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In this paper we study n -dimensional compact minimal submanifolds in S^{n+p} with scalar curvature S satisfying the pinching condition $S > n(n - 2)$. We show that for $p \leq 2$ these submanifolds are totally geodesic (cf. Theorem 3.2 and Corollary 3.1). However, for codimension $p \geq 2$, we prove the result under an additional restrictions on the curvature tensor corresponding to the normal connection (cf. Theorem 3.1 and Corollary 4.1). We also show that the scalar curvature S of a non-totally geodesic n -dimensional non-negatively curved minimal submanifold in S^{n+p} with flat normal connection satisfies $n(n - p - 1) \leq S \leq n(n - 2)$ (cf. Theorem 4.1). Since for a compact hypersurface M of S^{n+1} the normal connection is flat, we use the above estimate for a scalar curvature S of a non-negatively curved minimal hypersurface M in S^{n+1} to infer that either M is totally geodesic or else it is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right)$. As a consequence this result, we conclude that the only non-negatively curved compact minimal hypersurfaces in S^{n+1} which are diffeomorphic to S^n is totally geodesic sphere.

1. Introduction.

Let M be an n -dimensional compact minimal submanifold in a unit sphere S^{n+p} . One of the interesting questions in the geometry of the minimal submanifolds of S^{n+p} is to obtain conditions under which they are totally geodesic. These conditions generally involve the pinching of the sectional curvatures, Ricci curvatures, or the scalar curvature (or equivalently the square of the length of the second fundamental form). In [3] for $n \geq 4$, it is proved that if the Ricci curvatures of a minimal submanifold in S^{n+p} are greater than $n - 2$, then M is totally geodesic, where $n = \dim M$. For $n = 3$ also the above result holds and is proved in [6]. As a natural generalization of this result one may expect that if the scalar curvature S of an n -dimensional compact minimal submanifold of S^{n+p} satisfies $S > n(n - 2)$, then M is totally geodesic. This will improve all the existing results involving pinching of the scalar curvature for the minimal submanifolds of S^{n+p} , and specially

that in [2]. However, it is not known whether this result holds even in dimension 3. Recently in [6] a slightly courser (than expected above) result is proved in dimension 3, namely there it is shown that for a 3-dimensional compact minimal submanifold M of S^{3+p} , the condition $S > 4$ implies that M is totally geodesic.

In this paper we study n -dimensional compact minimal submanifolds in S^{n+p} with scalar curvature satisfying the pinching condition $S > n(n-2)$. We show that for $p \leq 2$ these submanifolds are totally geodesic (cf. Theorem 3.2 and Corollary 3.1). However, for codimension $p \geq 2$, we have to put additional restrictions on the curvature tensor corresponding to the normal connection to get the result (cf. Theorem 3.1 and Corollary 4.1). We also show that the scalar curvature S of a non-totally geodesic n -dimensional non-negatively curved minimal submanifold in S^{n+p} with flat normal connection satisfies $n(n-p-1) \leq S \leq n(n-2)$ (cf. Theorem 4.1). Since for a compact hypersurface M of S^{n+1} the normal connection is flat, we use the above estimate for the scalar curvature S of a non-negatively curved minimal hypersurface M in S^{n+1} to infer (cf. Corollary 4.2), that either M is totally geodesic or else it is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}}\right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}}\right)$ considered in [2].

As a consequence of Corollary 4.2, we conclude that the only non-negatively curved compact minimal hypersurface in S^{n+1} which is diffeomorphic to S^n is totally geodesic sphere, giving a result in the direction of a question asked by Yau in [7] (cf. p. 692, Problem 99).

