FINITE RELATIVE DETERMINATION AND RELATIVE STABILITY

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This paper is divided in three parts. The first part deals with the equivalence of finite determination on the right and finite relative determination (with respect to $S$) under some conditions on $S$. The second part deals with infinite determinacy (with respect to $S$, a germ of a closed set of $\mathbb{R}^n$). Both generalize results of P. Porto [P] for a big family of closed subsets $S$ of $\mathbb{R}^n$. The third part is a special case which is quite interesting, when $S$ coincides with the closure of its interior.

Introduction.

This paper continues the work done in [K]. In that paper there were proven results of finite relative determination for particular algebraic subsets of $\mathbb{R}^n$. Here we continue in this direction. In the first part we prove the equivalence of finite determination on the right and finite relative determination for a big family of algebraic subsets, generalizing the results of [P-L]. In the second part we continue with the concept of infinite determinacy and remarking the importance of quasihomogeneous polynomials. In the third part we generalize the results on relative stability in [P-L] and [P] for a broader family of closed subsets of $\mathbb{R}^n$, such as good semianalytic subsets.

Notation.

We shall work in $\mathcal{E}(n)$, the local algebra of $C^\infty$ function germs of $\mathbb{R}^n$ to $\mathbb{R}$ around the origin with maximal ideal $m(n)$. The powers of $m(n)$ will be denoted by $m(n)^k$ and $m(n) = \cap_{k=1}^\infty m(n)^k$. For $I = (i_1, \ldots, i_n)$ a multi-index of natural numbers and $x = (x_1, \ldots, x_n)$ we shall write $x^I = x_1^{i_1} \cdots x_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$, also for a germ $f$, $\frac{\partial^{|I|} f}{\partial x^I} = \frac{\partial^{i_1} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$.

For $S$ a subset of $\mathbb{R}^n$, $\emptyset \in S$, $\text{cl}(S)$ and $\text{int}(S)$ will denote the closure and the interior of $S$ respectively and $G_S$ will be the group of germs of diffeomorphisms $\phi$ at $\emptyset$, such that $\phi(x) = x \forall x \in S$. Also $d(x, S)$ will denote the usual distance from the point $x$ to the subset $S$.

Finally if $f$ is a germ, $\partial f / \partial x_i$ will be the partial derivatives of $f$ and $\langle \partial f / \partial x_i \rangle$ will be the ideal of $\mathcal{E}(n)$ generated by them. Also for a germ $f$,
$j^k f(x)$ will be the Taylor expansion of $f$ at the point $x$ up to degree $k$ and it is called the $k$-jet of $f$ at $x$. We will denote by $J^k(n, 1)$ the \( \mathbb{R} \)-vector space of all polynomials in $n$-coordinates up to degree $k$. In the case $k = \infty$ we understand $j^\infty f(x)$ the Taylor series of $f$ at $x$. Also $J^\infty(n, 1)$, the set of all these jets will be identified with the formal power series ring $\mathbb{R}[[x_1, \ldots, x_n]]$.

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1. Finite Relative Determination and Finite Determination on the Right.

**Definition 1.**

(a) Let $S$ be a germ of a subset of $\mathbb{R}^n$, $f$ be a germ with $f(\bar{0}) = 0$ and let $k \leq \infty$. We shall say that $f$ is $k$-determined relative to $G_S$ if whenever $g$ is a germ such that $j^k g(\bar{0}) = j^k f(\bar{0})$ and $g - f$ vanishes at $S$, there exists $h$ in $G_S$ with $g = f \circ h$. In the case $S = \{\bar{0}\}$ we say that $f$ is finitely or infinitely determined on the right according to $k$ is finite or not. In general if $k$ is finite, then we say that $f$ is finitely determined relative to $G_S$.

(b) Let $I$ be an ideal of $\mathcal{E}(n)$ and $S = z(I)$ the germ of the common zeroes of $I$ (we suppose $\bar{0} \in S$). We denote by $\text{rad } I$ the ideal of $\mathcal{E}(n)$ consisting of all germs vanishing at $S$, and we say that $I$ is radical if $I = \text{rad } I$.

**Lemma 2** (Artin-Rees). If $I$ is an ideal of $\mathbb{R}[[x]] = \mathbb{R}[[x_1, \ldots, x_n]]$, there exists $k$ such that $I \cap M^m = M^{m-k}(I \cap M^k) \ (\forall m \geq k)$.

We shall denote $\mathcal{A}(I)$ the minimum $k$ satisfying the equality of Lemma 2. Consider $\mathbb{R}[[x]]$ the algebra of formal power series, $M$ its maximal ideal, and the canonical projection $\pi : \mathcal{E}(n) \rightarrow \mathbb{R}[[x]]$ which sends a germ to its Taylor infinite series and $J$ an ideal of $\mathcal{E}(n)$, we will get by Artin-Rees lemma for $l = \mathcal{A}(\pi(J))$, $M^m \cap \pi(J) = M^{m-l}(M^l \cap \pi(J)), \forall m \geq l$. Hence applying $\pi^{-1}$ to the above equality and intersecting each member with $J$ we get

\[ (*) \quad J \cap m(n)^m = m(n)^{m-l}(J \cap m(n)^l) + J \cap m(n)^\infty \ (\forall m \geq l). \]

We shall denote $\mathcal{A}(J)$ the minimum $l$ satisfying this equality, therefore $\mathcal{A}(J) \leq \mathcal{A}(\pi(J))$.

Since $m(n)^k \supset \ker \pi$, then $\pi(J \cap m(n)^k) = \pi(J) \cap \pi(m(n)^k) = \pi(J) \cap M^k$. If we apply the epimorphism $\pi$ to the equality $(*)$, we get $M^m \cap \pi(J) = M^{m-l}(M^l \cap \pi(J)), \forall m \geq l$. Therefore $\mathcal{A}(\pi(J)) \leq \mathcal{A}(J)$ and hence $\mathcal{A}(J) = \mathcal{A}(\pi(J))$.

