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EXTENDING MAPS OF A CANTOR SET PRODUCT WITH AN ARC TO NEAR HOMEOMORPHISMS OF THE 2-DISK

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We prove that a positive entropy map of the product of a Cantor Set and an arc (which covers a homeomorphism) cannot be "embedded" into a near homeomorphism of the 2-disk. Thus a theorem of M. Brown cannot be used to embed the induced shift map on the corresponding inverse limit space into a 2-disk homeomorphism.

1. Introduction.

In 1990, M. Barge and J. Martin [**BM90**] proved that the shift map on the inverse limit space $([0,1], f)$, for any map $f : [0,1] \to [0,1]$, can be realized as a global attractor in the plane. In 1960, M. Brown [**Bro60**] proved that the inverse limit space of any near homeomorphism (Definition 1.2) of a compact metric space is homeomorphic to the original space. M. Barge and J. Martin prove that, for all such f, there exists an embedding $h : [0,1] \to D^2$ such that $h \circ f \circ h^{-1}$ can be extended to a near homeomorphism of the 2-disk, D^2. They then use M. Brown's theorem to extend the induced shift homeomorphism on $h([0,1])$ to a homeomorphism of D^2. With care in the construction of the near homeomorphism of D^2, the inverse limit space $(h([0,1]), h \circ f \circ h^{-1})$ becomes a global attractor.

The main goal of this paper is to show that analogous techniques for maps, $F : C \times [0,1] \to C \times [0,1]$, where C is a Cantor set, $F(x,y) = (F_1(x), F_2(x,y))$ is a surjective map with positive topological entropy (Definition 1.4), and F_1 is a homeomorphism, do not work; no near homeomorphic extension of $h \circ F \circ h^{-1}$ to D^2 exists for any embedding $h : C \times [0,1] \to D^2$ (Theorem 3.1). In our terminology, such F cannot be "embedded" into any 2-disk homeomorphism (Definition 1.1). In the proof of Theorem 3.1 one first assumes that h is a "tame" embedding (Definition 3.1). But in recent work, R. Walker proves that all embeddings of $C \times [0,1]$ into D^2 are tame [**Wal**].

Our study of maps of $C \times [0,1]$ and their embeddings has links to a central problem in the dynamical systems of positive entropy homeomorphisms of compact surfaces.

Does there exist a C^1 positive entropy 2-disk diffeomorphism without shifts?

In 1980, A. Katok [**Kat80**] proved that all $C^{1+\alpha}$, $\alpha > 0$, positive entropy diffeomorphisms of a compact surface, have transverse homoclinic points. So some power of such a diffeomorphism restricts to a shift map of finite type. The next year M. Rees announced a minimal positive entropy homeomorphism of the 2-torus [**Ree81**]. So her homeomorphism has no periodic orbits thus no shifts. Though not in print, it appears that techniques M. Rees used can be adapted to build a positive entropy 2-disk homeomorphism which has a fixed point and no other periodic orbits. The C^1 case remains open. In 1993, M. Barge and R. Walker built a chainable continuum which is the inverse limit space of a map of a Cantor comb [**BW93**]. The map restricted to each "tooth" was a tent map over an adding machine base map. The induced shift homeomorphism has positive entropy but all periodic orbits are period a power of 2. Thus no shifts are present. All chainable continua can be embedded into the 2-disk [**Bin62**]. Although their Cantor comb map can be embedded into a near homeomorphism of the 3-ball, it cannot be embedded into a near homeomorphism of the 2-disk. (To prove this M. Barge and R. Walker rely on properties of the adding machine base map.) So their induced shift homeomorphism cannot be used to build a new Rees-type 2-disk homeomorphism. By our Theorem 3.1, a much larger class of maps (all positive entropy maps of $C \times [0,1]$ which cover any homeomorphism) has the same drawback.

In Section 2 we show that if $F : C \times [0,1] \to C \times [0,1]$ is a surjective map such that $F(x,y) = (F_1(x), F_2(x,y))$, F_1 is a homeomorphism and $F_2(x_0,\cdot) : [0,1] \to [0,1]$ is nonmonotone (Definition 1.3) for some x_0, then there exists no embedding of F into a near homeomorphism (Definition of the 2-disk). We will show this by assuming such a near homeomorphism does exist and then obtaining a contradiction using a result of S. Schwartz [**Sch92**] (Theorem 1.1) concerning nonmonotone maps.

Unless otherwise specified X, and Y are compact metric spaces. And π_1 and π_2 on $X \times Y$ are the first and second coordinate projection maps.

Definition 1.1. A map $f : X \to X$ can be *embedded* into the map $F : Y \to Y$ if there exists a topological embedding $h : X \to Y$ such that $F|_{h(X)} = h \circ f \circ h^{-1}$.

Definition 1.2. A map $f : X \to Y$ is called a *near homeomorphism* provided there exists a sequence $\{f_k : X \to Y\}_{k=1}^{\infty}$ of homeomorphisms which uniformly converge to f.

Definition 1.3. A map $f : X \to Y$ is *monotone* provided $f^{-1}(V)$ is connected, whenever $V \subset Y$ is connected.

Theorem 1.1 (S. Schwarts [**Sch92**])**.** *Suppose that X is a locally connected compact metric space. If $f : X \to X$ is a near homeomorphism then f is monotone.*

As mentioned, in Section 3 we show that if $F : C \times [0,1] \to C \times [0,1]$ is a surjective map with positive topological entropy (Definition 1.4), which is embedded in the 2-disk, then F cannot be extended to a near homeomorphism of the disk. The proof uses theorems of R. Bowen (Theorem 1.2) [**Bow71**] and M. Barge (Theorem 1.3) [**Bar87**].

Definition 1.4 (Topological Entropy)**.** Suppose that $F : X \times Y \to X \times Y$ is a surjective map and has the form $F(x,y) = (F_1(x), F_2(x,y))$. Fix x_0 and let $\epsilon > 0$. A set $E \subset Y$ is (n,ϵ)*–separated* by $F|_{\pi_1^{-1}(x_0)}$ if for all $y_0, y_1 \in E$, $y_0 \neq y_1$, $d(\pi_2 F^k(x_0,y_0), \pi_2 F^k(x_0,y_1)) > \epsilon$ for some $k \in [0,n)$, where d is the Y–metric. Since Y is compact and $n < \infty$, card $E < \infty$. Let the maximum number of (n,ϵ)-separated orbits for each ϵ be

$$s(n,\epsilon) = \max\left\{ \text{card } E \;\middle|\; \begin{array}{l} E \subset Y \text{ such that} \\ E \text{ is } (n,\epsilon) - \text{separated by } F|_{\pi_1^{-1}(x_0)} \end{array} \right\}.$$

Now, let the growth rate of $s(n,\epsilon)$ (or ϵ-topological entropy) be

$$h_{\text{top}}\left(F|_{\pi_1^{-1}(x_0)}, \epsilon\right) = \limsup_{n\to\infty} \frac{\log s(n,\epsilon)}{n}.$$

Lastly we let $\epsilon \to 0$ and define *topological entropy* for $F|_{\pi_1^{-1}(x_0)}$.

$$h_{\text{top}}\left(F|_{\pi_1^{-1}(x_0)}\right) = \lim_{\epsilon\to 0} h_{\text{top}}\left(F|_{\pi_1^{-1}(x_0)}, \epsilon\right).$$

The topological entropy $h_{\text{top}}(F_1)$ of the homeomorphism F_1 is defined similarly (see [**Bow71**]).

