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Let f, g be transcendental entire functions and p, q be nonlinear polynomials with deg $p \neq 3, 6$. Suppose that f and p are prime and $f(p(z)) = g(q(z))$, then $f = g \circ L$ and $p = L^{-1} \circ q$, where L is a linear polynomial. Similar results for $p(f(z)) = q(g(z))$ are also obtained.

1. Introduction and Main Results.

A meromorphic function $F(z)$ is said to has a factorization with left factor f and right factor g provided

$$F(z) = f(g(z)), \tag{1}$$

where f is meromorphic and g is entire (g may be meromorphic when f is rational). A nonlinear meromorphic function $F(z)$ is called prime (pseudo - prime) if every factorization of form (1) implies that either f is bilinear or g is linear (either f is rational or g is a polynomial). Clearly, a prime function is an analogy of a prime number. Over the past thirty years, many classes of prime or pseudo-prime functions have been obtained (see [**2**]).

As an analogue of the unique factorizability of natural numbers, one can also define that concept for entire functions. Suppose an entire function F has two factorizations $f_1 \circ f_2 \circ \cdots \circ f_m(z)$ and $g_1 \circ g_2 \circ \cdots \circ g_n(z)$ into nonlinear entire factors. If $m = n$ and if there exist linear polynomials L_j ($j = 1, 2, 3, \ldots, n-1$) such that the relations

(2)
$$f_1(z) = g_1 \circ L_1^{-1}, \quad f_2(z) = L_1 \circ g_2 \circ L_2^{-1}, \quad \ldots\ldots \quad , \quad f_n(z) = L_{n-1} \circ g_n(z)$$

hold simultaneously, then the two factorizations are called equivalent. If any two factorizations of $F(z)$ into nonlinear, prime entire factors are equivalent to each other, then F is called uniquely factorizable in entire sense.

As far as just polynomial factors are concerned, it is easy to exhibit functions which are not uniquely factorizable in entire sense, for instance, $z^3 \circ z^2 = z^2 \circ z^3$.

Therefore, the following question is not without interest.

Problem (A). Suppose f and g are prime entire functions and one of them is transcendental, will $F(z) = f \circ g(z)$ be uniquely factorizable in entire sense?

Counter-example. Take $f(z) = z^2, g(z) = ze^{z^2}, f_1(z) = ze^{2z}$ and $g_1(z) = z^2$. All of them are prime functions (see [**2**]) and $f \circ g = f_1 \circ g_1$ are two nonequivalent factorizations of $z^2e^{2z^2}$.

In this paper, we shall consider the following problems. Let f and p be two prime entire functions where f is transcendental and p is a polynomial. Suppose that $f \circ p = g \circ q$ or $p \circ f = q \circ g$. Under what conditions on the entire functions g, q will these factorizations be equivalent?

From the above counterexample, it is clear that two factorizations of a function $F = h \circ k = h_1 \circ k_1$ may not be equivalent. Therefore, we need to have some further assumptions on these factors h, h_1, k and k_1.

With this in mind, we have come up with the following results. The functions f, g, p and q considered below are all entire and nonlinear.

Theorem 1. *Let f, p be two non-periodic prime entire functions and p be a polynomial. Suppose that $p \circ f = q \circ g$ and both f, g are transcendental. Then $p = q \circ L^{-1}$ and $f = L \circ g$, where L is a linear polynomial.*

Theorem 2. *Let f, p be two prime entire functions and f be transcendental. Suppose that $p \circ f = q \circ g$ and both p, q are polynomials. Then $p = q \circ L^{-1}$ and $f = L \circ g$, where L is a linear polynomial.*

Theorem 3. *Let f, p be two prime entire functions and f be transcendental. Suppose that $f \circ p = g \circ q$ and both p, q are polynomials with $\deg p \neq 3, 6$. Then $f = g \circ L$ and $p = L^{-1} \circ q$, where L is a linear polynomial.*

Theorem 1, 2 and 3 deal with the relationships between polynomials p and q, transcendental functions f and g when we have factorizations of the form $p \circ f = q \circ g$ or $f \circ p = g \circ q$. It is natural to investigate the case $f \circ p = q \circ g$.