In case $I$ is a radical ideal of $\mathcal{E}(n)$, we get in fact $I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k) \ \forall m \geq k.$
Theorem 3. Consider $I$ a finitely generated ideal of $\mathcal{E}(n)$. Then for any $k < \infty$, $I \cap m(n)^k$ is also finitely generated.

Proof. Consider $g_1, \ldots, g_s$ generators of $I$ and let $f = \sum_{i=1}^s h_i g_i$. Then we have

$$f = \sum_{i=1}^s h_i^{(k)} g_i + \sum_{i=1}^s h_i^{[k]} g_i,$$

where $h_i^{(k)}$ is the $(k - 1)$-jet of $h_i$ and $h_i^{[k]} = h_i - h_i^{(k)}$.

Hence as vector spaces $I = V + m(n)^k I$, where $V$ is the vector space generated by $\{x^t g_i\}$ with $|I| \leq k - 1$. Therefore $I \cap m(n)^k = V \cap m(n)^k + m(n)^k I$. It is clear that a basis of the subspace $V \cap m(n)^k$ of $V$ and the generators of $m(n)^k I$ generate $I \cap m(n)^k$ as an ideal of $\mathcal{E}(n)$.

The proofs of the above theorems can be found in [K], Theorems 11 and 15.

We can change Theorem 4 in the following way.

Theorem 4. Suppose $I$ is a radical ideal of $\mathcal{E}(n)$, $I \cap m(n)^k$ a finitely generated ideal and $I \cap m(n)^k \subseteq \text{Im}(n) \left\{ \frac{\partial f}{\partial x_i} \right\}$ with $k \geq A(I)$. Then $f$ is $k$-determined relative to $G_S$, where $S = z(I)$.

Theorem 5. Let $f$ be a $k$-determined germ relative to $G_S$, $S = z(I)$ and $I$ a radical finitely generated ideal. Then $I \cap m(n)^{k+1} \subseteq I \left\{ \frac{\partial f}{\partial x_i} \right\} + m(n)^{k+2} \cap I$.

Joining Theorems 4 and 5 we get for $I$ a finitely generated ideal, the following:

Theorem 6. Let $f$ be a germ, $I$ a finitely generated radical ideal, $S = z(I)$, and $k \geq A(I)$. Then $f$ is finitely determined relative to $G_S$ if and only if there exists a number $l$ greater or equal than $k$ such that $m(n)^l \cap I \subseteq I \left\{ \frac{\partial f}{\partial x_i} \right\}$.

The proofs of the above theorems can be found in [K], Theorems 11 and 15.

We can change Theorem 4 in the following way.

Theorem 7. Let $I$ be a radical ideal, $k = A(I)$ and suppose that $I \cap m(n)^k$ is finitely generated. Let $l$ be a natural number such that $m(n)^l I \subseteq m(n) I \left\{ \frac{\partial f}{\partial x_i} \right\}$. Then $f$ is $(k + l - 1)$ determined relative to $G_S$, where $S = z(I)$.

Proof. Let $g$ be a germ with $g \equiv f$ on $S$ and $j^{k+l-1} g(0) = j^{k+l-1} f(0)$.

If we define the trivial homotopy $F(x, t) = (1 - t) f(x) + t g(x)$ we get

$$\frac{\partial F}{\partial t} = g - f \in m(n)^{k+l} \cap I$$

and

$$\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + t \left( \frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i} \right).$$
Since \(m(n)^{k+l} \cap I \subseteq m(n)^I\) and \(Im(n)^p-1 \subseteq I \cap m(n)^p\ \forall\ p\), we get
\[
\left( m(n)^{k+l} \cap I \right) E(n+1) \subseteq m(n)I \left( \frac{\partial F}{\partial x_i} \right) E(n+1) \\
+ m(n) \left( m(n)^{k+l} \cap I \right) E(n+1).
\]

By Nakayama’s lemma we arrive to the inclusion:
\[
(m(n)^{k+l} \cap I)E(n+1) \subseteq m(n)I \left( \frac{\partial F}{\partial x_i} \right) E(n+1).
\]

Hence \(\frac{\partial F}{\partial t} = \sum h_i(x,t)\frac{\partial F}{\partial x_i}\) with \(h_i(x,t) \equiv 0\) for \(x \in S, t near t_0 (t_0 \text{ fixed})\). We now proceed in the usual way.

**Remark 1.**

(a) If \(I = m(n)\) then \(k = 1\) and we get that \(m(n)^{l+1} \subseteq m(n)^2\) \(\frac{\partial F}{\partial x_i}\) implies \(f\) is \(l\)-determined on the right ([M]).

(b) If \(I = \langle x_1, \ldots, x_s \rangle\) then \(k = 1\) and we get that \(m(n)^l I \subseteq m(n)I \langle \frac{\partial f}{\partial x_i} \rangle\) implies \(f\) is \(l\)-determined relative to \(G_S, S = \{0\} \times \mathbb{R}^{n-s} ([P-L])\).

**Corollary 8.** Let \(f\) be a germ, \(I\) a radical ideal, \(k = A(I)\) and \(I \cap m(n)^k\) be finitely generated. Suppose that \(m(n)^l \subseteq m(n)\langle \frac{\partial f}{\partial x_i} \rangle\). Then \(f\) is \((k+l-1)\)-determined relative to \(G_S\). Hence finite determination on the right implies finite relative determination.

**Proof.** Since \(m(n)^l \subseteq m(n)\langle \frac{\partial f}{\partial x_i} \rangle\) then \(m(n)^l I \subseteq m(n)I \langle \frac{\partial f}{\partial x_i} \rangle\). We now use Theorem 7.

We are now interested in determining for which ideals \(I\) we have the converse of Corollary 8. For this purpose we need the following:

**Theorem 9.** Let \(A\) be a commutative ring, \(I, J, K\) ideals of \(A\) with \(I = \langle g_1, \ldots, g_k \rangle\). Suppose that \(ag_i = 0\) for all \(i\) and \(a \in J^k + K\) implies \(a = 0\). Then if \(IJ \subset IK\) hence \(J^k \subseteq K\).

