EXTENDING MAPS OF A CANTOR SET PRODUCT WITH AN ARC TO NEAR HOMEOMORPHISMS OF THE 2-DISK

MICHAEL D. SANFORD AND RUSSELL B. WALKER
EXTENDING MAPS OF A CANTOR SET PRODUCT WITH AN ARC TO NEAR HOMEOMORPHISMS OF THE 2-DISK

MICHAEL D. SANFORD AND RUSSELL B. WALKER

We prove that a positive entropy map of the product of a Cantor Set and an arc (which covers a homeomorphism) cannot be “embedded” into a near homeomorphism of the 2-disk. Thus a theorem of M. Brown cannot be used to embed the induced shift map on the corresponding inverse limit space into a 2-disk homeomorphism.

1. Introduction.

In 1990, M. Barge and J. Martin [BM90] proved that the shift map on the inverse limit space \(([0, 1], f)\), for any map \(f : [0, 1] \to [0, 1]\), can be realized as a global attractor in the plane. In 1960, M. Brown [Bro60] proved that the inverse limit space of any near homeomorphism (Definition 1.2) of a compact metric space is homeomorphic to the original space. M. Barge and J. Martin prove that, for all such \(f\), there exists an embedding \(h : [0, 1] \to D^2\) such that \(h \circ f \circ h^{-1}\) can be extended to a near homeomorphism of the 2-disk, \(D^2\). They then use M. Brown’s theorem to extend the induced shift homeomorphism on \(h([0, 1])\) to a homeomorphism of \(D^2\). With care in the construction of the near homeomorphism of \(D^2\), the inverse limit space \((h([0, 1]), h \circ f \circ h^{-1})\) becomes a global attractor.

The main goal of this paper is to show that analogous techniques for maps, \(F : C \times [0, 1] \to C \times [0, 1]\), where \(C\) is a Cantor Set, \(F(x, y) = (F_1(x), F_2(x, y))\) is a surjective map with positive topological entropy (Definition 1.4), and \(F_1\) is a homeomorphism, do not work; no near homeomorphic extension of \(h \circ F \circ h^{-1}\) to \(D^2\) exists for any embedding \(h : C \times [0, 1] \to D^2\) (Theorem 3.1). In our terminology, such \(F\) cannot be “embedded” into any 2-disk homeomorphism (Definition 1.1). In the proof of Theorem 3.1 one first assumes that \(h\) is a “tame” embedding (Definition 3.1). But in recent work, R. Walker proves that all embeddings of \(C \times [0, 1]\) into \(D^2\) are tame [Wal].

Our study of maps of \(C \times [0, 1]\) and their embeddings has links to a central problem in the dynamical systems of positive entropy homeomorphisms of compact surfaces.

Does there exist a \(C^1\) positive entropy 2-disk diffeomorphism without shifts?
In 1980, A. Katok [Kat80] proved that all $C^{1+\alpha}$, $\alpha > 0$, positive entropy diffeomorphisms of a compact surface, have transverse homoclinic points. So some power of such a diffeomorphism restricts to a shift map of finite type. The next year M. Rees announced a minimal positive entropy homeomorphism of the 2-torus [Ree81]. So her homeomorphism has no periodic orbits thus no shifts. Though not in print, it appears that techniques M. Rees used can be adapted to build a positive entropy 2-disk homeomorphism which has a fixed point and no other periodic orbits. The $C^1$ case remains open. In 1993, M. Barge and R. Walker built a chainable continuum which is the inverse limit space of a map of a Cantor comb [BW93]. The map restricted to each “tooth” was a tent map over an adding machine base map. The induced shift homeomorphism has positive entropy but all periodic orbits are period a power of 2. Thus no shifts are present. All chainable continua can be embedded into the 2-disk [Bin62]. Although their Cantor comb map can be embedded into a near homeomorphism of the 3-ball, it cannot be embedded into a near homeomorphism of the 2-disk. (To prove this M. Barge and R. Walker rely on properties of the adding machine base map.) So their induced shift homeomorphism cannot be used to build a new Rees-type 2-disk homeomorphism. By our Theorem 3.1, a much larger class of maps (all positive entropy maps of $C \times [0,1]$ which cover any homeomorphism) has the same drawback.

In Section 2 we show that if $F : C \times [0,1] \to C \times [0,1]$ is a surjective map such that $F(x,y) = (F_1(x), F_2(x,y))$, $F_1$ is a homeomorphism and $F_2(x_0,\cdot) : [0,1] \to [0,1]$ is nonmonotone (Definition 1.3) for some $x_0$, then there exists no embedding of $F$ into a near homeomorphism (Definition of the 2-disk). We will show this by assuming such a near homeomorphism does exist and then obtaining a contradiction using a result of S. Schwartz [Sch92] (Theorem 1.1) concerning nonmonotone maps.

Unless otherwise specified $X$ and $Y$ are compact metric spaces. And $\pi_1$ and $\pi_2$ on $X \times Y$ are the first and second coordinate projection maps.

**Definition 1.1.** A map $f : X \to X$ can be embedded into the map $F : Y \to Y$ if there exists a topological embedding $h : X \to Y$ such that $F|_{h(X)} = h \circ f \circ h^{-1}$.

**Definition 1.2.** A map $f : X \to Y$ is called a near homeomorphism provided there exists a sequence $\{f_k : X \to Y\}_{k=1}^{\infty}$ of homeomorphisms which uniformly converge to $f$.

**Definition 1.3.** A map $f : X \to Y$ is monotone provided $f^{-1}(V)$ is connected, whenever $V \subset Y$ is connected.

**Theorem 1.1** (S. Schwartz [Sch92]). Suppose that $X$ is a locally connected compact metric space. If $f : X \to X$ is a near homeomorphism then $f$ is monotone.
As mentioned, in Section 3 we show that if $F : C \times [0, 1] \to C \times [0, 1]$ is a surjective map with positive topological entropy (Definition 1.4), which is embedded in the 2-disk, then $F$ cannot be extended to a near homeomorphism of the disk. The proof uses theorems of R. Bowen (Theorem 1.2) [Bow71] and M. Barge (Theorem 1.3) [Bar87].

