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We show that the two-sided inclusions of braid subfactors of type A coincide with the asymptotic inclusions for the one-sided pairs. We also include an elementary method for obtaining the higher relative commutants of the one-sided pairs, which were already computed by H. Wenzl.

Introduction.

In [**W-1**] and [**W-2**] Wenzl constructed new examples of subfactors of the hyperfinite II_1 factor. He considered unitary representations ρ of the infinite braid group $\mathbf{B}_\infty$, whose restrictions to finite braid groups $\mathbf{B}_n$ generate finite dimensional C*-algebras, with additional properties such as the existence of a positive Markov trace on the quotients. His pairs of subfactors are given by the von Neumann algebras generated by $\{g_i \ : \ i \in \mathbb{N}, \ i > m\}$ and by $\{g_i \ : \ i \in \mathbb{N}_0\}$ in the trace representation, where $g_i := (\pi_{tr} \circ \rho)(\sigma_i)$, σ_i are the braid generators for $i \in \mathbb{N}_0$, and $m \in \mathbb{N}_0$ is arbitrary.

These special unitary braid representations can be obtained in connection with the representation theory of the classical Lie algebras. The ones corresponding to the Lie type A factor through the Hecke algebra $H_\infty(q)$ for q a root of unity, [**W-1**]. The ones corresponding to the the types B,C,D factor through the Birman-Murakami-Wenzl algebra $C_\infty(r, q)$ for special values of r and q, [**W-2**]. Wenzl subfactors are a generalisation of Jones subfactors which arise in connection with the $sl(2)$ case, [**J**].

In [**E**] the two-sided versions of the Wenzl subfactors for classical Lie types were considered; for the Jones subfactors this had been done by M. Choda in [**Ch**]. The two-sided subfactors are defined by extending the unitary braid representations ρ to the infinite two-sided braid group with generators σ_i, with $i \in \mathbb{Z}$. Thus, the two-sided pairs are generated by $\{g_i \ : \ i \in \mathbb{Z} \backslash \{0, \dots, m\}\}$ and by $\{g_i \ : \ i \in \mathbb{Z}\}$.

It turns out that if we set $m = 0$ — where m is as in the first paragraph — then the asymptotic inclusions (in the sense of Ocneanu, [**O**]) for Wenzl's one-sided pairs associated with the Lie types A,B,C,D coincide with the corresponding two-sided versions. This was easily proved in [**E**] for the B,C,D types. In this paper we shall prove this fact for the remaining type A case. It has also been shown by S. Goto in [**G**] independently by a different method.

The method we employ here is fairly elementary. Roughly, we show that the iterations of the Jones' basic construction for the one-sided pair at the finite dimensional level are obtained by adding generators "to the left", and then by reducing by a special projection. In this way we can relate the Jones' basic iterations of the one-sided pair with the two-sided pair. For example, for large $n \in \mathbb{N}$, the first basic construction for

$$\langle g_1, \dots, g_n \rangle \subset \langle g_0, \dots, g_n \rangle$$

is isomorphic to the algebra

$$p\langle g_{-k+1}, \dots, g_n \rangle p = \langle e,\ p\langle g_0, \dots, g_n \rangle p \rangle. \tag{$*$}$$

Here, the parameter k corresponds to $sl(k)$, and p and e are special projections in $\langle g_{-k+1}, \dots, g_{-2} \rangle$ and in $\langle g_{-k}, \dots, g_{-2} \rangle$, respectively. The projection e behaves like the Jones' basic projection. When $q = 1$, e is precisely the antisymmetrizer a_k of the symmetric group algebra $\mathbb{C}\mathcal{S}_k$, acting on $V^{\otimes k}$ by permuting the tensor factors, where V is a k-dimensional vector space; and p coincides with the antisymmetrizer a_{k-1}, which projects $V^{\otimes k-1}$ onto the dual representation of V, regarded as an $Sl(k)$-module. The subfactors for this case, $q = 1$, were studied by A. Wassermann, [**Wa**].

At each iteration, we do a similar procedure as in $(*)$ for the first basic construction, with variations depending on its parity, and one obtains, in the limit, a reduced two-sided pair. Later, by applying some Bisch-McDuff results, we show that this reduced two-sided pair is in fact conjugate to the two-sided pair.

This paper is organised as follows: In the first section we include some results by Wenzl on his type A subfactors and definitions. In the second section we prove some technical lemmas and establish the relation between the Jones' iterations of the one-sided finite dimensional approximants and the two-sided approximants. In the last section, first part, we show that we can apply Bisch-McDuff results that imply that the pairs in consideration are stable, that is, that they split a copy of the hyperfinite II_1 factor. Later on, we express the asymptotic inclusion as some "infinitely reduced" version of the two-sided pair, and using inductive arguments and stableness we prove our main result. As a remark, the "infinitely reduced" two-sided pair is conjugate to the two-sided pair in a more general setting: In fact, in our proof we only assume the properties of more general braid-type subfactors as constructed in [**E**]. We also include in Section 3.2 an expression for the higher relative commutants of Wenzl's (finite depth) type A subfactors, which have already been computed by him, [**W-4**], and later also computed by D. Evans and Y. Kawahigashi in [**EK**].

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1. Preliminaries.

Let us recall that the finite dimensional Hecke algebra $H_n(q)$ is the free complex algebra with generators $1, T_0, \dots, T_{n-2}$, and relations, depending on a parameter $q \in \mathbb{C}$,

(B_1) $T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$, for $i = 0, \dots, n-3$,
(B_2) $T_iT_j = T_jT_i$, for $|i-j| \geq 2$,
(H) $T_i^2 = (q-1)T_i + q$, for $i = 0, \dots, n-2$.

It can be shown by induction that these complex algebras have dimension $n!$, independent of q. Set $H_\infty(q) = \bigcup H_n(q)$. The representations considered for defining the braid subfactors are interesting in the case that the parameter q is a root of unity, $q \neq 1$. So, we shall fix $q = e^{\pm \frac{2\pi i}{l}}$, with $l \geq 3$. We shall briefly summarise the parametrisations of the semisimple quotients of $H_n(q)$, with q as above, which are associated with $sl(k)$ for $1 < k < l$ (see [**W-1**]).

For $k \in \mathbb{N}$, and $k < l$, a (k,l) *Young diagram* λ of size n is a k-tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$, $\lambda_1 - \lambda_k \leq l - k$, and $\sum_{i=1}^k \lambda_i = n$. We denote the set of (k,l) diagrams of size n by $\Lambda_n^{(k,l)}$. We can also regard a (k,l) diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ of size n as k ordered rows of boxes with λ_i boxes in the i^{th} row. A (k,l) *tableau* t of shape λ of size n is a standard tableau, such that for each $j \leq n$, the subdiagram occupied by the numbers $\{1, \dots, j\}$ is an element of $\Lambda_j^{(k,l)}$. A (k,l) standard tableau t of shape λ can also be regarded as an increasing sequence of (k,l) diagrams $\emptyset = \lambda_0 \subset [1] = \lambda_1 \subset \lambda_2 \subset \dots \subset \lambda_n = \lambda$. Denote by $T_\lambda^{(k,l)}$ the set of (k,l) tableaux of shape λ.

For each diagram $\lambda \in \Lambda_n^{(k,l)}$ let V_λ be a vector space with basis $\{v_t\}$ labelled by $T_\lambda^{(k,l)}$. Wenzl defined in [**W-1**] an irreducible representation $\pi_\lambda^{(k,l)}$ of $H_n(q)$ on V_λ given by

$$\pi_\lambda^{(k,l)}(T_i)\vec{v}_t = b_d(q)\vec{v}_t + \frac{\{(1-q^{d+1})(1-q^{d-1})\}^{1/2}}{1-q^d}\ \vec{v}_{s_i(t)},$$

where $d = d(t,i) = c(i+1) - c(i) - r(i+1) + r(i)$, with $c(j)$ and $r(j)$ denoting the column and the row of the box containing j respectively, and where $b_d(q) = \frac{q^d(1-q)}{(1-q^d)}$, and $s_i(t)$ is the tableau obtained from t by interchanging the numbers i and $i+1$ (if the resulting tableau is not standard then it appears with zero coefficient).

Different diagrams in $\Lambda_n^{(k,l)}$ give inequivalent representations. Consider $H_{n-1}(q)$ as the subalgebra of $H_n(q)$ generated by $1, T_0, \dots, T_{n-3}$. The restriction rule is given by $\pi_\lambda^{(k,l)}|_{H_{n-1}(q)} \cong \bigoplus_{\lambda'<\lambda} \pi_{\lambda'}^{(k,l)}$, where $\lambda' < \lambda$ means that λ can be obtained by adding one box to $\lambda' \in \Lambda_{n-1}^{(k,l)}$. We can then define a representation $\pi_n^{(k,l)}$ of $H_n(q)$ by $\pi_n^{(k,l)}(x) = \bigoplus_{\lambda \in \Lambda_n^{(k,l)}} \pi_\lambda^{(k,l)}(x)$, for $x \in H_n(q)$. Its restriction to $H_{n-1}(q)$ is equivalent to $\pi_{n-1}^{(k,l)}$. Finally, we have a well defined representation $\pi^{(k,l)}$ of $H_\infty(q)$ given by $\pi^{(k,l)}(x) = \pi_n^{(k,l)}(x)$, if $x \in H_n(q)$. The representation $\pi^{(k,l)}$ is locally finite dimensional, i.e., the algebras $B_n := \pi^{(k,l)}(H_n(q))$ are finite dimensional C*-algebras. We have $B_n \cong \bigoplus_{\lambda \in \Lambda_n^{(k,l)}} M_{a_\lambda}(\mathbb{C})$, that is, the equivalence classes of minimal idempotents of B_n are labelled by $\Lambda_n^{(k,l)}$. The representation $\pi^{(k,l)}$ is also unitary (that is, $g_i := \pi_n^{(k,l)}(T_i)$ is a unitary element of $B_n = \langle g_0, \dots, g_{n-2} \rangle$, for $i = 0, \dots, n-2$, and every $n \in \mathbb{N}$), and has the following properties:

(i) *Any element $x \in B_{n+1}$ can be written as a sum of elements $a g_{n-1} b + c$ with $a, b, c \in B_n$.*

(ii) *The ascending sequence of finite dimensional C*-algebras (B_n) is periodic with period k, in the sense of Wenzl,* [**W-1**].

