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Central extensions of gyrocommutative gyrogroups (K-loops) are studied in order to clarify the status of a cocycle equation introduced by Smith and Ungar. A sufficient and necessary conditions under which a central invariant extension is a gyrocommutative gyrogroup are formulated in terms of a 2-cochain $f(x, y)$. In particular, it is shown that for central invariant extensions of gyrocommutative gyrogroups defined by Cartan decompositions of simple Lie algebras, the corresponding $f(x, y)$ satisfies the cocycle equation, provided an extension is a gyrocommutative gyrogroup.

1. Introduction.

There has been a renewal of an interest in loop theory in recent years, concerning a special non-associative loop structure called a *gyrocommutative gyrogroup,* known also under the name of a K-loop. It began with a paper by A. Ungar [**15**], who pointed it out that the addition law of relativistic velocities leads to an interesting algebraic structure on a unit ball in R^n, which he originally called a K-loop. Further properties of these structures under the name of *gyrogroups* and *gyrocommutative gyrogroups* were summarized in [**16**]. Concerning terminology, see also remarks in [**8**].

Independent studies by A. Kreuzer [**9**] and H. Karzel, and H. Wefelscheid [**4**] clarified the status of *gyrocommutative gyrogroups* within the framework of loop theory and provided some important constructions generalizing an example of Ungar.

In fact, *gyrocommutative gyrogroups* or similar structures were contemplated, although not explicitly under that name, by M. Kikkawa [**5**], in relation to symmetric spaces and by P. Miheev and L. Sabinin in relation to so called odular structures, [**10**]. That first aspect of *gyrocommutative gyrogroups* has been discussed by Y. Friedman and A. Ungar in [**2**] and recently by W. Krammer and H.K. Urbantke in [**8**].

One can obtain more examples of *gyrocommutative gyrogroups* by means of their extensions. Such extensions are in fact extensions of loops. These were discussed a long time ago by R.H. Bruck in [**1**]. They generalize extensions of groups (see for example [**12**]). As for groups, the simplest among their extensions are the ones with an abelian kernel, which include central extensions as well [**12**]. They are constructed, in principle, via a 2-cocycle, i.e. a 2-cochain $f(x,y)$ subject to a cocycle equation,

$$f(x_1,x_2)\cdot f(x_1\cdot x_2,x_3)=f(x_2,x_3)\cdot f(x_1,x_2\cdot x_3), \tag{1}$$

under the assumption that the group under consideration acts trivially on an abelian group.

Central invariant extensions of *gyrocommutative gyrogroups,* in a narrower sense, were defined in [**14**] and then employed with a particular purpose of reconstructing from a *gyrocommutative gyrogroup* of relativistic velocities, the Lorentz group, i.e. $SO_0(1,n)$; more specifically a standard matrix representation of that group. As far as central extensions are concerned it was assumed that the operators generating the left associant of an extension (in terminology of [**10**]) or the structure group of an extension (in terminology of [**9**]) are central (in the sense of [**14**]). That assumption resulted in an equation for the corresponding 2-cochain $f(x,y)$ which the authors called a cocycle equation. Thus, in principal the cocycle equation was incorporated into a definition of a central extension of a *gyrocommutative gyrogroup.* It reads,

$$f(x_1,x_2)\cdot f(x_2\cdot x_1,x_3)=f(x_2,x_3)\cdot f(x_1,x_2\cdot x_3). \tag{2}$$

That brings us to a question whether such an extension always has to arise from a 2-cochain satisfying the cocycle equation of [**14**]?

To answer that question we discuss matters in a context of central extensions of loops according to R.H. Bruck [**1**]. Basic definitions and properties of relevant concepts are presented in Sections 2 and 3. A multiplicative notation instead of an additive one, preferred by the authors of [**14**], is employed. Definitions concerning extensions of loops are, except of small modifications, faithful copies of the ones in [**1**]. Definitions of a *gyrogroup* and a *gyrocommutative gyrogroup* ($K-loop$) are equivalent to the ones of [**16**]; see Appendix. They emphasize the role of three identities, known as a Bol identity, an A-loop identity and an inverse automorphic identity (see for example [**10**]). In Section 4 a discussion is restricted to invariant, in the sense of [**14**], central extensions of a *gyrocommutative gyrogroup* M with a trivial action of M on an abelian group G. Also there, necessary and sufficient conditions under which a 2-cochain $f(x,y)$ determines an extension of M which is a *gyrogroup* or a *gyrocommutative gyrogroup* (Theorems 16 and 17) are provided. Theorem 20 of Section 5 is a generalization of the fact pointed out in [**14**], that central extensions of *gyrocommutative gyrogroups* corresponding to symmetric 2-cochains are *gyrocommutative gyrogroups* again.

In Section 6 we discuss *gyrocommutative gyrogroups* determined by Cartan decompositions of noncompact semisimple and in particular simple Lie algebras. It turns out that any central invariant extension of the latter structure, if it is a *gyrocommutative gyrogroup,* then it arises from a 2-cochain which satisfies a cocycle equation of [**14**], (2), (Theorem 29).

2. Definitions of basic concepts.

Let G be an abelian group, M a loop, [**11**], and χ a function, $\chi : M \to \mathrm{Aut}(G)$, which satisfies the following properties,

$$1\chi = \mathrm{id}_G, \tag{3}$$

where 1 and id_G are units of M and $\mathrm{Aut}(G)$ respectively, and

$$(x \cdot y)\chi = x\chi \cdot y\chi, \tag{4}$$

for all x, y ϵ M.

Given g ϵ G and x ϵ M, we denote the value of an automorphism $x\chi$ at g by gx.