2. Preliminaries.

Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} . We denote by g the Riemannian metric on S^{n+p} as well as that induced on M , and by $\bar{\nabla}$ and ∇ the Riemannian connections S^{n+p} on M and respectively. Then we have

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp Y,$$

$X, Y \in \chi(M), N \in \Gamma(\nu)$, where h is the second fundamental form, $\chi(M)$ is the Lie algebra of smooth vector fields on M , $\Gamma(\nu)$ is the space of smooth sections of the normal bundle ν of M , ∇^\perp is the normal connection and A_N is the Weingarten map corresponding to the normal $N \in \Gamma(\nu)$ which satisfies

$$(2.2) \quad g(A_N X, Y) = g(h(X, Y), N), \quad X, Y \in \chi(M), N \in \Gamma(\nu).$$

For the submanifold M the equations of Gauss, Codazzi and Ricci are respectively

$$(2.3) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + A_{h(Y, Z)}X - A_{h(X, Z)}Y$$

$$(2.4) \quad (\nabla h)(X, Y, Z) = (\nabla h)(Y, Z, X)$$

$$(2.5) \quad R^\perp(X, Y, N_1, N_2) = g([A_{N_1}, A_{N_2}](X), Y)$$

for $X, Y, Z \in \chi(M)$, $N_1, N_2 \in \Gamma(\nu)$, where R, R^\perp are the curvature tensors corresponding to the connections ∇ and ∇^\perp respectively and the covariant derivative $(\nabla h)(X, Y, Z)$ is given by

$$(\nabla h)(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The second covariant derivative $(\nabla^2 h)(X, Y, Z, W)$ is defined as

$$\begin{aligned} (\nabla^2 h)(X, Y, Z, W) &= \nabla_X^\perp (\nabla h)(Y, Z, W) - (\nabla h)(\nabla_X Y, Z, W) \\ &\quad - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W) \end{aligned}$$

and we have the Ricci identity

$$\begin{aligned} (2.6) \quad (\nabla^2 h)(X, Y, Z, W) &- (\nabla^2 h)(Y, X, Z, W) \\ &= R^\perp(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W). \end{aligned}$$

For a local orthonormal frame $\{e_1, \dots, e_n\}$ on M , as M is a minimal submanifold we have

$$(2.7) \quad \sum_i h(e_i, e_i) = 0, \quad \sum_i (\nabla h)(X, e_i, e_i) = 0, \quad \sum_i (\nabla^2 h)(X, Y, e_i, e_i) = 0.$$

Let Ric be the Ricci tensor of M . Then the Ricci operator Q is a symmetric operator defined by

$$\text{Ric}(X, Y) = g(Q(X), Y), \quad X, Y \in \chi(M).$$

The Gauss Equation (2.3) gives the following expression for the Ricci operator Q of the minimal submanifold M

$$(2.8) \quad Q(X) = (n-1)X - \sum_i A_{h(e_i, X)}e_i,$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M . The scalar curvature $S = \sum_i \text{Ric}(e_i, e_i)$ of the minimal submanifold is given by

$$(2.9) \quad S = n(n-1) - \|h\|^2,$$

where $\|h\|^2 = \sum_{ij} \|h(e_i, e_j)\|^2$ is the square of the length of the second fundamental form.

Lemma 2.1. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} . Then*

$$(2.10) \quad \int_M \left\{ \|\nabla h\|^2 + \sum_{ijk} \left[R^\perp(e_k, e_i; h(e_j, e_k), h(e_i, e_j)) \right. \right. \\ \left. \left. - R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) \right] + \sum_{ij} \text{Ric} \left(e_i, A_{h(e_i, e_j)} e_j \right) \right\} dv = 0.$$

Proof. Define $f : M \rightarrow R$ by $f = \frac{1}{2} \|h\|^2$. Then the Laplacian of f given by

$$\Delta f = \sum_k [e_k e_k(f) - \nabla_{e_k} e_k(f)],$$

can be shown to satisfy

$$\Delta f = \sum_{ijk} g((\nabla^2 h)(e_k, e_k, e_i, e_j), h(e_i, e_j)) + \sum_{ijk} \|(\nabla h)(e_i, e_j, e_k)\|^2.$$

Now, using the Ricci identity (2.6) and Equations (2.4), (2.7) in the above equation and integrating the resulting equation we get the integral formula (2.10).