**Proof.** Let \(m_1, \ldots, m_k\) be arbitrary elements of \(J\), then \(g_i m_i = \sum_{j=1}^k g_j d_{ij} \forall i\) with \(1 \leq i \leq k\) \((d_{ij} \in K)\). In matricial notation we can write
\[
C \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ where } C = (\delta_{ij} m_i - d_{ij}).
\]

If we multiply \((*)\) by the adjoint of \(C\) we get \((\det C)g_i = 0 \ \forall\ i\), but \(\det C = m_1 \cdots m_k + b\) with \(b \in K\). Hence by hypothesis \(\det C = 0\) and then \(m_1 \cdots m_k \in K\), therefore \(J^k \subseteq K\).
Corollary 10. Let $A = \mathcal{E}(n)$, $J = m(n)^1$, (or $J = m(n)^\infty$), $K = \langle \frac{\partial f}{\partial x_i} \rangle$ and $I$ ideal with $I = \langle g_1, \ldots, g_k \rangle$. Suppose that $hg_i = 0$ for all $i$ and $h \in m(n)^{1k} + \langle \frac{\partial f}{\partial x_i} \rangle$ (or $h \in m(n)^\infty + \langle \frac{\partial f}{\partial x_i} \rangle$) implies $h \equiv 0$. Then if $I m(n)^1 \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$ (or $I m(n)^\infty \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$) hence $m(n)^{1k} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle$ (or $m(n)^\infty \subseteq \langle \frac{\partial f}{\partial x_i} \rangle$) and $f$ is $(1k+1)$-determined on the right ($\infty$-determined on the right).

This result motivates us to find examples where $I$ is a finitely generated radical ideal satisfying the hypothesis of the above corollary.

Example 11. Let $I$ be a radical ideal generated by a non trivial analytic germ $g$. If $hg = 0$ then $h \equiv 0$ and we will have finite relative determination implies determination on the right ($hg \equiv 0 \implies h^{-1}(0) \cup g^{-1}(0) = \mathbb{R}^n$ but $g^{-1}(0)$ is a closed set with empty interior, therefore $h^{-1}(0) = \mathbb{R}^n$).

Example 12. Consider in $\mathcal{E}(3)$ the ideal $I$ generated by $\{x_1 x_2, x_1 x_3, x_2 x_3\}$. It is easy to see that $I$ is radical and $\mathcal{A}(I) = 2$. Moreover if we denote $P_1 = x_1 x_2$, $P_2 = x_1 x_3$ and $P_3 = x_2 x_3$, we get for $i \neq j$, $i \neq k$, $j \neq k$ that the closure of $z(P_i) \cap z(P_j) - z(P_k)$ is a plane and does not contain $z(I)$, which is the union of the three axes, hence it does not satisfy the hypothesis of Theorem 20 [K], but the conclusion is still true. We give a proof since it is important for the converse of Corollary 8.

Proposition 13. With the above notation, if $f$ is $m$-determined relative to $G_S$, where $S = z(I)$ are the coordinate axes, then $f$ is $(2m - 2)$-determined on the right.

Proof. By Theorem 15 ([K]) we know that $m(3)^{m+1} \cap I \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$ which in this case is equivalent to $Im(3)^{m-1} \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$. We shall show that $m(3)^{2m-1} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle m(3)^2$ and hence $f$ is $(2m - 2)$-determined on the right. Any mixed monomial of $m(3)^{2m-1}$ has a factor of $I$ times a monomial of degree $2m - 3$, hence for $m \geq 2$ it is contained in the Jacobian ideal. We now give the proof for $x_1^{2m-1}$, the other two are similar,

\[(*) \quad x_1^{m-1}(x_1 x_2) = x_1 x_2 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + x_1 x_3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{2j} + x_2 x_3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{3j}.
\]

If we denote $\phi = x_1^{m-1} - \sum_{i=1}^{3} \frac{\partial f}{\partial x_j} h_{1j}$ we get that the zeroes of $\phi$ contain \{ $x_3 = 0$ \} and the zeroes of $x_1 \phi$ contain \{ $x_3 = 0$ \} $\cup$ \{ $x_1 = 0$ \}, hence $x_1 \phi \subseteq I = I$. From $(*)$ and the definition of $\phi$, $x_1^{m} = x_1 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + x_1 \phi$, therefore $x_1^{2m-1} \in m(3)^{2} \langle \frac{\partial f}{\partial x_i} \rangle$ and $f$ is $(2m - 2)$-determined. $\square$
Remark 2. By Corollary 10, since \( I = \hat{I} = \langle x_1x_2, x_1x_3, x_2x_3 \rangle \) then \( \text{Im}(3)^{m-1} \subseteq I(\frac{\partial f}{\partial x_i}) \) implies \( m(3)^{m-1} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \) and \( f \) is \((3m-2)\)-determined on the right.

Definition 14. Let \( f_1, \ldots, f_r \) be germs in \( m(n) \). We say that they are linearly independent if their gradients denoted by \( \nabla f_1, \ldots, \nabla f_r \) are linearly independent at the origin.

Lemma 15. Let \( f_1, \ldots, f_r \) be linearly independent germs. Then the ideal \( I \) generated by them is radical.

Proof. Let \( H \) be the germ of the common zeroes of \( I \) and \( P_{r+1}, \ldots, P_n \) linear polynomials such that \( \nabla f_1(\hat{0}), \ldots, \nabla f_r(\hat{0}), \nabla P_{r+1}(\hat{0}), \ldots, \nabla P_n(\hat{0}) \) is a basis of \( \mathbb{R}^n \). Thus \( \phi = (f_1, \ldots, f_r, P_{r+1}, \ldots, P_n) \) is a germ of diffeomorphism. Let \( f \in \hat{I} \) hence \( f = 0 \) on \( H \) if and only if \( f \circ \phi^{-1} \equiv 0 \) on \( \{0\} \times \mathbb{R}^{n-r} \). By Hadamard’s lemma we get \( f \circ \phi^{-1}(x_1, \ldots, x_n) = \sum_{i=1}^r x_i g_i, \) therefore \( f = \sum_{i=1}^r f_i(g_i \circ \phi) \), and hence \( f \) belongs to the ideal \( I \).