Definition 1. A (G, M, χ)-extension (E,θ) is a pair consisting of a loop E and a homomorphism θ of E onto M, such that,

(i) $\ker(\theta) \subset A(E)$, where $A(E)$ is the associator of E, i.e. a subset $A(E)$ of E, such that $(e_1 \cdot e_2) \cdot e_3 = e_1 \cdot (e_2 \cdot e_3)$ if at least one of e_1, e_2, e_3 is in $A(E)$,
(ii) the center of $\ker(\theta)$ equals to G,
(iii) $g \cdot e = e \cdot (g\ x)$ for all $g \in G$, $e \in E$ and $x = e\theta$.

A (G, M, χ)-extension, (E, θ), is called central if $\ker(\theta) = G$.

Definition 2. A normalized 2-cochain f is a function $f : M^2 \to G$, with values $f(x_1, x_2)$, taking the value 1 whenever one of x_1 or x_2 is 1.

The 3-coboundary δf of f is the following normalized 3-cochain,

$$\begin{aligned} &\delta f(x_1, x_2, x_3) \\ &\qquad = [f(x_1, x_2)x_3] \cdot [f(x_2, x_3)]^{-1} \cdot f(x_1 \cdot x_2, x_3) \cdot [f(x_1, x_2 \cdot x_3)]^{-1}. \end{aligned} \tag{5}$$

Definition 3. Let f be a normalized 2-cochain. We define a central (G, M, χ) extension, (E, θ), as follows.

(i) E is the set of all ordered pairs (x, g), where $x \in M$ and $g \in G$,
(ii) $(x, g) \cdot (y, h) = (x \cdot y, f(x, y) \cdot (g\ y) \cdot h)$,
(iii) $(x, g)\theta = x$.

We denote this central extension by (G, M, χ, f).

Remark 1. In order to fulfill the conditions of Definition 1 G has to be identified with its homomorphic image in E under a natural injective homomorphism which sends $g \in G$ into $(1, g)$.

It is known, [**1**], that each central (G, M, χ)-extension is equivalent to at least one (G, M, χ, f)-extension.

Further discussion will concern extensions corresponding to a trivial function χ, i.e. the one that assigns to each $x \in M$ the identity automorphism of G. Such extensions will be referred to as (G, M) or (G, M, f)-extensions respectively.

To define a *gyrogroup* and a *gyrocommutative gyrogroup* we point out certain important identities.

Definition 4. Let M be a loop. The following identities,

$$L_x \circ L_y \circ L_x = L_{x \cdot (y \cdot x)}, \tag{6}$$

$$\ell(x, y) \circ L_z = L_{\ell(x,y)z} \circ \ell(x, y), \tag{7}$$

where $\ell(x, y) = L_{x \cdot y}^{-1} \circ L_x \circ L_y$, and

$$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}, \tag{8}$$

are called a B, an A and an I-identity respectively. B stands for Bol, A for an A-loop, and I for inverse automorphic. (For any $z \in M$, z^{-1} means its right inverse.)

For purposes of this paper we introduce the following terminology. A loop will be called a B-loop, a BA-loop or a BAI-loop, if a B, a B and an A or B, A and I-identities hold.

Remark 2. There are other equivalent forms of those identities. In particular, (7) means that the mapping $\ell(x, y)$ is an automorphism of M for all $x, y \in M$. We list below two identities equivalent to (6) and (7) correspondingly,

$$x \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot x)) \cdot z, \tag{9}$$

$$l(x, y)(z \cdot w) = (l(x, y)z) \cdot l(x, y)w. \tag{10}$$

Now we define a *gyrogroup* and a *gyrocommutative gyrogroup.*

Definition 5. Any BA-loop is called a *gyrogroup* and any BAI-loop is called a *gyrocommutative gyrogroup.*

We have also this natural definition.

Definition 6. A central (G, M) extension, (E, θ), of a *gyrocommutative gyrogroup* is called a B, a BA or a BAI-extension if E is a B, a BA or a BAI-loop respectively.

3. Central extensions of gyrocommutative gyrogroups.

We study now (G, M) central extensions of a *gyrocommutative gyrogroup* M. We follow the method of Bruck, [**1**].

Let $u(x)$ be a normalized transversal function of E. Thus, for each $x \in M$, $u(x)$ is an element of E, such that

$$u(x)\theta = x, \tag{11}$$

and

$$u(1) = 1\ . \tag{12}$$

Then every $e \in E$ has a unique representation of the form $e = u(x) \cdot g$, with $g \in G$ and $x = e\theta$. Moreover,

$$u(x) \cdot g \cdot u(y) \cdot h = u(x) \cdot u(y) \cdot g \cdot h = u(x \cdot y) \cdot f(x, y) \cdot g \cdot h, \tag{13}$$

where f is a normalized, because of (12), 2-cochain. (There is no need for additional parentheses in that identity since $g, h \in G \subset A(E)$, and G is abelian.) Then the mapping $u(x) \cdot g \mapsto (x, g)$ gives the equivalence of E and (G, M, f). (See [**1**].)

Thus, for all x, y $\in M$,

$$u(x) \cdot u(y) = u(x \cdot y) \cdot f(x, y). \tag{14}$$

We make now few simple observations.

Lemma 7. *Let (E, θ) be a central (G, M) extension and $u(x)$ a normalized transversal function of E. Then for all $g \in G$ and $e \in E$,*

$$g \cdot e = e \cdot g, \tag{15}$$

and

$$(e \cdot g)^{-1} = g^{-1} \cdot e^{-1} = e^{-1} \cdot g^{-1}. \tag{16}$$

If E is a B-loop, then

$$(u(x))^{-1} = u(x^{-1}) \cdot [f(x, x^{-1})]^{-1} = [f(x, x^{-1})]^{-1} \cdot u(x^{-1}), \tag{17}$$

$$f(x, x^{-1}) = f(x^{-1}, x), \tag{18}$$

$$\ell(u(a), u(b))u(c) = u(\ell(a, b)c) \cdot g_{a,b,c}, \tag{19}$$

where

$$\begin{aligned} g_{a,b,c} = {} & [f(a, b) \cdot f((a \cdot b)^{-1}, a \cdot b)]^{-1} \\ & \cdot f((a \cdot b)^{-1}, a \cdot (b \cdot c)) \cdot f(a, b \cdot c) \cdot f(b, c) \end{aligned} \tag{20}$$

and

$$\ell(u(a), u(b))g = g, \tag{21}$$

for all $g \in G$.