Theorem 4. *Let f and g be two transcendental entire functions, p and q be two nonlinear polynomials with degree n and m respectively. If $f \circ p = q \circ g$ and p is not a right factor of g, then $\deg p \leq \deg q$. In particular, the conclusion is true when g is prime.*

Remark 1. Let $f(z) = e^z$, $g(z) = e^{\frac{z^3}{2}}$, $p(z) = z^3$ and $q(z) = z^2$. Then $f \circ p = q \circ g$ and deg $p >$ deg q. Therefore, the condition that p is not a right factor of g is essential.

Definition 1. Let $F(z)$ be an nonconstant entire function. An entire function $g(z)$ is a generalized right factor of F (denoted by $g \leq F$) if there exists a function f ,which is analytic on the image of g, such that $F = f \circ g$. If such f is entire, g will be a right factor of F (denoted by $g|F$).

Definition 2. If $h \le f$ and $h \le g$, we say that h is a generalized common right factor of f and g. If $g \le F$ and $f \le F$, we say that F is a generalized common left multiple of f and g.

The existence and uniqueness problems of the greatest generalized common right factor and the least generalized common left multiple for a given pair of entire functions were solved by A. Eremenko and L.A. Rubel as follows.

Lemma 1 ([**4**]). *Any pair of non-constant entire functions has* (*up to a linear factor*) *a unique greatest generalized common right factor* h, *greatest in the sense that any generalized common right factor of* f *and* g *is a generalized right factor of* h.

Lemma 2 ([**4**]). *Suppose that* f *and* g *have a generalized common left multiple. Then* f *and* g *have* (*up to a linear factor*) *a unique least generalized common left multiple* F, *least in the sense that* F *is a generalized right factor of any generalized common left multiple of* f *and* g.

The proof of Theorem 1 is mainly based on the following lemma.

Lemma 3 ([**10**]). *Let* f *and* g *be two entire functions. Suppose that there exist two nonconstant complex functions* k *and* R *such that* $F = R \circ f = k \circ g$ *is meromorphic. If* g *is transcendental and* R *is rational, then there exists a transcendental entire function* h *satisfying* $h \le f$ *and* $h \le g$.

Proof of Theorem 1. By Lemma 3, there exists a transcendental entire function h satisfying $h \le f$ and $h \le g$. Hence, $f = h_1 \circ h$ and $g = h_2 \circ h$, where h_1, h_2 are analytic on the image of h. If the image of h is $\mathbf{C} - \{a\}$, then $h = a + e^k$ for some entire function k. Without loss of generality, we may assume $a = 0$ so that $f(z) = h_1(e^w) \circ k(z)$. The primeness of f will force k to be linear. This contradicts the assumption that f is not a periodic function. So the image of h must be the whole plane. This implies that both h_1, h_2 are entire and $p \circ h_1 = q \circ h_2$ on $\mathbf{C}$. Since $f = h_1 \circ h$ is prime, h_1 must be linear. From $p \circ h_1 = q \circ h_2$, h_2 must also be linear as p is prime. Take $L = h_1 \circ h_2^{-1}$ and we are done. □

The proof of Theorem 2 is similar, we simply apply Lemma 4 below instead of Lemma 3.

Lemma 4 ([**6**]). *Let* f *and* g *be two entire functions. Suppose that there exist two nonconstant polynomials* p *and* q *such that* $p \circ f(z) = q \circ g(z)$. *Then there exist an entire function* h *and rational functions* $U(z)$ *and* $V(z)$ *such that*

$$f(z) = U \circ h(z), \quad g(z) = V \circ h(z).$$

To prove Theorem 4, we need the following lemma which can be used to prove Lemma 3.

Lemma 5 ([**10**])**.** *Let f and g be two entire functions. Suppose that there exist two nonconstant functions h_1 and h_2 so that $F = h_1(f(z)) = h_2(g(z))$ and F is meromorphic. Suppose further that there exist $k \geq 2$ distinct points $z_1, \ldots.., z_k$ such that $F'(z_i) \neq 0, \infty$ for all i and*

$$\begin{cases} f(z_1) = f(z_2) = \ldots\ldots f(z_k) \\ g(z_1) = g(z_2) = \ldots\ldots g(z_k). \end{cases}$$

Then, there exists an entire function $h(z)$ (independent of k and $z_i's$) with $h \leq f$, $h \leq g$ and $h(z_1) = h(z_i)$ for all $2 \leq i \leq k$.