Lemma 16. Let \( I_1, \ldots, I_r \) be radical ideals in \( E(n) \). Then their intersection is also a radical ideal.

Proof. In general \( \text{rad}(\cap_{i=1}^r I_i) \subseteq \cap_{i=1}^r \text{rad} I_i \), hence we get
\[
\cap_{i=1}^r I_i \subseteq \text{rad} \cap_{i=1}^r I_i \subseteq \cap_{i=1}^r \text{rad} I_i = \cap_{i=1}^r I_i.
\]
Therefore the equality \( \cap_{i=1}^r I_i = \text{rad} \cap_{i=1}^r I_i \).

Lemmas 15 and 16 generate a special collection of algebraic sets. They are called bouquets of subspaces.

Example 17. Consider \( I \subseteq E(3) \) the ideal generated by \( x \) and \( yz \), hence \( z(I) \) is the union of the \( y \)-axis and \( z \)-axis, they are not in general position (in \( \mathbb{R}^3 \)). By Lemma 16, \( I \) is clearly a radical ideal since \( I = I_1 \cap I_2 \) where \( I_1 = \langle x, y \rangle \) and \( I_2 = \langle x, z \rangle \).

Definition 18. Let \( I \) be a finitely generated ideal of \( E(n) \), \( I = \langle g_1, \ldots, g_k \rangle \). We say that \( I \) is integral if \( S = \text{z}(I) \) is nowhere dense.

We now arrive at the main theorem of this section.

Theorem 19. Let \( I \) be a finitely generated ideal of \( E(n) \) which is radical. Then if \( f \) is finitely determined relative to \( G_S \), \( S = \text{z}(I) \), hence \( f \) is finitely determined on the right.

Proof. Suppose \( I = \langle g_1, \ldots, g_k \rangle \) and that \( hg_i \equiv 0 \forall i \). Therefore \( \text{z}(h) \cup \text{z}(g_i) = \mathbb{R}^n \forall i \) and hence \( \text{z}(h) \cup \text{z}(I) = \mathbb{R}^n \). Since \( I \) is an integral ideal, see \([R], \text{z}(h) = \mathbb{R}^n \) and hence \( h \equiv 0 \). On the other side there exists a natural number \( p \) such that \( m(n)^p I \subset \langle \frac{\partial f}{\partial x_i} \rangle I \). By Corollary 10, we get \( m(n)^{pk} \subset \langle \frac{\partial f}{\partial x_i} \rangle \) and therefore \( f \) is \((pk + 1)\)-determined on the right. \( \Box \)
Corollary 20. Consider \( I_1, \ldots, I_r \) ideals each of them generated by linearly independent linear polynomials and \( S \) the union of their common zeroes (bouquet of subspaces). Then a germ \( f \) is finitely determined on the right if and only if \( f \) is finitely determined relative to \( G_S \).

We finish this section with an observation about homogeneous polynomials.

Proposition 21. Let \( h_1, \ldots, h_k \) be homogeneous polynomials of degree \( s_1, \ldots, s_k \) respectively and let \( s \) be the maximum of these degrees. Hence if \( I \) is the ideal generated by \( h_1, \ldots, h_k \) we get \( \mathcal{A}(I) \leq s \).

Proof. We have to show that \((I \cap m(n)^{s})m(n)^r = I \cap m(n)^{s+r} \forall r \geq 0\). Let \( f \in I \cap m(n)^{s+r} \), hence we have the following equalities.

\[
(*) \quad f = h_1 g_1 + \ldots + h_k g_k
\]

\[
(**) \quad 0 = j^{s+r-1} f(0) = h_1 j^{s_r-1-s_1} g_1(0) + \cdots + h_k j^{s_r-1-s_k} g_k(0).
\]

Subtracting \((**)\) from \((*)\) we get \( f = h_1 \tilde{g}_1 + \cdots + h_k \tilde{g}_k \), where \( \tilde{g}_i \in m(n)^{r+s-s_i} \).

Hence each \( \tilde{g}_i \) is a sum of elements of the form \( h_i^j h_i^j \), with \( h_i^j \in m(n)^r \) and \( h_i^j \) is a homogeneous monomial of degree \( s - s_i \).

Therefore \( f \) is a sum of elements of the form \( (h_i h_i^j)^j \), with \( (h_i h_i^j)^j \in I \cap m(n)^s \), so \( f \in (I \cap m(n)^s)m(n)^r \). We have shown that \( I \cap m(n)^{s+r} \subseteq (I \cap m(n)^s)m(n)^r \forall r \geq 0 \). The other inclusion is obvious. \( \square \)

2. Infinite determinacy on germs of closed subsets of \( \mathbb{R}^n \).

In this section we will assume that \( S \) is a germ of a closed subset of \( \mathbb{R}^n \) such that the origin is an accumulation point of \( S \).

Definition 22. Let \( S \subseteq \mathbb{R}^n \) be a germ of a closed set such that \( \tilde{\emptyset} \) is an accumulation point of \( S \). We say that a germ \( f \) in \( \mathcal{E}(n) \) is \( S \)-infinitely determined if given a germ \( g \) such that \( j^{\infty} g(x) = j^{\infty} f(x) \forall x \in S \) there exists a germ of a diffeomorphism \( \phi \) such that \( g = f \circ \phi \).

We denote by \( \mathcal{E}(S, n) \) the ideal of \( \mathcal{E}(n) \) consisting of the germs \( f \) such that \( j^{\infty} f(x) = 0 \) for all \( x \in S \). If \( f \) is a germ in this ideal, we can write \( f = gh \) where \( \{g, h\} \subseteq m(n)^{\infty} \) and \( h(x) > 0 \) for \( x \neq 0 \). Then \( j^{\infty} g(x) = 0 \forall x \neq 0, x \in S \) and therefore \( \mathcal{E}(S, n) \subseteq \mathcal{E}(S, n) m(n)^{\infty} \). We get in fact the equality.