**Definition 1.4** (Topological Entropy). Suppose that $F : X \times Y \to X \times Y$ is a surjective map and has the form $F(x, y) = (F_1(x), F_2(x, y))$. Fix $x_0$ and let $\epsilon > 0$. A set $E \subset Y$ is $(n, \epsilon)$-separated by $F|_{\pi^{-1}_1(x_0)}$ if for all $y_0, y_1 \in E$, $y_0 \neq y_1$, $d(\pi_2 F^k(x_0, y_0), \pi_2 F^k(x_0, y_1)) > \epsilon$ for some $k \in [0, n]$, where $d$ is the $Y$-metric. Since $Y$ is compact and $n < \infty$, card $E < \infty$. Let the maximum number of $(n, \epsilon)$-separated orbits for each $\epsilon$ be

$$s(n, \epsilon) = \max \left\{ \text{card } E \left| E \subset Y \text{ such that } E \text{ is } (n, \epsilon) - \text{separated by } F|_{\pi^{-1}_1(x_0)} \right. \right\}.$$ 

Now, let the growth rate of $s(n, \epsilon)$ (or $\epsilon$-topological entropy) be

$$h_{\text{top}} \left( F|_{\pi^{-1}_1(x_0)} \right) = \limsup_{n \to \infty} \frac{\log s(n, \epsilon)}{n}.$$

Lastly we let $\epsilon \to 0$ and define topological entropy for $F|_{\pi^{-1}_1(x_0)}$:

$$h_{\text{top}} \left( F|_{\pi^{-1}_1(x_0)} \right) = \lim_{\epsilon \to 0} h_{\text{top}} \left( F|_{\pi^{-1}_1(x_0)} \right).$$

The topological entropy $h_{\text{top}}(F_1)$ of the homeomorphism $F_1$ is defined similarly (see [Bow71]).

**Theorem 1.2** (R. Bowen [Bow71]). If $F : X \times Y \to X \times Y$ has the form $F(x, y) = (F_1(x), F_2(x, y))$ then

$$h_{\text{top}}(F) \leq h_{\text{top}}(F_1) + \sup_{x \in X} \left\{ h_{\text{top}} \left( F|_{\pi^{-1}_1(x)} \right) \right\}.$$ 

If $h_{\text{top}}(F_1) = 0$ then $h_{\text{top}}(F) = \sup_{x \in X} \left\{ h_{\text{top}} \left( F|_{\pi^{-1}_1(x)} \right) \right\}$. 

**Theorem 1.3** (M. Barge [Bar87]). If $F : X \times [0, 1] \to X \times [0, 1]$ has the form $F(x, y) = (F_1(x), F_2(x, y))$, $F_2(x, \cdot) : [0, 1] \to [0, 1]$ is monotone for each $x$ and $h_{\text{top}}(F_1) = 0$, then $h_{\text{top}}(F) = 0$.

2. Nonmonotone Maps of the Cantor Set Cross the Interval.

2.1. Preliminaries. Let $C \subset [0, 1]$ be a Cantor set. Let $C \times [0, 1]$ and $\{\alpha\} \times [0, 1] \subset \mathbb{R}^2$ for $\alpha \in C$. The goal of this section is to prove Theorem 2.1 to follow. But first some preliminaries.

2.1.0.1. Assume $F : C \times [0, 1] \to C \times [0, 1]$ is a surjective map that has the form

$$F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$$
where $F_1 : C \to C$ is a homeomorphism. Furthermore, for a given $\alpha_0 \in C$, $F_2(\alpha_0, y) = t(y)$ where $t : [0, 1] \to [0, 1]$ is a continuous nonmonotone map (see Figure 1 for an example). Let $\lambda_0 = F_1(\alpha_0)$.

![Figure 1. Example of a nonmonotone map.](image)

It will be needed later, that because $t$ is nonmonotone we can find a point that has at least two points in the the pre-image that can be separated by disjoint epsilon balls. We introduce this idea at this point so that we can use the notation developed here throughout.

2.1.0.2. Since $t$ is nonmonotone and continuous, there is an $a \in (0, 1)$ such that $t^{-1}(a)$ is closed and not connected. Thus, there is an interval $(m, M) \subset [0, 1] \setminus t^{-1}(a)$ such that $a = t(m) = t(M)$, and $t([m, M]) = [a, b]$ (or $[b, a]$) for some $b \neq a$. Without loss of generality we will assume that $a < b$. Let $\tau \in t^{-1}(b)$. By the intermediate value theorem, $t([m, M]) = [t(m), t(\tau)]$. Now choose $c = \frac{1}{2}(a + b)$. Since $t$ is continuous there are $s_1 \in (m, \tau)$ and $s_2 \in (\tau, M)$ such that $c = t(s_1) = t(s_2)$ (see Figure 1).

2.1.0.3. By the continuity of $F$, for any $\epsilon > 0$ there is a $\delta_1 = \delta_1(\epsilon) > 0$ such that $F(x, y) \in B_c(\lambda_0, t(y))$ when $d(\alpha_0, x) < \delta_1$ and $y \in [0, 1]$. Suppose $K_1 = K_1(\epsilon) \in \mathbb{N}$ is such that $\frac{1}{K_1} < \delta_1$.

Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4\}$. Now let $h_0 : C \times [0, 1] \to D$ be an arbitrary topological embedding. Then there is a homeomorphism $h_1 : D \to D$ such that $(h_1 \circ h_0)(\alpha_0, y) = (\alpha_0, y)$ and $(h_1 \circ h_0)(\lambda_0, y) = (\lambda_0, y)$ for all $y \in [0, 1]$. So $h_1$ “straightens out” $h_0(\alpha_0 \times [0, 1])$ and $h_0(\lambda_0 \times [0, 1])$ in a strong sense. Notice that $C \times [0, 1] \to D$.