(iii) *The unique positive faithful trace tr on $\bigcup B_n$ has the Markov property:*

$$\operatorname{tr}(g_{n-1}x) = \eta \operatorname{tr}(x),$$

for all $x \in B_n$, for all $n \in \mathbb{N}$, where $\eta \in \mathbb{C}$ is fixed. Given condition **(i)**, *the Markov condition implies the multiplicativity property for the trace:*

(iii)′ *If x and y are in $*$-subalgebras generated by disjoint subsets of generators g_i, then*

$$\operatorname{tr}(xy) = \operatorname{tr}(x) \operatorname{tr}(y).$$

(iv) *Existence of a projection $p \in B_k$, where k is the periodicity, with the contraction property: $p \in B_k$ has the contraction property if for all $n \in \mathbb{N}$,*

$$pB_{n+k}p \cong pB_{k+1,n+k+1} \cong B_{k+1,n+k+1},$$

where $B_{s,t}$ is the algebra generated by $\{g_s, \dots, g_{t-2}\}$.

Property **(iv)** is equivalent to a special property in connection with the structure coefficients for the multiplication in $\bigoplus_n K_0(B_n)$. More precisely, given a locally finite dimensional representation of the braid group $\mathbf{B}_\infty = \bigcup_n \mathbf{B}_n$, one has an associative, commutative, graded product on $\bigoplus_n K_0(B_n)$ defined as follows (see [**GW**] for more details). For projections $x \in B_n$ and $y \in B_m$ define $x \otimes y = x(\rho \circ \operatorname{shift}_n)(y) \in B_{n+m}$, where $\operatorname{shift}_n : \mathbb{C}\mathbf{B}_m \to \mathbb{C}\mathbf{B}_{n+m}$ is determined by $\sigma_i \mapsto \sigma_{i+n}$. Then, $[x] \otimes [y] = [x \otimes y]$

defines the multiplication on $\bigoplus_n K_0(B_n)$. Denote the structure constants of this multiplication by $c^{\nu}_{\lambda\mu}$. That is, if p_λ and p_μ are minimal projections in the classes labelled by $\lambda \in \Lambda_n^{(k,l)}$ and $\mu \in \Lambda_m^{(k,l)}$ respectively, then $[p_\lambda] \otimes [p_\mu] = \sum_{\nu \in \Lambda_{n+m}^{(k,l)}} c^{\nu}_{\lambda\mu}[p_\nu]$. The existence of a projection p with the contraction property is equivalent to **(v)** or **(vi)** below:

(v) *For all $n \in \mathbb{N}_0$ there exists an injective map $j : \Lambda_n^{(k,l)} \to \Lambda_{n+k}^{(k,l)}$ that preserves the structure coefficients for the multiplication in $\bigoplus_n K_0(B_n)$, that is, such that $c^{\nu}_{\lambda\mu} = c^{j(\nu)}_{j(\lambda)\mu} = c^{j(\nu)}_{\lambda j(\mu)}$ for all $\lambda \in \Lambda_n^{(k,l)}$, $\mu \in \Lambda_m^{(k,l)}$, $\nu \in \Lambda_{n+m}^{(k,l)}$, and also such that $c^{\epsilon}_{\lambda j(\mu)} = 0$ if $\epsilon \notin j(\Lambda_{n+k}^{(k,l)})$.*

(vi) *There exists a projection $p \in B_k$ such that for every minimal projection $p_\lambda \in B_n$, and for all $n \in \mathbb{N}$, the projection $p \otimes p_\lambda$ remains minimal in B_{n+k}. Moreover, if $\lambda \neq \lambda'$ then $p \otimes p_\lambda$ and $p \otimes p_{\lambda'}$ are not equivalent.*

For a proof of these see [**W-3**], [**E**]. An implication of these last conditions is that the injective map j preserves the coefficients of the inclusion matrices for the pairs $B_n \subset B_{n+1}$, $G_n = (g_{\lambda\mu})_{\lambda \in \Lambda_n^{(k,l)}, \mu \in \Lambda_{n+1}^{(k,l)}}$, that is, $g_{\lambda\mu} = g_{j(\lambda)j(\mu)}$, since $g_{\lambda\mu} = c^{\mu}_{[1]\lambda}$. It is also easy to show that the periodicity condition on (B_n) forces the injective map $j : \Lambda_n^{(k,l)} \to \Lambda_{n+k}^{(k,l)}$ to be a bijection for large n (see [**E**]). It was shown in [**GW**] that an idempotent $p_{[1^k]} \in B_k$ in the equivalence class given by the diagram $[1^k]$ (a column of height k boxes) has the contraction property: The map j maps an element $\mu \in \Lambda_n^{(k,l)}$ to a diagram $j(\mu) \in \Lambda_{n+k}^{(k,l)}$ obtained by adding to μ a column of k boxes to the left. Let us remark that the minimal projections $p_{[1^s]} \in B_s$ correspond to "q-antisymmetrizers": If $q = 1$, $p_{[1^s]}$ is just the antisymmetrizer of $\mathbb{C}S_s$ acting on $V^{\otimes s}$ – where V is an s-dimensional vector space – by permuting tensor factors.

We include here a well-known lemma related to the structure coefficients that will be needed in the next section:

Lemma 1.

(a) $c^{\sigma}_{\eta[1]} = c^{j(\eta)}_{[1^{k-1}]\sigma}$, *for $\eta \in \Lambda_n^{(k,l)}$ and $\sigma \in \Lambda_{n+1}^{(k,l)}$, where the integer coefficients $c^{\mu}_{\nu\rho}$ are the structure coefficients for the associative graded multiplication $\otimes$ on $\bigoplus_n K_0(B_n)$.*

(b) $c^{j(\eta)}_{[1^{k-1}]\sigma} \neq 0 \iff \eta < \sigma$.

Proof. (a) The coefficients $c^{\sigma}_{\eta[1]}$ and $c^{j(\eta)}_{[1^{k-1}]\sigma}$ coincide with the classical Littlewood-Richardson coefficients by the Young-Pieri Rules (see [**GW**]). So we only need to apply the classical Littlewood-Richardson rule, which says that the (classical) $c^{\nu}_{\lambda\mu}$ coefficient is the number of ways the Young diagram λ can be expanded to the Young diagram ν by strict μ-expansion (check for these rules e.g. in [**FH**]). By this rule,

$$c^{\sigma}_{\eta[1]} = \begin{cases} 1 & \text{if} \quad \sigma > \eta \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

If $c^{\sigma}_{\eta[1]} = 1$ then $\sigma > \eta$, i.e., $\sigma -$ {one box} $= \eta$, so that $j(\sigma -$ {one box}$) = j(\eta)$, where $\sigma -$ {one box} means a diagram obtained by taking away one box of σ. Thus, one can see that $j(\eta)$ is a strict σ-expansion of $[1^{k-1}]$, and it follows that $c^{j(\eta)}_{[1^{k-1}]\sigma} = 1$. Conversely, if $c^{j(\eta)}_{[1^{k-1}]\sigma} > 0$ then $j(\eta)$ can be obtained by adding the boxes of σ to $[1^{k-1}]$. Since $j(\eta)$ has at least one column with k boxes, then $j(\eta)$ is formed by adding one box of σ to $[1^{k-1}]$ to obtain $[1^k]$, and then "attaching" the resulting $\sigma -$ {one box} to the "right" of $[1^k]$. Then, $\eta = \sigma -$ {one box}, so that $c^{\sigma}_{\eta[1]} = 1$. It follows that $c^{\sigma}_{\eta[1]} = c^{j(\eta)}_{[1^{k-1}]\sigma}$.

(b) It is an obvious consequence of (a) and (1.1). □

The so-called one-sided and two-sided inclusions of hyperfinite II_1 factors corresponding to the representation $\pi^{(k,l)}$ of $H_\infty(q)$ are defined as follows: For $r \in \mathbb{N}$ set

$$A_r := \langle g_1, \dots , g_{r-2} \rangle, \quad B_r := \langle g_0, \dots , g_{r-2} \rangle.$$

The pair $A_r \subset B_r$ is conjugate to $B_{r-1} \subset B_r$. The inclusion of ascending sequences $(A_r) \subset (B_r)$ is periodic, and the Wenzl one-sided pair is given by the inclusion of hyperfinite II_1 factors $A \subset B$, with A and B the weak closures of $\bigcup_r A_r$ and $\bigcup_r B_r$ respectively, in the GNS representation with respect to the unique trace (see [**W-1**]). For the two-sided inclusion (see [**E**]), consider the ascending sequences of finite dimensional C^*-algebras

$$\tilde{C}_r := \langle g_{-r+1}, \dots , g_{-1}, g_1, \dots , g_{r-1} \rangle,$$
$$\tilde{D}_r := \langle g_{-r+1}, \dots , g_{-1}, g_0, g_1, \dots , g_{r-1} \rangle,$$

after extending our representation to the two-sided infinite braid group. We have the identifications

$$\tilde{C}_r \cong B_r \otimes B_r, \quad \tilde{D}_r \cong B_{2r},$$

via the trace preserving automorphism $g_{-i}g_j \mapsto g_{r-i} \otimes g_j$, and $g_i \mapsto g_{r+i}$. The pair $\tilde{C}_r \subset \tilde{D}_r$ is conjugate to $B_r \otimes B_r \subset B_{2r}$, and the inclusion here depends on the structure coefficients for the graded multiplication on $\bigoplus_n K_0(B_n)$, as explained in properties **(iv)-(vi)**. The inclusion of ascending sequences $(\tilde{C}_r) \subset (\tilde{D}_r)$ is periodic as well, and the two-sided pair is given by the inclusion of hyperfinite II_1 factors $C \subset D$ with C and D the weak closures of $\bigcup_r \tilde{C}_r$ and $\bigcup_r \tilde{D}_r$ respectively, in the GNS representation with respect to the unique trace.

In the rest of the paper we shall assume a basic knowledge on Wenzl's Hecke algebra article, [**W-1**].

2. Some technical lemmas and the basic iterations for $A_r \subset B_r$.

In this section we shall first prove some rather technical lemmas. These will be used later in this section to show how the iterations of the basic construction for $A_r \subset B_r$ are related to the two-sided pair. We use the same notation as in the preliminaries, for $i, j \in \mathbb{Z}$, $i < j-1$,

$$B_{i,j} := \langle g_i, \dots , g_{j-2} \rangle,$$

so that $B_r = B_{0,r}$. The algebra $B_{i,j} = \text{shift}_i(B_{j-i})$ is just the shifting "to the right (or left) by $|i|$" of the algebra B_{j-i}. If k is the periodicity constant, $n \in \mathbb{N}$, and if

$$k(n) := \begin{cases} -ik+1 & \text{if } n = 2i-1 \\ -ik & \text{if } n = 2i \end{cases},$$

define the projections

$$\begin{cases} p^{(n)} := \text{shift}_{k(n)}(p_{[1^{k-1}]}) & \in \text{shift}_{k(n)}(B_{k-1}) \\ q^{(n)} := \text{shift}_{k(n)}(p_{[1^k]}) & \in \text{shift}_{k(n)}(B_k) \end{cases}.$$

By this we mean that $p^{(n)}$ and $q^{(n)}$ are the q-antisymmetrizers $p_{[1^{k-1}]} \in B_{k-1}$ and $p_{[1^k]} \in B_k$, but with the appropriate shifting of the generators' indexes depending on the parity of n. Set also

$$P^{(i)} := \prod_{s=1}^{i} p^{(2s-1)} = p^{(1)}p^{(3)} \dots p^{(2i-1)}.$$

Below we list some properties which will be used frequently in the next section. The reference for these is basically [**W-1**].