Remark 3. If \( f \in \mathcal{E}(S, n) \) then for all multi-index \( I, \frac{\partial^{|I|} f}{\partial x^I} \in \mathcal{E}(S, n) \).
Definition 23. A germ $f$ is $S$-infinitesimally stable if $\mathcal{E}(S, n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S, n)$.

Theorem 24. Let $S$ be a germ of a closed subset of $\mathbb{R}^n$ such that the origin is an accumulation point of $S$. If $f$ is $S$-infinitesimally stable then $f$ is $S$-infinitely determined.

Proof. Let $g(x)$ be a germ such that $j^\infty g(x) = j^\infty f(x) \forall x \in S$. We define the homotopy $F(x, t) = tg(x) + (1 - t) f(x)$ . Consider the following $\mathcal{E}(n + 1)$-modules $N = \mathcal{E}(n + 1)\langle \frac{\partial f}{\partial x_i} \rangle$ and $K = \mathcal{E}(n + 1)\langle \frac{\partial^2 f}{\partial x_i^2} \rangle$. If $h \in N$, we have $h(x, t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) h_i(x, t) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x, t) h_i(x, t) + t \sum_{i=1}^{n} \frac{\partial (f - g)}{\partial x_i}(x) h_i(x, t)$. Since $\frac{\partial (f - g)}{\partial x_i}(x) \in \mathcal{E}(S, n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S, n)$, we get $N \subseteq K + \mathcal{E}(S, n)N$, and by Nakayama’s lemma, $N \subseteq K$ and hence $\mathcal{E}(S \times \mathbb{R}, n + 1) \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(n + 1) \subseteq \mathcal{E}(S \times \mathbb{R}, n + 1)\langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(n + 1)$. Since $g - f \in \mathcal{E}(S \times \mathbb{R}, n + 1) \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(n + 1)$, hence $\frac{\partial f}{\partial x_i} \in \mathcal{E}(S \times \mathbb{R}, n + 1)\langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(n + 1)$. We now proceed in the usual way. □

Proposition 25. If $f$ is a germ, finitely (infinitely) determined on the right, it is $S$-infinitesimally stable and therefore $S$-infinitely determined.

Proof. By our hypothesis we have

$$m(n)^\infty \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \quad \text{and} \quad \mathcal{E}(S, n) \subseteq \mathcal{E}(S, n)m(n)^\infty \subseteq \mathcal{E}(S, n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle.$$ □

Example 26.

(a) Let $f$ be a germ and $k$ a natural number, denote by $I_k$ the ideal generated by $f^k$. Suppose $\mathcal{E}(S, n) \subseteq I_k$, $f^k \in \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ and $j^\infty f(x) \neq 0 \forall x \in T$, where the closure of $T$ is $S$. If $h \in \mathcal{E}(S, n)$, $h = f^k g$ where $g \in \mathcal{E}(S, n)$. Therefore $\mathcal{E}(S, n) \subseteq \mathcal{E}(S, n)\langle f^k \rangle \subseteq \mathcal{E}(S, n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle$, and thus $f$ is $S$-infinitely determined.

(b) In particular let $S = \{(x_1, \ldots, x_n) | x_1 \leq 0\}$. Then for the germs $f_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$, and $f_2(x_1, \ldots, x_n) = x_1$, we get $j^\infty f_1(x) \neq 0$ and $j^\infty f_2(x) \neq 0 \forall x \in \text{int } S$. Since $S = cl(\text{int } S)$ and $\mathcal{E}(S, n) \subseteq \langle f_i^k \rangle$ for $i = 1, 2$ (Proposition 5.4 of Chapter V, [T]), and clearly $f_i^k \in \left\langle \frac{\partial f_i}{\partial x_j} \right\rangle$ for $i = 1, 2$, we get that $f_1^k, f_2^k$ are $S$-infinitely determined. ([P-L]).

Definition 27.

(a) Let $S$ be a closed subset of $\mathbb{R}^n$ (containing the origin) and $f$ a germ with $f(\bar{0}) = 0$. We say that $f$ satisfies a Lojasiewicz inequality for $S$ if for any $K$, a germ of a compact set with $\bar{0} \in K$, there exist constants $c > 0$ and $\alpha \geq 0$ such that $|f(x)| \geq cd(x, S)^\alpha$ for all $x \in K$. 

(b) Let \( I \) be a finitely generated ideal of \( \mathcal{E}(n) \) and \( S \) the germ of its common zeroes. We say that \( I \) is a Lojasiewicz ideal if there exists \( f \) in \( I \) satisfying a Lojasiewicz inequality for \( S \).

(c) Let \( f \in m(n) \) and \( S \) a closed subset of \( \mathbb{R}^n \), we say that \( f \) satisfies a Jacobi-Lojasiewicz condition for \( S \) if \( |\nabla f| \) satisfies a Lojasiewicz inequality for \( S \).

**Remark 4.** If \( \{f_1, \ldots, f_s\} \) is a set of generators of a Lojasiewicz ideal \( I \),

Then \( \sum_{i=1}^s f_i^2, \sum_{i=1}^s |f_i| \) and \( \max \{f_1^2, \ldots, f_s^2\} \) also satisfy a Lojasiewicz inequality for \( S \).

**Definition 28.** Let \( (b_i) \) be a sequence of positive real numbers converging to zero. We say that a sequence of real numbers \( (a_i) \) is flat along \( (b_i) \) if given \( r > 0 \) there exists a natural number \( N = N(r) \) such that \( |a_i| \leq b_i^r \) for \( i \geq N \). Sequences of vectors, matrices, jets are flat along a sequence \( (b_i) \) if each entry is flat along \( (b_i) \). A sequence is flat along a sequence \( (x_i) \) of nonzero vectors in \( \mathbb{R}^n \) converging to the vector \( 0 \) if it is flat along the sequence \( (|x_i|) \). In the case of \( \infty \)-jets, we ask for a uniform \( N = N(r) \) for all entries. Here we are identifying \( \sum_\alpha a_\alpha (x-x_0)^\alpha \) with \( (a_\alpha) \).

**Remark 5.** We can change \( r > 0 \) for \( r = n, n \) a natural number since for \( n > r \), we get \( b_i^n \leq b_i^n (0 \leq b_i \leq 1) \).