Proof. Indeed, (15) follows directly from (iii) of Definition 1. (16) is a consequence of the fact that $G \subset A(E)$ and (15). e^{-1} is understood here as the right inverse of e in the loop E.

For B-loops right and left inverses coincide [**11**]. Due to (14) we have,

$$u(x) \cdot u(x^{-1}) = f(x, x^{-1}), \tag{22}$$

and consequently

$$u(x^{-1}) = [u(x)]^{-1} \cdot f(x, x^{-1}). \tag{23}$$

Next,

$$u(x^{-1}) \cdot u(x) = f(x^{-1}, x), \tag{24}$$

$$u(x) = [u(x^{-1})]^{-1} \cdot f(x^{-1}, x) \tag{25}$$

and due to (16),

$$[u(x)]^{-1} = [f(x^{-1}, x)]^{-1} \cdot u(x^{-1}). \tag{26}$$

Then from (23) and (26) we get (18) and (17). To prove (19) we employ (15) and the other just proven identities. The method of proving (21) is similar. □

We prove now this fact.

Proposition 8. *Let M be a gyrocommutative gyrogroup. The central extension $E = (G, M, f)$ is a B-extension, if and only if,*

$$f(x, y \cdot (x \cdot z)) \cdot f(y, x \cdot z) \cdot f(x, z) = f(x \cdot (y \cdot x), z) \cdot f(x, y \cdot x) \cdot f(y, x). \tag{27}$$

Proof. It is clear, due to (9) and (13), (see also [**1**]), that a B-identity for E is equivalent to,

$$u(x) \cdot (u(y) \cdot (u(x) \cdot u(z))) = (u(x) \cdot (u(y) \cdot u(x))) \cdot u(z). \tag{28}$$

Now, using (14), (15), the fact that $G \subset A(E)$ and commutativity of G, one obtains,

$$\begin{aligned} u(x) \cdot (u(y) \cdot (u(x) \cdot u(z))) \\ = u(x \cdot (y \cdot (x \cdot z))) \cdot f(x, y \cdot (x \cdot z)) \cdot f(y, x \cdot z) \cdot f(x, z) \end{aligned} \tag{29}$$

and

$$\begin{aligned} (u(x) \cdot (u(y) \cdot u(x))) \cdot u(z) \\ = u((x \cdot (y \cdot x)) \cdot z) \cdot f(x \cdot (y \cdot x), z) \cdot f(x, y \cdot x) \cdot f(y, x). \end{aligned} \tag{30}$$

That shows equivalence of (28) and (27), since M is a B-loop. □

As far as BA-extensions are concerned we have,

Proposition 9. *Let M be a gyrocommutative gyrogroup. The central extension $E = (G, M, f)$ is a BA-extension if and only if it is a B-extension and,*

$$f(z,w)\cdot g_{x,y,z\cdot w} = f(\ell(x,y)z, \ell(x,y)w)\cdot g_{x,y,z}\cdot g_{x,y,w}. \tag{31}$$

Proof. An $A-identity$ for E is equivalent to (see (10)),

$$\ell(u(x),u(y))\cdot(u(z)\cdot u(w)) = [\ell(u(x),u(y))u(z)]\cdot\ell(u(x),u(y))u(w). \tag{32}$$

To put it into an equivalent form of (31), it suffices to apply (19), (20) and (21). Indeed, one can rewrite then (32) into,

$$\begin{aligned} &u(\ell(x,y)(z\cdot w))\cdot f(z,w)\cdot g_{x,y,z\cdot w} \\ &\qquad = u(\ell(x,y)z\cdot\ell(x,y)w)\cdot f(\ell(x,y)z,\ell(x,y)w)\cdot g_{x,y,z}\cdot g_{x,y,w}, \end{aligned} \tag{33}$$

which due to the fact that M is an A-loop is equivalent to (31). □

Finally we have a Proposition concerning central BAI-extensions.

Proposition 10. *Let M be a gyrocommutative gyrogroup. The central extension $E = (G, M, f)$ is a BAI-extension if and only if it is a BA-extension and,*

$$f(x^{-1},x)\cdot f(y^{-1},y) = f(x^{-1},y^{-1})\cdot f(x,y)\cdot f((x\cdot y)^{-1}, x\cdot y). \tag{34}$$

Proof. An I-identity for E is equivalent to,

$$(u(x)\cdot u(y))^{-1} = (u(x))^{-1}\cdot(u(y))^{-1}. \tag{35}$$

Next one derives, by a rather straightforward process, which employs Lemma 7, these identities,

$$[u(x)\cdot u(y)]^{-1} = u((x\cdot y)^{-1})\cdot[f(x,y)\cdot f((x\cdot y)^{-1},x\cdot y)]^{-1}, \tag{36}$$

and

$$[u(x)]^{-1}\cdot[u(y)]^{-1} = u(x^{-1}\cdot y^{-1})\cdot f(x^{-1},y^{-1})\cdot[f(x^{-1},x)\cdot f(y^{-1},y)]^{-1}. \tag{37}$$

Feeding them back into (35) and making use of the fact that M satisfies an I-identity, one infers equivalence of (35) and (34). □

4. Central invariant extensions of gyrocommutative gyrogroups.

Definition 11. A central extension $E = (G, M, f)$ of a loop M is called invariant if

$$f(\ell(a,b)x, \ell(a,b)y) = f(x,y) \tag{38}$$

for all $a, b, x, y \in M$.