Proof of Theorem 4. By Lemma 1, there exists a generalized greatest common right factor k of p and g. Since, p is a polynomial, k is actually the greatest common right factor of p and g. Let p_1 and g_1 be entire functions such that $p = p_1 \circ k$ and $g = g_1 \circ k$. Hence, $f \circ p_1 = q \circ g_1$ on $\mathbf{C}$ and p_1, g_1 do not have any nonlinear common right factor. p_1 is nonlinear as p is not a right factor of g. If we can show that deg $p_1 \leq$ deg q_1, then deg $p \leq$ deg q. Therefore, we may assume that p and g do not have any nonlinear common right factor. Suppose that $n > m$. Define $E = \{p(z) | F'(z) = 0\}$,where $F = f \circ p$. Then E is a countable set. Therefore, we can choose $A \in \mathbf{C} - E$ so that the equation $p(z) = A$ has $n \geq 2$ distinct roots $z_1, \ldots, z_n$. Since $f(A) = f(p(z_i)) = q(g(z_i))$, $g(z_i)$ are roots of the equation $q(z) = f(A)$ which has at most m roots. $n > m$ implies that there exist two distinct roots z_i, z_j such that $g(z_i) = g(z_j)$. Note that $p(z_i) = p(z_j) = A$ and $F'(z_i), F'(z_j) \neq 0$. By Lemma 5, there exists an entire function h with $h \leq p$, $h \leq g$ and $h(z_i) = h(z_j)$. Clearly h is a polynomial. Hence, there exists a nonlinear h such that $h|p$ and $h|g$. This is impossible and we must have $n \leq m$.

In Theorem 3, we only assume that p and q are polynomials. If we further restrict p and q to have deg $p =$ deg $q \geq 3$, then the conclusion of Theorem 3 can be drawn directly from the following lemma.

Lemma 6 ([**5**])**.** *Let p and q be two polynomials with the same degree. Suppose there exist entire functions f and g such that $f \circ p = g \circ q$. Then one of the following two cases holds:*

(a) *$p(z) = L \circ q(z)$ where L is a linear polynomial.*
(b) *$p(z) = (r(z))^2 + a$ and $q = b(r(z) + c)^2 + d$, where a, b, c, d are complex numbers.*

The above type of results were first investigated by I.N. Baker and F. Gross in [**1**] and then L. Flatto in [**5**]. Finally, S.A. Lysenko in [**8**] gives an algebraic necessary and sufficient condition for the existence of meromorphic f and g satisfy $f \circ p = g \circ q$.

The proof of Theorem 3 is based on a method developed by S.A. Lysenko in [**8**] which depends on a fundamental result of local holomorphic dynamics.

2. Local holomorphic dynamics.

Let X be a Riemann surface and let $f : (X, a) \to (X, a)$ denote a mapping defined in some neighbourhood of a point a on X with $f(a) = a$. A germ of a mapping $f : (X, a) \to (X, a)$ is defined to be the equivalent class of all mappings which coincide with f in some neighbourhood of a and it is denoted by $[f]$. We say that f is conformal at a if f is analytic in some neighbourhood of a and $f'(a) \neq 0$. In this case f will have an inverse f^{-1} in a neighbourhood of a. Let $\Gamma(X, a)$ be the set of all germs of conformal mapping $(X, a) \to (X, a)$. We define $[f] \circ [g]$ by $[f \circ g]$. Note that if $[f] = [f_1]$, then $f \equiv f_1$ on any region for which both f and f_1 are analytic. Hence, the binary operation $\circ$ is well-defined. Clearly, the inverse of $[f]$ under $\circ$ is $[f^{-1}]$. Therefore, $(\Gamma(X, a), \circ)$ is a group. Note that two germs in $(\Gamma(X, a), \circ)$ are the same if they have the same Talyor series expansions about a. Therefore, from time to time, we shall simply denote the germ $[f]$ by its Talyor series.

