SIMPLE CONNECTIVITY OF THE MARKOV PARTITION SPACE

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In Wagoner, 1987 the simplicial complex $P_A$ of Markov partitions was introduced as a tool for studying the group of automorphisms of a subshift of finite type $(X_A, \sigma_A)$ built from a zero-one transition matrix $A$. Triangles in $P_A$ led to the matrix Triangle Identities in Wagoner, Pac. Journal, 1990 which have been used in Wagoner, 1990, 1990, 1990, 1992, Kim, Roush & Wagoner, 1992, and the Williams Conjecture counterexample paper Kim & Roush, to appear.

A key fact about $P_A$ is that it is contractible. See Wagoner, 1987. The purpose of this note is to correct the proof on pp. 99-100 in Wagoner, 1987 that $P_A$ is simply connected and in the process to improve the bound in Proposition 2.13 of Wagoner, 1987.

Proposition. A closed path in $P_A$ with $L$ edges can be spanned by a (possibly singular) triangulated 2-disc in $P_A$ having at most $8L^2 + L$ triangles.

The difficulty with the proof of (2.13) in [W1] occurs in the diagram of Step 2 on p. 99, because it may not be the case that $V_{i-1} \cap V_{i+1}$ is a Markov partition.

To correct this, it is better to change to a more straightforward notation and let $U \rightarrow V$ rather than $V \rightarrow U$ mean $V < U < \sigma_A(V) \cap V$. Recall from [W1] that $U \rightarrow V$ means $U < V < U \cap \sigma^{-1}_A(U)$. Then Definition 2.10 of [W1] becomes

$$U \rightarrow V \text{ iff } U \rightarrow U \cap V \rightarrow V.$$

In particular, now $U \rightarrow V$ implies that $U \rightarrow V$ in $P_A$ but with extra information, whereas in [W1] the notation $V \rightarrow U$ implied $U \rightarrow V$, which is somewhat contrary. Here are some properties of the arrows $U \rightarrow V$ and $U \rightarrow V$.

1) If $U \rightarrow V$ and $U$ is a Markov partition, then $V$ is a Markov partition.

1) If $U \rightarrow V$ and $V$ is a Markov partition, then $U$ is a Markov partition.
2) If $U \to V$ and $U$ and $V$ are Markov partitions, then $U \cap V$ is a Markov partition.
3) If $U \to V$, $W$ and $U$ is a Markov partition, then $U \to V \cap W$ and $V \cap W$ is a Markov partition. If $U, V \to W$ and $W$ is a Markov partition, then $U \cap V \to W$ and $U \cap V$ is a Markov partition.
4) If $U \to X \to V$ and $U, X, \text{ and } V$ are Markov partitions, then $U \to U \cap V \to V$ and $U \cap V$ is a Markov partition.

For completeness, we recall Definition 2.11 of [W1] giving the simplicial structure on $P_A$. Namely, an $n$-simplex of $P_A$ is an ordered $(n + 1)$-tuple $\langle V_0, V_1, \ldots, V_n \rangle$ such that $V_i \to V_j$ whenever $i \leq j$.

The next step is to replace the diagram in Step 2 on p. 99 with the following diagram:

\[
\begin{array}{cccc}
V_i & \to & V_{i+1} \cap V_i \cap V_{i+3} & \to \\
\downarrow & & \downarrow & \\
V_{i+1} & \leftarrow & V_i \cap V_{i+1} \cap V_{i+2} & \\
\downarrow & & \downarrow & \\
V_i & \to & V_{i-1} \cap V_i \cap V_{i+1} & \\
\downarrow & & \downarrow & \\
V_{i-1} & \leftarrow & V_{i-2} \cap V_{i-1} \cap V_i & \\
\downarrow & & \downarrow & \\
V_{i-2} & \to & V_{i-3} \cap V_{i-2} \cap V_{i-1} & \\
\end{array}
\]

We can now deform a closed path of length $L$ to a constant path as follows: Step 1 on p. 99 deforms the closed path of length $L$ to an alternating...
closed path of length $2L$ with vertices $V_0, V_1, \ldots, V_{2L}$. The number of triangles in this deformation is at most $L$. Then the above diagram deforms the alternating closed path of length $2L$ on the left to an alternating closed path of length $2L$ on the right with vertices of the form $V_{i-1} \cap V_i \cap V_{i+1}$. Repeating the deformation in the diagram $L-1$ more times produces an alternating closed path of length $2L$ with vertices of the form

$$V_{i-L} \cap V_{i-L+1} \cap \ldots \cap V_{i-1} \cap V_i \cap V_{i+1} \ldots \cap V_{i+L-1} \cap V_{i+L}.$$

Thus all the vertices in this path are equal to

$$V_0 \cap V_1 \cap \ldots \cap V_{2L}.$$

The total number of triangles in this deformation is at most $8L^2 + L$.

**Remark.** The argument in [W1] that $H_n(P_A) = 0$ for $n \geq 2$ avoids the $V_{i-1} \cap V_{i+1}$ type difficulty, because all intersections of Markov partitions encountered in the proof are Markov partitions as a consequence of properties (1) through (4) above. There is a typographical change on p. 102, l.10: $V_p \cap V_q$ should read $V_{ps} \cap V_{qs}$.

**References**


Received May 5, 1998. The authors were partially supported by NSF Grant # DMS 9322498.

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