Definition 12. For any 2-cochain f we define a 3-cochain Δf,

$$\Delta f(x,y,z) = f(x,y)\cdot[f(y,z)]^{-1}\cdot f(y\cdot x,z)\cdot[f(x,y\cdot z)]^{-1}. \tag{39}$$

The expression for Δf is different from the one for δf, (5). Indeed, those two are related by,

$$\delta f(x,y,z) = \Delta f(x,y,z) \cdot f(x \cdot y, z) \cdot [f(y \cdot x, z)]^{-1}. \tag{40}$$

However, it is Δf rather than δf , which is important in analysis of identities of Propositions 8-10. We have now a sequence of a Proposition and two Theorems that correspond to Propositions 8-10 in case of central invariant extensions.

Proposition 13. *Let M be a gyrocommutative gyrogroup. The central invariant extension $E = (G, M, f)$ is a B-extension if and only if,*

$$\Delta f(x, y \cdot x, \ell(y,x)z) \cdot \Delta f(y,x,z) = 1. \tag{41}$$

Proof. We rewrite (27) as,

$$f(v, u \cdot (v \cdot w)) \cdot f(u, v \cdot w) \cdot f(v,w) = f(v \cdot (u \cdot v), w) \cdot f(v, u \cdot v) \cdot f(u,v). \tag{42}$$

From (39) we infer that

$$f(u, v \cdot w) \cdot f(v,w) \cdot \Delta f(u,v,w) = f(u,v) \cdot f(v \cdot u, w). \tag{43}$$

Hence (42) can be put into,

$$f(v, u \cdot (v \cdot w)) \cdot f(v \cdot u, w) = f(v \cdot (u \cdot v), w) \cdot f(v, u \cdot v) \cdot \Delta f(u,v,w). \tag{44}$$

However,

$$u \cdot (v \cdot w) = (u \cdot v) \cdot \ell(u,v)w \tag{45}$$

and

$$f(v \cdot u, w) = f(\ell(u,v)(v \cdot u), \ell(u,v)w) = f(u \cdot v, \ell(u,v)w), \tag{46}$$

since M is a *gyrocommutative gyrogroup* and (38) holds. Consequently, (44) can be rewritten as,

$$\begin{aligned} f(x, (y \cdot x) \cdot \ell(y,x)z) \cdot f(y \cdot x, \ell(y,x)z) \\ = f(x \cdot (y \cdot x), z) \cdot f(x, y \cdot x) \cdot \Delta f(y,x,z). \end{aligned} \tag{47}$$

We simplify (47) further by means of (43) applied to the left-hand side of (47) with $u = x$, $v = y \cdot x$ and $w = \ell(y,x)z$. The result is,

$$\begin{aligned} f((y \cdot x) \cdot x, \ell(y,x)z) \\ = f(x \cdot (y \cdot x), z) \cdot \Delta f(y,x,z) \cdot \Delta f(x, y \cdot x, \ell(y,x)z). \end{aligned} \tag{48}$$

Finally, we observe that

$$\begin{aligned} f(x \cdot (y \cdot x), z) &= f(\ell(y,x)\ell(x, y \cdot x)((y \cdot x) \cdot x), \ell(y,x)z) \\ &= f((y \cdot x) \cdot x, \ell(y,x)z), \end{aligned} \tag{49}$$

where the last inference is based on the identity,

$$\ell(y,x)\ell(x,y\cdot x)=\ell(y,x)\ell(x,y)=id_M, \tag{50}$$

satisfied in any *gyrocommutative gyrogroup* M (see [**16**], [**14**]). Feeding (49) back into (48) reduces (48) to a desired form of (41). It is clear that an outlined process of inference of (41) from (27) can be reversed. Therefore (41) and (27) are indeed equivalent. □

Before we discuss BA-extensions, we need an identity.

Lemma 14. *Let M be a gyrocommutative gyrogroup and $E=(G,M,f)$ an invariant central B-extension. Then,*

$$f((a\cdot b)^{-1},a\cdot(b\cdot c))\cdot f(b\cdot a,c)=f(a\cdot b,(a\cdot b)^{-1}). \tag{51}$$

Proof. We employ (42) with $v=(a\cdot b)^{-1}$, $u=a\cdot b$ and $w=(a\cdot b)\cdot\ell(a,b)c$. That leads to,

$$f((a\cdot b)^{-1},(a\cdot b)\cdot\ell(a,b)c)\cdot f(a\cdot b,\ell(a,b)c)=f(a\cdot b,(a\cdot b)^{-1}). \tag{52}$$

However,

$$f((a\cdot b)^{-1},(a\cdot b)\cdot\ell(a,b)c)=f((a\cdot b)^{-1},a\cdot(b\cdot c)), \tag{53}$$

and

$$f(a\cdot b,\ell(a,b)c)=f(\ell(a,b)(b\cdot a),\ell(a,b)c)=f(b\cdot a,c). \tag{54}$$

Feeding (53) and (54) back into (52) one arrives at (51). □

Now we prove this Lemma.

Lemma 15. *Let $\Phi:M\to G$ be a homomorphism of a loop M into a group G. Then for all $x,y,z\in M$,*

$$\Phi(\ell(x,y)z)=\Phi(z). \tag{55}$$

Proof. Indeed,

$$\Phi(x\cdot(y\cdot z))=\Phi(x)\cdot\Phi(y)\cdot\Phi(z) \tag{56}$$

and

$$\Phi(x\cdot(y\cdot z))=\Phi((x\cdot y)\cdot\ell(x,y)z)=\Phi(x)\cdot\Phi(y)\cdot\Phi(\ell(x,y)z). \tag{57}$$