Next we define $\|R\|^2$ the square of the length of the curvature tensor field of M by

$$\|R\|^2 = \sum_{ijk} \|R(e_i, e_j) e_k\|^2,$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M . Then using Equations (2.3) and (2.7) we immediately get:

Lemma 2.2. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} . Then*

$$(2.11) \quad \|R\|^2 = 2n(n-1) - 4\|h\|^2 + 2\|A_h\|^2 \\ - 2 \sum_{ijk} g \left(A_{h(e_j, e_k)} e_i, A_{h(e_i, e_k)} e_j \right)$$

where $\|A_h\|^2 = \sum_{ijk} \|A_{h(e_i, e_j)} e_k\|^2$ and $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M .

Lemma 2.3. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} . Then*

$$(2.12) \quad \sum_{ijk} R \left(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k \right) = \frac{1}{2} \|R\|^2 - S,$$

where S is the scalar curvature of M and $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M .

Proof. Equation (2.3) implies

$$A_{h(e_i, e_j)} e_k = R(e_k, e_i) e_j + A_{h(e_k, e_j)} e_i - \delta_{ij} e_k + \delta_{kj} e_i.$$

Taking inner product with $A_{h(e_i, e_j)} e_k$ in the above equation and noticing that

$$\sum_{ijk} R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) = - \sum_{ijk} R(e_k, e_i; e_j, A_{h(e_k, e_j)} e_i)$$

we get (2.12).

3. Minimal submanifolds with pinched scalar curvature.

Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} . We choose a local orthonormal frames $\{e_1, \dots, e_n\}$ on M and $\{N_1, \dots, N_p\}$ that of normals and define the function $K^\perp : M \rightarrow R$ by

$$K^\perp = \sum_{ij\alpha\beta} \left[R^\perp(e_i, e_j, N_\alpha, N_\beta) \right]^2$$

and call it the normal curvature of the minimal submanifold. We also define a function $\varphi : M \rightarrow R$ by

$$\varphi = 2 \sum_{\alpha < \beta} \|A_\alpha\|^2 \|A_\beta\|^2,$$

where $A_\alpha = A_{N_\alpha}$ is the Weingarten map in the direction of the normal vector N_α .

Theorem 3.1. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} . If the scalar curvature S and the normal curvature K^\perp of M satisfy $S > n(n-2)$ and $K^\perp \leq \varphi$ respectively, then M is totally geodesic.*

Proof. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on M . Then Equations (2.3) and (2.8) imply

$$(3.1) \quad A_{h(e_j, e_k)} e_i = R(e_i, e_k) e_j - \delta_{kj} e_i + \delta_{ij} e_k + A_{h(e_i, e_j)} e_k.$$

$$(3.2) \quad \sum_k A_{h(e_j, e_k)} e_k = (n-1) e_j - Q(e_j).$$

Taking inner product in (3.1) with $A_{h(e_i, e_j)} e_k$, we arrive at

$$(3.3) \quad \begin{aligned} & \sum_{ijk} g \left(A_{h(e_i, e_j)} e_k, A_{h(e_j, e_k)} e_i \right) \\ &= \|A_h\|^2 + \sum_{ijk} R(e_i, e_k; e_j, A_{h(e_i, e_j)} e_k) - \|h\|^2. \end{aligned}$$

Similarly Equation (3.2) gives

$$(3.4) \quad \sum_{ijk} g \left(A_{h(e_j, e_k)} e_k, A_{h(e_i, e_j)} e_i \right) = (n-1) \|h\|^2 - \sum_{ij} \text{Ric} \left(e_j, A_{h(e_i, e_j)} e_i \right).$$

Now, using Equations (3.3) and (3.4) in (2.5) we arrive at

$$(3.5) \quad R^\perp(e_k, e_i, h(e_i, e_j), h(e_j, e_k)) = \sum_{ijk} R(e_i, e_k; e_j, A_{h(e_i, e_j)} e_k) + \sum_{ij} \text{Ric}(e_j, A_{h(e_i, e_j)} e_i) + \|A_h\|^2 - n \|h\|^2.$$