For example, elements of $\Gamma(\mathbf{CP^1}, \infty)$ are of the form $a_1 z + a_0 + \dfrac{a_{-1}}{z} + \dfrac{a_{-2}}{z^2} + \cdots$ with $a_1 \neq 0$. While elements of $\Gamma(\mathbf{C}, 0)$ are of the form $a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ with $a_1 \neq 0$.

We simply denote $\Gamma(\mathbf{CP^1}, \infty)$ by Γ.

Definition 3. Let p be a nonconstant polynomial. Since $p^{-1}(\{\infty\}) = \{\infty\}$, we can define a group $T_p = \{g \in \Gamma \mid p \circ g = p\}$. Then, it can be shown that T_p is a cyclic subgroup of Γ and its order equals to deg p.

Example 1. $T_{z^n} = \{\lambda z \mid \lambda^n = 1\}$ and $T_{(z+1)^m} = \{\delta z + \delta - 1 \mid \delta^m = 1\}$.

T_p is so-called a discrete invariant subgroup of Γ. In fact, we have the following definition.

Definition 4. A subgroup G of Γ is discrete invariant if there exists a non-constant function F, meromorphic in a punctured neighbourhood of infinity in $\mathbf{C}$, such that $F(g(z)) = F(z)$ for all $g \in G$.

In [**11**], A.A. Shcherbakov proved that if $G \subset \Gamma$ is discrete invariant, then G is a solvable group.

We also need another important necessary condition for $G \subset \Gamma$ to be discrete. Define $\Gamma_1 = \left\{g \in \Gamma \mid g = z + a_0 + \dfrac{a_{-1}}{z} + \dfrac{a_{-2}}{z^2} + \cdots\right\}$ and $\Gamma_0 = \left\{g \in \Gamma \mid g = z + \dfrac{a_{-1}}{z} + \dfrac{a_{-2}}{z^2} + \cdots\right\}$. Clearly, Γ_1/Γ_0 is isomorphic to $(\mathbf{C}, +)$.

Lemma 7 ([**8**]). *Let $G \subset \Gamma$, $G_1 = G \cap \Gamma_1$ and $G_0 = G \cap \Gamma_0$. If G is discrete invariant, then G_1/G_0 is isomorphic to a discrete subgroup of* $(\mathbf{C}, +)$.

Example 2. Let f,g be nonconstant meromorphic functions and p, q be nonconstant polynomials. Suppose that $F(z) = f(p(z)) = g(q(z))$, then the group generated by T_p and T_q, denoted by $[T_p, T_q]$, is a discrete invariant subgroup of Γ. Hence, $[T_p, T_q]$ is solvable. If we take $p(z) = z^n$, $q(z) = (z+1)^m$ and $G = [T_{z^n}, T_{(z+1)^m}]$, then $G_1 \subset \{T_b(z) = z + b \mid b \in \mathbf{C}\}$ and $G_0 = \{z\}$. Now $G_1 \cong G_1/G_0$ which is isomorphic to a discrete subgroup of $(\mathbf{C}, +)$.

T_p and $[T_p, T_q]$ are the main objects we shall study. The following two lemmas which were proved by using Galois Theory will be needed in the proof of Theorem 3.

Lemma 8 ([**8**]). *Let p and q be two nonconstant polynomials. Define $H_{p,q} = \{ \sigma \in T_p \mid \rho\sigma = \sigma\rho$ for all $\rho \in T_q\}$. Then $H_{p,q} = T_{p_1}$, where p_1 is a right factor of p.*

Lemma 9 ([**8**]). *If $[T_p, T_q]$ is finite, then there exist two nonconstant rational functions R_1, R_2 such that $R_1 \circ p(z) = R_2 \circ q(z)$.*

If $[T_p, T_q]$ is infinite, then $[T_p, T_q]$ must be non-Abelian as both T_p and T_q are cyclic. Moreover, if $[T_p, T_q]$ is also solvable, then we can construct some groups that are isomorphic to $[T_p, T_q]$. These groups come from local holomorphic dynamics and are easier to deal with.