Theorem 1.2 (R. Bowen [**Bow71**])**.** *If $F : X \times Y \to X \times Y$ has the form $F(x,y) = (F_1(x), F_2(x,y))$ then*

$$h_{\text{top}}(F) \leq h_{\text{top}}(F_1) + \sup_{x\in X}\left\{h_{\text{top}}\left(F|_{\pi_1^{-1}(x)}\right)\right\}.$$

If $h_{\text{top}}(F_1) = 0$ then $h_{\text{top}}(F) = \sup_{x\in X}\left\{h_{\text{top}}\left(F|_{\pi_1^{-1}(x)}\right)\right\}.$

Theorem 1.3 (M. Barge [**Bar87**])**.** *If $F : X \times [0,1] \to X \times [0,1]$ has the form $F(x,y) = (F_1(x), F_2(x,y))$, $F_2(x,\cdot) : [0,1] \to [0,1]$ is monotone for each x and $h_{\text{top}}(F_1) = 0$, then $h_{\text{top}}(F) = 0$.*

2. Nonmonotone Maps of the Cantor Set Cross the Interval.

2.1. Preliminaries. Let $C \subset [0,1]$ be a Cantor set. Let $C \times [0,1]$ and $\{\alpha\} \times [0,1] \subset \mathbb{R}^2$ for $\alpha \in C$. The goal of this section is to prove Theorem 2.1 to follow. But first some preliminaries.

2.1.0.1. Assume $F : C \times [0,1] \to C \times [0,1]$ is a surjective map that has the form

$$F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$$

where $F_1 : C \to C$ is a homeomorphism. Furthermore, for a given $\alpha_0 \in C$, $F_2(\alpha_0, y) = t(y)$ where $t : [0,1] \to [0,1]$ is a continuous nonmonotone map (see Figure 1 for an example). Let $\lambda_0 = F_1(\alpha_0)$.

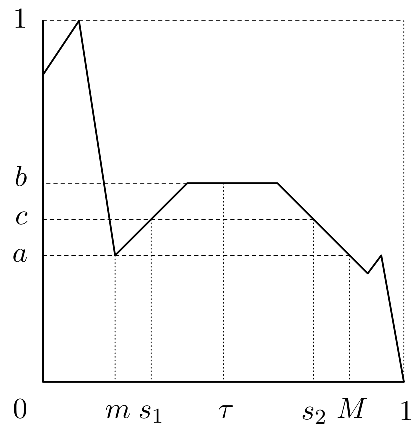

Figure 1. Example of a nonmonotone map.

It will be needed later, that because t is nonmonotone we can find a point that has at least two points in the the pre-image that can be seperated by disjoint epsilon balls. We introduce this idea at this point so that we can use the notation developed here throughout.

2.1.0.2. Since t is nonmonotone and continuous, there is an $a \in (0,1)$ such that $t^{-1}(a)$ is closed and not connected. Thus, there is an interval $(m, M) \subset [0,1] \setminus t^{-1}(a)$ such that $a = t(m) = t(M)$, and $t([m, M]) = [a, b]$ (or $[b, a]$) for some $b \neq a$. Without loss of generality we will assume that $a < b$. Let $\tau \in t^{-1}(b)$. By the intermediate value theorem, $t([m, M]) = [t(m), t(\tau)]$. Now choose $c = \frac{1}{2}(a + b)$. Since t is continuous there are $s_1 \in (m, \tau)$ and $s_2 \in (\tau, M)$ such that $c = t(s_1) = t(s_2)$ (see Figure 1).

2.1.0.3. By the continuity of F, for any $\epsilon > 0$ there is a $\delta_1 = \delta_1(\epsilon) > 0$ such that $F(x, y) \in \mathcal{B}_\epsilon(\lambda_0, t(y))$ when $d(\alpha_0, x) < \delta_1$ and $y \in [0,1]$. Suppose $K_1 = K_1(\epsilon) \in \mathbb{N}$ is such that $\frac{1}{K_1} < \delta_1$.

Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4\}$. Now let $h_0 : C \times [0,1] \to D$ be an arbitrary topological embedding. Then there is a homeomorphism $h_1 : D \to D$ such that $(h_1 \circ h_0)(\alpha_0, y) = (\alpha_0, y)$ and $(h_1 \circ h_0)(\lambda_0, y) = (\lambda_0, y)$ for all $y \in [0,1]$. So h_1 "straightens out" $h_0(\alpha_0 \times [0,1])$ and $h_0(\lambda_0 \times [0,1])$ in a strong sense. Notice that $C \times [0,1] \to D$.

2.1.0.4. By the uniform continuity of $h_1 \circ h_0$, for all $\epsilon > 0$ there is a $\delta_2 = \delta_2(\epsilon) > 0$ such that for all $y \in [0,1]$, $h_1 \circ h_0(x, y) \in \mathcal{B}_\epsilon(\alpha_0, y)$ and $h_1 \circ h_0(x', y) \in \mathcal{B}_\epsilon(\lambda_0, y)$, for all $(x, y) \in \mathcal{B}_{\delta_2(\epsilon)}(\alpha_0, y)$ and $(x', y) \in \mathcal{B}_{\delta_2(\epsilon)}(\lambda_0, y)$. Let $K_2 = K_2(\epsilon) \in \mathbb{N}$ be such that $\frac{1}{K_2} < \delta_2$.