Remarks.

(1) For every $n \in \mathbb{N}$, $p^{(n)} \in \text{shift}_{k(n)}(B_{k-1})$ and $q^{(n)} \in \text{shift}_{k(n)}(B_k)$ are minimal. ([**W-1**, p. 368].)

(2) For $i \in \mathbb{N}$, by the braid relations, the $p^{(2i-1)}$'s are mutually commuting, and the $p^{(2i)}$'s are also mutually commuting. In particular, $P^{(i)}$ is a projection.

(3) For every $n \in \mathbb{N}$, $q^{(n)} \leq p^{(n)}$. Furthermore, if we regard $p^{(n)}$ in $\text{shift}_{k(n)}(B_k)$, then $p^{(n)}$ splits orthogonally as $p^{(n)} = q^{(n)} \oplus \tilde{q}^{(n)}$, where $\tilde{q}^{(n)} \in \text{shift}_{k(n)}(B_k)$ is also minimal. ([**W-1**, p. 368].)

(4) $q^{(2i)} \leq p^{(2i-1)}$: Since $\text{shift}_1(p_{[1^{k-1}]}) \sim p_{[1^{k-1}]}$ in B_k, $p_{[1^k]}$ is a rank one minimal central projection in B_k, and $p_{[1^k]} \leq p_{[1^{k-1}]}$ by (3), then $p_{[1^k]} \leq \text{shift}_1(p_{[1^{k-1}]})$. Thus, $q^{(2i)} = \text{shift}_{-ik}(p_{[1^k]}) \leq \text{shift}_{-ik}(\text{shift}_1(p_{[1^{k-1}]})) = p^{(2i-1)}$.

(5) By the braid relations, $[q^{(n)}, B_{s,t}] = 0$ for any $n \in \mathbb{N}$, and $s,t \in \mathbb{Z}$ such that $-(i-1)k+1 \leq s \leq t-2$, or such that $-(i-1)k \leq s \leq t-2$ (depending on parity of n). In particular, $q^{(2i-1)}$ and $q^{(2i)}$ commute with $p^{(2j-1)}$ for $j = 1, \dots, i-1$.

(6) $q^{(n)} \in \text{shift}_{k(n)}(B_k)$ remains minimal in $\text{shift}_{k(n)}(B_{k+1})$: This can be shown by using the contraction property of $p_{[1^k]}$, or by using the properties for $p_{[1^k]}$ stated in [**W-1**, p. 368].

(7) Let $A \subset B$ be finite dimensional C* algebras acting on a Hilbert space H with a faithful trace on B, and e be a projection on H. Then, if $(*) ebe = e\epsilon_A(b)e$, where ϵ_A is the trace preserving conditional expectation onto A, and $(**) eA \cong A$, then $\langle B, e\rangle \cong Q \oplus K$, where Q is isomorphic to the basic construction for $A \subset B$, and K is isomorphic to a subalgebra of B. [**W-1**, Theorem 1.1, (i)].

Next in Lemma 2 we prove some relations involving the projections $q^{(1)}$, $q^{(2)}$, and trace preserving conditional expectations. This will enable us to show that in fact the projections $q^{(n)}$ behave as the Jones' basic projections.

Lemma 2. *If $N \subset M$ are von Neumann algebras and $p \in M$, set N_p to be the von Neumann algebra generated by $\{pxp \ : \ x \in N\}$. So, for $r \in \mathbb{N}$ consider the inclusions $A_{r_{p^{(1)}}} \subset B_{r_{p^{(1)}}} \subset B_{-k+1, r_{p^{(1)}}}$. Let $E_1 : B_{r_{p^{(1)}}} \to A_{r_{p^{(1)}}}$ and $E_2 : B_{-k+1, r_{p^{(1)}}} \to B_{r_{p^{(1)}}}$ denote the conditional expectations with respect to tr onto $A_{r_{p^{(1)}}}$ and $B_{r_{p^{(1)}}}$, respectively. We have:*

(a) *$q^{(1)}$ commutes with $A_{r_{p^{(1)}}}$.*

(b) *$q^{(2)}$ commutes with $B_{r_{p^{(1)}}}$.*

(c) *$q^{(1)} b q^{(1)} = E_1(b) q^{(1)}$, for every $b \in B_{r_{p^{(1)}}}$.*

(d) *$q^{(2)} q^{(1)} q^{(2)} = \alpha q^{(2)}$, with $\alpha \in \mathbb{C}$ such that $E_2(p^{(1)} q^{(1)} p^{(1)}) = \alpha p^{(1)}$.*

(e) *$\langle q^{(1)}, B_{r_{p^{(1)}}}\rangle$ contains a subalgebra, Q_1, isomorphic to $\langle A_{r_{p^{(1)}}}, e_1\rangle$.*

Proof. Parts (a) and (b) are true by remarks (3), (4) and (5).

(c) Any element $x \in B_r$ can be written as a linear combination of elements of the form $ug_0v + w$, with $u, v, w \in A_r$ (see [**W-1**]). Since $p^{(1)}$ commutes with B_r, then any element in $B_{r_{p^{(1)}}}$ can be written as a linear combination of elements of the form $ug_0v + w$, with $u, v, w \in A_{r_{p^{(1)}}}$. By the bimodule property of the conditional expectation ($E_1(uzv) = uE_1(z)v$ if $u, v \in A_{r_{p^{(1)}}}$), if $u, v, w \in A_{r_{p^{(1)}}}$ then $E_1(ug_0v + w) = \alpha uv + w$, where $\alpha = \text{tr}(g_0)$ (the last equality follows from multiplicativity property of tr; see **(iii)′** in the preliminaries). As $q^{(1)}$ commutes with A_r and $q^{(1)}p^{(1)} = q^{(1)} = p^{(1)}q^{(1)}$, the claim follows as soon as we show $q^{(1)} g_0 q^{(1)} = \alpha q^{(1)}$. Since $q^{(1)} \in B_{-k+1,2}$ is minimal then $q^{(1)} g_0 q^{(1)} = \beta q^{(1)}$, for some $\beta \in \mathbb{C}$. By the Markov property $\beta = \text{tr}(g_0) = \alpha$.

(d) $q^{(2)}$ remains minimal in $B_{-k,1}$ and $q^{(2)}p^{(1)} = q^{(2)}$. (Remarks (4),(6).) In particular, if $x \in B_{-k+1,1}$, $q^{(2)}p^{(1)}xp^{(1)}q^{(2)} = \alpha_x q^{(2)}$, with $\alpha_x := \frac{\mathrm{tr}\,(q^{(2)}x)}{\mathrm{tr}\,(q^{(2)})}$. Also, for $y \in B_r$, $\mathrm{tr}\,(p^{(1)}xp^{(1)}yp^{(1)}) = \beta_x \mathrm{tr}\,(p^{(1)}yp^{(1)})$, where $\beta_x := \frac{\mathrm{tr}\,(p^{(1)}x)}{\mathrm{tr}\,(p^{(1)})}$, by multiplicativity of tr, so that $E_2(p^{(1)}xp^{(1)}) = \beta_x p^{(1)}$. We shall show that $\beta_x = \alpha_x$ for every $x \in B_{-k+1,r}$. For $x \in B_{-k+1,0}$, $p^{(1)}xp^{(1)} = \beta_x p^{(1)}$, by minimality of $p^{(1)}$ in $B_{-k+1,0}$. Then, $\alpha_x q^{(2)} = \beta_x q^{(2)}$, so that $\alpha_x = \beta_x$. For $x = g_{-1}$, $\alpha_x = \beta_x$ by multiplicativity of tr. Finally, because $\alpha_x = \beta_x$ for $x \in B_{-k+1,0}$ and for $x = g_{-1}$, then the equality holds for every $x \in B_{-k+1,1}$, since any element in $B_{-k+1,1}$ is a linear combination of elements of the form $ug_{-1}v + w$, with $u, v, w \in B_{-k+1,0}$. (For $x = ug_{-1}v$, $\alpha_x = \frac{\mathrm{tr}\,(q^{(2)}ug_{-1}v)}{\mathrm{tr}\,(q^{(2)})} = \frac{\mathrm{tr}\,(vq^{(2)}ug_{-1})}{\mathrm{tr}\,(q^{(2)})} = \frac{\mathrm{tr}\,(vq^{(2)}u)\mathrm{tr}\,(g_{-1})}{\mathrm{tr}\,(q^{(2)})} = \alpha_{uv}\mathrm{tr}\,(g_{-1}) = \beta_{uv}\mathrm{tr}\,(g_{-1}) = \beta_x$.) It now follows that for $x \in B_{-k+1,1}$, $q^{(2)}\big(p^{(1)}xp^{(1)}\big)q^{(2)} = E_2(p^{(1)}xp^{(1)})q^{(2)}$. In particular, (d) follows if we take $x = q^{(1)}$.

(e) The projection $q^{(1)}$ and the algebra $B_{r_{p^{(1)}}}$ verify the conditions of Remark (7) (conditions (a) and (c) of this lemma). Therefore, $\langle q^{(1)}, B_{r_{p^{(1)}}}\rangle \cong \langle e_1, B_{r_{p^{(1)}}}\rangle \oplus K_1^r$, where K_1^r is isomorphic to a subalgebra of $B_{r_{p^{(1)}}}$, and where e_1 is the Jones' basic projection for $A_{r_{p^{(1)}}} \subset B_{r_{p^{(1)}}}$. $\square$

The lemmas above and the following results have the following purpose: We want to obtain all the iterations of the Jones' basic construction for the pair $A_r \subset B_r$, in a way that they can be related to the two-sided construction. The odd iterations will be obtained by adding $k-1$ "reduced" generators on the "left" of the previous iteration, and then by reducing by the q-antisymmetrizer $p^{(n)}$ for the appropriate n. The even iterations will be obtained by just adding one reduced generator on the left of the previous iteration. We shall now show this for the first two iterations, and afterwards for the general case, by induction.