Definition 5. Let w be a holomorphic vector field on $V \subset \mathbf{C}$. Associated with w, it is well known that there exists a unique local phase flow $g_w : U \times V \to \mathbf{C}$ which is a solution of the Cauchy problem

$$\frac{d}{dt} g_w(t,z) = w(g_w(t,z)), \qquad g_w(0,z) = z\,, \tag{3}$$

where $U \subset \Re$ is a sufficiently small neighbourhood of 0. For brevity, we denote $g_w(t,z)$ by $g_w^t(z)$ the time-t transformation for the flow of the holomorphic vector field w. Moreover, we have the following important property:

$$g_w^{t+s}(z) = g_w^t(g_w^s(z)), \tag{4}$$

in the sense that if one side of (4) is defined, so is the other, and they are equal. If we extend the definition of $g_w^t(z)$ for all $t \in \mathbf{C}$, then $g_w^t(z)$ (possibly divergent) will be a formal solution of Equation (3), which will be denoted as $\widehat{g_w^t}(z)$.

Definition 6. If $f : V \to W$ is a bijective conformal mapping, then the forward image $f_* w$ of the vector field w on V is defined as

$$(f_* w)(z) = f'(f^{-1}(z)) \times w(f^{-1}(z)),$$

for all $z \in W$.

Let k be a natural number. We denote by $g_{z^{k+1}}^t$ the time-t transformation for the flow of the holomorphic vector field $z^{k+1}\frac{\partial}{\partial z}$. Express $g_{z^{k+1}}^t$ as $a_0(t) + a_1(t)z + a_2(t)z^2 + \cdots$ and substitute it into Equation (3). Comparing the coefficient of the constant term, we have $a_0'(t) = a_0^{k+1}(t)$, $\quad a_0(0) = 0$.

Hence, $a_0(t) \equiv 0$ on some neighbourhood of zero. By repeating this process, it is easy to check that $g^t_{z^{k+1}}(z) = z + tz^{k+1} + \cdots$. Therefore, for each sufficiently small real t, $g^t_{z^{k+1}}(z)$ is conformal in some neighbourhood of zero with $g^t_{z^{k+1}}(0) = 0$. Note that for complex number $|t| < 1$, we have $g^t_{z^2}(z) = z + tz^2 + t^2z^3 + t^3z^4 + \cdots$ is conformal in some neighbourhood of zero.

Now, we consider the set of germs

$$G(k) = \{\lambda g^t_{z^{k+1}} : (\mathbf{C}, 0) \to (\mathbf{C}, 0) \mid \lambda \in \mathbf{C}^* = \mathbf{C} - \{0\}, t \in \mathbf{C}\}.$$

We shall show that $G(k)$ under composition is a group. For brevity, denote $\lambda g^t_{z^{k+1}}$ by (λ, t). For any $\mu \in \mathbf{C}^*$, let $\mu(z) = \mu z$, it is easy to check that $\mu^{-1} \circ g^t_{\mu_* w} \circ \mu$ satisfies condition (3) and hence $g^t_{\mu_* w} \circ \mu = \mu \circ g^t_w$. Similarly, we have $g^t_{z^{k+1}} = g^{\mu^{-k}t}_{\mu_* z^{k+1}}$. Now,

$$g^t_{z^{k+1}} \circ \mu = g^{\mu^{-k}t}_{\mu_* z^{k+1}} \circ \mu = \mu \circ g^{\mu^{-k}t}_{z^{k+1}}. \tag{5}$$

(4) and (5) imply that $G(k)$ is a group under composition. From (4) and (5), the multiplication table for $G(k)$ has the following form:

$$(\lambda, t) \times (\mu, s) = (\lambda\mu, t\mu^{-k} + s).$$

With the above formula, it is easy to prove that the subgroup $C(k) = \{\lambda z = \lambda g^0_{z^{k+1}} \in G(k) | \lambda^k = 1\}$ is the center of $G(k)$ (i.e., set of element commutes with all elements of $G(k)$).

Definition 7. Let G and G_1 be two groups of germs of conformal mappings $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$. G and G_1 is said to be formally equivalent if there exists an isomorphism $K : G \to G_1$ and a formal series $\widehat{h}$ whose constant term is zero and the linear term is nonzero, such that for any $f \in G$,

$$\widehat{h^{-1}} \circ f \circ \widehat{h} = \widehat{Kf}.$$

The hat over a symbol stands for the corresponding formal series.