2.1.0.4. By the uniform continuity of $h_1 \circ h_0$, for all $\epsilon > 0$ there is a $\delta_2 = \delta_2(\epsilon) > 0$ such that for all $y \in [0, 1]$, $h_1 \circ h_0(x, y) \in B_c(\alpha_0, y)$ and $h_1 \circ h_0(x', y) \in B_c(\alpha_0, y)$, for all $(x, y) \in B_{\delta_2(\epsilon)}(\alpha_0, y)$ and $(x', y) \in B_{\delta_2(\epsilon)}(\alpha_0, y)$. Let $K_2 = K_2(\epsilon) \in \mathbb{N}$ be such that $\frac{1}{K_2} < \delta_2$. 
2.1.0.5. With $a, b$ defined as in [2.1.0.2], let $\hat{d} = \min\{a, 1 - b, |a - b|\}$. For $0 < \epsilon_0 < \frac{\hat{d}}{100}$ choose $0 < \delta_0 \leq \min\{\delta_1(\epsilon_0), \delta_2(\epsilon_0), \frac{M - m}{100}\}$. So in particular [2.1.0.3] and [2.1.0.4] are satisfied. Note that $t([m, M]) \subset [\epsilon_0, 1 - \epsilon_0]$. Let $K_0 \geq \max\{K_1(\epsilon_0), K_2(\epsilon_0)\}$ be such that $\frac{1}{K_0} < \delta_0$. Since $C$ is perfect, there is a sequence $\{\alpha_k\} \subset C$ such that $\alpha_k \to \alpha_0$ as $k \to \infty$, and $\alpha_k \times [0, 1] \subset N_{\delta_0}(\alpha_0 \times [0, 1])$, for all $k > K_0$. Let $\lambda_k = F_1(\alpha_k)$. (Note that $N_{\delta}(S)$ is a $\delta$–neighborhood of $S$.) It follows that $\lambda_k \to \lambda_0$ as $k \to \infty$ and $\lambda_k \times [0, 1] \subset N_{\epsilon_0}(\lambda_0 \times [0, 1])$ for all $k > K_0$. For a possibly larger $K_0$, also called $K_0$, and $o_k \in B_{\epsilon_0}(\lambda_0, c), k > K_0$, there exist $q_1(k)$ and $q_2(k)$ such that $\{q_1(k), q_2(k)\} \subset F^{-1}(o_k), q_1(k) \in B_{\epsilon_0}(\alpha_0, s_1)$ and $q_2(k) \in B_{\delta_0}(\alpha_0, s_2)$.

We now state our first theorem.

### 2.1.1. Nonmonotone Nonextension Theorem.

**Theorem 2.1.** Let $F : C \times [0, 1] \to C \times [0, 1]$ be a map of the form $F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$ where $F_1 : C \to C$ is a homeomorphism. Furthermore, assume $F_2(\alpha_0, \cdot) : [0, 1] \to [0, 1]$ is surjective but not monotone for some $\alpha_0$. Then there exists no extension of $h_0 \circ F \circ h_0^{-1}$ to a near homeomorphism of the disk $D$, for any topological embedding $h_0 : C \times [0, 1] \to D$.

**Proof.** Assume $h, K_0, \epsilon_0, \delta_0, \{\alpha_k\}, \{\lambda_k\}, q_1(k)$, and $q_2(k)$ are defined as in [2.1.0.1-5]. Suppose that $H_0 : D \to D$ is a near homeomorphism such that $H_0|_{h_0(C \times [0, 1])} = h_0 \circ F \circ h_0^{-1}$. And let $H_1 : D \to D$ be given by $H_1 = h_1 \circ H_0 \circ h_1^{-1}$. Thus $H_1$ is also a near homeomorphism. So the diagram in Figure 2 commutes.

\[
\begin{array}{ccc}
C \times [0, 1] & \xrightarrow{F} & C \times [0, 1] \\
\downarrow h_0 & & \downarrow h_0 \\
D & \xrightarrow{H_0} & D \\
\downarrow h_1 & & \downarrow h_1 \\
D & \xrightarrow{H_1} & D \\
\end{array}
\]

**Figure 2.** Commuting Diagram.

2.1.1. Let $\Lambda(\alpha) = h_1 \circ h_0(\alpha \times [0, 1])$ for all $\alpha \in C$. By [2.1.0.3] $h_1 \circ h_0$ is a homeomorphism and if $\{\alpha\} \times [0, 1] \bigcap \{\lambda\} \times [0, 1] = \emptyset$ (when $\alpha \neq \lambda$), then $\Lambda(\alpha) \bigcap \Lambda(\lambda) = \emptyset$. Let $\ell_\beta$ be the horizontal line $\{y = \beta\}$. And let $\ell_\beta(k) =$
Λ(α_k) \cap \ell_{\beta} and \ell^\lambda_{\beta}(k) = \Lambda(\lambda_k) \cap \ell_{\beta}. Because h_1 \circ h_0 (\alpha_k, 0) \in B_{\delta_0}(\alpha_0, 0), h_1 \circ h_0 (\alpha_k, 1) \in B_{\delta_0}(\alpha_0, 1), and \Lambda(\alpha_k) is connected, then \ell^\alpha_{\beta}(k) \neq \emptyset and all \ k \geq K_0 (see Figure 3 and [2.1.0.2]). Similarly \ell^\lambda_{\beta}(k) \neq \emptyset, for all \ \beta \in [\epsilon_0, 1-\epsilon_0] and \ k \geq K_0. Note that if \ p \in \ell^\lambda_{\beta}(k) for given \ k \geq K_0 then \ p \in B_{\epsilon_0}(\lambda_0, \beta).

Lemma 2.1 follows from the continuity of \ h_1, h_0, and \ \pi_1.

**Lemma 2.1.** Choose \ p_k \in \ell^\alpha_{\beta}(k) for each \ k. Then \ \pi_1 p_k \rightarrow \alpha_0 as \ k \rightarrow \infty.

Notice that \ \pi_1(h_1 \circ h_0)(\alpha_k, \frac{1}{2}) \neq \alpha_0 for sufficiently large \ k. So either

\[
\text{card}\left\{ k \mid \pi_1\left(h_1 \circ h_0\left(\alpha_k, \frac{1}{2}\right)\right) > \alpha_0 \right\} = \infty
\]

or

\[
\text{card}\left\{ k \mid \pi_1\left(h_1 \circ h_0\left(\alpha_k, \frac{1}{2}\right)\right) < \alpha_0 \right\} = \infty.
\]

2.1.1.2. So without loss of generality we may assume there exist distinct \ \{k_n\}_{n=1}^{\infty} such that \ k_n \rightarrow \infty as \ n \rightarrow \infty, and

\[
\pi_1\left(h_1 \circ h_0\left(\alpha_{k_n}, \frac{1}{2}\right)\right) > \alpha_0.
\]

2.1.1.3. For the sake of simplicity we denote \ \alpha_{k_n} by \ \alpha_n, \ \Lambda(\alpha_{k_n}) by \ \Lambda(\alpha_n), \ \Lambda(\lambda_{k_n}) by \ \Lambda(\lambda_n) and \ \ell^\lambda_{\beta} by \ \ell^\alpha_{\beta}.