We state an interesting equivalence.

**Lemma 29.** A germ \( g \) does not satisfy a Lojasiewicz inequality for a closed subset \( S \) if and only if there exists a sequence of vectors \( x_i \in \mathbb{R}^n - S \) converging to the vector \( 0 \) such that \( (g(x_i)) \) is flat along \( (d(x_i,S)) \).

**Remark 6.** For a germ \( g \) not identically zero we can choose \( g(x_i) \neq 0 \forall i \).

**Definition 30.** Let \( S \) be a closed subset of \( \mathbb{R}^n \). Then \( M(S,n) \) is the set of maps \( \phi : \mathbb{R}^n - S \longrightarrow \mathbb{R} \) such that if \( K \) is a germ of a compact set and \( I \) is a multi-index of natural numbers, there exist constants \( c > 0 \) and \( \alpha > 0 \) such that

\[
|\frac{\partial^\alpha \phi}{\partial x^\alpha}(x)| \leq cd(x,S)^{-\alpha}
\]

for all \( x \in K - S \).

We state the following proposition (Chapter IV, Proposition 4.2 of [T]).

**Proposition 31.** Let \( \phi \in M(S,n) \) and \( f \in \mathcal{E}(S,n) \). Then we can extend \( \phi f \) in a unique way to a germ in \( \mathcal{E}(S,n) \), denoted also by \( \phi f \).

**Theorem 32.** Let \( f \) be a germ, \( S \) a germ of a closed subset of \( \mathbb{R}^n \) such that \( \hat{0} \) is an accumulation point of \( S \). Suppose that \( f \) satisfies a Jacobi-Lojasiewicz condition for \( S \). Then \( f \) is \( S \)-infinitesimally stable and therefore \( S \)-infinitely determined.

**Proof.** Consider \( g = |\nabla f|^2 \), we shall show that \( \mathcal{E}(S,n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S,n) \).

Let \( K \) be a germ of a compact subset and \( g_1 \) be a representative of \( g \); for
each $I$ multi-index there exists $C_I$ constant such that \[
\left| \frac{\partial^{|I|}(g)}{\partial x^{|I|}} \right| \leq \frac{C_I}{|g_1(x)|^{(|I|+1)}} \]
$\forall x \in K$. Since $g_1$ satisfies a Lojasiewicz inequality for $S$, there exist $c > 0$ and $\alpha \geq 0$ such that \[|g_1(x)| \geq c(d(x,S))^\alpha \forall x \in K - S \] and therefore \[
\left| \frac{\partial^{|I|}(g)}{\partial x^{|I|}} \right| \leq \frac{C_I}{c^{|I|+1} d(x,S)^{\alpha(|I|+1)}} \forall x \in K - S, \] hence $\frac{1}{g_1} \in M(S, n)$. Now for $h \in \mathcal{E}(S, n)$ and $x \notin S$ we have $h(x) = \frac{h(x)}{g_1(x)} g_1(x)$, extend $\frac{h(x)}{g_1(x)}$ to a germ $H$ in $\mathcal{E}(S, n)$ and $H = Hg_1$ in $\mathcal{E}(S, n)\left( \frac{\partial f}{\partial x} \right)$. Therefore we get $\mathcal{E}(S, n) \subseteq \mathcal{E}(S, n)\left( \frac{\partial f}{\partial x} \right)$ and $f$ is $S$-infinitesimally stable.

**Lemma 33** ([W, Lemma 3.3]). Suppose there exist a sequence $(w_i)$ in $J^k(n, 1)$, $k \leq \infty$, a sequence $(x_i)$ in $\mathbb{R}^n - \{0\}$ converging to the origin and a germ $f$ such that $q_i = w_i - j^k f(x_i)$ is flat along $(x_i)$. Then there exists a germ $g$ such that $j^k g(x_i) = w_i$ holds for $(x_i)$ subsequence of $(x_i)$, and $j^\infty g(0) = j^\infty f(0)$.

**Lemma 34.** Suppose there exist a sequence $(w_i)$ in $J^k(n, 1)$, $k \leq \infty$, a sequence $(x_i)$ in $\mathbb{R}^n - S$ converging to zero and a germ $f \in \mathcal{E}(n)$ such that $(q_i) = (w_i - j^k f(x_i))$ is flat along $(d(x_i, S))$, where $S$ is a closed subset of $\mathbb{R}^n (0 \in S)$. Then there exists a germ $g \in \mathcal{E}(n)$, such that $j^\infty g(x) = j^\infty f(x) \forall x \in S$ and $j^k g(x_i) = w_i$ holds for a subsequence of $(x_i)$.

**Proof.** If $k$ is finite, then we transform $q_i$ into an $\infty$-jet in such a way that all the terms of order greater than $k$ of $q_i$ are zero. Thus we will assume $k = \infty$.

We define $Q$, a Taylor field, by $q_i$ at $x_i$ and by the zero series on $S$. We want to show that $Q$ is a $C^\infty$ Whitney field. It is enough to show (Proposition 1.5 of Chapter IV, [T]) for each $m$ and each multi-index $I$ with $|I| \leq m$, that $(R^m(y)Q)^I(x) = o(|x - y|^{m-|I|})$, where $(R^m_y Q)^I(x) = Q^I(x) - \sum_{|I| = m-|I|} Q^{I+L(y)}(x-y)^L$.

If $\{x, y\} \subseteq S$ then the proof is obvious. In the case $\{x, y\} \subseteq \{x_i\} \cup \{0\}$ we proceed as in the proof of Lemma 3.3 of [W]. If $\{x, y\} = \{x_j, s\}$, $s \in S$, we use the flatness of $(q_i)$ along $(d(x_i, S))$ to obtain for each natural number $l$ another $N(l)$ such that $(R^m_{x_j})^I(x_j) = |q_j^I| \leq d(x_j, S)^I \leq d(x_j, s)^I$ and $|(R^m_{x_j})^I(s)| \leq \sum_{|L| = m-|I|} q_j^{I+L}(x_j)^L \leq C d(x_j, s)^I$ for $j \geq N(l)$, where $C$ is a positive real number depending only on $m$ and $I$. Let $l = m + 1$.