Proposition 3. *For large $r \in \mathbb{N}$, consider the inclusion $A_{r_{p^{(1)}}} \subset B_{r_{p^{(1)}}}$ as before:*

(i) *The 1^{st} basic construction is isomorphic to $B_{-k+1,r_{p^{(1)}}} = \langle q^{(1)}, B_{r_{p^{(1)}}}\rangle$.*

(ii) *The 2^{nd} basic construction is isomorphic to $B_{-k,r_{p^{(1)}}} = \langle q^{(2)}, B_{-k+1,r_{p^{(1)}}}\rangle$.*

Proof. (i) By Lemma 2 (e), the algebra $\langle q^{(1)}, B_{r_{p^{(1)}}}\rangle$ contains a subalgebra, Q_1, isomorphic to the basic construction for $A_{r_{p^{(1)}}} \subset B_{r_{p^{(1)}}}$. Consider the inclusions:

$$B_{r_{p^{(1)}}} \subset^{G^t} Q_1, \quad \text{and} \quad B_{r_{p^{(1)}}} \subset^{L} B_{-k+1,r_{p^{(1)}}},$$

where G is the inclusion matrix for $A_r \subset B_r$. Our goal is to show that $L = G^t$, for then we shall have $\dim Q_1 = \dim B_{-k+1,r_{p^{(1)}}}$, and so the desired equality $Q_1 = \langle q^{(1)}, B_{r_{p^{(1)}}}\rangle = B_{-k+1,r_{p^{(1)}}}$. A minimal idempotent in $B_{r_{p^{(1)}}}$ is given by $p^{(1)}p_\lambda p^{(1)} = p^{(1)}p_\lambda$, where p_λ is a minimal idempotent in B_r in the central summand labelled by the diagram $\lambda \in \Lambda_r^{(k,l)}$. We can write the class of $p^{(1)}p_\lambda$ in $B_{-k+1,r_{p^{(1)}}}$ as follows, using Lemma 1 (a): $[p^{(1)}p_\lambda] = [p_{[1^{k-1}]} \otimes p_\lambda] = \sum_{\nu\in\Lambda_{r-1}^{(k,l)}} c^\lambda_{[1]\nu} \;\; [p_{j(\nu)}]$, where we use that the map $j : \Lambda_n^{(k,l)} \to \Lambda_{n+k}^{(k,l)}$ as in the preliminaries is a bijection for large n. Therefore, the inclusion matrix L is given by $L_{\lambda\nu} = c^\lambda_{[1]\nu}$ for $\lambda \in \Lambda_r^{(k,l)}$, $\nu \in \Lambda_{r-1}^{(k,l)}$, which coincides with the matrix G^t.

(ii) Using (i) and Lemma 2 (d), we shall show that $\langle q^{(2)}, B_{-k+1,r_{p^{(1)}}}\rangle$ contains a subalgebra, Q_2, isomorphic to the basic construction for $B_{r_{p^{(1)}}} \subset B_{-k+1,r_{p^{(1)}}}$. By Remark (7), Lemma 2 (b), and Remark (5) we just need to show that $q^{(2)}bq^{(2)} = E_2(b)q^{(2)}$ for all $b \in B_{-k+1,r_{p^{(1)}}}$, where E_2 is the unique trace preserving conditional expectation onto $B_{r_{p^{(1)}}}$. The equation above is true for $b \in B_{r_{p^{(1)}}}$, and by Lemma 2 (d) it also holds for $b = q^{(1)}$. Since $B_{-k+1,r_{p^{(1)}}} = \langle q^{(1)}, B_{r_{p^{(1)}}}\rangle$ by part (i), then the equation holds for all $b \in B_{-k+1,r_{p^{(1)}}}$ by the bimodule property of E_2, and because any element of $\langle q^{(1)}, B_{r_{p^{(1)}}}\rangle$ is a linear combination of elements of the form $xq^{(1)}y + z$, with $x, y, z \in B_{r_{p^{(1)}}}$. Consider the inclusions:

$$B_{-k+1,r_{p^{(1)}}} \subset^G Q_2 \quad \text{and} \quad B_{-k+1,r_{p^{(1)}}} \subset^H B_{-k,r_{p^{(1)}}},$$

where G is again the inclusion matrix for $A_r \subset B_r$. Since $Q_2 \subseteq B_{-k,r_{p^{(1)}}}$, it is enough to show that $G = H$ in order to prove that $Q_2 = B_{-k,r_{p^{(1)}}}$. By Lemma 1 and the fact that the map $j : \Lambda_n^{(k,l)} \to \Lambda_{n+k}^{(k,l)}$ is a bijection for large n, it is easy to show that the equivalent class of $p^{(1)}$ in $B_{-k+1,r}$ is given by $[p_{[1^{k-1}]} \otimes 1_r] = \sum_{\lambda\in\Lambda_{r-1}^{(k,l)}} \alpha_\lambda \;\; [p_{j(\lambda)}]$, with $\alpha_\lambda \neq 0$ for every $\lambda \in \Lambda_{r-1}^{(k,l)}$. Therefore, the classes of minimal idempotents in $B_{-k+1,r}$ are given by $[p_{j(\lambda)}]$ with $\lambda \in \Lambda_{r-1}^{(k,l)}$. The class of $p_{j(\lambda)}$ in $B_{-k,r_{p^{(1)}}}$ is $[p_{j(\lambda)}] = \sum_{\delta\in\Lambda_{r+k}^{(k,l)}} c^\delta_{j(\lambda)[1]}[p_\delta] = \sum_{\mu\in\Lambda_r^{(k,l)}} c^\mu_{\lambda[1]}[p_{j(\mu)}]$, so that the inclusion matrix H is given by $H_{\lambda\mu} = c^\mu_{\lambda[1]}$ for $\lambda \in \Lambda_{r-1}^{(k,l)}$ and $\mu \in \Lambda_r^{(k,l)}$. So H coincides with G. ☐

Lemma 4. *Let $1 \in A \subset B$ be finite dimensional C^* algebras acting on a Hilbert space H with a faithful trace tr on B. Let $p \in A$ be a full projection (i.e., the central support of p is 1). Then, the algebra $\langle B_p, pe_A\rangle$ is isomorphic to the basic construction for $A_p \subset B_p$, where e_A is the Jones projection for $A \subset B$.*

Proof. The algebras A_p, B_p and pe_A satisfy the conditions of Remark (7). Therefore, $\langle B_p, pe_A\rangle \cong \langle B_p, e_{A_p}\rangle \oplus K$, where K is isomorphic to a subalgebra of B_p. Because p is full in A, then the inclusion matrix for $A_p \subset B_p$ coincides with that for $A \subset B$, G. Thus, $\frac{1}{\mathrm{tr}(p)}\vec{t}_A = \vec{t}_{A_p} \geq GG^t\vec{t}_{\langle B_p, pe_A\rangle} = \frac{1}{\mathrm{tr}(p)}GG^t\vec{t}_{\langle B, e_A\rangle} = \frac{1}{\mathrm{tr}(p)}\vec{t}_A$, so that an equality holds, and by faithfulness of tr, K must be zero. $\square$

We shall obtain now all the iterations of the basic construction, by doing the same procedure as in the last proposition, by induction. Define

$$B^r(2i-1) := B_{-ki+1,r}$$
$$B^r(2i) := B_{-ki,r}.$$

Corollary 5. *For $r \in \mathbb{N}$ large, the n^{th} basic construction for $(A_r)_{P([\frac{n+1}{2}])} \subset (B_r)_{P([\frac{n+1}{2}])}$ is isomorphic to*

$$B^r(n)_{P([\frac{n+1}{2}])} = \left\langle q^{(n)}, B^r(n-1)_{P([\frac{n+1}{2}])}\right\rangle.$$

Proof. The statement is true for the cases $n = 1, 2$ by Proposition 3. We shall proceed now by induction in $n \in \mathbb{N}$. Suppose that the statement is true for $m \leq n$. We want to show that then it is also true for $m = n+1$. By inductive hypothesis, the $n-1^{\mathrm{st}}$ basic construction for $(A_r)_{P([\frac{n}{2}])} \subset (B_r)_{P([\frac{n}{2}])}$ is isomorphic to $B^r(n-1)_{P([\frac{n}{2}])}$, and the n^{th} basic construction for $(A_r)_{P([\frac{n+1}{2}])} \subset (B_r)_{P([\frac{n+1}{2}])}$ is isomorphic to $B^r(n)_{P([\frac{n+1}{2}])}$. Thus, the $n+1^{st}$ basic construction for $(A_r)_{P([\frac{n+2}{2}])} \subset (B_r)_{P([\frac{n+2}{2}])}$ is isomorphic to the first basic construction for for $B^r(n-1)_{P([\frac{n+2}{2}])} \subset B^r(n)_{P([\frac{n+2}{2}])}$. (Here we also use that $P^{([\frac{n+2}{2}])} = P^{([\frac{n}{2}])}p$, with $p = p^{(n)}$ or $p = p^{(n+1)}$ according to parity.)

(a) If n is odd, write $n+1 = 2i$. The inclusion $B^r(2i-2)_{P^{(i)}} \subset B^r(2i-1)_{P^{(i)}}$ is conjugate, via $\mathrm{shift}_{(i-1)k}$, to $(B^{r+(i-1)k}(0)_{p^{(1)}})_{\tilde{p}} \subset (B^{r+(i-1)k}(1)_{p^{(1)}})_{\tilde{p}}$, with $\tilde{p} = \mathrm{shift}_{(i-1)k}(\prod_{s=1}^{i-1} p^{(2s-1)})$. By Proposition 3 (ii) and Lemma 4, the first basic construction for the latter is given by $(B^{r+(i-1)k}(2)_{p^{(1)}})_{\tilde{p}} = \langle q^{(2)}, B^{r+(i-1)k}(1)_{p^{(1)}}\rangle_{\tilde{p}}$. Then, by applying the inverse of $\mathrm{shift}_{(i-1)k}$, $\mathrm{shift}_{-(i-1)k}$, the basic construction for $B^r(2i-2)_{P^{(i)}} \subset B^r(2i-1)_{P^{(i)}}$ is isomorphic to $B^r(2i)_{P^{(i)}} = \langle q^{(2i)}, B^r(2i-1)_{P^{(i)}}\rangle$.

(b) If n is even, write $n+1 = 2i-1$. As in (a), apply $\mathrm{shift}_{(i-1)k}$ to obtain that $B^r(2i-3)_{P^{(i)}} \subset B^r(2i-2)_{P^{(i)}}$ is conjugate to $(A_{r+(i-1)})_{p^{(1)}\tilde{p}} \subset (B_{r+(i-1)})_{p^{(1)}\tilde{p}}$, with basic construction isomorphic to $(B^{r+(i-1)k}(1)_{p^{(1)}})_{\tilde{p}} = \langle q^{(1)}, (B_{r+(i-1)k})_{p^{(1)}}\rangle_{\tilde{p}}$. Thus, the basic construction for $B^r(2i-3)_{P^{(i)}} \subset B^r(2i-2)_{P^{(i)}}$ is isomorphic to $B^r(2i-1)_{P^{(i)}} = \langle q^{(2i-1)}, B^r(2i-2)_{P^{(i)}}\rangle$. $\square$

Section 3.