2.1.0.5. With a, b defined as in [2.1.0.2], let $\hat{d} = \min\{a, 1-b, |a-b|\}$. For $0 < \epsilon_0 < \frac{\hat{d}}{100}$ choose $0 < \delta_0 \leq \min\left\{\delta_1(\epsilon_0), \delta_2(\epsilon_0), \frac{M-m}{100}\right\}$. So in particular [2.1.0.3] and [2.1.0.4] are satisfied. Note that $t([m, M]) \subset [\epsilon_0, 1-\epsilon_0]$. Let $K_0 \geq \max\{K_1(\epsilon_0), K_2(\epsilon_0)\}$ be such that $\frac{1}{K_0} < \delta_0$. Since C is perfect, there is a sequence $\{\alpha_k\} \subset C$ such that $\alpha_k \to \alpha_0$ as $k \to \infty$, and $\alpha_k \times [0,1] \subset \mathcal{N}_{\delta_0}(\alpha_0 \times [0,1])$, for all $k > K_0$. Let $\lambda_k = F_1(\alpha_k)$. (Note that $\mathcal{N}_\delta(S)$ is a $\delta-$neighborhood of S.) It follows that $\lambda_k \to \lambda_0$ as $k \to \infty$ and $\lambda_k \times [0,1] \subset \mathcal{N}_{\epsilon_0}(\lambda_0 \times [0,1])$ for all $k > K_0$. For a possibly larger K_0, also called K_0, and $o_k \in \mathcal{B}_{\epsilon_0}(\lambda_0, c)$, $k > K_0$, there exist $q_1(k)$ and $q_2(k)$ such that $\{q_1(k), q_2(k)\} \subset F^{-1}(o_k)$, $q_1(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)$ and $q_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2)$.

We now state our first theorem.

2.1.1. Nonmonotone Nonextension Theorem.

Theorem 2.1. *Let $F : C \times [0,1] \to C \times [0,1]$ be a map of the form $F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$ where $F_1 : C \to C$ is a homeomorphism. Furthermore, assume $F_2(\alpha_0, \cdot) : [0,1] \to [0,1]$ is surjective but not monotone for some α_0. Then there exists no extension of $h_0 \circ F \circ h_0^{-1}$ to a near homeomorphism of the disk D, for any topological embedding $h_0 : C \times [0,1] \to D$.*

Proof. Assume $h, K_0, \epsilon_0, \delta_0, \{\alpha_k\}, \{\lambda_k\}, q_1(k)$, and $q_2(k)$ are defined as in [2.1.0.1-5]. Suppose that $H_0 : D \to D$ is a near homeomorphism such that $H_0|_{h_0(C \times [0,1])} = h_0 \circ F \circ h_0^{-1}$. And let $H_1 : D \to D$ be given by $H_1 = h_1 \circ H_0 \circ h_1^{-1}$. Thus H_1 is also a near homeomorphism. So the diagram in Figure 2 commutes.

$$\begin{tikzcd}
C \times [0,1] \arrow[r, "F"] \arrow[d, "h_0"'] & C \times [0,1] \arrow[d, "h_0"] \\
D \arrow[r, "H_0"] \arrow[d, "h_1"'] & D \arrow[d, "h_1"] \\
D \arrow[r, "H_1"] & D
\end{tikzcd}$$

Figure 2. Commuting Diagram.

2.1.1.1. Let $\Lambda(\alpha) = h_1 \circ h_0 (\alpha \times [0,1])$ for all $\alpha \in C$. By [2.1.0.3] $h_1 \circ h_0$ is a homeomorphism and if $(\{\alpha\} \times [0,1]) \bigcap (\{\lambda\} \times [0,1]) = \emptyset$ (when $\alpha \neq \lambda$), then $\Lambda(\alpha) \bigcap \Lambda(\lambda) = \emptyset$. Let ℓ_β be the horizontal line $\{y = \beta\}$. And let $\ell^\alpha_\beta(k) =$

$\Lambda(\alpha_k)\bigcap \ell_\beta$ and $\ell^\lambda_\beta(k) = \Lambda(\lambda_k)\bigcap \ell_\beta$. Because $h_1 \circ h_0\,(\alpha_k, 0) \in \mathcal{B}_{\delta_0}(\alpha_0, 0)$, $h_1 \circ h_0\,(\alpha_k, 1) \in \mathcal{B}_{\delta_0}(\alpha_0, 1)$, and $\Lambda(\alpha_k)$ is connected, then $\ell^\alpha_\beta(k) \neq \emptyset$ and all $k \geq K_0$ (see Figure 3 and [2.1.0.2]). Similarly $\ell^\lambda_\beta(k) \neq \emptyset$, for all $\beta \in [\epsilon_0, 1-\epsilon_0]$ and $k \geq K_0$. Note that if $p \in \ell^\lambda_\beta(k)$ for given $k \geq K_0$ then $p \in \mathcal{B}_{\epsilon_0}(\lambda_0, \beta)$.

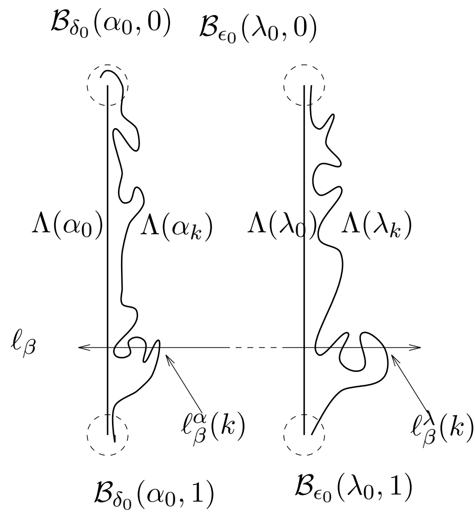

Figure 3. Intersection of $\Lambda(\alpha_k)$ with ℓ_β.

Lemma 2.1 follows from the continuity of h_1, h_0, and π_1.

Lemma 2.1. *Choose $p_k \in \ell^\alpha_\beta(k)$ for each k. Then $\pi_1 p_k \to \alpha_0$ as $k \to \infty$.*

Notice that $\pi_1(h_1 \circ h_0)\left(\alpha_k, \frac{1}{2}\right) \neq \alpha_0$ for sufficiently large k. So either

$$\text{card}\left\{k \middle| \pi_1\left(h_1 \circ h_0\left(\alpha_k, \frac{1}{2}\right)\right) > \alpha_0\right\} = \infty$$

or

$$\text{card}\left\{k \middle| \pi_1\left(h_1 \circ h_0\left(\alpha_k, \frac{1}{2}\right)\right) < \alpha_0\right\} = \infty.$$

2.1.1.2. So without loss of generality we may assume there exist distinct $\{k_n\}_{n=1}^\infty$ such that $k_n \to \infty$ as $n \to \infty$, and

$$\pi_1\left(h_1 \circ h_0\left(\alpha_{k_n}, \frac{1}{2}\right)\right) > \alpha_0.$$

2.1.1.3. For the sake of simplicity we denote α_{k_n} by α_n, $\Lambda(\alpha_{k_n})$ by $\Lambda(\alpha_n)$, $\Lambda(\lambda_{k_n})$ by $\Lambda(\lambda_n)$ and $\ell^{k_n}_\beta$ by ℓ^n_β.