Hence, using Whitney Extension Theorem (Theorem 3.1 of Chapter IV, [T]), there exists a smooth germ $q$ such that $j^\infty q(x) = 0 \forall x \in S$ and $j^\infty q(x_i) = q_i$. If $g = f + q$, we see that $g$ has the desired properties.

**Theorem 35.** Let $f$ be a germ, $S$ a closed subset of $\mathbb{R}^n$ and $\hat{0}$ an accumulation point of $S$. Hence if $f$ is $S$-infinitely determined, then $f$ satisfies a Jacobi-Lojasiewicz condition for $S$. 
Proof. We shall prove the theorem by contradiction. Then there is a sequence \((x_j)\) in \(\mathbb{R}^n - S\) converging to the origin such that \(|\nabla f(x_j)|\) is flat along \((d(x_j, S))\). Choose \((y_j)\) a sequence of regular values of \(f\) converging to zero and such that \((f(x_j) - y_j)\) is flat along \((d(x_j, S))\). It clearly follows that \((y_j, 0) - (f(x_j), \nabla f(x_j))\) is flat along \((d(x_j, S))\).

If we denote \(j = (y_j, 0) - (f(x_j), \nabla f(x_j))\) and setting \(k = 1\) in the previous lemma, there exists a germ \(g\) such that \(j^1 g(x_j) = (y_j, 0)\) and \(g - f \in \mathcal{E}(S, n)\). Now since \(f\) is \(S\)-infinitely determined, \(f\) and \(g\) must have the same critical and regular values, which is not the case, since the points \(y_j\) are regular values for \(f\) but critical values for \(g\).

As a consequence of Theorems 24, 32 and 35 we get the main theorem of part II:

**Theorem 36.** Let \(f \in \mathcal{E}(n)\). The concepts of \(S\)-infinitesimally stability, \(S\)-infinite determinacy and the Jacobi-Łojasiewicz condition at \(S\) are equivalent for the germ \(f\) and \(S\) a germ of a closed subset of \(\mathbb{R}^n\) with \(\hat{0}\) an accumulation point of \(S\).

### 3. A special case.

**Definition 37.** Let \(S\) be a germ of a closed subset of \(\mathbb{R}^n\) such that \(\hat{0} \in cl(\text{int})\). We say that a germ \(f\) is \(S\)-stable, if given a germ \(g\) such that \(g(x) = f(x) \forall x \in S\), there exists a germ of a diffeomorphism \(\phi \in G_S\) such that \(g = f \circ \phi\).

Note that if \(cl(\text{int}) = S\), the previous definition is apparently much stronger than Definition 23. In this case \(f(x) = g(x) \forall x \in S\) and \(j^\infty g(x) = j^\infty f(x) \forall x \in S\) are equivalent but now we restrict ourselves to the group \(G_S\), hence the diffeomorphism must be the identity on \(S\).

**Example 38.**

(a) Let \(S = \{(x, y) \in \mathbb{R}^2 | x \leq 0 \text{ and } y = 0\}\), then \(S\) is closed but \(\hat{0} \notin cl(\text{int}S)\).

(b) Let \(S = \{(x, y) \in \mathbb{R}^2 | x^4 - x^3 - xy^2 \geq 0\}\), in this case \(S = cl(\text{int}S)\).

(c) Let \(S = \{(x, y, z) \in \mathbb{R}^3 | (x^2 + y^2) - x^3 \leq 0\}\), in this case \(\hat{0} \in cl(\text{int}S)\) but clearly \(cl(\text{int}S) \neq S\).

For \(S\) any germ of subset of \(\mathbb{R}^n\)containing the origin, we let \(C_S(\mathbb{R}^n)\) be the \(\mathbb{R}\)-algebra of germs constant at \(S\). It is a local algebra with maximal ideal \(m(S)\) consisting of germs of \(C_S(\mathbb{R}^n)\) vanishing at \(S\). In fact \(m(S)\) is an ideal of \(\mathcal{E}(n)\).

**Remark 7.** If \(f \in m(S)\) and \(S = cl(\text{int}S)\) we have \(f \in m(n)^\infty\) and \(\frac{\partial |f|}{\partial x_I} \in m(S)\) for all multi-index \(I\). We also get in this case the equality \(m(S) = m(n)^\infty m(S)\).
**Lemma 39.** Let $S$ a subset of $\mathbb{R}^n$. Suppose $S_0$ is a nonempty open subset of $S$. Then $\text{cl}(S_0) = \text{cl}(\text{int} S)$ if and only if $\text{int}(S - S_0) \subseteq \text{cl}(S_0)$.

**Proof.** We decompose int $S$ in the following way: $\text{int} S = S_0 \cup (S - S_0) \cup T$, where $\text{int} T = \emptyset$. Then $\text{cl}(\text{int} S) = \text{cl}(S_0) \cup \text{cl}(\text{int}(S - S_0))$, since $\text{cl}(T) \subseteq \text{cl}(S_0 \cup \text{int}(S - S_0))$. Hence $\text{cl}(\text{int} S) = \text{cl}(S_0)$ if and only if $\text{cl}(\text{int}(S - S_0)) \subseteq \text{cl}(S_0)$ and this is equivalent to $\text{int}(S - S_0) \subseteq \text{cl}(S_0)$.

**Definition 40.** Let $A$ be a closed subset of $\mathbb{R}^n$. We say that $A$ is good if there exists a locally finite partition $P$ of $A$ into $C^0$-submanifolds of $\mathbb{R}^n$, called strata, such that if $X \in P$ and $\dim X < n$, then there exists a nonvoid open stratum $Y \in P$ such that $X \subset \text{cl}(Y)$.