3.1. Denote by $B^{(n)}$ the Jones' n^{th} basic construction for the one-sided pair $A \subset B$, (see [**J**]). The *asymptotic inclusion* for $A \subset B$ is the inclusion $A \vee (A' \cap B^{(\infty)}) \subset B^{(\infty)}$, where $B^{(\infty)} := (\bigcup_n B^{(n)})''$. We shall proceed first by showing that both the two-sided pair $C \subset D$ and the asymptotic inclusion for the one-sided pair $A \subset B$ satisfy the conditions of some splitting results due to D. McDuff and D. Bisch (see [**McD**], [**B**]).

Lemma 6.

(a) *The asymptotic inclusion $A \vee (A' \cap B^{(\infty)}) \subset B^{(\infty)}$ of the one-sided pair $A \subset B$ has two non-trivial non-commuting central sequences.*

(b) *The two-sided pair $C \subset D$ has two non-trivial non-commuting central sequences.*

Proof. Let us recall first that a central sequence (x_n) for a II_1 factor M is a bounded sequence in M with the property that $\|[x_n, x]\|_2 \to 0$, for all $x \in M$, where $\| \,.\, \|_2$ is the norm defined on M via its trace. A central sequence $(x_n) \in M$ is said to be trivial if $\|x_n - \lambda_n.1\|_2 \to 0$, where $\lambda_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. A central sequence for a pair of II_1 factors $N \subset M$ is a central sequence (x_n) for M which is contained in N.

(a) The candidates are the sequences $(e_{B^{(n)}})_{n\in\mathbb{N}}$ and $(e_{B^{(n+1)}})_{n\in\mathbb{N}}$, where $e_{B^{(n)}}$ is the projection corresponding to the n^{th} Jones' basic construction for $A \subset B$. It is obvious that these sequences lie in $A \vee (A' \cap B^{(\infty)})$. To show that they are non-trivial, consider a sequence (λ_n) of complex numbers. For all $n \in \mathbb{N}$,

$$\|e_{B^{(n)}} - \lambda_n.1\|_2^2 = \tau|1-\lambda_n|^2 + (1-\tau)|\lambda_n|^2 \geq \frac{1}{2}\min\{\tau, (1-\tau)\} > 0,$$

because $\mathrm{tr}\,(e_{B^{(n)}}) = \tau \in (0,1)$ for all $n \in \mathbb{N}$. To show that $(e_{B^{(n)}})_n$ and $(e_{B^{(n+1)}})_n$ are non-commuting, write

$$\begin{aligned}\|[e_{B^{(n)}}, e_{B^{(n+1)}}]\|_2^2 &= -2\mathrm{tr}\,(\eta e_{B^{(n)}} e_{B^{(n+1)}}) + 2\mathrm{tr}\,(e_{B^{(n)}} e_{B^{(n+1)}}) \\ &= 2\eta(1-\eta) \equiv c > 0,\end{aligned}$$

since $e_{B^{(n+1)}} e_{B^{(n)}} e_{B^{(n+1)}} = \eta e_{B^{(n+1)}}$, with $0 < \eta = tr(e_{B^{(n)}})$ for all $n \in \mathbb{N}$. Finally, to show that $(e_{B^{(n)}})_n$ and $(e_{B^{(n+1)}})_n$ are central with respect to $B^{(\infty)}$ fix an element $x \in B^{(\infty)}$. There exists a sequence (x_n) with $x_n \in B^{(n)}$ which converges to x in the $\| \,.\, \|_2$ norm. Given $\epsilon > 0$, choose some $j_0 \in \mathbb{N}$ with $\|x - x_{j_0}\|_2 < \epsilon$. Then, $[e_{B^{(n)}}, x_{j_0}] = 0$ for $n > j_0 + 1$, so that

$$\|[e_{B^{(n)}}, x]\|_2 \leq 2\|e_{B^{(n)}}\|.\|x - x_{j_0}\|_2 + \|e_{B^{(n)}} x_{j_0} - x_{j_0} e_{B^{(n)}}\|_2 < 2\epsilon.$$

(b) For the two-sided pair $C \subset D$ consider the sequences $(e_n)_{n\in\mathbb{N}}$ and $(e_{n+1})_{n\in\mathbb{N}}$, where e_n is the image (under the braid representation considered) of the spectral projection corresponding to the eigenvalue -1 of the generator T_n of the Hecke algebra. In other words, for $q \neq -1$, $e_n = \frac{q-g_n}{q+1}$ (see [**W-1**]). These sequences belong to C (since none of its terms is e_0), and they are non-trivial: As in (a), for a complex sequence (λ_n),

$$\|e_n - \lambda_n.1\|_2^2 = d|1-\lambda_n|^2 + (1-d)|\lambda_n|^2 \geq \frac{1}{2}\min\{d,(1-d)\} > 0,$$

for all $n \in \mathbb{N}$, since $d := \mathrm{tr}(e_n) \in (0,1)$ for all $n \in \mathbb{N}$. For showing that they are non-commuting we use one of the defining relations for the Hecke algebra generators, namely, $e_n e_{n+1} e_n - \alpha e_n = e_{n+1} e_n e_{n+1} - \alpha e_{n+1}$, for $0 \neq \alpha = \frac{q}{(1+q)^2}$. Hence,

$$\begin{aligned}\|[e_n, e_{n+1}]\|_2^2 &= -2\mathrm{tr}\,(e_n e_{n+1} e_n e_{n+1} - \alpha e_n e_{n+1} + \alpha e_n e_{n+1}) + 2\mathrm{tr}\,(e_n e_{n+1}) \\ &= 2\alpha\mathrm{tr}\,(e_n) - 2\alpha\mathrm{tr}\,(e_n)^2 \equiv \kappa > 0,\end{aligned}$$

where we also use the fact that $\mathrm{tr}\,(e_n e_{n+1}) = \mathrm{tr}(e_n)^2$, which is constant for all $n \in \mathbb{N}$ because of the Markov property of tr. Finally, we need to show that these are central sequences with respect to D. This follows from a similar procedure as in (a). Let x be an element in D. Then, there exists a sequence (x_n) with $x_n \in \langle e_{-n}, \ldots, e_n\rangle$, which converges to x in the $\|\ .\ \|_2$ norm. For $\epsilon > 0$, take $j_0 \in \mathbb{N}$ such that $\|x - x_j\|_2 < \epsilon$. Then, $[e_n, x_{j_0}] = 0$ for $n > j_0 + 1$, because $x_{j_0} \in \langle e_{-j_0}, \ldots, e_{j_0}\rangle$ and $[e_i, e_j] = 0$ for $|i-j| > 1$. Therefore,

$$\|[e_n, x]\|_2 \leq 2\|e_n\|\ \|x - x_{j_0}\|_2 + \|e_n x_{j_0} - x_{j_0} e_n\|_2 < 2\epsilon.$$

$\square$

Now we are able to use the Bisch-McDuff splitting results mentioned above: In [**McD**], McDuff gave necessary and sufficient conditions for a $\|\ .\ \|_2$-separable II_1 factor M to be isomorphic to $M\bar{\otimes}R$, where R is the separable II_1 factor. In [**B**, Theorem 3.1], based on the work by McDuff, Bisch gave necessary and sufficient conditions for a pair $N \subset M$ of separable II_1 factors to be stable, that is, for $N \subset M$ to be conjugate to $N\bar{\otimes}R \subset M\bar{\otimes}R$ (i.e., existence of an isomorphism $\Phi : M \to M\bar{\otimes}R$ with $\Phi(N) = N\bar{\otimes}R$). The equivalent condition that we shall use is the following, namely, that the algebra $M' \cap N^\omega$ is non-commutative, where ω is a free ultrafilter in $\mathbb{N}$. In particular, it is enough to show that there exist two non-trivial non-commutative central sequences for M which are contained in N. Therefore, as a result of Lemma 6:

Corollary 7. *Both the asymptotic pair $A \vee (A' \cap B^{(\infty)}) \subset B^{(\infty)}$ for the one-sided pair $A \subset B$ and the two-sided pair $C \subset D$ are stable.*

Remark. Let $R^{\otimes\infty}$ be the II_1 factor given by the inductive limit of $R^{\otimes n}$ with the canonical embeddings and trace. It can easily be shown that it is hyperfinite, so that it is isomorphic to R. For a non-zero projection $p \in R$, let

$R_p^{\otimes\infty}$ be the II_1 factor given by the inductive limit of the factors $R_p^{\otimes n}$, with the embeddings $R_p^{\otimes n} \cong R_p^{\otimes n} \otimes p \subset R_p^{\otimes n+1}$, together with an appropriate renormalisation of the canonical trace at each step. It can also be easily shown that $R_p^{\otimes\infty}$ is hyperfinite, and so isomorphic to R. Therefore, if a pair $N \subset M$ splits a copy of R, then it also splits a copy of $R^{\otimes n}$, $R^{\otimes\infty}$, or $R_p^{\otimes\infty}$.

3.2. In Section 2 we obtained all the iterations of the Jones' basic constructions for $A_r \subset B_r$ in a way which relates them to the two-sided pair $C \subset D$. Corollary 5 says that for *large* r, and for every s, the $2s+1^{\text{st}}$ basic construction for the pair $(A_r)_{P(s)} \subset (B_r)_{P(s)}$ is isomorphic to $B^r(2s-1)_{P(s)}$. Then, $B^{(\infty)}$ can be identified with the II_1 factor defined by the inductive limit of $(D_s)_{P(s)}$, with

$$D_s := \langle g_{-sk+1}, g_{-sk}, \dots \rangle'',$$

and the embeddings $\psi_s : (D_s)_{P(s)} \to (D_{s+1})_{P(s+1)}$ given by $(D_s)_{P(s)} \cong (D_s)_{P(s+1)} \subset (D_{s+1})_{P(s+1)}$ (note that $[p^{(s+1)}, D_s] = 0$ for all $s \in \mathbb{N}$), together with an appropriate renormalisation of the trace tr at each step; that is, define $\mathrm{tr}^{(s)}$ on each $(D_s)_{P(s)}$ by $\mathrm{tr}^{(s)} := \frac{1}{\mathrm{tr}(p_{[1k-1]})^s}\,\mathrm{tr}$. With the same identification as above, the subalgebra of $B^{(2s-1)}$ given by $A \vee (A' \cap B^{(2s-1)})$ is isomorphic to $(C_s)_{P(s)}$, where

$$C_s := \langle g_{-sk+1}, \dots, g_{-1}, \hat{g}_0, g_1, \dots \rangle'' :$$

For showing this, one should prove that

$$\langle g_{-sk+1}, \cdots, g_{-1} \rangle_{P(s)} = \langle g_1, \dots \rangle''_{P(s)} \cap \langle g_{-sk+1}, \dots \rangle''_{P(s)}.$$

