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In Wagoner, 1987 the simplicial complex P_A of Markov partitions was introduced as a tool for studying the group of automorphisms of a subshift of finite type (X_A, σ_A) built from a zero-one transition matrix A. Triangles in P_A led to the matrix Triangle Identities in Wagoner, Pac. Journal, 1990 which have been used in Wagoner, 1990, 1990, 1990, 1992, Kim, Roush & Wagoner, 1992, and the Williams Conjecture counterexample paper Kim & Roush, to appear.

A key fact about P_A is that it is contractible. See Wagoner, 1987. The purpose of this note is to correct the proof on pp. 99-100 in Wagoner, 1987 that P_A is simply connected and in the process to improve the bound in Proposition 2.13 of Wagoner, 1987.

Proposition. *A closed path in P_A with L edges can be spanned by a (possibly singular) triangulated 2-disc in P_A having at most $8L^2 + L$ triangles.*

The difficulty with the proof of (2.13) in [**W1**] occurs in the diagram of Step 2 on p. 99, because it may not be the case that $V_{i-1} \cap V_{i+1}$ is a Markov partition.

To correct this, it is better to change to a more straightforward notation and let $U \underset{-}{\longrightarrow} V$ rather than $V \underset{-}{\longrightarrow} U$ mean $V < U < \sigma_A(V) \cap V$. Recall from [**W1**] that $U \underset{+}{\longrightarrow} V$ means $U < V < U \cap \sigma_A^{-1}(U)$. Then Definition 2.10 of [**W1**] becomes

$$U \to V \quad \text{iff} \quad U \underset{+}{\longrightarrow} U \cap V \underset{-}{\longrightarrow} V.$$

In particular, now $U \underset{-}{\longrightarrow} V$ implies that $U \to V$ in P_A but with extra information, whereas in [**W1**] the notation $V \underset{-}{\longrightarrow} U$ implied $U \to V$, which is somewhat contrary. Here are some properties of the arrows $U \underset{+}{\longrightarrow} V$ and $U \underset{-}{\longrightarrow} V$. See [**W1**].

1) If $U \underset{+}{\longrightarrow} V$ and U is a Markov partition, then V is a Markov partition.
   If $U \underset{-}{\longrightarrow} V$ and V is a Markov partition, then U is a Markov partition.

2) If $U \to V$ and U and V are Markov partitions, then $U \cap V$ is a Markov partition.
3) If $U \underset{+}{\longrightarrow} V, W$ and U is a Markov partition, then $U \underset{+}{\longrightarrow} V \cap W$ and $V \cap W$ is a Markov partition. If $U, V \underset{-}{\longrightarrow} W$ and W is a Markov partition, then $U \cap V \underset{-}{\longrightarrow} W$ and $U \cap V$ is a Markov partition.
4) If $U \underset{-}{\longrightarrow} X \underset{+}{\longrightarrow} V$ and U, X, and V are Markov partitions, then $U \underset{+}{\longrightarrow} U \cap V \underset{-}{\longrightarrow} V$ and $U \cap V$ is a Markov partition.

For completeness , we recall Definition 2.11 of [**W1**] giving the simplicial structure on P_A. Namely, an n-simplex of P_A is an ordered $(n+1)$-tuple $\langle V_0, V_1, \ldots, V_n\rangle$ such that $V_i \to V_j$ whenever $i \leq j$.

The next step is to replace the diagram in Step 2 on p. 99 with the following diagram

$$
\begin{array}{ccccc}
\vdots & & & & \vdots \\
V_{i+2} & & \underset{+}{\longrightarrow} & & V_{i+1}\cap V_{i+2}\cap V_{i+3} \\
 & \searrow + & & + \nearrow & \\
\downarrow & & V_{i+1}\cap V_{i+2} & & \uparrow \\
 & \swarrow - & & - \nwarrow & \\
V_{i+1} & & \underset{-}{\longleftarrow} & & V_{i}\cap V_{i+1}\cap V_{i+2} \\
 & \nwarrow - & & - \swarrow & \\
\uparrow & & V_{i}\cap V_{i+1} & & \downarrow \\
 & \nearrow + & & + \searrow & \\
V_{i} & & \underset{+}{\longrightarrow} & & V_{i-1}\cap V_{i}\cap V_{i+1} \\
 & \searrow + & & + \nearrow & \\
\downarrow & & V_{i-1}\cap V_{i} & & \uparrow \\
 & \swarrow - & & - \nwarrow & \\
V_{i-1} & & \underset{-}{\longleftarrow} & & V_{i-2}\cap V_{i-1}\cap V_{i} \\
 & \nwarrow - & & - \swarrow & \\
\uparrow & & V_{i-2}\cap V_{i-1} & & \downarrow \\
 & \nearrow + & & + \searrow & \\
V_{i-2} & & \underset{+}{\longrightarrow} & & V_{i-3}\cap V_{i-2}\cap V_{i-1} \\
\vdots & & & & \vdots
\end{array}
$$

We can now deform a closed path of length L to a constant path as follows: Step 1 on p. 99 deforms the closed path of length L to an alternating closed path of length 2L with vertices V_0, V_1, ..., V_{2L}. The number of triangles in this deformation is at most L. Then the above diagram deforms the alternating closed path of length 2L on the left to an alternating closed path of length 2L on the right with vertices of the form $V_{i-1} \cap V_i \cap V_{i+1}$. Repeating the deformation in the diagram L-1 more times produces an alternating closed path of length 2L with vertices of the form

$$V_{i-L} \cap V_{i-L+1} \cap \ldots \cap V_{i-1} \cap V_i \cap V_{i+1} \ldots \cap V_{i+L-1} \cap V_{i+L}.$$

Thus all the vertices in this path are equal to

$$V_0 \cap V_1 \cap \ldots \cap V_{2L}.$$

The total number of triangles in this deformation is at most $8L^2 + L$.

Remark. The argument in [**W1**] that $H_n(P_A) = 0$ for $n \geq 2$ avoids the $V_{i-1} \cap V_{i+1}$ type difficulty, because all intersections of Markov partitions encountered in the proof are Markov partitions as a consequence of properties (1) through (4) above. There is a typographical change on p. 102, l.10: $V_{p_s} \cap V_{p_s}$ should read $V_{p_s} \cap V_{q_s}$.

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