Hence (55) follows. □

Theorem 16. *Let M be a gyrocommutative gyrogroup. The central invariant extension $E=(G,M,f)$ is a BA-extension if and only if*

$$\Delta f(x,y\cdot x,z)\cdot\Delta f(y,x,z)=1 \tag{58}$$

and

$$\Delta f(x,y,z\cdot w)=\Delta f(x,y,z)\cdot\Delta f(x,y,w). \tag{59}$$

Proof. Let E be a BA-extension. We show first that the condition (31) of Proposition 9 implies (59). Indeed, due to (38), (31) reads,

$$g_{x,y,z\cdot w} = g_{x,y,z} \cdot g_{x,y,w}. \tag{60}$$

Making use of (39), with $u = a$, $v = b$, and $w = c$ in (20), one obtains,

$$f((a\cdot b)^{-1}, a\cdot b) \cdot \Delta f(a,b,c) \cdot g_{a,b,c} = f(b\cdot a, c) \cdot f((a\cdot b)^{-1}, a\cdot (b\cdot c)). \tag{61}$$

Employing now (61) and (51) in (60), one arrives at (59). Now, having established that, we employ Lemma 15 to prove that (41) implies (58). Conversely, if E is a central invariant extension and satisfies (58) and (59), then by Lemma 15 it satisfies (41) as well. Thus E is a B-extension. Consequently, (61) holds and because of (51) it reads $g_{a,b,c} = [\Delta f(a,b,c)]^{-1}$. Next, (59) can be rewritten into (60) which for invariant extensions is (31). □

Theorem 17. *Let M be a gyrocommutative gyrogroup. The central invariant extension $E = (G, M, f)$ is a BAI-extension if and only if,*

$$\Delta f(x, y\cdot x, z) \cdot \Delta f(y, x, z) = 1, \tag{62}$$

$$\Delta f(x, y, z\cdot w) = \Delta f(x, y, z) \cdot \Delta f(x, y, w) \tag{63}$$

and

$$f(x,y) \cdot \Delta f(y, x, x^{-1}) = f(y,x) \cdot \Delta f(x\cdot y, x^{-1}, y^{-1}). \tag{64}$$

Proof. Due to Theorem 16 it suffices to prove that a central invariant BA-extension is a BAI-extension if and only if (64) holds. We prove that (64) is equivalent to (34).

From (43) with $u = x^{-1}\cdot y^{-1}$, $v = x$ and $w = y$, we get,

$$\begin{aligned} f(x^{-1}\cdot y^{-1}, x\cdot y) \cdot f(x,y) \cdot \Delta f(x^{-1}\cdot y^{-1}, x, y) \\ = f(x^{-1}\cdot y^{-1}, x) \cdot f(y^{-1}, y). \end{aligned} \tag{65}$$

Next, using (65) and an I-identity for M in (34), one arrives at,

$$f(x^{-1}, x) \cdot \Delta f(x^{-1}\cdot y^{-1}, x, y) = f(x^{-1}, y^{-1}) \cdot f(x^{-1}\cdot y^{-1}, x). \tag{66}$$

Employing again (43), with $u = y^{-1}$, $v = x^{-1}$, $w = x$, we arrive at,

$$f(y^{-1}, x^{-1}) \cdot f(x^{-1}\cdot y^{-1}, x) = f(x^{-1}, x) \cdot \Delta f(y^{-1}, x^{-1}, x). \tag{67}$$

Feeding it back into (66) leads to,

$$f(x^{-1}, y^{-1}) \cdot \Delta f(y^{-1}, x^{-1}, x) = f(y^{-1}, x^{-1}) \cdot \Delta f(x^{-1}\cdot y^{-1}, x, y), \tag{68}$$

which is equivalent to (64). □

5. Central symmetric and invariant extensions of a gyrocommutative gyrogroup.

Definition 18. A central extension $E = (G, M, f)$ is called symmetric if,

$$f(x, y) = f(y, x). \tag{69}$$

We prove this Proposition.

Proposition 19. *Let M be a gyrocommutative gyrogroup and let $E = (G, M, f)$ be a central, symmetric, invariant, BA-extension. Then, E is a BAI-extension. In particular,*

$$\Delta f(z, y, x) = (\Delta f(x, y, z))^{-1}, \tag{70}$$

$$\Delta f(x, y, y) = 1, \tag{71}$$

and

$$\Delta f(a, b, c) = \Delta f(c, a, b) = \Delta f(b, c, a). \tag{72}$$

Proof. We prove (70)-(72) first. In order to prove (70) it suffices to apply a definition of Δf, and (69). Next we employ identities, (58) and (59) of Theorem 16. In particular, in (58) we substitute $x = a, y = b/a$ and $z = c$. Since M is a B-loop (even a *gyrocommutative gyrogroup*), then (see [**10**], [**13**]),

$$b/a = a^{-1} \cdot ((a \cdot b) \cdot a^{-1}). \tag{73}$$

Feeding that back into (58), applying (70) and (59), one arrives at,

$$\Delta f(a, b, c) = \Delta f(c, a, b) \cdot \Delta f(c, a, a^{-1}). \tag{74}$$

Putting $b = 1$, one infers that $\Delta f(c, a, a^{-1}) = 1$, which due to (59) is equivalent to $\Delta f(c, a, a) = 1$. That in turn is equivalent to (71). Now, (74) reads,

$$\Delta f(a, b, c) = \Delta f(c, a, b), \tag{75}$$

which implies (72).

Next, it is not difficult to prove the principal assertion of the Proposition. Indeed, it suffices to prove the identity (64) of Theorem 17. That identity is a straightforward consequence of (69), (59) and (70)-(72). □

We close this section with a Theorem.