One inclusion,

$$\langle g_{-sk+1}, \cdots, g_{-1} \rangle_{P(s)} \subset \langle g_1, \dots \rangle''_{P(s)} \cap \langle g_{-sk+1}, \dots \rangle''_{P(s)},$$

is obvious. For the other inclusion, one needs to apply an estimate, [**W-1**, Theorem 1.6] — $\dim A' \cap B^{(2s-1)} \leq \dim p(A'_r \cap B_r^{(2s-1)})$, for any non-zero projection $p \in A_r$ and sufficiently large r — as it was used in [**W-1**, Theorem 3.7, (b)], where $p \in A_r$ is chosen to be a projection with the contraction property. So, as a corollary we re-obtain:

Wenzl's Theorem. *The n^{th}-higher relative commutant $A' \cap B^{(n)}$ of the pair $A \subset B$ is isomorphic to $B^1(n)_{P([\frac{n+1}{2}])}$.*

Proof. It is a corollary of Corollary 5, and the above identification. (We include at the end of this article a rule of how to draw the inclusion diagrams for $A' \cap B^{(n)} \subset A' \cap B^{(n+1)}$ and a description of the principal graphs, together with two examples.) □

We have just seen that the asymptotic pair $A \vee (A' \cap B^{(\infty)}) \subset B^{(\infty)}$ for the one-sided inclusion $A \subset B$ is conjugate to the von Neumann algebra inductive limit of $(C_s)_{P^{(s)}} \subset (D_s)_{P^{(s)}}$ — which we shall denote by $C_{p^\infty} \subset D_{p^\infty}$ for convenience — with the embeddings and the renormalisation of the trace at each step as above. We need to show that $C_{p^\infty} \subset D_{p^\infty}$ is conjugate to the two-sided pair $C \subset D$. In other words, we need to "eliminate" the projections $P^{(s)}$. That is the reason in the next results we need consider larger von Neumann algebras, of the form $(C_s)_{P^{(s)}} \otimes R^{\otimes s}$.

Lemma 8. *Consider s large enough so that $P^{(s)}$ is full in $B^1(2s-1) = B_{-sk+1,1}$. Then, there exist a projection $\tilde{P}^{(s)} \in C_s$ with $\tilde{P}^{(s)} \leq P^{(s)}$, and a unitary $u_s \in C_s \otimes R^{\otimes s}$ such that:*

(a) $\widehat{u}_s(\tilde{P}^{(s)} \otimes 1_{R^{\otimes s}}) := u_s(\tilde{P}^{(s)} \otimes 1_{R^{\otimes s}})u_s^* = 1_C \otimes q_s$, *with q_s a projection in $R^{\otimes s}$.*

(b) *$\tilde{P}^{(s)}$ is "maximal" with the property described in* (i). *That is, if there is a projection ${P^{(s)}}' \in C_s$ with ${P^{(s)}}' \leq P^{(s)}$, and a unitary $u_s' \in C_s \otimes R^{\otimes s}$ with $\widehat{u}_s'({P^{(s)}}' \otimes 1_{R^{\otimes s}}) = 1_C \otimes q_s'$ for some $q_s' \in R^{\otimes s}$, then ${P^{(s)}}' \precsim \tilde{P}^{(s)}$.*

Proof. (a) Let Λ be the finite set of (k, l) Young diagrams that label the minimal central projections $\tilde{z}_\lambda$ of $B^1(2s-1)$, and $z_\lambda = \tilde{z}_\lambda \otimes 1$ the minimal central projections in $C_s = \langle g_{-sk+1}, \ldots, g_{-1}, g_1, \ldots \rangle''$. For $\lambda \in \Lambda$ let q_λ be a projection in $R^{\otimes s}$ with $\mathrm{tr}(q_\lambda) = \frac{\mathrm{tr}(P^{(s)} z_\lambda)}{\mathrm{tr}(z_\lambda)}$. Since in a II_1 factor projections are unitarily equivalent if and only if their traces coincide, there is a unitary $\tilde{u}_{\lambda,s}$ in the factor $(C_s \otimes R^{\otimes s})_{z_\lambda \otimes 1_{R^{\otimes s}}}$ such that $\widehat{\tilde{u}}_{\lambda,s}(P^{(s)} z_\lambda \otimes 1_{R^{\otimes s}}) = \tilde{u}_{\lambda,s}(P^{(s)} z_\lambda \otimes 1_{R^{\otimes s}})\tilde{u}_{\lambda,s}^* = z_\lambda \otimes q_\lambda$. Take $q_s \in R^{\otimes s}$ with $\mathrm{tr}(q_s) = \min_{\lambda \in \Lambda}\{\mathrm{tr}(q_\lambda)\}$, so that $z_\lambda \otimes q_\lambda \succsim z_\lambda \otimes q_s$ in $(C_s \otimes R^{\otimes s})_{z_\lambda \otimes 1_{R^{\otimes s}}}$ for all λ. It follows that there is a projection $\tilde{P}^{(s)} \in C_s \otimes R^{\otimes s}$ and a unitary $u_{\lambda,s}$ in $(C_s \otimes R^{\otimes s})_{z_\lambda \otimes 1_{R^{\otimes s}}}$ with $\tilde{P}^{(s)} z_\lambda \leq P^{(s)} z_\lambda$ and $\widehat{u}_{\lambda,s}(\tilde{P}^{(s)} z_\lambda \otimes 1_{R^{\otimes s}}) = z_\lambda \otimes q_s$. Thus, if $u_s := \sum_\lambda u_{\lambda,s}$, then $u_s \in C_s \otimes R^{\otimes s}$ is a unitary and $\widehat{u}_s(\tilde{P}^{(s)} \otimes 1_{R^{\otimes s}}) = 1_C \otimes q_s$.

(b) Suppose there exist ${P^{(s)}}' \in C_s$ with ${P^{(s)}}' \leq P^{(s)}$ and a unitary $u_s' \in C_s \otimes R^{\otimes s}$ such that $\widehat{u}_s'({P^{(s)}}' \otimes 1_{R^{\otimes s}}) = 1_C \otimes q_s'$ for some $q_s' \in R^{\otimes s}$. Then for all $\lambda \in \Lambda$

$$\mathrm{tr}(z_\lambda) tr(q_s') = \mathrm{tr}({P^{(s)}}' z_\lambda) \leq \mathrm{tr}(P^{(s)} z_\lambda) = \mathrm{tr}(z_\lambda)\mathrm{tr}(q_\lambda),$$

so that $\mathrm{tr}(q_s') \leq \mathrm{tr}(q_\lambda)$. Thus, $\mathrm{tr}(q_s') \leq \min_{\lambda \in \Lambda} \mathrm{tr}(q_\lambda) = \mathrm{tr}(q_s)$, and so ${P^{(s)}}' \precsim \tilde{P}^{(s)}$. □

Lemma 9. *Fix s large. There is a projection $\tilde{P}^{(2s)} \in C_{2s}$ with $\tilde{P}^{(s)} \otimes \tilde{P}^{(s)} \leq \tilde{P}^{(2s)} \leq P^{(2s)}$, a projection $q_{2s} \in R^{\otimes 2s}$ with $q_{2s} \geq q_s \otimes q_s$ ($\tilde{P}^{(s)}$ and q_s as in the lemma above), and a unitary u_{2s} in $C_{2s} \otimes R^{\otimes 2s}$ such that:*

(a) *As in Lemma* 8, (a), *for* $2s$.

(b) *As in Lemma* 8, (b), *for* $2s$.
(c) *The following is a commuting diagram:*

$$\begin{array}{ccc} (C_s \otimes R^{\otimes s})_{(\tilde{P}^{(s)}\otimes 1_{R^{\otimes s}})} & \xrightarrow{\widehat{u}_s} & (C_s \otimes R^{\otimes s})_{(1_C\otimes q_s)} \\ \widetilde{\iota}_s \downarrow & & \downarrow \widetilde{\jmath}_s \qquad (A) \\ (C_{2s} \otimes R^{\otimes 2s})_{(\tilde{P}^{(s)}\otimes \tilde{P}^{(s)}\otimes 1_{R^{\otimes 2s}})} & \xrightarrow{\widehat{u_s\otimes u_s}} & (C_{2s} \otimes R^{\otimes 2s})_{(1_C\otimes q_s\otimes q_s)} \\ \widetilde{\widetilde{\iota}}_s \downarrow & & \downarrow \widetilde{\widetilde{\jmath}}_s \qquad (B) \\ (C_{2s} \otimes R^{\otimes 2s})_{(\tilde{P}^{(2s)}\otimes 1_{R^{\otimes 2s}})} & \xrightarrow{\widehat{u}_{2s}} & (C_{2s} \otimes R^{\otimes 2s})_{(1_C\otimes q_{2s})}, \end{array}$$

with the embeddings

$$\begin{cases} \widetilde{\iota}_s = \text{ reduction by the projection } \tilde{P}^{(s)}\otimes\tilde{P}^{(s)}\otimes 1_{R^{\otimes 2s}}, \\ \widetilde{\jmath}_s = \text{ reduction by the projection } 1_C\otimes q_s\otimes q_s, \\ \widetilde{\widetilde{\iota}}_s = \text{ reduction by the projection } \tilde{P}^{(2s)}\otimes 1_{R^{\otimes 2s}}, \\ \widetilde{\widetilde{\jmath}}_s = \text{ reduction by the projection } 1_C\otimes q_{2s}. \end{cases}$$

(*Note that the maps above are embeddings since* $\tilde{P}^{(s)}\otimes\tilde{P}^{(s)}\otimes 1_{R^{\otimes 2s}}$ *and* $1_C\otimes q_s\otimes q_s$ *commute with* $\tilde{P}^{(s)}\otimes 1_{R^{\otimes s}}$ *and with* $1_C\otimes q_s$ *respectively, and also* $\tilde{P}^{(2s)}\otimes 1_{R^{\otimes 2s}} \geq \tilde{P}^{(s)}\otimes\tilde{P}^{(s)}\otimes 1_{R^{\otimes 2s}}$, *and* $1_C\otimes q_{2s} \geq 1_C\otimes q_s\otimes q_s$).

Proof. Note that we identify $B^1(2s-1)\otimes B^1(2s-1)$ with a subalgebra of $B^1(4s-1)$ according to the multiplication rules for the structure coefficients that describe the embeddings $B_r\otimes B_r \subset B_{2r}$, as in the preliminaries. By Lemma 8 we can find $\widetilde{P^{(2s)}} \leq P^{(2s)}$, with $\widetilde{P^{(2s)}}\otimes 1_{R^{\otimes 2s}} \in C_{2s}\otimes R^{\otimes 2s}$ maximal with respect to the property of being unitarily equivalent to a projection of the form $1_C\otimes \tilde{q}_{2s}$, for some $\tilde{q}_{2s}\in R^{\otimes 2s}$.

