^{n - 3} < n^{(n - 1)/2},$$

$$n - 3 < (n - 1)/2,$$

$$n < 5.$$

This contradicts our assumption that $n \geq 10$, so that $\dim(\mathcal{V}) \leq 2n^4$ or $k < n/3 - 1$.

Case 2. We assume that $k < n/3 - 1$ and that $k$ is odd. Lemma 4.1 and Frobenius reciprocity imply $\dim(M) \leq D[H : C_k]$. Since $D \geq \frac{\ell}{2k}$ and $[H : C_k] = \frac{n!}{2(3^k)(n - 3k - 2)!}$, we have $\dim(M) \leq \frac{\ell}{2k} \frac{n!}{3^k(n - 3k - 2)!}$. Combining with (1) we get:

$$2^k \binom{\ell}{k} \leq \frac{\ell}{2k} \frac{n!}{2(3^k)(n - 3k - 2)!},$$

$$2^k \binom{\ell - 1}{k - 1} \leq \frac{n^{3k+2}}{3^{k-1}},$$

$$2^k \frac{(\ell - k + 1)^{k-1}}{(k - 1)^{k-1}} \leq \frac{n^{3(k-1)n^5}}{3^{k-1}},$$

$$2 \frac{\ell - k}{k - 1} \leq \frac{n^3}{3} n^{5/(k-1)}.$$

Observing that $(k - 1)n^{5/(k-1)} < n^3$ when $k \geq 3$ and $n \geq 10$, we have

$$2\ell < \frac{n^6 + 2n}{3} < n^6.$$

Case 3. Finally we assume that $k < n/3 - 1$ and that $k$ is even. Again Lemma 4.1 and Frobenius reciprocity imply $\dim(M) \leq [H : C_k] \leq \frac{n!}{2(3^k)(n - 3k - 2)!}$. Combining with (1) we get:

$$2^k \binom{\ell}{k} \leq \frac{n!}{3^k(n - 3k - 2)!}.$$
\[ 2^k \left( \ell - k + 1 \right)^k < \frac{n^{3k+2}}{3^k} = \frac{n^{3k}}{3^k} n^2, \]
\[ 2^{\ell - k} < \frac{n^3}{3} n^{2/k}, \]
\[ 2\ell < \frac{n^5 + 3n}{9}. \]

In all cases, \( 2\ell < n^6 \), which implies that \( \dim(\mathcal{V}) \leq n^6 \). This completes the proof of Theorem 1.2. \( \square \)

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