Now, we can state the main lemma as follows.

Lemma 10 ([**3**]). *A finitely generated non-Abelian solvable group of all germs of conformal mapping* $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$ *is formally equivalent to a finitely generated subgroup of* $G(k)$ *for some* k.

Remark 2. Let $J(z) = 1/z$ and G be a subgroup of $\Gamma(\mathbf{CP^1}, \infty)$. Then $J^{-1}GJ = \{J^{-1} \circ g \circ J | g \in G\}$ is a subgroup of $\Gamma(\mathbf{C}, 0)$. Clearly G and $J^{-1}GJ$ are isomorphic and from now on, we shall identify G with $J^{-1}GJ$ frequently. For example, T_{z^n} is identified with $J^{-1}T_{z^n}J = \{\lambda z \mid \lambda^n = 1\} = \{\lambda g^0_{z^2} \mid \lambda^n = 1\}$ and $T_{(z+1)^m}$ is identified with $J^{-1}T_{(z+1)^m}J = \{\delta z + \delta(\delta - 1)z^2 + \delta(\delta - 1)^2z^3 + \delta(\delta - 1)^3z^4 + \cdots \mid \delta^m = 1\} = \{\delta g^{\delta-1}_{z^2} \mid \delta^m = 1\}$.

3. Proof of Theorem 3.

Let $F(z) = f(p(z)) = g(q(z))$. From Example 2, we know that $[T_p, T_q]$ is solvable. We shall consider two cases: i) $[T_p, T_q]$ is finite and ii) $[T_p, T_q]$ is infinite.

Suppose that $[T_p, T_q]$ is finite, then by Lemma 9, there exist two nonconstant rational functions R_1, R_2 such that $R_1 \circ p(z) = R_2 \circ q(z)$. Express R_i as $\frac{P_i}{Q_i}$, where P_i and Q_i are polynomials and do not have any common zero. Without loss of generality, we may assume that P_1 is nonconstant. Since P_i and Q_i do not have any common zero, we have $F_1 = P_1(p(z)) = AP_2(q(z))$ for some nonzero constant A. By Lemma 2, there exists a nonconstant entire function F_2, which is the least generalized common left multiple of p and q, such that $F_2 \leq F_1$ and $F_2 \leq F$. From $F_2 \leq F_1$, it follows that F_2 is a polynomial and hence $F_2|F_1$ and $F_2|F$. Now, we can let $F_2 = h \circ p = k \circ q$ for some polynomials h, k. Note that $F_2|F$ which implies $h|f$. Since f is prime and transcendental, h must be linear. Therefore, $p = h^{-1} \circ k \circ q$, where $h^{-1} \circ k$ is linear because p is prime and q is nonlinear. So, we are done for case i).

If $[T_p, T_q]$ is infinite, then it is non-Abelian as both T_p, T_q are finite order cyclic groups. Since $[T_p, T_q]$ is also solvable, it follows from Lemma 10 that $[T_p, T_q]$ is formally equivalent to a subgroup of $G(k)$ for some natural number k. Let $\mathrm{d} = \mathrm{lcm}(\mathrm{n}, \mathrm{m})$ where $\mathrm{n} = \deg\ p$ and $\mathrm{m} = \deg\ q$. Let $\lambda g^t_{z^{k+1}}$ and $\mu g^s_{z^{k+1}}$ be the generators of T_p and T_q respectively. From the multiplication table of $G(k)$, $\lambda^n = 1$ and $\mu^m = 1$. Hence, all elements of $[T_p, T_q]$ are in $G_d(k) = \{\lambda g^t_{z^{k+1}} \in G(k) | \lambda^d = 1\}$. Therefore, $[T_p, T_q]$ is actually formally equivalent to a subgroup of $G_d(k)$.