Lemma 2.2. *Let N_0 be such that $k_n \geq K_0$ for all $n \geq N_0$. Then*

$$\Lambda(\alpha_n) \bigcap \left\{ \left(x, \frac{1}{2} \right) \middle| x < \alpha_0 \right\} = \emptyset.$$

Proof. Fix $n \geq N_0$ and assume there exists

$$p_1 \in \Lambda(\alpha_n) \bigcap \left\{ \left(x, \frac{1}{2} \right) \middle| x < \alpha_0 \right\},$$

and let $p_2 = \left(h_1 \circ h_0 \left(\alpha_{k_n}, \frac{1}{2}\right)\right)$. By [2.1.1.2] $\pi_1(p_2) > 0$. Let A be the arc in $\Lambda(\alpha_n)$ with end points p_1 and p_2. By [2.1.1.1], $p_1, p_2 \in \mathcal{B}_{\epsilon_0}\left(\alpha_0, \frac{1}{2}\right)$. So by [2.1.0.5],

$$d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_2)) < \delta_0.$$

Since $\Lambda(\alpha_n) \bigcap \Lambda(\alpha_0) = \emptyset$, then using a Jordan Curve argument, it follows

$$A \bigcap \{(\alpha_0, y) | y > 1 \text{ or } y < 0\} \neq \emptyset.$$

Let $p_3 \in A \bigcap \{(0, y) | y > 1 \text{ or } y < 0\}$. So $d(p_1, p_3) > \frac{1}{2}$. But because $p_3 \in A$, either

$$\pi_2 \circ (h_1 \circ h_0)^{-1}(p_1) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_2)$$

or

$$\pi_2 \circ (h_1 \circ h_0)^{-1}(p_2) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_1).$$

In either case $d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_3)) < \delta_0$. And so $d(p_1, p_3) < \epsilon_0$ which is a contradiction. □

2.1.1.4. Assume $n \geq N_0$. Let $g_n : [0, 1] \to \Lambda(\alpha_n)$ be the parameterization of $\Lambda(\alpha_n)$ defined by $g_n(\beta) = h_1 \circ h_0 (\alpha_n, \beta)$. Using τ, m from [2.1.0.2] $\Lambda(\alpha_n) \bigcap \ell_\tau \neq \emptyset$ and $\Lambda(\alpha_n) \bigcap \ell_m \neq \emptyset$ (see Figure 4) so by Lemma [2.2] and the connectivity of $\Lambda(\alpha_n)$ there is a largest β, call it β_n^-, such that $g_n(\beta_n^-) \in \ell_m$. Let $a_n = g_n(\beta_n^-)$ (see Figure 4). Similarly there is a smallest β, call it β_n^+, such that $g_n(\beta_n^+) \in \ell_m$. Let $b_n = g_n(\beta_n^+)$.

2.1.1.5. If necessary, renumber the k_n's so that if $k_n < k_{n+1}$ then $\pi_1(a_n) > \pi_1(a_{n+1})$. It follows by an argument similar to that of Lemma [2.2] that $\pi_1(b_n) > \pi_1(b_{n+1})$. (Because $h_1 \circ h_0$ may have scrambled the $C \times [0, 1]$ order in the first coordinate, it may be necessary to relabel the k_n's so that $\Lambda(\alpha_{k_n})$ to be "between" $\Lambda(\alpha_{k_{n-1}})$ and $\Lambda(\alpha_{k_{n+1}})$.)

Considering [2.1.1.5] and [2.1.1.2] and to simplify the notation assume

$$\text{card} \left\{ k | \pi_1 (h_1 \circ h_0) \left(\alpha_k, \frac{1}{2} \right) > \alpha_0 \right\} = \infty$$

and $\pi_1(a_k) > \pi_1(a_{k+1})$ for all k.

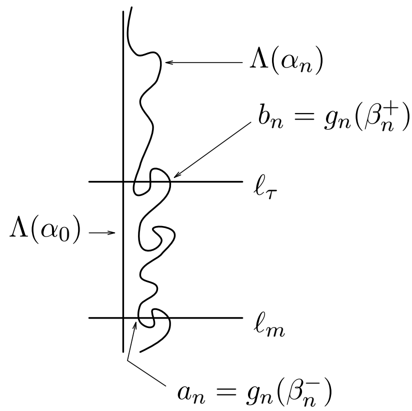

Figure 4. First and Last Intersections.

Using [2.1.1.4], for $k \geq N_0$ define the four curves $I\,(k,m)$, $I\,(k,\tau)$, J_{k-1}, and J_{k+1} in the following manner (see Figure 5). Let $I\,(k,m)$ be the line segment in ℓ_m between a_{k+1} and a_{k-1} and $I\,(k,\tau)$ be the line segment in ℓ_τ between b_{k+1} and b_{k-1}. Let

$$J_{k-1} = \left\{ g_{k-1}(\beta) \,\middle|\, \beta_{k-1}^- \leq \beta \leq \beta_{k-1}^+ \right\}, \text{ and}$$
$$J_{k+1} = \left\{ g_{k+1}(\beta) \,\middle|\, \beta_{k+1}^- \leq \beta \leq \beta_{k+1}^+ \right\}.$$

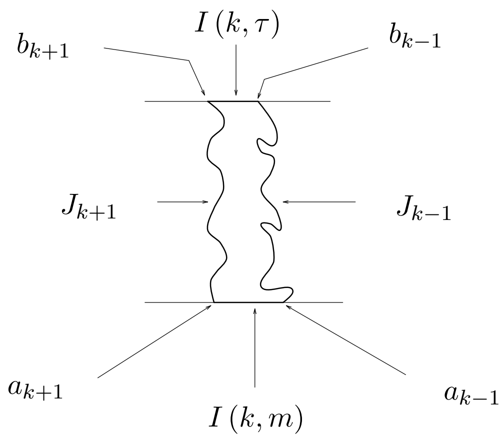

Figure 5. The Boundary of R_k.