**Figure 3.** Intersection of \ \Lambda(\alpha_k) with \ \ell_{\beta}.
Lemma 2.2. Let $N_0$ be such that $k_n \geq K_0$ for all $n \geq N_0$. Then

$$\Lambda(\alpha_n) \cap \left\{ \left( x, \frac{1}{2} \right) \, | \, x < \alpha_0 \right\} = \emptyset.$$  

Proof. Fix $n \geq N_0$ and assume there exists

$$p_1 \in \Lambda(\alpha_n) \cap \left\{ \left( x, \frac{1}{2} \right) \, | \, x < \alpha_0 \right\},$$

and let $p_2 = (h_1 \circ h_0(\alpha_k, \frac{1}{2}))$. By [2.1.1.2] $\pi_1(p_2) > 0$. Let $A$ be the arc in $\Lambda(\alpha_n)$ with end points $p_1$ and $p_2$. By [2.1.1.1], $p_1, p_2 \in B_{\epsilon_0}(\alpha_0, \frac{1}{2})$. So by [2.1.0.5],

$$d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_2)) < \delta_0.$$  

Since $\Lambda(\alpha_n) \cap \Lambda(\alpha_0) = \emptyset$, then using a Jordan Curve argument, it follows

$$A \cap \{(a_0, y) | y > 1 \text{ or } y < 0 \} \neq \emptyset.$$  

Let $p_3 \in A \cap \{(0, y) | y > 1 \text{ or } y < 0 \}$. So $d(p_1, p_3) > \frac{1}{2}$. But because $p_3 \in A$, either

$$\pi_2 \circ (h_1 \circ h_0)^{-1}(p_1) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_2)$$

or

$$\pi_2 \circ (h_1 \circ h_0)^{-1}(p_2) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_1).$$

In either case $d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_3)) < \delta_0$. And so $d(p_1, p_3) < \epsilon_0$ which is a contradiction. □

2.1.1.4. Assume $n \geq N_0$. Let $g_n : [0,1] \rightarrow \Lambda(\alpha_n)$ be the parameterization of $\Lambda(\alpha_n)$ defined by $g_n(\beta) = h_1 \circ h_0(\alpha_n, \beta)$. Using $\tau$, $m$ from [2.1.0.2] $\Lambda(\alpha_n) \cap \ell_\tau \neq \emptyset$ and $\Lambda(\alpha_n) \cap \ell_m \neq \emptyset$ (see Figure 4) so by Lemma [2.2] and the connectivity of $\Lambda(\alpha_n)$ there is a largest $\beta$, call it $\beta_n^-$, such that $g_n(\beta_n^-) \in \ell_m$. Let $\alpha_n = g_n(\beta_n^-)$ (see Figure 4). Similarly there is a smallest $\beta$, call it $\beta_n^+$, such that $g_n(\beta_n^+) \in \ell_m$. Let $b_n = g_n(\beta_n^+)$.

2.1.1.5. If necessary, renumber the $k_n$’s so that if $k_n < k_{n+1}$ then $\pi_1(a_n) > \pi_1(a_{n+1})$. It follows by an argument similar to that of Lemma [2.2] that $\pi_1(b_n) > \pi_1(b_{n+1})$. (Because $h_1 \circ h_0$ may have scrambled the $C \times [0,1]$ order in the first coordinate, it may be necessary to relabel the $k_n$’s so that $\Lambda(\alpha_k)$ to be “between” $\Lambda(\alpha_{k-1})$ and $\Lambda(\alpha_{k+1})$.)

Considering [2.1.1.5] and [2.1.1.2] and to simplify the notation assume

$$\text{card} \left\{ k | \pi_1(h_1 \circ h_0)(\alpha_k, \frac{1}{2}) > \alpha_0 \right\} = \infty$$

and $\pi_1(a_k) > \pi_1(a_{k+1})$ for all $k$.  

Using [2.1.4], for \( k \geq N_0 \) define the four curves \( I(k, m) \), \( I(k, \tau) \), \( J_{k-1} \), and \( J_{k+1} \) in the following manner (see Figure 5). Let \( I(k, m) \) be the line segment in \( \ell_m \) between \( a_{k+1} \) and \( a_{k-1} \) and \( I(k, \tau) \) be the line segment in \( \ell_\tau \) between \( b_{k+1} \) and \( b_{k-1} \). Let

\[
J_{k-1} = \{ g_{k-1}(\beta) \mid \beta_{k-1}^- \leq \beta \leq \beta_{k-1}^+ \}, \text{ and }
\]

\[
J_{k+1} = \{ g_{k+1}(\beta) \mid \beta_{k+1}^- \leq \beta \leq \beta_{k+1}^+ \}.
\]

\[a_n = g_n(\beta^-_n)\]

\[b_n = g_n(\beta^+_n)\]

**Figure 4.** First and Last Intersections.

**Lemma 2.3.** \( I(k, \tau) \cup J_{k-1} \cup I(k, m) \cup J_{k+1} \) is a simple closed curve.

**Proof.** Since \( \Lambda(\alpha_{k-1}) \cap \Lambda(\alpha_{k+1}) = \emptyset \) we have \( J_{k-1} \cap J_{k+1} = \emptyset \). By [2.1.1] \( I(k, m) \cap I(k, \tau) = \emptyset \). And by [2.1.4] we have that

\[a_{k-1} = J_{k-1} \cap I(k, m) \] and \( a_{k+1} = J_{k+1} \cap I(k, m) \]
and

\[ b_{k-1} = J_{k-1} \cap I(k, \tau) \text{ and } b_{k+1} = J_{k+1} \cap I(k, \tau). \]

And so the lemma follows. \( \square \)

Let \( R_k \) be the closed and bounded set with boundary

\[ I(k, m) \bigcup J_{k-1} \bigcup I(k, \tau) \bigcup J_{k+1} \]

(see Figure 5). Recall from [2.1.0.2] that \( s_1 \in [m, \tau] \) and from [2.1.1.1] that \( \ell_{s_1}^\alpha(k) = \Lambda(\alpha_k) \cap \ell_{s_1} \) (see Figure 3).