We clearly have the next:

**Proposition 41.** Suppose that $S$ is a good subset of $\mathbb{R}^n$. Then $\text{cl}(S) = \text{cl}(\text{int} S)$.

Joining Lemma 39 and Proposition 41 we get the following:

**Proposition 42.** Let $P_1, \ldots, P_s$ be real continuous functions on $\mathbb{R}^n$ such that $S = \{x \mid P_i(x) \leq 0 \; \forall \; i\}$ is good and define $S_0 = \{x \mid P_i(x) < 0 \; \forall \; i\}$. Suppose $\text{int}(S - S_0) \subseteq \text{cl}(S_0)$. Then $\text{cl}(S_0) = S$.

**Remark 8.** If $P_1, \ldots, P_s$ are real analytic functions on $\mathbb{R}^n$ then $S = \{x \mid P_i(x) \leq 0 \; \forall \; i\}$ will be good if we have for a decomposition of $S$, that whenever $T$ is a stratum of lower dimension, then there exists a nonempty open stratum $T'$ such that $T \subset \text{cl}(T')$. Obviously there are more examples of good sets than the semianalytical ones. For this purpose see for instance Sections 1 and 2 of [V-M].

We remind here the following:

**Definition 43.** Suppose that $S$ is a closed subset of $\mathbb{R}^n$ containing the origin and such that $S = \text{cl}(\text{int} S)$. We say that $f$ is $S$-infinitesimally stable if $m(S) \subseteq \langle \frac{\partial f}{\partial x_1} \rangle m(S)$.

**Theorem 44.** Suppose $S$ is a closed subset of $\mathbb{R}^n$ such that $\overline{0} \in S$ and $S = \text{cl}(\text{int} S)$. If $f$ is $S$-infinitesimally stable then $f$ is $S$-stable.

**Proof.** Following the proof of Theorem 24 we start with $g$ a germ such that $g(x) = f(x) \; \forall \; x \in S$, therefore $\frac{\partial g}{\partial x'}(x) = \frac{\partial f}{\partial x'}(x) \; \forall \; x \in S$, and we arrive to the inclusion $m(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x'} \rangle C_{S \times \mathbb{R}}(\mathbb{R}^n \times \mathbb{R}) \subseteq m(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x'} \rangle C_{S \times \mathbb{R}}(\mathbb{R}^n \times \mathbb{R})$.

Since $\frac{\partial f}{\partial t} = g - f \in m(S \times \mathbb{R}) \subseteq m(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x'} \rangle C_{S \times \mathbb{R}}(\mathbb{R}^n \times \mathbb{R})$, then $\frac{\partial f}{\partial t}(x,t) = \sum_{i=1}^n h_i(x,t) \frac{\partial f}{\partial x_i}(x,t)$, with $h_i(x,t) \in m(S \times \mathbb{R})$, hence $h_i(x,t) = 0 \; \forall \; (x,t) \in S \times \mathbb{R}$. When we integrate, the required diffeomorphism will belong to $G_S$. 

$\square$
Proposition 45. If \( f \in \mathcal{E}(n) \) is a finitely (infinitely) determined on the right, then \( f \) is \( S \)-infinitesimally stable and therefore \( S \)-stable for \( S = \text{cl}(\text{int} \ S) \).

Proof. Since \( m(S) = m(n)\infty m(S) \) and \( m(n)^k \subseteq \left( \frac{\partial f}{\partial x_i} \right) \) for some \( k \leq \infty \), we get that \( m(S) \subseteq \left( \frac{\partial f}{\partial x_i} \right) m(S) \). We now use Theorem 44. □

Definition 46. Let \( P \) be a polynomial in variables \( x_1, \ldots, x_n \). We say that \( P \) is quasihomogeneous of degree \( l \) and weights \( k_1, \ldots, k_n \) if \( P(t^{k_1}x_1, \ldots, t^{k_n}x_n) = t^l P(x_1, \ldots, x_n) \).

For \( P \) quasihomogeneous we get \( \frac{\partial P}{\partial x_j}(t^{k_1}x_1, \ldots, t^{k_n}x_n) = t^{l-kj} \frac{\partial P}{\partial x_j}(x_1, \ldots, x_n) \).

Also if we write \( P = \sum a_I x^I \), for a quasihomogeneous polynomial we obtain for any multi-index \( I = (i_1, \ldots, i_n) \), \( i_1k_1 + \ldots + i_nk_n = l \) \( (a_I \neq 0) \).

Theorem 47. Let \( P(x) \) be a quasihomogeneous polynomial and \( S \) a closed subset of \( \mathbb{R}^n \) containing the origin and such that \( S = \text{cl}(\text{int} \ S) \). Suppose that \( m(S) \subseteq \langle P \rangle \) and that \( z(P) \cap \text{int} \ S = \phi \). Then \( P \) is \( S \)-infinitesimally stable.

In the case \( S = \{ x | P(x) \leq 0 \} \) is a good semialgebraic set, we can skip the equality \( z(P) \cap \text{int} \ S = \phi \).

Proof. By hypothesis we get \( m(S) \subseteq \langle P \rangle \) and \( P \in \left( \frac{\partial P}{\partial x_i} \right) \), this together with \( z(P) \cap \text{int} \ S = \phi \) give the result using Example 26. For the second part it is obvious that \( z(P) \cap \text{int} \ S = \phi \) since \( S \) is a good semialgebraic set. □

As in the previous section, we get the following:

Theorem 48. Let \( f \in \mathcal{E}(n) \), \( S \) be a closed subset of \( \mathbb{R}^n \) such that the origin is an accumulation point of \( S \) and \( S = \text{cl}(\text{int} \ S) \). Then the concepts for \( f \) of \( S \)-infinitesimally stability, \( S \)-stability and the Jacobi-Lojasiewicz condition for \( S \) are equivalent.

Proof. Our Theorem 44 shows that \( S \)-infinitesimally stability implies \( S \)-stability. Now as in Theorem 32, we show that the Jacobi-Lojasiewicz condition at \( S \) implies \( S \)-infinitesimally stability. Since Lemma 34 is true for any closed subset of \( \mathbb{R}^n \), the proof of Theorem 35 will be true in the case \( S = \text{cl}(\text{int} \ S) \), and hence \( S \)-stability implies the Jacobi-Lojasiewicz condition of \( f \) for \( S \).

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