Lemma 2.3. *$I\,(k,\tau) \bigcup J_{k-1} \bigcup I\,(k,m) \bigcup J_{k+1}$ is a simple closed curve.*

Proof. Since $\Lambda(\alpha_{k-1}) \bigcap \Lambda(\alpha_{k+1}) = \emptyset$ we have $J_{k-1} \bigcap J_{k+1} = \emptyset$. By [2.1.1.1] $I\,(k,m) \bigcap I\,(k,\tau) = \emptyset$. And by [2.1.1.4] we have that

$$a_{k-1} = J_{k-1} \bigcap I\,(k,m) \text{ and } a_{k+1} = J_{k+1} \bigcap I\,(k,m)$$

and

$$b_{k-1} = J_{k-1} \bigcap I(k,\tau) \text{ and } b_{k+1} = J_{k+1} \bigcap I(k,\tau).$$

And so the lemma follows. □

Let R_k be the closed and bounded set with boundary

$$I(k,m) \bigcup J_{k-1} \bigcup I(k,\tau) \bigcup J_{k+1}$$

(see Figure 5). Recall from [2.1.0.2] that $s_1 \in [m,\tau]$ and from [2.1.1.1] that $\ell_{s_1}^{\alpha}(k) = \Lambda(\alpha_k) \bigcap \ell_{s_1}$ (see Figure 3).

Lemma 2.4. *$R_k \bigcap \ell_{s_1}^{\alpha}(k) \neq \emptyset$.*

Proof. Let γ_k be the arc $\{g_k(\beta) | 0 \leq \beta \leq \beta_k^+\}$. Let $S_k = R_k \bigcap \pi_2^{-1}[s_1,\tau]$ (see Figure 6). So $\partial S_k \supset I(k,\tau)$ and by [2.1.1.5] $b_k \in I(k,\tau)$. But b_k is not an endpoint of $I(k,\tau)$ because the endpoints of $I(k,\tau)$ are b_{k-1} and b_{k+1}. And so there is an $\eta > 0$ such that if $p \in \mathcal{B}_\eta(b_k)$ and $\pi_2(p) < \tau$ then $p \in \text{int } S_k$. Now, if $q \in \gamma_k \setminus \{b_k\}$ then $\pi_2(q) < \frac{1}{2}$. And since γ_k connects $h_1 \circ h_0(\alpha_k, 0)$ to b_k, we have that $(\gamma_k \bigcap \mathcal{B}_\eta(b_k)) \setminus \{b_k\} \neq \emptyset$. Thus there exists $p_0 \in \gamma_k \bigcap \mathcal{B}_\eta(b_k) \bigcap \text{int} S_k$. Let $A_k \subset \gamma_k$ be the arc with endpoints p_0 and $h_1 \circ h_0(\alpha_k, 0)$ (see Figure 6).

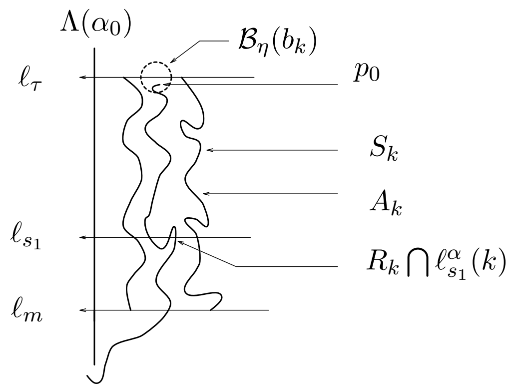

Figure 6. The Arc A_k.

Because $p_0 \in \text{int} S_k$ and $h_1 \circ h_0(\alpha_k, 0) \notin S_k$ then $A_k \bigcap \partial S_k \neq \emptyset$. Since $A_k \bigcap \Lambda(\alpha_{k-1}) = \emptyset$, $A_k \bigcap \Lambda(\alpha_{k+1}) = \emptyset$, $A_k \bigcap I(n,\tau) = \emptyset$ and $\ell_{s_1} \bigcap R_k \subset \partial S_k$, we have that $A_k \bigcap \ell_{s_1} \bigcap R_k \neq \emptyset$, or $R_k \bigcap \ell_{s_1}^{\alpha}(k) \neq \emptyset$. □

2.1.1.6. Note that since $\ell_{s_1}^{\alpha}(k) \bigcap \partial R_k = \emptyset$ then $\ell_{s_1}^{\alpha}(k) \subset \text{int} R_k$.

Lemma 2.5. $\left[\Lambda(\alpha_l) \bigcap H_1^{-1}(h_1 \circ h_0(\lambda_k, y))\right] = \emptyset$ *for* $k \neq l$.

Proof. Suppose that $\rho \in \Lambda(\alpha_l) \bigcap H_1^{-1}(h_1 \circ h_0(\lambda_k, y))$ for $k \neq l$. Then $H_1(\rho) = h_1 \circ h_0(\lambda_k, y)$ But $H_1\Lambda(\alpha_l) = \Lambda(\lambda_l)$. Thus $H_1(\rho) \in \Lambda(\lambda_l)$ So $h_1 \circ h_0(\lambda_k, y) \in \Lambda(\alpha_l)$ Or $(\lambda_k, y) \in C_{\lambda_l}$. Which contradicts, [2.1.1.1] since $k \neq l$. $\square$

Proof of Theorem 2.1 continued. By Lemma [2.4] there exists $p_1(k) \in R_k \bigcap \ell^{\alpha}_{s_1}(k)$ for all $k \geq N_0$. By [2.1.0.5] $(h_1 \circ h_0)^{-1}(p_1(k)) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)$. Using [2.1.0.5], let $o_k = F \circ (h_1 \circ h_0)^{-1}(p_1(k))$. So there exists $\{q_1(k), q_2(k)\} \subset F^{-1}(o_k)$ such that $q_1(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)$ and $q_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2)$. Choose $q_1(k)$ so that $p_1(k) = h_1 \circ h_0(q_1(k))$. And let $p_2(k) = h_1 \circ h_0(q_2(k))$ and $r_k = h_1 \circ h_0(o_k)$ (see Figure 7). Because $H_1 \circ h_1 \circ h_0 = h_1 \circ h_0 \circ F$ then $\{p_1(k), p_2(k)\} \in H_1^{-1}(r_k)$. By the size of δ_0 chosen in [2.1.0.5], $p_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2) \not\subset R_k$.

Recall that H_0 and H_1 are near homeomorphisms. Near homeomorphisms are monotone on locally connected compact metric spaces ([**Sch92**]). Thus pre-images of connected sets under H_1 are connected. So $H_1^{-1}(r_k)$ is a connected set which contains $p_2(k) \notin R_k$ and by [2.1.1.6] $p_1(k) \in \text{int}R_k$. Then $H_1^{-1}(r_k) \bigcap \partial R_k \neq \emptyset$. By Lemma [2.5] either $H_1^{-1}(r_k) \bigcap I(k, \tau) \neq \emptyset$ or $H_1^{-1}(r_k) \bigcap I(k, m) \neq \emptyset$. So there is an infinite sequence $\{\rho_{k_j}\}$ such that either $\rho_{k_j} \in I(k, \tau) \bigcap H_1^{-1}(r_{k_j})$ or $\rho_{k_j} \in I(k, m) \bigcap H_1^{-1}(r_{k_j})$ for all j (see Figure 7).