Theorem 20. *Let M be a gyrocommutative gyrogroup and $E = (G, M, f)$ a symmetric, invariant, BA-extension. E is a BAI-extension if and only if,*

$$\Delta f(x, y, z \cdot w) = \Delta f(x, y, z) \cdot \Delta f(x, y, w), \tag{76}$$

$$\Delta f(z, y, x) = (\Delta f(x, y, z))^{-1}, \tag{77}$$

$$\Delta f(x,y,y)=1, \tag{78}$$

and

$$\Delta f(x,y,z)=\Delta f(z,x,y)=\Delta f(y,z,x). \tag{79}$$

Proof. Indeed, necessity of these conditions follows from Theorem 16 and Proposition 19 . Conversely, it is not difficult to infer from them (58). Thus, according to Theorem 16, E is a BA-extension and by Proposition 19, it is a BAI-extension. □

6. Gyrocommutative gyrogroups determined by Cartan decompositions of noncompact semisimple Lie algebras.

For definitions of related concepts the reader is referred to [**3**].

Let $\mathfrak{g}$ be a noncompact semisimple Lie algebra over $\mathbf{R}$ and let $\mathfrak{g}=\mathfrak{t}+\mathfrak{p}$, be a Cartan decomposition of $\mathfrak{g}$. Then the mapping $\theta : T+X \mapsto T-X$, where $T\in\mathfrak{t}$ and $X\in\mathfrak{p}$ is an involutive automorphism of $\mathfrak{g}$.

We say that a pair (G,H) is associated with $(\mathfrak{g},\theta)$ if G is a connected Lie group with Lie algebra $\mathfrak{g}$ and H is a Lie subgroup of G with Lie algebra $\mathfrak{t}$. We shall refer to (G,H) as to a pair of a noncompact type.

Theorem 21 (Helgason [**3**])**.** *Let (G,H) be a pair of a noncompact type associated with $(\mathfrak{g},\theta)$. Then:*

(i) *There exists an involutive, analytic automorphism Θ of G whose fixed point set is H and whose differential at the identity of G is θ. In particular H is closed.*

(ii) *The mapping $\varphi : \mathfrak{p}\times H \to G$, defined by*

$$\varphi(X,h)=(\exp X)h \tag{80}$$

is a diffeomorphism.

Remark 3. According to that Theorem, $G=PH$, where $P=\exp\mathfrak{p}$, is an exact decomposition of G, i.e. any element $g\in G$ has a unique representation $g=ph$, where $p\in P$ and $h\in H$. We refer to those p and h as to P- and H-factors of g respectively.

Next, there is a natural binary operation $\star$ on P determined by the exact decomposition of G. Indeed, $\forall p_1,p_2\in P$, one defines $p_1\star p_2$ to be a unique P-factor of p_1p_2. Thus,

$$p_1p_2=(p_1\star p_2)h(p_1,p_2), \tag{81}$$

where $h(p_1,p_2)$ is an $H-factor$ of p_1p_2. It turns out that this operation provides a *gyrocommutative gyrogroup* structure on P. The origins of this fact can be traced in papers of M. Kikkawa [**5**]; see also [**10**], [**2**] and for the most recent discussion of the subject, [**8**]. Notice also that the inverse in the group G of an element $p\in P$, which we denote by p^{-1}, is an element of P and it is the inverse of p in the loop $(P, \star)$. Indeed, $1 = pp^{-1} = (p \star p^{-1})h(p, p^{-1})$. Consequently, $p \star p^{-1} = 1$. The same is true for $p^{-1} \star p$.

Theorem 22. *Let (G, H) be a pair of a noncompact type associated with $(\mathfrak{g}, \theta)$, $G = PH$ the corresponding exact decomposition of G and $\star$ a binary operation on P determined by the decomposition. Then $(P, \star)$ is a smooth gyrocommutative gyrogroup.*

The following fact will be useful in a sequel.

Lemma 23. *Let n be a positive integer, $n \geq 2$ and $p_1, \dots, p_n \in P$. Then,*

$$p_n p_{n-1} \cdots p_2 p_1 = q_n h_n, \tag{82}$$

where q_n, h_n are elements of P and H respectively, determined recursively by,

$$q_1 = p_1,\ h_1 = 1, \tag{83}$$

and (*see* (81)),

$$q_{k+1} = p_{k+1} \star q_k,\ h_{k+1} = h(p_{k+1}, q_k)h_k, \tag{84}$$

for $k = 1, \dots, n-1$.

Proof. By induction. For $n = 2$ it follows from (81). Suppose the assertion is true for $n = m$. Take $n = m + 1$. Then,

$$p_{m+1}p_m \cdots p_1 = p_{m+1}q_m h_m = (p_{m+1} \star q_m)h(p_{m+1}, q_m)h_m,$$

where the last equality is inferred from (81). Hence $q_{m+1} = p_{m+1} \star q_m$, and $h_{m+1} = h(p_{m+1}, q_m)h_m$. Thus the assertion is true for $n = m + 1$. $\square$

Now we can prove the following Proposition.

Proposition 24. *Let (G, H) be a pair of noncompact type associated with $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is simple. Let $G = PH$ be the corresponding exact decomposition of G. Then, the group G is generated by the set P and the subgroup H is generated by the set*

$$S = \{h(p_1, p_2) :\ p_1, p_2 \in P\}, \tag{85}$$

in the sense that any element of G or H is a product of a finite number of elements of P or S respectively.

Proof. Indeed, the homogeneous space G/H of the pair (G, H) is reductive (see [**7**], p. 27), because $Ad(h)\mathfrak{p} \subset \mathfrak{p}$, for all $h \in H$ (see [**3**]). Then, ([**7**], p. 27), $\mathfrak{l} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a nontrivial ideal of $\mathfrak{g}$ and the corresponding connected normal subgroup of G is generated by the set $P = \{\exp(X) : X \in \mathfrak{p}\}$. However, since $\mathfrak{g}$ is simple $\mathfrak{l} = \mathfrak{g}$ and G itself is generated by the set P.

To prove the second part of this Proposition, assume that $h \in H$. Then according to the first part of the Proposition there exists a finite sequence $p_1, \dots, p_n$ of elements of P, such that $h = p_n \cdots p_1$. Employing Lemma

23 we infer that $h = q_n h_n$, where $q_n \in P, h_n \in H$ and h_n is a product of elements from S. Moreover, since $G = PH$ is an exact decomposition, we must have $q_n = 1$ and $h = h_n$. $\square$

We need yet another definition ([**10**]).