By Lemma 8 and the fact that p is prime, $H_{p,q} = T_p$ or T_{id}. If $H_{p,q} = T_p$, then $[T_p, T_q]$ must be abelian which is impossible. So, we have $H_{p,q} = T_{id} = \{z\}$. It is easy to check that if $h \in G_k(k)$ is an element of finite order, then $h \in C(k)$. Hence, $T_p \cap G_k(k) \subset C(k)$. Note that $C(k)$ is the center of $G(k)$ and so $T_p \cap G_k(k) \subset H_{p,q} = \{z\}$. Now, we claim that $\mathrm{g} = \gcd(\mathrm{n}, \mathrm{k}) = 1$. Let (λ, t) be a generator of T_p. Then, it is very easy to check that $(\lambda, t)^{\frac{n}{g}}$ is an element of $T_p \cap G_k(k)$. Therefore, $(\lambda, t)^{\frac{n}{g}} = (1, 0)$ and hence $\frac{n}{g} = n$. We get $\mathrm{g} = \gcd(\mathrm{n}, \mathrm{k}) = 1$.

We first consider the case that q is prime. Then, we also have $\gcd(\mathrm{m}, \mathrm{k}) = 1$. So, if $\mathrm{d} = \mathrm{lcm}(\mathrm{n}, \mathrm{m})$, then $\gcd(\mathrm{d}, \mathrm{k}) = 1$. We define a map $f : G_d(k) \to G_d(1)$ by $f(\lambda g^t_{z^{k+1}}) = \lambda^k g^t_{z^2}$. Clearly, f is a group homorphism and surjective. The condition that $\gcd(\mathrm{d}, \mathrm{k}) = 1$ implies that f is also injective. Therefore $[T_p, T_q]$ is isomorphic to a subgroup of $G_d(1)$.

Let $\lambda g^t_{z^2}$ and $\delta g^s_{z^2}$ be the elements of $G_d(1)$ corresponding to generators of T_p and T_q respectively. Note that

$$(1,0) = \mathrm{id} = \lambda g^t_{z^2} \circ \lambda g^t_{z^2} \cdots \circ \lambda g^t_{z^2} (n \text{ times}) = (\lambda^n, t(\lambda^{-(n-1)} + \cdots + \lambda^{-1} + 1)).$$

So, λ (respectively δ) is a primitive nth root of unity (respectively a primitive mth root of unity).

By choosing a suitable number r, we have $(1,r) \times (\lambda,t) \times (1,-r) = (\lambda,0)$. Therefore, with this conjugation, we may assume $t = 0$ and this implies that $s \neq 0$, for otherwise $[T_p, T_q]$ will be abelian. By using the automorphism $\lambda g^t_{z^2} \to \lambda g^{ct}_{z^2} (c \neq 0)$ of $G_d(1)$, we may also assume that $s = \delta - 1$. Hence the generators are of the form $\lambda g^0_{z^2}$ and $\delta g^{\delta-1}_{z^2}$. From Remark 2, we know that they generate T_{z^n} and $T_{(z+1)^m}$ respectively. Therefore $[T_p, T_q]$ is isomorphic to $G = [T_{z^n}, T_{(z+1)^m}]$. From Example 2, $G_1 \cong (G_1/G_0) \cong ([T_p,T_q] \cap \Gamma_1)/([T_p,T_q] \cap \Gamma_0)$ which is isomorphic to a discrete subgroup of $(\mathbf{C},+)$ by Lemma 7.

Suppose $T_b \in G_1$, then $T_{\delta b}$ is also in G_1. It is because $z + \delta b = (\delta z + \delta - 1) \circ (z+b) \circ (\delta^{-1} z + \delta^{-1} - 1)$. Similarly, $T_{\lambda b} \in G_1$ and hence $T_{\epsilon b} \in G_1$, where ϵ is a d th root of unity with $d =$ lcm (n,m). Since G_1 is isomorphic to a nontrivial discrete subgroup of $(\mathbf{C},+)$, it is easy to show that either $G_1 = \{T_{na} \mid n \in \mathbf{Z}\}$ or $G_1 = \{T_{nb+mc} \mid n,m \in \mathbf{Z}\}$ for some $a,b,c \in \mathbf{C}$ and b/c being irrational (see [**9**], p. 63). We consider the first case: $T_a \in G_1$, which implies $T_{\epsilon^2 a + a} \in G_1$. Hence, $T_{2a\cos\frac{2\pi}{d}} = T_{\epsilon^{-1} \times (\epsilon^2 a + a)} \in G_1$. Thus, 2 $\cos\frac{2\pi}{d}$ is some integer which can only be 0, ± 1 or ± 2. So, it follows that $d \in \{2,3,4,6\}$. With similar argrument, we can have the same conclusion for the second case.