For a local orthonormal frame $\{N_1, \dots, N_p\}$ of normals we have

$$(3.6) \quad \begin{aligned} \sum_{ij} \text{Ric}(e_j, A_{h(e_i, e_j)} e_i) &= \sum_{\alpha ij} g(A_\alpha e_i, e_j) \text{Ric}(e_j, A_\alpha e_i) \\ &= \sum_{\alpha i} g(A_\alpha e_i, e_j) \text{Ric} \left(\sum_j g(A_\alpha e_i, e_j) e_j, A_\alpha e_i \right) \\ &= \sum_{\alpha i} \text{Ric}(A_\alpha e_i, A_\alpha e_i). \end{aligned}$$

Now, using (2.8) in (3.6) we arrive at

$$(3.7) \quad \begin{aligned} \sum_{ij} \text{Ric}(e_j, A_{h(e_i, e_j)} e_i) &= (n-1) \sum_{\alpha i} \|A_\alpha e_i\|^2 - \sum_{\alpha ij} \|h(e_j, A_\alpha e_i)\|^2 \\ &= (n-1) \|h\|^2 - \sum_{\alpha\beta} \|A_\alpha A_\beta\|^2. \end{aligned}$$

Also we observe that

$$(3.8) \quad \sum_{ijk} g \left(A_{h(e_j, e_k)} e_i, A_{h(e_i, e_k)} e_j \right) = \sum_{ij\alpha\beta} g(A_\alpha e_i, A_\beta e_j) g(A_\alpha e_j, A_\beta e_i)$$

and

$$(3.9) \quad \begin{aligned} \sum_{\alpha\beta} \|A_\alpha A_\beta - A_\beta A_\alpha\|^2 &= \sum_{ij\alpha\beta} g((A_\alpha A_\beta - A_\beta A_\alpha)(e_i), e_j)^2 \\ &= -2 \sum_{ij\alpha\beta} g(A_\alpha e_i, A_\beta e_j) g(A_\alpha e_j, A_\beta e_i) \\ &\quad + 2 \sum_{\alpha\beta} \|A_\alpha A_\beta\|^2 \end{aligned}$$

where we have used $\|A_\alpha A_\beta\|^2 = \|A_\beta A_\alpha\|^2$ which follows from the fact that A_α and A_β are symmetric. We use Equations (3.8) and (3.9) in (2.11) to arrive at

$$\begin{aligned} \|R\|^2 &= 2n(n-1) - 4\|h\|^2 + 2\|A_h\|^2 \\ &\quad + \sum_{\alpha\beta} \|A_\alpha A_\beta - A_\beta A_\alpha\|^2 - 2 \sum_{\alpha\beta} \|A_\alpha A_\beta\|^2 \end{aligned}$$

and consequently Lemma 2.3 gives

$$\begin{aligned} 2 \sum_{ijk} R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) &= 2n(n-1) - 4\|h\|^2 + 2\|A_h\|^2 \\ (3.10) \quad &\quad - 2S + \sum_{\alpha\beta} \|A_\alpha A_\beta - A_\beta A_\alpha\|^2 - 2 \sum_{\alpha\beta} \|A_\alpha A_\beta\|^2. \end{aligned}$$

The integral formula (2.10) in view of (3.5) takes the form

$$\begin{aligned} (3.11) \quad \int_M \left\{ \|\nabla h\|^2 + \|A_h\|^2 - n\|h\|^2 - 2 \sum_{ijk} R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) \right. \\ \left. + 2 \sum_{ij} \text{Ric}(e_j, A_{h(e_i, e_j)} e_i) \right\} dv = 0. \end{aligned}$$

Thus using (3.7), (3.10) and (2.9) in (3.11), we conclude

$$(3.12) \quad \int_M \left\{ \|\nabla h\|^2 + n\|h\|^2 - \|A_h\|^2 - \sum_{\alpha\beta} \|A_\alpha A_\beta - A_\beta A_\alpha\|^2 \right\} dv = 0.$$