Definition 25. Let M be a loop. Then its left associant, $as_l(M)$ is a subgroup of the group of left multiplications of M, $LMlt(M)$, generated by the set, $\{\ell(x, y) : x, y \in M\}$.

A relation between $as_l(P)$ and H is clarified in the following Proposition.

Proposition 26. *Under the assumptions of Proposition* 24, *the group* $as_l(P)$ *is identical to the group of conjugations of* P *by elements of* H.

Proof. Indeed, for all $p_1, p_2, p_3 \in P$,

$$\ell(p_1, p_2)p_3 = (p_1 \star p_2)^{-1} \star (p_1 \star (p_2 \star p_3)).$$

However, employing (81) one obtains,

$$p_1 p_2 p_3 = (p_1 \star (p_2 \star p_3))h,$$
$$(p_1 \star p_2)^{-1} = h_{12} p_2^{-1} p_1^{-1},$$

where $h, h_{12} \in H$ and $h_{12} = h(p_1, p_2) \in S$. (Concerning the inverses of elements of P see Remark 3.) Hence,

$$h_{12} p_3 = (p_1 \star p_2)^{-1}(p_1 \star (p_2 \star p_3))h = [\ell(p_1, p_2)p_3]h_1,$$

where h_1 is yet another element of H.

Consequently, $\ell(p_1, p_2)p_3 = (h_{12} p_3 h_{12}^{-1})(h_{12} h_1^{-1})$. Since the left-hand side of the previous equation is an element of P, and the factors of its right-hand side are in P and H respectively, therefore $h_{12} h_1^{-1} = 1$ and,

$$\ell(p_1, p_2)p_3 = h_{12} p_3 h_{12}^{-1}.$$

Notice that h_{12} depends on p_1 and p_2 only. Therefore $\ell(p_1, p_2)$ is indeed a conjugation of P by $h_{12} \in H$. Consequently, any element of $as_l(P)$ is a conjugation of P by an element of H. Conversely, given $h \in H$, it can be represented as a product of elements of the form $h(p, q)$, where $p, q \in P$, (85). Therefore a conjugation by h is a product of conjugations by such elements. Since a conjugation by $h(p, q)$ equals to $\ell(p, q)$, therefore a conjugation by h is an element of $as_l(P)$. $\square$

Now we arrive at this result.

Proposition 27. *Under the assumptions of Proposition* 24, *let* $\Phi : P \to A$, *be a homomorphism of a gyrocommutative gyrogroup* P *into an abelian group* A. *Then there exists a homomorphism* $\Psi : G \to A$, *such that* $\Psi \mid_P = \Phi$ *and for all* $h \in H$, $\Psi(h) = 1$.

Proof. Indeed, we define the mapping $\Psi : G \to A$ by, $\Psi(g) = \Phi(p)$ for all $g \in G$, where p is a unique P-factor of g. Next we verify that Ψ is a homomorphism. Indeed, for $p_1, p_2 \in P$ and $h_1, h_2 \in H$, there exists, due to (81), $h \in H$ such that,

$$p_1h_1p_2h_2 = p_1p_2^{h_1}h_1h_2 = (p_1 \star p_2^{h_1})h,$$

where $p_2^{h_1} = h_1p_2h_1^{-1}$.

Consequently,

$$\Psi(p_1h_1p_2h_2) = \Phi(p_1 \star p_2^{h_1}) = \Phi(p_1)\Phi(p_2^{h_1}). \tag{86}$$

However, according to Proposition 26, the conjugation of P by h_1 is an element of $as_l(P)$. Then, employing Lemma 15 we can rewrite (86) into,

$$\Psi(p_1h_1p_2h_2) = \Phi(p_1)\Phi(p_2) = \Psi(p_1h_1)\Psi(p_2h_2).$$

□

Finally we point out the following fact.

Proposition 28. *Let $\psi : \mathfrak{g} \to \mathfrak{a}$, be a homomorphism of a simple Lie algebra $\mathfrak{g}$, with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, into an abelian Lie algebra $\mathfrak{a}$, such that $\psi \mid_{\mathfrak{k}} = 0$. Then, $\psi = 0$.*

Proof. Indeed, $\ker \psi \neq \{0\}$ and $\ker \psi$ is an ideal of $\mathfrak{g}$. But $\mathfrak{g}$ is simple. Therefore $\ker \psi = \mathfrak{g}$. □

We arrive now at the main assertion of this section.

Theorem 29. *Let (G, H) be a pair of a noncompact type associated with $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is simple. Let $G = PH$ be the corresponding exact decomposition of G and $\star$ a binary operation on P determined by that decomposition. Let $E = (A, P, f)$, where A is an abelian Lie group and $f(p, q)$ a smooth 2-cochain, be a central, invariant extension of P. Then:*

(i) *E is a BA-extension if and only if $\Delta f(p, q, r) = 1$, for all $p, q, r \in P$.*
(ii) *E is a BAI-extension if and only if $\Delta f(p, q, r) = 1$ and $f(p, q) = f(q, p)$, for all $p, q \in P$.*

Proof. The conditions (59) of Theorem 16 or (63) of Theorem 17 are crucial. Indeed, let $\forall p, q \in P, \Phi_{p,q}$ be a mapping of P into A, defined by

$$\Phi_{p,q}(r) = \Delta f(p, q, r),$$

for all $r \in P$. Now, if that mapping is a homomorphism of a *gyrocommutative gyrogroup* P into a group A, then it can be extended to a homomorphism $\Psi_{p,q}$ of the group G into A such that $\Psi_{p,q} \mid_P = \Phi_{p,q}$ and $\Psi_{p,q}(h) = 1$, for all $h \in H$ (Proposition 27). $\Psi_{p,q}$ induces, in turn, a homomorphism $\psi_{p,q}$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ ($\psi_{p,q}$ is a differential of $\Psi_{p,q}$ at $1 \in G$). It must be a trivial homomorphism, by Proposition 28; $\forall p, q \in P, \forall Z \in \mathfrak{g}$, $\psi_{p,q}Z = 0$. However, the group G is generated by elements of the form $\exp Z$, where $Z \in \mathfrak{g}$, and