Lemma 2.4. \( R_k \cap \ell_{s_1}^\alpha(k) \neq \emptyset \).

Proof. Let \( \gamma_k \) be the arc \( \{g_k(\beta)|0 \leq \beta \leq \beta_k^+\} \). Let \( S_k = R_k \cap \pi_2^{-1}[s_1, \tau] \) (see Figure 6). So \( \partial S_k \supset I(k, \tau) \) and by [2.1.1.5] \( b_k \in I(k, \tau) \). But \( b_k \) is not an endpoint of \( I(k, \tau) \) because the endpoints of \( I(k, \tau) \) are \( b_{k-1} \) and \( b_{k+1} \). And so there is an \( \eta > 0 \) such that if \( p \in B_\eta(b_k) \) and \( \pi_2(p) < \tau \) then \( p \in \text{int} S_k \). Now, if \( q \in \gamma_k \setminus \{b_k\} \) then \( \pi_2(q) < \frac{1}{2} \). And since \( \gamma_k \) connects \( h_1 \circ h_0(\alpha_k, 0) \) to \( b_k \), we have that \( \gamma_k \cap B_\eta(b_k) \setminus \{b_k\} \neq \emptyset \). Thus there exists \( p_0 \in \gamma_k \cap B_\eta(b_k) \cap \text{int} S_k \). Let \( A_k \subset \gamma_k \) be the arc with endpoints \( p_0 \) and \( h_1 \circ h_0(\alpha_k, 0) \) (see Figure 6).

![Figure 6. The Arc \( A_k \).](image)

Because \( p_0 \in \text{int} S_k \) and \( h_1 \circ h_0(\alpha_k, 0) \not\in S_k \) then \( A_k \cap \partial S_k \neq \emptyset \). Since \( A_k \cap \Lambda(\alpha_{k-1}) = \emptyset, A_k \cap \Lambda(\alpha_{k+1}) = \emptyset, A_k \cap I(n, \tau) = \emptyset \) and \( \ell_{s_1} \cap R_k \subset \partial S_k \), we have that \( A_k \cap \ell_{s_1} \cap R_k \neq \emptyset \), or \( R_k \cap \ell_{s_1}^\alpha(k) \neq \emptyset \).

2.1.1.6. Note that since \( \ell_{s_1}^\alpha(k) \cap \partial R_k = \emptyset \) then \( \ell_{s_1}^\alpha(k) \subset \text{int} R_k \).

Lemma 2.5. \([\Lambda(\alpha_l) \cap H^{-1}_1(h_1 \circ h_0(\lambda_k, y))] = \emptyset \) for \( k \neq l \).
Proof. Suppose that \( \rho \in \Lambda(\alpha_l) \cap H_{-1}^{-1}(h_1 \circ h_0(\lambda_k, y)) \) for \( k \neq l \). Then \( H_1(\rho) = h_1 \circ h_0(\lambda_k, y) \) But \( H_1 \Lambda(\alpha_l) = \Lambda(\lambda_l) \). Thus \( H_1(\rho) \in \Lambda(\lambda_l) \) So \( h_1 \circ h_0(\lambda_k, y) \in \Lambda(\alpha_l) \) Or \( (\lambda_k, y) \in C_{\lambda_l} \). Which contradicts, \([2.1.1.1]\) since \( k \neq l \).

Proof of Theorem 2.1 continued. By Lemma \([2.4]\) there exists \( p_1(k) \in R_k \cap \ell_{s_1}(k) \) for all \( k \geq N_0 \). By \([2.1.0.5]\) \((h_1 \circ h_0)^{-1}(p_1(k)) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)\). Using \([2.1.0.5]\), let \( o_k = F \circ (h_1 \circ h_0)^{-1}(p_1(k)) \). So there exists \( \{q_1(k), q_2(k)\} \subset F^{-1}(o_k) \) such that \( q_1(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1) \) and \( q_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2) \). Choose \( q_1(k) \) so that \( p_1(k) = h_1 \circ h_0(q_1(k)) \). And let \( p_2(k) = h_1 \circ h_0(q_2(k)) \) and \( r_k = h_1 \circ h_0(o_k) \) (see Figure 7). Because \( H_1 \circ h_1 \circ h_0 = h_1 \circ h_0 \circ F \) then \( \{p_1(k), p_2(k)\} \in H_1^{-1}(r_k) \). By the size of \( \delta_0 \) chosen in \([2.1.0.5]\), \( p_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2) \not\subseteq R_k \).

Recall that \( H_0 \) and \( H_1 \) are near homeomorphisms. Near homeomorphisms are monotone on locally connected compact metric spaces (\[Sch92\]). Thus pre-images of connected sets under \( H_1 \) are connected. So \( H_1^{-1}(r_k) \) is a connected set which contains \( p_2(k) \not\subseteq R_k \) and by \([2.1.1.6]\) \( p_1(k) \in \text{int}R_k \). Then \( H_1^{-1}(r_k) \cap \partial R_k \neq \emptyset \). By Lemma \([2.5]\) either \( H_1^{-1}(r_k) \cap I(k, \tau) \neq \emptyset \) or \( H_1^{-1}(r_k) \cap I(k, m) \neq \emptyset \). So there is an infinite sequence \( \{\rho_{k_j}\} \) such that either \( \rho_{k_j} \in I(k, \tau) \cap H_1^{-1}(r_{k_j}) \) or \( \rho_{k_j} \in I(k, m) \cap H_1^{-1}(r_{k_j}) \) for all \( j \) (see Figure 7).