If $n = m = 3,4,6$, then it follows from Lemma 6 that $p = L \circ q$, where L is linear. Hence, $[T_p, T_q] = T_p$ is finite, which is a contradiction.

If $n = m = 2$, without loss of generality we may assume that $p(z) = z^2$ and $q(z) = (z+c)^2$. Then we have $F_1 = \cos\sqrt{z} \circ p = \cos(\sqrt{z} - c) \circ q$. By Lemma 2, there exists a nonconstant entire function F_2, which is the least generalized common left multiple of p and q, such that $F_2 \leq F_1$ and $F_2 \leq F$. Let $F_2 = h \circ p = k \circ q$, it follows that $h \leq f$ and $h \leq \cos\sqrt{z}$. Thus h is not periodic. By similar argrument used in the proof of Theorem 1, we have $h|f$. Since f is prime, h is linear or $h = L \circ f$ for some linear function L. h is linear implies $p = h^{-1} \circ k \circ q$ which is impossible again. Therefore, $h = L \circ f$. Hence, $\cos\sqrt{z}$ has a prime transcendental right factor f. Write $\cos\sqrt{z}$ as $h_1 \circ f$. Thus $\cos z = h_1 \circ f(z^2)$. From Theorem 3.10 in [**2**], $f(z^2) = \cos\frac{z}{n}$ which implies $f(z) = \cos\frac{\sqrt{z}}{n}$. This is impossible as $\cos\frac{\sqrt{z}}{n}$ is not a prime function.

Now, we can assume that $n \neq m$ and hence $d \neq 2,3$. $d = 4$ implies that one of n,m equals to 2. We may assume without loss of generality that $n = 2$ and $p = z^2$, $q(z) = z^4 + a_3 z^3 + a_2 z^2 + a_1 z$. Since $f(p(z)) = f(p(-z))$,

$g(q(z)) = g(q(-z))$, and because q is prime, Lemma 6 implies that $q(z) = L \circ q(-z)$. Note that L is linear, then $a_3 = a_1 = 0$ and hence q is not prime which is impossible. If $d = 6$, n can only be 2,3 or 6. The case for $n = 2$ can be treated similiar as above and the case $n = 3, 6$ are excluded from our considerations.

For general q, we can express q as $q_2 \circ q_1$ where q_1 is prime. From the above discussion, we have $f = g \circ q_2 \circ L^{-1}$ and $p = L \circ q_1$. Thus, f is prime implies that q_2 is linear and we are done.

4. Further discussions.

In Theorem 3, we assume that both the right factors p, q have polynomial growth. We can also restrict the left factors f, g to have comparable growth rate and ask the following question.

Problem (B). Let f and p be two prime entire functions and p is a polynomial. Suppose that $F = f \circ p = g \circ q$ and both f, g are transcendental. Are the two factorizations of F equivalent?

This problem is closely related to Problem C below (proposed by C.C. Yang, see e.g., [**7**], p. 124), which remains unsolved for more than a decade.

Problem (C). Let f be a pseudo-prime transcendental meromorphic function and p be a polynomial of degree ≥ 2. Must $f(p(z))$ be pseudo-prime?

If the answer to Problem C is positive, then the function q in Problem B must be a polynomial and this reduces to the case handled in Theorem 3. One may try to solve Problem C for the special case that $p(z) = z^n$, where n is a prime number.

Similarly, we can ask:

Problem (D). Let f be a pseudo-prime transcendental meromorphic function and p a polynomial of degree ≥ 3, which has no quadratic right factor. Must $p(f(z))$ be pseudo-prime?

In [**12**], G.D. Song and J. Huang proposed the above problem and solved it for the case that $p(z) = z^n$ with n being an odd number. We proved in [**10**] that it is true if f is not of the form $H \circ q$, where H is an entire periodic function and q is a polynomial. One may try to solve Problem D for deg p is odd first.

Finally, we ask whether the answer of Problem A is yes if both f and g are assumed to be transcendental?

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