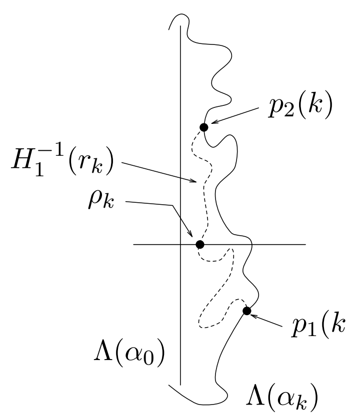

Figure 7. Subsequence and Pre-image.

Now by Lemma [2.1] either $\rho_{k_j} \to h_1 \circ h_0(\alpha_0, \tau)$ or $\rho_{k_j} \to h_1 \circ h_0(\alpha_0, m)$ as $j \to \infty$. Since H_1 is continuous for all j, either

$$H_1\rho_{k_j} \to H_1 \circ h_1 \circ h_0(\alpha_0, \tau) \quad \text{or} \quad H_1\rho_{k_j} \to H_1 \circ h_1 \circ h_0(\alpha_0, m).$$

Because $H_1 \circ h_1 \circ h_0 = h_1 \circ h_0 \circ F$, then either

$$r_{k_j} \to h_1 \circ h_0(\lambda_0, t(\tau)) \text{ or } r_{k_j} \to h_1 \circ h_0(\lambda_0, t(m)).$$

Since $h_1 \circ h_0$ is a homeomorphism either

$$o_{k_j} \to (\lambda_0, b) \quad \text{or} \quad o_{k_j} \to (\lambda_0, a)$$

which is a contradiction since $\{o_{k_j}\} \subset \mathcal{B}_{\epsilon_0}(\lambda_0, c)$. □

3. Positive Entropy Maps of $C \times [0,1]$.

3.1. Introduction. Let $C \subset \mathbb{R}$ be a Cantor set. In this chapter we use the results of Chapter 2 to prove the following:

Theorem 3.1. *Let $F : C \times [0,1] \to C \times [0,1]$ be a surjective map such that $F(a,y) = (F_1(a), F_2(a,y))$, where $F_1 : C \to C$ is a homeomorphism. If $h_{\text{top}}(F) > 0$ then there exists no topological embedding $h_0 : C \times [0,1] \to D \subset \mathbb{R}^2$ such that $h_0 \circ F \circ h_0^{-1}$ extends to a near homeomorphism of the disk D.*

Recall that $\pi_1 : C \times [0,1] \to C$ is the projection map onto the first coordinate. By work of R. Bowen [**Bow71**] we know that $h_{\text{top}}(F) \leq h_{\text{top}}(F_1) + \sup_{a \in C} \left\{ h_{\text{top}} \left(F|_{\pi_1^{-1}(a)} \right) \right\}$. It has been shown by M. Barge and R. Walker [**BW93**] that any near homeomorphism that extends $h_0 \circ F \circ h_0^{-1}$ to the disk must preserve a certain local order on the set of fibers $\{h_0(a \times [0,1]) \,|\, a \in C\}$. But we will show that if $h_{\text{top}}(F_1) > 0$ no such local order is preserved. So in fact $h_{\text{top}}(F_1) = 0$. Using [**Bow71**] and a result of M. Barge [**Bar87**], if $h_{\text{top}}(F) > 0$ then for some $a_0 \in C$, $F_2(a_0, \cdot)$ is a nonmonotone map. Thus by Theorem 2.1, $h_0 \circ F \circ h_0^{-1}$ cannot be extended to a near homeomorphism of the disk.

3.2. Proof of Theorem 3.1.

Definition 3.1 (Tame Embedding)**.** $h_0 : C \times [0,1] \to D \subset \mathbb{R}$ is a *tame embedding* provided there is a homeomorphism $h_1 : D \to D$ such that for all $a \in C$, $h_1 \circ h_0(\{a\} \times [0,1]) = (\{a'\} \times [0,1])$ for some $\{a'\}$. If h_0 is a tame embedding, using a theorem of E. Moise [**Moi77**], we may further require that h_1 has the property: $h_1 \circ h_0(\{a\}, i) = (\{a'\}, i)$ for all a and $i = 0, 1$.

For more information concerning tame embeddings see [**Rus73**] or [**Bin54**].

3.2.1. Proof of Theorem 3.1. All topological embeddings of $C \times [0,1]$ into D^2 are tame [**Wal**]. So it is enough to prove the theorem for all tame embeddings, h_0.

Let h_1 be as in Definition 3.1. Denote by Λ the set $h_1 \circ h_0(C \times [0,1])$ and by $\Lambda(a)$ the set $h_1 \circ h_0(a \times [0,1])$. Note that $\pi_1(\Lambda(a)) = a'$ for some $a' \in \mathbb{R}$. Assume there is a near homeomorphism $H : D \to D$ such that on $C \times I$, $h_1 \circ h_0 \circ F = H \circ h_1 \circ h_0$. Before continuing with the proof, we stop to define a local ordering on $\{\Lambda(a) \,|\, a \in C\}$ and prove a lemma.

3.2.2. Order Definitions and Lemmas. Here we show that H preserves the local order of fibers as defined by M. Barge and R. Walker [**BW93**], which we will write as $<_{bw}$. And it will follow that $F_1 : C \to C$ is a "local order preserving homeomorphism."

Note: Since h_0 is tame one could use the order on $\{\Lambda(a)|a \in C\}$ induced by π_1 in place of $<_{bw}$. That is, $\Lambda(a) < \Lambda(b)$ if $\pi_1\Lambda(a) < \pi_1\Lambda(b)$. Although h_1-dependent, this order may be more natural than the $<_{bw}$ order, and is locally equivalent to it. But in order to show that H preserves such a local order on fibers, one must cycle through the definition of $<_{bw}$ in any case.