![Figure 7. Subsequence and Pre-image.](image)

Now by Lemma \([2.1]\) either \( \rho_{k_j} \to h_1 \circ h_0(\alpha_0, \tau) \) or \( \rho_{k_j} \to h_1 \circ h_0(\alpha_0, m) \) as \( j \to \infty \). Since \( H_1 \) is continuous for all \( j \), either

\[
H_1 \rho_{k_j} \to H_1 \circ h_1 \circ h_0(\alpha_0, \tau) \quad \text{or} \quad H_1 \rho_{k_j} \to H_1 \circ h_1 \circ h_0(\alpha_0, m).
\]

Because \( H_1 \circ h_1 \circ h_0 = h_1 \circ h_0 \circ F \), then either

\[
r_{k_j} \to h_1 \circ h_0(\lambda_0, t(\tau)) \quad \text{or} \quad r_{k_j} \to h_1 \circ h_0(\lambda_0, t(m)).
\]
Since $h_1 \circ h_0$ is a homeomorphism either
\[ o_{k_j} \to (\lambda_0, b) \quad \text{or} \quad o_{k_j} \to (\lambda_0, a) \]
which is a contradiction since \( \{o_{k_j}\} \subset B_c(\lambda_0, c) \).
\[ \square \]

3. Positive Entropy Maps of \( C \times [0, 1] \).

3.1. Introduction. Let \( C \subset \mathbb{R} \) be a Cantor set. In this chapter we use the results of Chapter 2 to prove the following:

**Theorem 3.1.** Let \( F : C \times [0, 1] \to C \times [0, 1] \) be a surjective map such that \( F(a,y) = (F_1(a), F_2(a,y)) \), where \( F_1 : C \to C \) is a homeomorphism. If \( h_{\text{top}}(F) > 0 \) then there exists no topological embedding \( h_0 : C \times [0, 1] \to D \subset \mathbb{R}^2 \) such that \( h_0 \circ F \circ h_0^{-1} \) extends to a near homeomorphism of the disk \( D \).

Recall that \( \pi_1 : C \times [0, 1] \to C \) is the projection map onto the first coordinate. By work of R. Bowen [Bow71] we know that \( h_{\text{top}}(F) \leq h_{\text{top}}(F_1) + \sup_{a \in C} \{ h_{\text{top}}(F|_{\pi_1^{-1}(a)}) \} \). It has been shown by M. Barge and R. Walker [BW93] that any near homeomorphism that extends \( h_0 \circ F \circ h_0^{-1} \) to the disk must preserve a certain local order on the set of fibers \( \{h_0(a \times [0,1]) \mid a \in C \} \). But we will show that if \( h_{\text{top}}(F_1) > 0 \) no such local order is preserved. So in fact \( h_{\text{top}}(F_1) = 0 \). Using [Bow71] and a result of M. Barge [Bar87], if \( h_{\text{top}}(F) > 0 \) then for some \( a_0 \in C \), \( F_2(a_0, \cdot) \) is a nonmonotone map. Thus by Theorem 2.1, \( h_0 \circ F \circ h_0^{-1} \) cannot be extended to a near homeomorphism of the disk.

3.2. Proof of Theorem 3.1.

**Definition 3.1** (Tame Embedding). \( h_0 : C \times [0,1] \to D \subset \mathbb{R} \) is a tame embedding provided there is a homeomorphism \( h_1 : D \to D \) such that for all \( a \in C \), \( h_1 \circ h_0(\{a\} \times [0,1]) = \{(a') \times [0,1]\} \) for some \( \{a'\} \). If \( h_0 \) is a tame embedding, using a theorem of E. Moise [Moi77], we may further require that \( h_1 \) has the property: \( h_1 \circ h_0(\{a\}, i) = (\{a'\}, i) \) for all \( a \) and \( i = 0, 1 \).

For more information concerning tame embeddings see [Rus73] or [Bin54].

3.2.1. Proof of Theorem 3.1. All topological embeddings of \( C \times [0,1] \) into \( D^2 \) are tame [Wal]. So it is enough to prove the theorem for all tame embeddings, \( h_0 \).

Let \( h_1 \) be as in Definition 3.1. Denote by \( \Lambda \) the set \( h_1 \circ h_0(\{a\} \times [0,1]) \) and by \( \Lambda(a) \) the set \( h_1 \circ h_0(a \times [0,1]) \). Note that \( \pi_1(\Lambda(a)) = a' \) for some \( a' \in \mathbb{R} \). Assume there is a near homeomorphism \( H : D \to D \) such that on \( C \times I \), \( h_1 \circ h_0 \circ F = H \circ h_1 \circ h_0 \). Before continuing with the proof, we stop to define a local ordering on \( \{\Lambda(a) \mid a \in C\} \) and prove a lemma.
3.2.2. Order Definitions and Lemmas. Here we show that $H$ preserves the local order of fibers as defined by M. Barge and R. Walker [BW93], which we will write as $<_{bw}$. And it will follow that $F_1 : C \to C$ is a “local order preserving homeomorphism.”

Note: Since $h_0$ is tame one could use the order on $\{\Lambda(a) | a \in C\}$ induced by $\pi_1$ in place of $<_{bw}$. That is, $\Lambda(a) < \Lambda(b)$ if $\pi_1 \Lambda(a) < \pi_1 \Lambda(b)$. Although $h_1$-dependent, this order may be more natural than the $<_{bw}$ order, and is locally equivalent to it. But in order to show that $H$ preserves such a local order on fibers, one must cycle through the definition of $<_{bw}$ in any case.

**Barge-Walker order:**

**Definition 3.2.** For $a, b \in C$ suppose that $\gamma_-$ and $\gamma_+$ are arcs in the plane with the properties:

- $\gamma_-$ has endpoints $h_1 \circ h_0(a, 0)$ and $h_1 \circ h_0(b, 0)$, and $\gamma_-$ is otherwise disjoint from $\Lambda(a) \cup \Lambda(b)$; $\gamma_+$ has endpoints $h_1 \circ h_0(a, 1)$ and $h_1 \circ h_0(b, 1)$ and $\gamma_+$ is otherwise disjoint from $\Lambda(a) \cup \Lambda(b)$; and

\[
\left(\gamma_- \cup \gamma_+\right) \cap \left([0, 2] \times \left\{\frac{1}{2}\right\}\right) = \emptyset.
\]

Such arcs $\gamma_-$ and $\gamma_+$ will be called *admissible arcs* joining $\Lambda(a)$ and $\Lambda(b)$.