Note that in view of Equation (2.5) we have

$$\begin{aligned} (3.13) \quad K^\perp &= \sum_{ij\alpha\beta} \left[R^\perp(e_i, e_j, N_\alpha, N_\beta) \right]^2 = \sum_{ij\alpha\beta} g([A_\alpha, A_\beta](e_i), e_j)^2 \\ &= \sum_{\alpha\beta} \left[\sum_{ij} g((A_\alpha A_\beta - A_\beta A_\alpha)(e_i), e_j)^2 \right] \\ &= \sum_{\alpha\beta} \|A_\alpha A_\beta - A_\beta A_\alpha\|^2. \end{aligned}$$

Also we have

$$\begin{aligned}
\|A_h\|^2 &= \sum_{ijk} \left\| A_{h(e_i, e_j)} e_k \right\|^2 = \sum_{ijk\alpha} g(A_\alpha e_i, e_j)^2 \|A_\alpha e_k\|^2 \\
&= \sum_{ij\alpha} \|A_\alpha\|^2 g(A_\alpha e_i, e_j)^2 = \sum_{\alpha} \|A_\alpha\|^4 \\
&= \left(\sum_{\alpha} \|A_\alpha\|^2 \right)^2 - 2 \sum_{\alpha < \beta} \|A_\alpha\|^2 \|A_\beta\|^2,
\end{aligned}$$

which gives

$$(3.14) \quad \|A_h\|^2 = \|h\|^4 - 2 \sum_{\alpha < \beta} \|A_\alpha\|^2 \|A_\beta\|^2.$$

Using Equations (3.13) and (3.14) in (3.12), we arrive at

$$(3.15) \quad \int_M \left\{ \|\nabla h\|^2 + (n - \|h\|^2) \|h\|^2 + (\varphi - K^\perp) \right\} dv = 0.$$

Now using $S > n(n-2)$ with (2.4), we conclude that $\|h\|^2 < n$ and hence the integral formula (3.15) implies that M is totally geodesic.

Theorem 3.2. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+2} . If the scalar curvature S of M satisfies $S > n(n-2)$, then M is totally geodesic.*

Proof. In codimension 2 the integral formula (3.12) together with (3.14) gives

$$\int_M \left\{ \|\nabla h\|^2 + (n - \|h\|^2) \|h\|^2 + 2 \|A_1\|^2 \|A_2\|^2 - \|A_1 A_2 - A_2 A_1\|^2 \right\} dv = 0,$$

where $\{N_1, N_2\}$ is a local orthonormal frame of normals on M and $A_\alpha = A_{N_\alpha}$, $\alpha = 1, 2$. Then using Lemma 1 in ([2], p. 64), and the hypothesis $S > n(n-2)$, we get from above integral formula that $\|h\|^2 = 0$, that is M is totally geodesic.

Remark 3.1. For Veronese surface M in S^4 , one can show that the normal curvature K^\perp and the function φ appearing in Theorem 3.1 satisfy $K^\perp = \varphi = \frac{8}{9}$. Indeed we can choose a local orthonormal frame $\{e_1, e_2\}$ on M and a local orthonormal frame $\{N_1, N_2\}$ of normals such that the Weingarten maps A_{N_1}, A_{N_2} take the form (cf. [2])

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark 3.2. For a compact minimal hypersurface M in S^{n+1} we have $K^\perp = 0$ and $\varphi = 0$ and consequently the integral formula (3.15) takes the form

$$\int_M \left\{ \|\nabla A\|^2 + \left(n - \|A\|^2 \right) \|A\|^2 \right\} dv = 0,$$

where A is the Weingarten map of the hypersurface. Thus we have:

Corollary 3.1. *Let M be an n -dimensional compact minimal hypersurface of the unit sphere S^{n+1} . If the scalar curvature S of M satisfies $S \geq n(n-2)$, then M is either totally geodesic or else is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right)$.*

Proof. If M is not totally geodesic, then the above integral formula gives that the Weingarten map A is parallel and $\|A\|^2 = n$, consequently as in [2], M is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right)$ of S^{n+1} .

4. Minimal submanifolds with flat normal connection.

In this section we study compact minimal submanifolds of the unit sphere S^{n+p} which have flat normal connection, that is $R^\perp = 0$. First, we prove the following theorem which estimates the scalar curvature of such submanifolds which are non-totally geodesic and non-negatively curved.