We can also define the unitary '$u_s\otimes u_s$' $\in C_{2s}\otimes R^{\otimes 2s}$ since, by the construction of $u_s\in C_s\otimes R^{\otimes s}$ in the proof of Lemma 8, one has that in fact $u_s\in B^1(2s-1)\otimes R^{\otimes s}$ (so that $u_s\otimes u_s\in B^1(2s-1)\otimes B^1(2s-1)\otimes R^{\otimes 2s}\subset C_{2s}\otimes R^{\otimes 2s}$). For $\tilde{P}^{(s)}$ as in Lemma 8, $\tilde{P}^{(s)}\otimes\tilde{P}^{(s)} \leq P^{(s)}\otimes P^{(s)} = P^{(2s)}$, and $P^{(s)}\otimes P^{(s)}\otimes 1_{R^{\otimes 2s}}$ is unitarily equivalent to $1_C\otimes q_s\otimes q_s$ via $u_s\otimes u_s$, so that $\tilde{P}^{(s)}\otimes\tilde{P}^{(s)} \precsim \widetilde{P^{(2s)}}$ by maximality of $\widetilde{P^{(2s)}}$. Since this last relation holds in each factor summand of $C_{2s}\otimes R^{\otimes 2s}$, then for every minimal central projection $z_\lambda\in C_{2s}$ there is a projection $\tilde{P}^{(2s)}z_\lambda$ such that $\tilde{P}^{(2s)}z_\lambda \sim \widetilde{P^{(2s)}}z_\lambda$ and $(\tilde{P}^{(s)}\otimes\tilde{P}^{(s)})z_\lambda \leq \tilde{P}^{(2s)}z_\lambda \leq P^{(2s)}z_\lambda$. Thus, $\tilde{P}^{(s)}\otimes\tilde{P}^{(s)} \leq \tilde{P}^{(2s)} \leq P^{(2s)}$, and $\tilde{P}^{(2s)}\otimes 1_{R^{\otimes 2s}}$ is maximal with respect to being unitarily equivalent to a projection of the form $1_C\otimes q_{2s}\in C_{2s}\otimes R^{\otimes 2s}$, where we can assume that $q_{2s}\geq q_s\otimes q_s$. Since their traces agree in each factor summand, the projections $(\tilde{P}^{(2s)} - \tilde{P}^{(s)} \otimes \tilde{P}^{(s)}) \otimes 1_{R^{\otimes 2s}}$ and $1_C \otimes (q_{2s} - q_s \otimes q_s)$ are unitarily equivalent via some unitary $w_{2s} \in C_{2s} \otimes R^{\otimes 2s}$.

Note that the maps $\widetilde{\widetilde{\iota_s}}$ and $\widetilde{\widetilde{\jmath_s}}$ are clearly embeddings, and that the fact the morphisms $\widetilde{\iota_s}$ and $\widetilde{\jmath_s}$ are injective follows from the faithfulness and multiplicativity of the trace (property **(iii)**$'$ in the preliminaries).

To find a unitary $u_{2s} \in C_{2s} \otimes R^{\otimes 2s}$ that makes the diagram commute, take the partial isometry

$$v_{2s} := w_{2s}\big((\tilde{P}^{(2s)} - \tilde{P}^{(s)} \otimes \tilde{P}^{(s)}) \otimes 1_{R^{\otimes 2s}}\big) + (u_s \otimes u_s)(\tilde{P}^{(s)} \otimes \tilde{P}^{(s)} \otimes 1_{R^{\otimes 2s}}),$$

and a partial isometry v'_{2s} with initial projection given by $(1_C - \tilde{P}^{(2s)}) \otimes 1_{R^{\otimes 2s}}$ and final projection given by $1_C \otimes (1_{R^{\otimes 2s}} - q_{2s})$. (The latter can be defined since the trace of both projections agree in each factor summand of $C_{2s} \otimes R^{\otimes 2s}$.) One can see easily that $u_{2s} := v_{2s} + v'_{2s}$ is a unitary and that $\widehat{u}_{2s}(\tilde{P}^{(2s)} \otimes 1_{R^{\otimes 2s}}) = 1_C \otimes q_{2s}$. Finally, by the definitions of the embeddings and the unitaries, the diagrams (A) and (B) commute. $\square$

Lemma 10. *Fix s large. The pairs of von Neumann algebra inductive limits*

$$\varinjlim_n \, (C_{s2^n})_{\tilde{P}^{(s2^n)}} \otimes R^{\otimes s2^n} \subset \varinjlim_n \, (D_{s2^n})_{\tilde{P}^{(s2^n)}} \otimes R^{\otimes s2^n},$$

and

$$\varinjlim_n \, C_{s2^n} \otimes (R^{\otimes s2^n})_{q_{s2^n}} \subset \varinjlim_n \, D_{s2^n} \otimes (R^{\otimes s2^n})_{q_{s2^n}}$$

are conjugate with the embeddings as above, and the trace normalisation at each step as described at the beginning of this section.

Proof. By Lemma 8 and Lemma 9, for s large and fixed, we can inductively define sequences of projections $(\tilde{P}^{(s2^n)})_n$, with $\tilde{P}^{(s2^n)} \in C_{s2^n} \otimes R^{\otimes s2^n}$ and $\tilde{P}^{(s2^n)} \leq P^{(s2^n)}$, and $(q_{s2^n})_n$ with $q_{s2^n} \in R^{\otimes s2^n}$, and also a sequence of unitaries $(u_{s2^n})_n$ with $u_{s2^n} \in C_{s2^n} \otimes R^{\otimes s2^n}$, such that the diagrams below commute for all n:

$$
\begin{tikzcd}
(C_{s2^n})_{\tilde{P}(s2^n)} \otimes R^{\otimes s2^n} \arrow[rrr, "\widehat{u}_{s2^n}"] \arrow[dr] \arrow[ddd, "\iota_{s2^n}"] & & & C_{s2^n} \otimes (R^{\otimes s2^n})_{q_{s2^n}} \arrow[dl] \arrow[ddd, "\jmath_{s2^n}"'] \\
& (D_{s2^n})_{\tilde{P}(s2^n)} \otimes R^{\otimes s2^n} \arrow[r, "\widehat{u}_{s2^n}"] \arrow[d, "\iota_{s2^n}"] & (D_{s2^n})_{\tilde{P}(s2^n)} \otimes (R^{\otimes s2^n})_{q_{s2^n}} \arrow[d, "\jmath_{s2^n}"'] & \\
& (D_{s2^{n+1}})_{\tilde{P}(s2^{n+1})} \otimes R^{\otimes s2^{n+1}} \arrow[r, "\widehat{u}_{s2^{n+1}}"'] & D_{s2^{n+1}} \otimes (R^{\otimes s2^{n+1}})_{q_{s2^{n+1}}} & \\
(C_{s2^{n+1}})_{\tilde{P}(s2^{n+1})} \otimes R^{\otimes s2^{n+1}} \arrow[rrr, "\widehat{u}_{s2^{n+1}}"'] \arrow[ur] & & & C_{s2^{n+1}} \otimes (R^{\otimes s2^{n+1}})_{q_{s2^{n+1}}} . \arrow[ul]
\end{tikzcd}
$$

□

Proposition 11. *The inclusion of inductive limits obtained via "reducing" by the subprojection sequence $(\tilde{P}^{(s2^n)})_{n\geq 1}$,*

$$\varinjlim_n (C_{s2^n})_{\tilde{P}(s2^n)} \otimes R^{\otimes s2^n} \subset \varinjlim_n (D_{s2^n})_{\tilde{P}(s2^n)} \otimes R^{\otimes s2^n},$$

is conjugate to that obtained via reducing by the sequence of projections $(P^{(s2^n)})_{n\geq 1}$,

$$\varinjlim_n (C_{s2^n})_{P(s2^n)} \otimes R^{\otimes s2^n} \subset \varinjlim_n (D_{s2^n})_{P(s2^n)} \otimes R^{\otimes s2^n}.$$

For proving this we shall first need the following lemma:

Lemma 12.

$$\lim_{n\to\infty} \frac{\operatorname{tr}(P^{(n)} z_\lambda^{(n)})}{\operatorname{tr}(P^{(n)})\ \operatorname{tr}(z_\lambda^{(n)})} = 1,$$

for any Young diagram $\lambda \in \Lambda_{nk}^{(k,l)}$, where $z_\lambda^{(n)}$ is the minimal central idempotent of $B^1(2n-1)$ corresponding to λ, and where $z_\lambda^{(n+1)}$ is $z_{j(\lambda)}^{(n+1)}$, with $j(\lambda) \in \Lambda_{(n+1)k}^{(k,l)}$ obtained by adding to to the left of λ a column of height k.

Proof. Let G be the inclusion matrix for $B^1(2n-1) \subset B^1(2n+1)$. Consider $n > n_0$ large, and note that G does not depend on n by periodicity. Let $\tilde{G}$ be the one for $B^1(2n-1)_{P^{(n)}} \subset B^1(2n+1)_{P^{(n+1)}}$, where the embedding is as before: For $x \in B^1(2n-1)$, $P^{(n)}xP^{(n)} \mapsto P^{(n+1)}xP^{(n+1)}$. The trace is renormalised at $B^1(2n-1)_{P^{(n)}}$ as before: $\mathrm{tr}^{(n)} = \frac{1}{\mathrm{tr}(P^{(n)})}\mathrm{tr}$, where tr is the usual trace.

Because n is large, $P^{(n)}$ is full in $B^1(2n-1)$, and $\tilde{G}$ does not depend on n (the graph is periodic) and is "like" G, with the difference that the multiplicity of some edges of G may be larger than the corresponding ones for $\tilde{G}$. By [**W-1**], both $\tilde{G}$ and G are normal. In particular, the Perron-Frobenius eigenvectors for G and G^t (resp. for $\tilde{G}$ and $\tilde{G}^t$) coincide. Also, by fullness of $P^{(n)}$, the trace weight vectors $\vec{t}^{\,(n)}$ and $\vec{\tilde{t}}^{\,(n)}$ for $B^1(2n-1)$ and for $B^1(2n-1)_{P^{(n)}}$ are multiples of each other; moreover, $\vec{\tilde{t}}^{\,(n)} = \frac{\vec{t}^{\,(n)}}{tr(P^{(n)})}$. Let $\vec{a}^{\,(n)}$ and $\vec{\tilde{a}}^{\,(n)}$ be the dimension vectors for $B^1(2n-1)$ and for $B^1(2n-1)_{P^{(n)}}$, respectively. By P-F theory (see [**W-1**]), if β (resp. $\tilde{\beta}$) is the P-F eigenvalue for G (resp. $\tilde{G}$), then

$$\lim_{n\to\infty}\left(\frac{G^t}{\beta}\right)^n \vec{a}^{\,(n_0)} = c\,\vec{t}^{\,(n_0)} \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{\tilde{G}^t}{\tilde{\beta}}\right)^n \vec{\tilde{a}}^{\,(n_0)} = \tilde{c}\,\vec{\tilde{t}}^{\,(n_0)},$$