Barge-Walker order:

Definition 3.2. For $a, b \in C$ suppose that γ_- and γ_+ are arcs in the plane with the properties:

γ_- has endpoints $h_1 \circ h_0\,(a, 0)$ and $h_1 \circ h_0\,(b, 0)$, and γ_- is otherwise disjoint from $\Lambda(a) \bigcup \Lambda(b)$; γ_+ has endpoints $h_1 \circ h_0\,(a, 1)$ and $h_1 \circ h_0\,(b, 1)$ and γ_+ is otherwise disjoint from $\Lambda(a) \bigcup \Lambda(b)$; and

$$\left(\gamma_- \bigcup \gamma_+\right) \bigcap \left([0, 2] \times \left\{\frac{1}{2}\right\}\right) = \emptyset.$$

Such arcs γ_- and γ_+ will be called *admissible arcs* joining $\Lambda(a)$ and $\Lambda(b)$.

Definition 3.3. Given $a, b \in C$, $a \neq b$, then $\Lambda(a) <_{bw} \Lambda(b)$ if there are admissible arcs joining $\Lambda(a)$ and $\Lambda(b)$, as above, and the orientation $\gamma_- \to \Lambda(b) \to \gamma_+ \to \Lambda(a)$ is positive (counterclockwise) on the simple closed curve $\gamma_- \bigcup \Lambda(b) \bigcup \gamma_+ \bigcup \Lambda(a)$. (See Figure 8.)

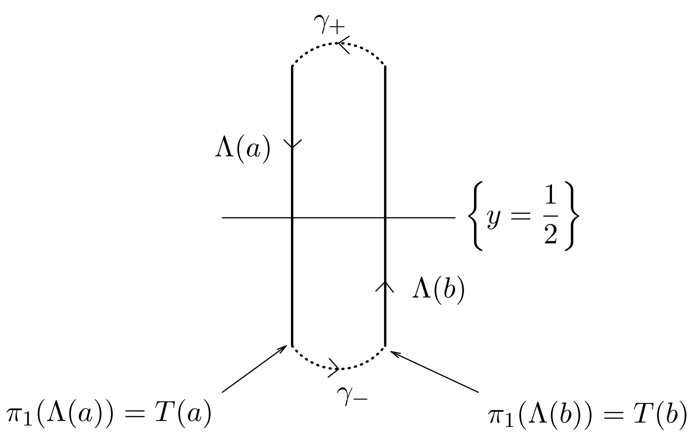

Figure 8. Barge-Walker Ordering on Cantor Fibers.

Definition 3.4. $<_X$ is a *local ordering* on X if for all $x \in X$ there is a $\delta > 0$ such that $<_X$ is an order relation on $\mathcal{B}_\delta(x)$. $(X, <_X)$ is a *locally ordered metric space.*

In [**BW93**] it is shown that if a and b are sufficiently close, $a \neq b$, then such admissible arcs exist. So either $\Lambda(a) <_{bw} \Lambda(b)$ or $\Lambda(b) <_{bw} \Lambda(a)$. Furthermore $<_{bw}$ is a local ordering on $\Lambda = \{\Lambda(a) \,|a \in C\}$ where we use the metric $d(\Lambda(a), \Lambda(b)) = d(a, b)$.

Definition 3.5. Let $a, b \in C$. Then $a <_C b$ provided $\Lambda(a) <_{bw} \Lambda(b)$.

It follows from the proceeding remark and that $h_1 \circ h_0$ is uniformly continuous, that $<_C$ is a local ordering on C.

Definition 3.6. Let $(X, <_X)$ and $(Y, <_Y)$ be locally ordered metric spaces. Let $G : (X, <_X) \to (Y, <_Y)$ be a homeomorphism. G is a *local order preserving homeomorphism*, if there is a $\delta > 0$ such that if $x_0, x_1 \in X$, $|x_0 - x_1| < \delta$, and $x_0 <_X x_1$, then $G(x_0) <_Y G(x_1)$.

Denote by $[x, y] = \{z \in C | x \leq_C z \leq_C y\}$. We next show $<_C$ on C is $\mathbb{R}$-like in the following sense.

Lemma 3.1. *Given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in C$ and $|x - y| < \delta$, then for all $z \in [x, y]$, $|x - z| < \epsilon$ and $|y - z| < \epsilon$.*

Proof. Suppose that $x, y, z \in C$ and $x <_C z <_C y$. By Definition 3.5 there are admissible arcs γ_1^+, γ_1^-, γ_2^+, and γ_2^- such that $\Lambda(z) \to \gamma_1^+ \to \Lambda(x) \to \gamma_1^-$ and $\Lambda(y) \to \gamma_2^+ \to \Lambda(z) \to \gamma_2^-$ have positive orientation.

Sublemma 3.1. *For $\epsilon > 0$ there is a $\delta_1 > 0$ such that if*

$$\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$$

then $|x - z| < \epsilon$.

Proof. By the continuity of $(h_1 \circ h_0)^{-1}$, if $\epsilon > 0$ there is a $\delta_1 > 0$ such that if $d(p, q) < \delta_1$ where $p, q \in \Lambda$ then $d((h_1 \circ h_0)^{-1}(p), (h_1 \circ h_0)^{-1}(q)) < \epsilon$. So if $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$ there is $p \in \Lambda(x)$, $q \in \Lambda(z)$ such that $d(p, q) < \delta_1$. Thus $|x-z| = |\pi_1((h_1 \circ h_0)^{-1}(p)) - \pi_1((h_1 \circ h_0)^{-1}(q))| \leq d((h_1 \circ h_0)^{-1}(p), (h_1 \circ h_0)^{-1}(q)) < \epsilon$. ☐

Choose $\delta_1 > 0$ smaller so that if

$$\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset \text{ and}$$

$$\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(y)) \neq \emptyset$$

then $|x - z| < \epsilon$ and $|y - z| < \epsilon$.

By the continuity of $(h_1 \circ h_0)$ there is $\delta > 0$ such that if $|x - y| < \delta$ then $\Lambda(x) \subset \mathcal{N}_{\delta_1}(\Lambda(y))$ and $\Lambda(y) \subset \mathcal{N}_{\delta_1}(\Lambda(x))$.

Suppose that $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \bigcap \mathcal{N}_{\delta_1}(\Lambda(y)) = \emptyset$. So either $\pi_1\Lambda(z) < \pi_1\Lambda(x)$ or $\pi_1\Lambda(y) < \pi_1\Lambda(z)$. Thus either $\Lambda(z) \to \gamma_1^+ \to \Lambda(x) \to \gamma_1^-$ has negative orientation or $\Lambda(z) \to \gamma_2^+ \to \Lambda(y) \to \gamma_2^-$ has positive orientation which contradicts $x <_C z <_C y$.