Theorem 4.1. *Let M be an n -dimensional compact non-negatively curved minimal submanifold of the unit sphere S^{n+p} with flat normal connection. Then either M is totally geodesic or else the scalar curvature S of M satisfies $n(n-p-1) \leq S \leq n(n-2)$.*

Proof. Since the normal connection is flat we have $R^\perp = 0$ and that all the Weingarten maps A_N , $N \in \Gamma(\nu)$, can be diagonalized with respect to the same local orthonormal frame $\{e_1, \dots, e_n\}$ (cf. [1], p. 127). Choose a local orthonormal frame $\{N_1, \dots, N_p\}$ of normals such that $A_\alpha(e_i) = \lambda_i^\alpha e_i$, where

λ_i^α are the smooth functions and $\alpha = 1, \dots, p$. Then we have

$$\begin{aligned}
& \sum_{ij} Ric(e_j, A_{h(e_i, e_j)} e_j) - \sum_{ijk} R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) \\
&= \sum_{\alpha} \left[\sum_{ij} g(A_{\alpha} e_i, e_j) Ric(e_i, A_{\alpha} e_j) - \sum_{ijk} g(A_{\alpha} e_i, e_j) R(e_k, e_i; e_j, A_{\alpha} e_k) \right] \\
&= \sum_{\alpha} \left[\sum_j Ric(A_{\alpha} e_j, A_{\alpha} e_j) - \sum_{jk} R(e_k, A_{\alpha} e_j; e_j, A_{\alpha} e_k) \right] \\
&= \sum_{\alpha} \left[\sum_{jk} R(e_k, A_{\alpha} e_j; A_{\alpha} e_j, e_k) - \sum_{jk} R(e_k, A_{\alpha} e_j; e_j, A_{\alpha} e_k) \right] \\
&= \sum_{\alpha jk} ((\lambda_j^\alpha)^2 K_{kj} - \lambda_j^\alpha \lambda_k^\alpha K_{kj})
\end{aligned}$$

where $K_{kj} = R(e_k, e_j; e_j, e_k)$ is the sectional curvature of the plane section spanned by $\{e_k, e_j\}$. Consequently we have

$$\begin{aligned}
(4.1) \quad & \sum_{ij} Ric(e_i, A_{h(e_i, e_j)} e_j) - \sum_{ijk} R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) \\
&= \frac{1}{2} \left[\sum_{\alpha jk} 2(\lambda_j^\alpha)^2 K_{kj} - 2 \sum_{\alpha jk} \lambda_j^\alpha \lambda_k^\alpha K_{kj} \right] \\
&= \frac{1}{2} \left[\sum_{\alpha jk} (\lambda_j^\alpha)^2 K_{kj} + \sum_{\alpha jk} (\lambda_k^\alpha)^2 K_{kj} - 2 \sum_{\alpha jk} \lambda_j^\alpha \lambda_k^\alpha K_{kj} \right] \\
&= \frac{1}{2} \left[\sum_{\alpha jk} (\lambda_j^\alpha - \lambda_k^\alpha)^2 K_{kj} \right] \geq 0.
\end{aligned}$$

Then the integral formula (2.10) with $R^\perp = 0$ and (4.1) gives

$$\begin{aligned}
(4.2) \quad & \|\nabla h\| = 0, \\
& \sum_{ij} Ric(e_i, A_{h(e_i, e_j)} e_j) - \sum_{ijk} R(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k) = 0.
\end{aligned}$$

Now Equation (3.5) with $R^\perp = 0$ and the second equation in (4.2) yields

$$(4.3) \quad \|A_h\|^2 - n \|h\|^2 = 0.$$

Also, we have

$$\begin{aligned}
 (4.4) \quad \|A_h\|^2 &= \sum_{\alpha} \|A_{\alpha}\|^4 = \left(\sum_{\alpha} \|A_{\alpha}\|^2 \right)^2 - 2 \sum_{\alpha < \beta} \|A_{\alpha}\|^2 \|A_{\beta}\|^2 \\
 &= \|h\|^4 - 2 \sum_{\alpha < \beta} \|A_{\alpha}\|^2 \|A_{\beta}\|^2.
 \end{aligned}$$