**Definition 3.3.** Given $a, b \in C$, $a \neq b$, then $\Lambda(a) <_{bw} \Lambda(b)$ if there are admissible arcs joining $\Lambda(a)$ and $\Lambda(b)$, as above, and the orientation $\gamma_- \to \Lambda(b) \to \gamma_+ \to \Lambda(a)$ is positive (counterclockwise) on the simple closed curve $\gamma_- \cup \Lambda(b) \cup \gamma_+ \cup \Lambda(a)$. (See Figure 8.)

![Figure 8. Barge-Walker Ordering on Cantor Fibers.](image)

**Definition 3.4.** $<_X$ is a *local ordering* on $X$ if for all $x \in X$ there is a $\delta > 0$ such that $<_X$ is an order relation on $B_\delta(x)$. $(X, <_X)$ is a *locally ordered metric space*. 
Lemma 3.1. Given $x < y$ which contradicts $z$ negative orientation or \( \Lambda(z) \). Furthermore $<_{bw}$ is a local ordering on $\Lambda = \{ \Lambda(a) \mid a \in C \}$ where we use the metric $d(\Lambda(a), \Lambda(b)) = d(a, b)$.

**Definition 3.5.** Let $a, b \in C$. Then $a <_{C} b$ provided $\Lambda(a) <_{bw} \Lambda(b)$.

It follows from the proceeding remark and that $h_{1} \circ h_{0}$ is uniformly continuous, that $<_{C}$ is a local ordering on $C$.

**Definition 3.6.** Let $(X, <_{X})$ and $(Y, <_{Y})$ be locally ordered metric spaces. Let $G : (X, <_{X}) \to (Y, <_{Y})$ be a homeomorphism. $G$ is a local order preserving homeomorphism, if there is a $\delta > 0$ such that if $x_{0}, x_{1} \in X$, $|x_{0} - x_{1}| < \delta$, and $x_{0} <_{X} x_{1}$, then $G(x_{0}) <_{Y} G(x_{1})$.

Denote by $[x, y] = \{ z \in C \mid x \leq_{C} z \leq_{C} y \}$. We next show $<_{C}$ on $C$ is $\mathbb{R}$-like in the following sense.

**Lemma 3.1.** Given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in C$ and $|x - y| < \delta$, then for all $z \in [x, y]$, $|x - z| < \epsilon$ and $|y - z| < \epsilon$.

**Proof.** Suppose that $x, y, z \in C$ and $x <_{C} z <_{C} y$. By Definition 3.5 there are admissible arcs $\gamma_{1}^{+}, \gamma_{1}^{-}, \gamma_{2}^{+}, \gamma_{2}^{-}$ such that $\Lambda(z) \to \gamma_{1}^{+} \to \Lambda(x) \to \gamma_{1}^{-}$ and $\Lambda(y) \to \gamma_{2}^{+} \to \Lambda(z) \to \gamma_{2}^{-}$ have positive orientation.

**Sublemma 3.1.** For $\epsilon > 0$ there is a $\delta_{1} > 0$ such that if

\[
\Lambda(z) \cap N_{\delta_{1}}(\Lambda(x)) \neq \emptyset
\]

then $|x - z| < \epsilon$.

**Proof.** By the continuity of $(h_{1} \circ h_{0})^{-1}$, if $\epsilon > 0$ there is a $\delta_{1} > 0$ such that if $d(p, q) < \delta_{1}$ where $p, q \in \Lambda$ then $d((h_{1} \circ h_{0})^{-1}(p), (h_{1} \circ h_{0})^{-1}(q)) < \epsilon$. So if $\Lambda(z) \cap N_{\delta_{1}}(\Lambda(x)) \neq \emptyset$ there is $p \in \Lambda(x)$, $q \in \Lambda(z)$ such that $d(p, q) < \delta_{1}$. Thus $|x - z| = |\pi_{1}((h_{1} \circ h_{0})^{-1}(p)) - \pi_{1}((h_{1} \circ h_{0})^{-1}(q))| \leq d((h_{1} \circ h_{0})^{-1}(p), (h_{1} \circ h_{0})^{-1}(q)) < \epsilon$.

Choose $\delta_{1} > 0$ smaller so that if

\[
\Lambda(z) \cap N_{\delta_{1}}(\Lambda(x)) \neq \emptyset
\]

then $|x - z| < \epsilon$ and $|y - z| < \epsilon$.

By the continuity of $(h_{1} \circ h_{0})$ there is $\delta > 0$ such that if $|x - y| < \delta$ then $\Lambda(x) \subset N_{\delta_{1}}(\Lambda(y))$ and $\Lambda(y) \subset N_{\delta_{1}}(\Lambda(x))$.

Suppose that $\Lambda(z) \cap N_{\delta_{1}}(\Lambda(x)) \cap N_{\delta_{1}}(\Lambda(y)) = \emptyset$. So either $\pi_{1}\Lambda(z) < \pi_{1}\Lambda(x)$ or $\pi_{1}\Lambda(y) < \pi_{1}\Lambda(z)$. Thus either $\Lambda(z) \to \gamma_{1}^{+} \to \Lambda(x) \to \gamma_{1}^{-}$ has negative orientation or $\Lambda(z) \to \gamma_{2}^{+} \to \Lambda(y) \to \gamma_{2}^{-}$ has positive orientation which contradicts $x <_{C} z <_{C} y$. 

In [BW93] it is shown that if $a$ and $b$ are sufficiently close, $a \neq b$, then such admissible arcs exist. So either $\Lambda(a) <_{bw} \Lambda(b)$ or $\Lambda(b) <_{bw} \Lambda(a)$.
Thus $\Lambda(z) \cap N_{\delta_1}(\Lambda(x)) \neq \emptyset$ and $\Lambda(z) \cap N_{\delta_1}(\Lambda(y)) \neq \emptyset$. So by the choice of $\delta$ then $|x - z| < \epsilon$ and $|y - z| < \epsilon$ as desired. □

**Lemma 3.2.** Let $f : (C, <_C) \to (C, <_C)$ be a local order preserving homeomorphism. Then there is a $\delta > 0$ such that if $|x - y| < \delta$ then $f([x, y]) = [f(x), f(y)]$.