$\Psi_{p,q}(\exp Z) = \exp \psi_{p,q} Z = 1$. Thus $\Psi_{p,q}$ is itself trivial. Consequently, if $\Delta f(p, q, r)$ satisfies (59) of Theorem 16, then $\Delta f = 1$, which already implies identity (58). Similarly, (63) of Theorem 16, implies $\Delta f = 1$, (62) and reduces (64) into, $f(p, q) = f(q, p)$.

Conversely, given that $\Delta f = 1$, one infers using Theorem 16 that E is a BA-extension. Likewise, assuming $\Delta f = 1$ and the identity $f(p, q) = f(q, p)$, one employs Theorem 17 to infer that E is a BAI-extension. □

Remark 4. The equation, $\Delta f = 1$ (2) has been called in [**14**] a cocycle equation. Together with a condition of invariance for a cochain f, they were employed there to study central extensions of *gyrocommutative gyrogroups.* Our slightly more general treatment of central extensions shows that at least for a class of *gyrocommutative gyrogroups* discussed in this section (Theorem 29), a cocycle equation emerges in a natural way. This class of *gyrocommutative gyrogroups* is determined by Cartan decompositions of simple Lie algebras. Many of classical Lie algebras are of that type. (For their list see [**6**], Appendix C.) In particular, a *gyrocommutative gyrogroup*, which has been one of the main objects studied in ([**14**]), and which arises from the pair $(SO_0(1, n), SO(n))$, belongs to that class.

Appendix.

For completeness of presentation we clarify here how the definitions of *gyrogroup* and *gyrocommutative gyrogroup* used in this paper correspond to the original ones.

According to [**16**] a *gyrogroup* is a grupoid $(M, \cdot)$ satisfying the following axioms.

$(G1)$ There is an element $1 \in M$, such that,

$$1 \cdot x = x,$$

for all $x \in M$.

$(G2)$ For each $x \in M$ there is an $a \in M$ such that

$$a \cdot x = 1.$$

$(G3)$ For any $x, y, z \in M$ there exists a unique element $gyr[x, y]z \in M$ such that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot gyr[x, y]z.$$

$(G4)$ If $gyr[x, y]$ denotes the map of M into M, given by $z \mapsto gyr[x, y]z$ then $gyr[x, y] \in Aut(M, \cdot)$.

$(G5)$ For all $x, y \in M$

$$gyr[x, y] = gyr[x \cdot y, y].$$

From these axioms one can infer (see [**16**]) that the equation $x \cdot y = z$ has a unique solution for x or y, given the other two elements. Thus $(M, \cdot)$ is a loop, [**11**].

Proposition 30. *Let $(M, \cdot)$ be a loop. $(M, \cdot)$ is a gyrogroup if and only if it is an A-loop and it is a left Bol loop,* [**11**].

Proof. Indeed, suppose that $(M, \cdot)$ is a gyrogroup. Then $(G3)$ can be rewritten into $gyr[x, y]z = L_{x\cdot y}^{-1} L_x L_y z = \ell(x, y)z$. Hence $gyr[x, y] = \ell(x, y)$. Therefore $(G3)$ and $(G4)$ are equivalent to the statement that $(M, \cdot)$ is an A-loop. To prove *Bol*-identity we calculate $L_x L_y L_x z = x \cdot (y \cdot (x \cdot z)) = x \cdot ((y \cdot x) \cdot \ell(y, x)z) = (x \cdot (y \cdot x)) \cdot \ell(x, y \cdot x)\ell(y, x)z$. However, (see [**16**]), $\ell(x, y \cdot x) = \ell(x, y)$ and $\ell(y, x) = (\ell(x, y))^{-1}$. Hence $L_x L_y L_x = L_{x\cdot(y\cdot x)}$, which is a Bol identity, (6).

Conversely, let $(M, \cdot)$ be an $A - loop$ and a Bol loop. Then $(G1) - (G4)$ hold automatically. It suffices to prove $(G5)$. We obtain, $L_{(x\cdot y)\cdot y} = L_{(x\cdot y)\cdot(x^{-1}\cdot(x\cdot y))} = L_{x\cdot y} L_{x^{-1}} L_{x\cdot y}$, where we have employed the fact that for Bol loops $\ell(x, x^{-1}) = \ell(x^{-1}, x) = L_{x^{-1}} L_x = id_M$, [**11**]. Therefore $L_{(x\cdot y)\cdot y}^{-1} L_{x\cdot y} L_y = L_{x\cdot y}^{-1} L_{x^{-1}}^{-1} L_{x\cdot y}^{-1} L_{x\cdot y} L_y = L_{x\cdot y}^{-1} L_x L_y$, which is equivalent to $\ell(x \cdot y, y) = \ell(x, y)$, which is $(G5)$. □

Now, according to [**16**], $(M, \cdot)$ is a *gyrocommutative gyrogroup* if it is a *gyrogroup,* which satisfies an additional axiom,

$$(G6) \qquad x \cdot y = gyr[x, y](y \cdot x).$$

As it has been indicated in [**16**], this identity in a *gyrogroup* is equivalent to an inverse automorphic identity, (8).

Therefore the following is true.

Proposition 31. *Let $(M, \cdot)$ be a loop. $(M, \cdot)$ is a gyrocommutative gyrogroup if and only if it is an A-loop, a left Bol loop and it satisfies an inverse automorphic identity.*

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