Combining (4.3) and (4.4), we arrive at

$$(4.5) \quad \|h\|^2 \left(\|h\|^2 - n \right) \geq 0.$$

Also from (4.3), we have $\sum_{\alpha} \left(\|A_{\alpha}\|^4 - n \|A_{\alpha}\|^2 \right) = 0$, or equivalently

$$(4.6) \quad \sum_{\alpha} \left(\|A_{\alpha}\|^2 - \frac{n}{2} \right)^2 = \frac{n^2 p}{4}.$$

Now using Schwarz inequality, we get

$$(4.7) \quad \sum_{\alpha} \left(\|A_{\alpha}\|^2 - \frac{n}{2} \right)^2 \geq \frac{1}{p} \left[\sum_{\alpha} \left(\|A_{\alpha}\|^2 - \frac{n}{2} \right) \right]^2 = \frac{1}{p} \left(\|h\|^2 - \frac{np}{2} \right)^2.$$

Combining (4.6) and (4.7), we conclude that

$$(4.8) \quad \|h\|^2 \left(\|h\|^2 - np \right) \leq 0.$$

This proves that either M is totally geodesic or $\|h\|^2 \leq np$. If M is not totally geodesic, then from $\|h\|^2 \leq np$ together with (4.5) we get $n \leq \|h\|^2 \leq np$, which by Equation (2.9) is equivalent to $n(n-p-1) \leq S \leq n(n-2)$.

Next we prove the following:

Theorem 4.2. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} with flat normal connection. If the scalar curvature S of M satisfies $S \geq n(n-2)$, then M is either totally geodesic or is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right)$.*

Proof. Since $R^{\perp} = 0$, Equation (3.5) gives

$$\begin{aligned}
 (4.9) \quad \sum_{ij} \text{Ric} \left(e_i, A_{h(e_i, e_j)} e_j \right) - \sum_{ijk} R \left(e_k, e_i; e_j, A_{h(e_i, e_j)} e_k \right) \\
 = n \|h\|^2 - \|A_h\|^2.
 \end{aligned}$$

Thus the integral formula (2.10) in view of (4.9) and $R^{\perp} = 0$, takes the form

$$(4.10) \quad \int_M \left\{ \|\nabla h\|^2 + n \|h\|^2 - \|A_h\|^2 \right\} dv = 0.$$

For a local orthonormal frame $\{N_1, \dots, N_p\}$ of normals, using the expression for $\|A_h\|^2$ as given in (4.4) into (4.10) we arrive at

$$(4.11) \quad \int_M \left\{ \|\nabla h\|^2 + (n - \|h\|^2) \|h\|^2 + 2 \sum_{\alpha < \beta} \|A_\alpha\|^2 \|A_\beta\|^2 \right\} dv = 0.$$

Since $S \geq n(n-2)$, that is $\|h\|^2 \leq n$, above integral gives $\|\nabla h\| = 0$, $\|h\|^2 (n - \|h\|^2) = 0$, $\sum_{\alpha < \beta} \|A_\alpha\|^2 \|A_\beta\|^2 = 0$. The last two equations give either M is totally geodesic or else $\|h\|^2 = n$ and $A_\beta = 0$, $\beta = 2, \dots, p$. In the second case, we have as in [2] that M is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}}\right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}}\right)$ of S^{n+1} .

Now as applications of these theorems, first we have:

Corollary 4.1. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} with flat normal connection. If the scalar curvature S of M satisfies $S > n(n-2)$, then M is totally geodesic.*

Corollary 4.2. *Let M be a non-negatively curved compact minimal hypersurface of S^{n+1} . Then M is either totally geodesic or is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}}\right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}}\right)$.*

Proof. Since M is hypersurface, the normal connection is flat, by Theorem 4.1, we have either M is totally geodesic or the scalar curvature is given by $S = n(n-2)$. If M is non-totally geodesic, we have $\|h\|^2 = n$ and consequently by the result in [2], M is isometric to the hypersurface $S^m \left(\sqrt{\frac{m}{n}}\right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}}\right)$.