**Proof.** By Definition 3.6 there is an $\epsilon > 0$ such that for any $x, y \in C$ if $|x - y| < \epsilon$, and $x <_C y$ then $f(x) <_C f(y)$. By Lemma 3.1 there is a $\delta > 0$ such that if $x <_C z <_C y$ and $|x - y| < \delta$ then $|x - z| < \epsilon$ and $|y - z| < \epsilon$. Thus $f(x) <_C f(z)$ and $f(z) <_C f(y)$. □

The proof of the following lemma was suggested by M. Barge.

**Lemma 3.3.** Let $f : (C, <) \to (C, <)$ be a local order preserving homeomorphism. Then $h_{\text{top}}(f) = 0$.

**Proof.** Recall that $S \subset C$ is an $(n, \epsilon)$-spanning set, for $f$ if for all $x \in C$ there is a $y \in S$ such that $|f^k(x) - f^k(y)| < \epsilon$ for all $k = 0, 1, 2, \ldots n - 1$. Then $(h_{\text{top}})_\epsilon (f) = \limsup_{\epsilon \to 0} \frac{\log \text{card } S(n, \epsilon)}{n}$, and $h_{\text{top}}(f) = \lim_{\epsilon \to 0} (h_{\text{top}})_\epsilon (f)$.

Choose $\delta$ as in Lemma 3.2 and suppose that $S \subset C$ is an $(n, \epsilon)$-spanning set where $0 < \epsilon \leq \delta$ (from the lemma). Let $X$ be a finite set of $C$ that is $\epsilon$-dense, let $N = \text{card } X$. Before proceeding with the proof of Lemma 3.3 we prove the following sublemma.

**Sublemma 3.2.** $S \bigcup f^{-n}(X)$ is an $(n + 1, \epsilon)$-spanning set.

**Proof.** Let $x \in C$. Suppose that $y \in S$ is such that $|f^k(x) - f^k(y)| < \epsilon$ for $k = 0, 1, 2, \ldots n - 1$. There is a $z \in X$ such that either $z \in [f^n(x), f^n(y)]$ or $z \in [f^n(y), f^n(x)]$, and such that $|f^n(x) - z| < \epsilon$. Then we have that $f^{-n}(z) \in S \bigcup f^{-n}(X)$ and $z$ satisfies $|f^k(x) - f^k(z)| < \epsilon$ for $k = 0, 1, 2, \ldots n$ as desired. □

Continuing with proof of Lemma 3.3, it follows from Sublemma 3.2 that there exists a constant $K > 0$ such that for all $n$, $\text{card } S(n, \epsilon) \leq K + nN$. Thus,

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} (h_{\text{top}})_\epsilon (f)$$

$$= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log \text{card } S(n, \epsilon)}{n}$$

$$= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(K + nN)}{n} = 0.$$ □

**Lemma 3.4.** Either $H$ or $H^2$ locally preserves $<_{bw}$ on $\{\Lambda(a) | a \in C\}$. 

Proof. By Theorem 2.1, $H_{|\Lambda(c)}$ is monotone for all $c \in C$. Fix $a_0 \in C$ and assume that $h_1 \circ h_0(\{a\} \times \{i\}) \subset \ell_i$ and $H \circ h_1 \circ h_0(\{a\} \times \{i\}) \subset \ell_i$ for $i = 0$ or 1. (The other cases are similar.) For all $a \neq a_0$ there exists an admissible arc, $\gamma_a^-$ linking $h_1 \circ h_0(\{a\} \times \{0\})$ to $h_1 \circ h_0(\{a\} \times \{0\})$ and an admissible arc, $\gamma_a^+$ linking $h_1 \circ h_0(\{a\} \times \{1\})$ to $h_1 \circ h_0(\{a\} \times \{1\})$. Now $H$ is monotone on the simple closed curve $\Gamma = \Lambda(a_0) \cup \gamma_a^- \cup \Lambda(a) \cup \gamma_a^+$. Thus $H$ can be approximated by a homeomorphism $H' : D \to D$ such that $H'\Lambda(a_0) = H(\Lambda(a_0))$, $H'\Lambda(a) = H(\Lambda(a))$, $H'\gamma_a^- = H(\gamma_a^-)$, and $H'\gamma_a^+ = H(\gamma_a^+)$. So the orientation of $H(\Gamma)$ is identical to the orientation of $H'(\Gamma)$. For a sufficiently close to $a_0$, $H'$ (or $(H')^2$) preserves $<_{bw}$ between $\Lambda(a_0)$ and $\Lambda(a)$ [BW93]. Thus $H$ (or $(H)^2$) does so as well. \hfill \square

Proof of Theorem 3.1 continued. We now complete the proof of Theorem 3.1. First suppose that $F_1$ and $F_1^2$ do not locally preserve $<_C$. Then by Definition 3.5 $H$ and $H^2$ cannot locally preserve $<_{bw}$ on the fibers $\{\Lambda(a)\mid a \in C\}$, contradicting Lemma 3.4.

Next suppose $F_1$ locally order preserves $<_C$. Then by Lemma 3.3 we have that $h_{top}(F_1) = 0$. And if $F_1^2$ locally preserves $<_C$, then $h_{top}(F_1^2) = 0$, thus $h_{top}(F_1) = 0$. So by [Bow71] $h_{top}(F) = h_{top}(F_1) + \sup_{a \in C} \{h_{top}(F|_{\pi_1^{-1}(a)})\} = \sup_{a \in C} \{h_{top}(F|_{\pi_1^{-1}(a)})\}$. But if $h_{top}(F) > 0$ there is an $a_0 \in C$ such that $h_{top}(F|_{\pi_1^{-1}(a)}) > 0$. Thus by Theorem 1.3 ([Bar87]) $F_2|_{a_0 \times [0,1]}$ is not monotone. So by Theorem 2.1 no such near homeomorphism extension $H$ of $h_1 \circ h_0 \circ F \circ (h_1 \circ h_0)^{-1}$ exists. \hfill \square

References


Received July 18, 1996 and revised December 16, 1996. This research was supported in part by NSF-OSR grant #9350546.

262 South Main St.
Lodi, NJ 07644

*E-mail address:* sanfordm@inet.felician.edu

Montana State University
Bozeman, MT 59715

*E-mail address:* walker@math.montana.edu