Thus $\Lambda(z)\bigcap\mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$ and $\Lambda(z)\bigcap\mathcal{N}_{\delta_1}(\Lambda(y)) \neq \emptyset$. So by the choice of δ then $|x-z|<\epsilon$ and $|y-z|<\epsilon$ as desired. $\square$

Lemma 3.2. *Let $f:(C,<_C)\to(C,<_C)$ be a local order preserving homeomorphism. Then there is a $\delta>0$ such that if $|x-y|<\delta$ then $f([x,y])=[f(x),f(y)]$.*

Proof. By Definition 3.6 there is an $\epsilon>0$ such that for any $x,y\in C$ if $|x-y|<\epsilon$, and $x<_C y$ then $f(x)<_C f(y)$. By Lemma 3.1 there is a $\delta>0$ such that if $x<_C z<_C y$ and $|x-y|<\delta$ then $|x-z|<\epsilon$ and $|y-z|<\epsilon$. Thus $f(x)<_C f(z)$ and $f(z)<_C f(y)$. $\square$

The proof of the following lemma was suggested by M. Barge.

Lemma 3.3. *Let $f:(C,<)\to(C,<)$ be a local order preserving homeomorphism. Then $h_{\text{top}}(f)=0$.*

Proof. Recall that $S\subset C$ is an *(n,ϵ)-spanning set*, for f if for all $x\in C$ there is a $y\in S$ such that $|f^k(x)-f^k(y)|<\epsilon$ for all $k=0,1,2,\dots n-1$. Then $(h_{\text{top}})_\epsilon(f)=\limsup_{n\to\infty}\dfrac{\log\text{card } S(n,\epsilon)}{n}$, and $h_{\text{top}}(f)=\lim_{\epsilon\to0}(h_{\text{top}})_\epsilon(f)$.

Choose δ as in Lemma 3.2 and suppose that $S\subset C$ is an (n,ϵ)-spanning set where $0<\epsilon\le\delta$ (δ from the lemma). Let X be a finite set of C that is ϵ-dense, let $N=\text{card } X$. Before proceeding with the proof of Lemma 3.3 we prove the following sublemma.

Sublemma 3.2. *$S\bigcup f^{-n}(X)$ is an $(n+1,\epsilon)$-spanning set.*

Proof. Let $x\in C$. Suppose that $y\in S$ is such that $|f^k(x)-f^k(y)|<\epsilon$ for $k=0,1,2,\dots n-1$. There is a $z\in X$ such that either $z\in[f^n(x),f^n(y)]$ or $z\in[f^n(y),f^n(x)]$, and such that $|f^n(x)-z|<\epsilon$. Then we have that $f^{-n}(z)\in S\bigcup f^{-n}(X)$ and z satisfies $|f^k(x)-f^k(z)|<\epsilon$ for $k=0,1,2,\dots n$ as desired. $\square$

Continuing with proof of Lemma 3.3, it follows from Sublemma 3.2 that there exists a constant $K>0$ such that for all n, card $S(n.\epsilon)\le K+nN$. Thus,

$$\begin{aligned} h_{\text{top}}(f) &= \lim_{\epsilon\to0}(h_{\text{top}})_\epsilon(f)\\ &= \lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{\log\text{ card } S(n,\epsilon)}{n}\\ &= \lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{\log(K+nN)}{n}=0. \end{aligned}$$

$\square$

Lemma 3.4. *Either H or H^2 locally preserves $<_{bw}$ on $\{\Lambda(a)|a\in C\}$.*

Proof. By Theorem 2.1, $H|_{\Lambda(c)}$ is monotone for all $c \in C$. Fix $a_0 \in C$ and assume that $h_1 \circ h_0(\{a\} \times \{i\}) \subset \ell_i$ and $H \circ h_1 \circ h_0(\{a_0\} \times \{i\}) \subset \ell_i$ for $i = 0$ or 1. (The other cases are similar.) For all $a \neq a_0$ there exists an admissible arc, γ_a^- linking $h_1 \circ h_0(\{a_0\} \times \{0\})$ to $h_1 \circ h_0(\{a\} \times \{0\})$ and an admissible arc, γ_a^+ linking $h_1 \circ h_0(\{a_0\} \times \{1\})$ to $h_1 \circ h_0(\{a\} \times \{1\})$. Now H is monotone on the simple closed curve $\Gamma = \Lambda(a_0) \bigcup \gamma_a^- \bigcup \Lambda(a) \bigcup \gamma_a^+$. Thus H can be approximated by a homeomorphism $H' : D \to D$ such that $H'\Lambda(a_0) = H(\Lambda(a_0))$, $H'\Lambda(a) = H(\Lambda(a))$, $H\gamma_a^- = H(\gamma_a^-)$, and $H\gamma_a^+ = H(\gamma_a^+)$. So the orientation of $H(\Gamma)$ is identical to the orientation of $H'(\Gamma)$. For a sufficiently close to a_0 H' (or $(H')^2$) preserves $<_{bw}$ between $\Lambda(a_0)$ and $\Lambda(a)$ [**BW93**]. Thus H (or $(H)^2$) does so as well. □

Proof of Theorem 3.1 continued. We now complete the proof of Theorem 3.1. First suppose that F_1 and F_1^2 do not locally preserve $<_C$. Then by Definition 3.5 H and H^2 cannot locally preserve $<_{bw}$ on the fibers $\{\Lambda(a)|\ a \in C\}$, contradicting Lemma 3.4.

Next suppose F_1 locally order preserves $<_C$. Then by Lemma 3.3 we have that $h_{\text{top}}(F_1) = 0$. And if F_1^2 locally preserves $<_C$, then $h_{\text{top}}\left(F_1^2\right) = 0$, thus $h_{\text{top}}(F_1) = 0$. So by [**Bow71**] $h_{\text{top}}(F) = h_{\text{top}}(F_1) + \sup_{a \in C}\left\{h_{\text{top}}\left(F|_{\pi_1^{-1}(a)}\right)\right\} = \sup_{a \in C}\left\{h_{\text{top}}\left(F|_{\pi_1^{-1}(a)}\right)\right\}$. But if $h_{\text{top}}(F) > 0$ there is an $a_0 \in C$ such that $h_{\text{top}}\left(F|_{\pi_1^{-1}(a_0)}\right) > 0$. Thus by Theorem 1.3 ([**Bar87**]) $F_2|_{a_0 \times [0,1]}$ is not monotone. So by Theorem 2.1 no such near homeomorphism extension H of $h_1 \circ h_0 \circ F \circ (h_1 \circ h_0)^{-1}$ exists. □

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262 South Main St.
Lodi, NJ 07644
E-mail address: sanfordm@inet.felician.edu

Montana State University
Bozeman, MT 59715
E-mail address: walker@math.montana.edu