where $c, \tilde{c} \in \mathbb{C}$. We can compute these constants:

$$\begin{aligned}1 = \mathrm{tr}(1) = \sum_\lambda \vec{t}_\lambda^{\,(n+n_0)}\vec{a}_\lambda^{\,(n+n_0)} &= \sum_\lambda \vec{t}_\lambda^{\,(n+n_0)}\beta^n \left(\left(\frac{G^t}{\beta}\right)^n \vec{a}^{\,(n_0)}\right)_\lambda \\ \underset{n\to\infty}{\longrightarrow} \sum_\lambda \vec{t}_\lambda^{\,(n_0)} c\,\vec{t}_\lambda^{\,(n_0)} &= c\,\left\|\vec{t}^{\,(n_0)}\right\|^2,\end{aligned}$$

so that $c = \|\vec{t}^{\,(n_0)}\|^{-2}$. Similarly, $\tilde{c} = \mathrm{tr}(P^{(n_0)})^2\|\vec{t}^{\,(n_0)}\|^{-2}$. Therefore, if $\lambda \in \Lambda^{(k,l)}_{(n+n_0)k}$,

$$\lim_{n\to\infty}\frac{\mathrm{tr}(P^{(n+n_0)}z_\lambda^{(n+n_0)})}{\mathrm{tr}(P^{(n+n_0)})\;\mathrm{tr}(z_\lambda^{(n+n_0)})} = \lim_{n\to\infty}\frac{\vec{\tilde{t}}_\lambda^{\,(n+n_0)}\vec{\tilde{a}}_\lambda^{\,(n+n_0)}}{\vec{t}_\lambda^{\,(n+n_0)}\vec{a}_\lambda^{\,(n+n_0)}} = \frac{\tilde{c}\,\vec{\tilde{t}}_\lambda^{\,(n_0)\,2}}{c\,\vec{t}_\lambda^{\,(n_0)\,2}} = 1.$$

□

Proof of Proposition 11. By definition, if n_0 is large and $n > n_0$, we have $\mathrm{tr}(\tilde{P}^{(n)}) = \frac{\mathrm{tr}(P^{(n)}z_\lambda^{(n)})}{\mathrm{tr}(z_\lambda^{(n)})}$, for some $z_\lambda^{(n)} \in \Lambda^{(k,l)}_{nk}$. Since we can identify all the finite sets $\Lambda^{(k,l)}_{nk}$ via the bijection j, it follows by Lemma 12 that $\lim_{n\to\infty}\frac{\mathrm{tr}(\tilde{P}^{(n)})}{\mathrm{tr}(P^{(n)})} = 1$. Consider the canonical embeddings given by $\phi_n : (D_{s2^n})_{P^{(s2^n)}} \otimes R^{\otimes s2^n} \to (D_{s2^n})_{\tilde{P}^{(s2^n)}} \otimes R^{\otimes s2^n}$, with $\phi_n(P^{(s2^n)}xP^{(s2^n)} \otimes r) = \tilde{P}^{(s2^n)}x\tilde{P}^{(s2^n)} \otimes r$ for

$x \in D_{s2^n}, r \in R^{\otimes s2^n}$. Then

$$\phi = (\phi_n) : \varinjlim_n \, (D_{s2^n})_{P^{(s2^n)}} \otimes R^{\otimes s2^n} \to \varinjlim_n \, (D_{s2^n})_{\tilde{P}^{(s2^n)}} \otimes R^{\otimes s2^n}$$

is an isomorphism of the inductive limits which conjugates our pairs: ϕ is clearly surjective. Since $\lim_{n\to\infty} \frac{\mathrm{tr}(\tilde{P}^{(n)})}{\mathrm{tr}(P^{(n)})} = 1$, then $\phi((P^{(s2^n)} \otimes 1)_n) = (\tilde{P}^{(s2^n)} \otimes 1)_n$, and injectivity is a consequence of this fact. □

Theorem 13. *The asymptotic inclusion for the one-sided inclusion of subfactors $A \subset B$, $A \vee (A' \cap B^{(\infty)}) \subset B^{(\infty)}$, is conjugate to the two-sided inclusion $C \subset D$.*

Proof. By the identification shown at the beginning of Section 3.2, before Lemma 8, we have $A \vee (A' \cap B^{(\infty)}) \subset B^{(\infty)} \cong C_{p^\infty} \subset D_{p^\infty}$. The statement follows from this observation, Corollary 7, Lemma 10 and Proposition 11. □

Remark. If $C \subset D$ is a two-sided braid inclusion of hyperfinite II_1 factors defined as in [**E**] (i.e., via unitary braid representations which are locally finite dimensional and have properties as **(i)**-**(vi)** in the preliminaries), then $C \subset D$ is conjugate to $C_{p^\infty} \subset D_{p^\infty}$, where the latter is an "infinitely reduced" version of $C \subset D$ by a descending sequence of projections (p_n), with $p_n \in C_n$ and $tr(p^n) = c^n$ for some fixed $c \in \mathbb{C}$. This is just the observation that for proving Theorem 13 we only used these assumptions.

3.3. We include here the description for the inclusions $A' \cap B^{(n)} \subset A' \cap B^{(n+1)}$ mentioned in the proof of Wenzl's theorem. The embeddings are given by the ψ_n's as at the beginning of **3.2.** To draw the inclusion diagrams for $A' \cap B^{(n)} \subset A' \cap B^{(n+1)}$ in general, for any $1 < k < l$, $l \geq 3$, one alternates according to the parity of n in the following way:

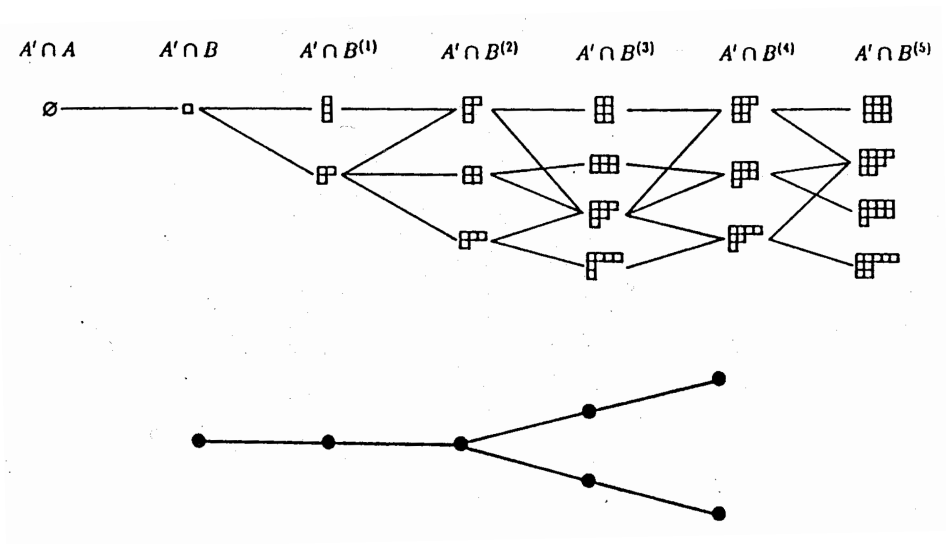
If n is odd, then $A' \cap B^{(n)} \subset A' \cap B^{(n+1)}$ is given by simply considering the diagram for $B_{(\frac{n+1}{2})k} \subset B_{(\frac{n+1}{2})k+1}$, and restricting it to the simple ideals of $B_{(\frac{n+1}{2})k}$ which correspond to classes of minimal idempotents p_λ, with $p_\lambda \leq p_{[1^{k-1}]}{}^{\otimes \frac{n+1}{2}}$.

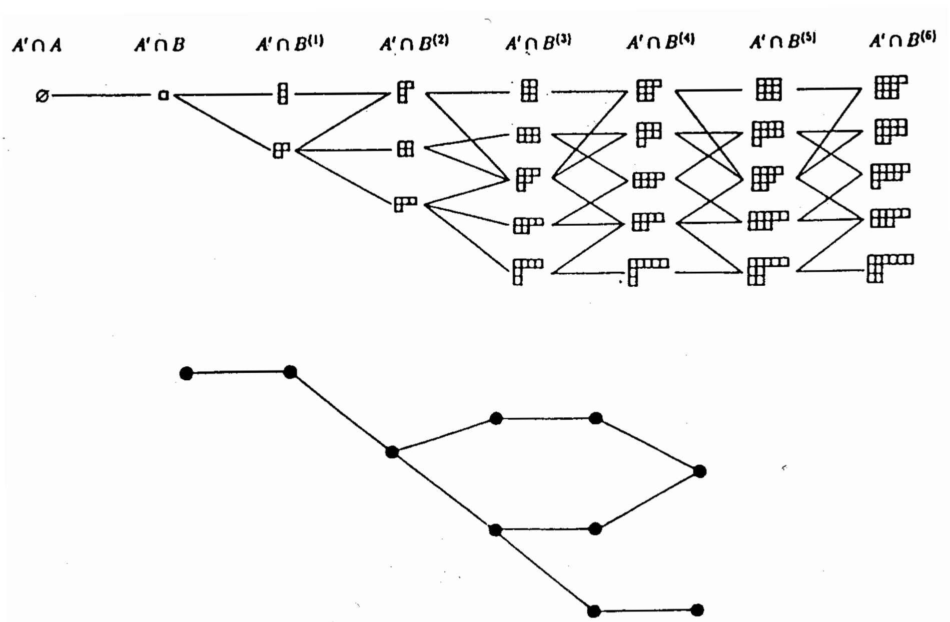
If n is even, then $A' \cap B^{(n)} \subset A' \cap B^{(n+1)}$ is given by a subgraph of the graph for $B_{(\frac{n}{2})k+1} \subset B_{(\frac{n}{2}+1)k}$, which is obtained by restricting it to the simple ideals of $B_{(\frac{n}{2})k+1}$ which correspond to classes of minimal idempotents p_λ, with $p_\lambda \leq p_{[1^{k-1}]}{}^{\otimes \frac{n}{2}}$, and to the simple ideals of of $B_{(\frac{n}{2}+1)k}$ which correspond to classes of minimal idempotents p_μ, with $p_\mu \leq p_{[1^{k-1}]}{}^{\otimes \frac{n}{2}+1}$.

The principal graphs in general can be described the following simple way: Fix n_0 large. For the (k,l) case the principal graph is given by the inclusion diagram for $B_{n_0 k} \subset B_{n_0 k+1}$, where the even vertices are labelled by (k,l) diagrams of $n_0 k$ boxes, and the odd vertices by (k,l) diagrams of $n_0 k + 1$

boxes. The $*$-vertex is given by the $k \times n_0$ rectangle. Let us also observe that the dual principal graph is the same, which follows by applying similar arguments.

For example, the Bratteli diagrams and principal graphs in the cases $k = 3$ and $l = 6, 7$ are the following:





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