Remark 4.1. In [2], it is asked that the values of $\|h\|^2$ should determine the compact hypersurfaces up to rigid motion in the ambient sphere S^{n+1} (cf. p. 75). The above Corollary shows that for non-negatively curved hypersurfaces in S^{n+1} , $\|h\|^2$ takes only two values 0 and n and the corresponding hypersurfaces are known. Thus the Corollary 4.2 can be thought of as a result in the direction of this question.

We also have the following consequence of Corollary 4.2:

Corollary 4.3. *Let M be a non-negatively curved compact minimal hypersurface of S^{n+1} . If M is diffeomorphic to S^n , then M is totally geodesic.*

Remark 4.2. One of the important questions concerning the hypersurfaces of S^{n+1} is to show that “the only embedded minimal hypersurface in S^{n+1} which is diffeomorphic to S^n is the totally geodesic sphere” (cf. [7], p. 692, Problem-99). Corollary 4.3 can be considered as a result in this direction for the class of non-negatively curved minimal hypersurfaces in S^{n+1} .

Finally, we prove:

Theorem 4.3. *Let M be an n -dimensional compact minimal submanifold of the unit sphere S^{n+p} with flat normal connection. If M is of constant sectional curvature $c \geq 0$, then either M is flat ($c = 0$) and $p \geq (n - 1)$, or totally geodesic.*

Proof. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on M . Then for the minimal submanifold M of constant sectional curvature c , Equation (2.3) can be restated as

$$(c - 1)(\delta_{jk}e_i - \delta_{ik}e_j) = A_{h(e_j, e_k)}e_i - A_{h(e_i, e_k)}e_j.$$

Taking inner product with $A_{h(e_j, e_k)}e_i$ and summing the resulting equation we arrive at

$$(4.12) \quad (1 - c) \|h\|^2 = \|A_h\|^2 - \sum_{ijk} g \left(A_{h(e_i, e_k)}e_j, A_{h(e_j, e_k)}e_i \right).$$

Now as the normal connection is flat, $R^\perp = 0$ and with the help of Equation (2.5) we compute

$$\begin{aligned} \sum_{ijk} g \left(A_{h(e_i, e_k)}e_j, A_{h(e_j, e_k)}e_i \right) &= \sum_{ijk} g \left(A_{h(e_j, e_k)}e_j, A_{h(e_i, e_k)}e_i \right) \\ &= \sum_{ik} g \left(\sum_j A_{h(e_j, e_k)}e_j, A_{h(e_i, e_k)}e_i \right) \\ &= \sum_{ik} g \left((n - 1)e_k - Q(e_k), A_{h(e_i, e_k)}e_i \right) \\ &= (n - 1) \|h\|^2 - \sum_{ik} \text{Ric}(e_k, A_{h(e_i, e_k)}e_i). \end{aligned}$$

However for M the Ricci curvature satisfies $\text{Ric}(X, Y) = (n - 1)cg(X, Y)$ and consequently,

$$\sum_{ijk} g \left(A_{h(e_i, e_k)}e_j, A_{h(e_j, e_k)}e_i \right) = (n - 1)(1 - c) \|h\|^2.$$

Using this equation in (4.12), we get

$$(4.13) \quad \|A_h\|^2 = n(1 - c) \|h\|^2.$$

Note that Equation (4.3) holds for non-negatively curved compact minimal submanifolds of S^{n+p} with flat normal connection. Thus combining (4.3) and (4.13), we conclude that $c \|h\|^2 = 0$ and this proves that either $c = 0$ or M is totally geodesic. However if $c = 0$, then $S = 0$ and consequently by Theorem 4.1, $n(n - p - 1) \leq 0$ (as M is non-totally geodesic), which implies $p \geq n - 1$.

Corollary 4.4. *There is no compact flat minimal hypersurface in S^4 .*

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