EMBEDDINGS OF $\mathbb{Z}_2$-HOMOLOGY 3-SPHERES IN $\mathbb{R}^5$ UP TO REGULAR HOMOTOPY

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Let $F : M^3 \hookrightarrow R^5$ be an embedding of an (oriented) $\mathbb{Z}_2$-homology 3-sphere $M^3$ in $\mathbb{R}^5$. Then $F$ bounds an embedding of an oriented manifold $W^4$ in $\mathbb{R}^5$. It is well known that the signature $\sigma(W^4)$ of $W^4$ is equal to the $\mu$-invariant of $M^3$ modulo 16. In this paper we prove that $\sigma(W^4)$ itself completely determines the regular homotopy class of $F$.

1. Introduction.

Let $\text{Imm}[X,Y]$ be the set of regular homotopy classes of immersions of a manifold $X$ in a manifold $Y$, and $\text{Emb}[X,Y]$ denote the subset of $\text{Imm}[X,Y]$ consisting of all regular homotopy classes containing an embedding. Smale [6] has given a 1-1 correspondence (the Smale invariant) $s : \text{Imm}[S^n, R^N] \rightarrow \pi_n(V_{N,n})$, where $V_{N,n}$ is the Stiefel manifold of all $n$-frames in $R^N$. Hirsch [2] has generalized this to the case of immersions of an arbitrary manifold in an arbitrary manifold. These results solve the problem of the number of regular homotopy classes in terms of homotopy theory, but do not succeed in finding representatives for each class or determining which classes are represented by an embedding.

According to Hughes [4], $\text{Imm}[S^n, R^N]$ has a group structure under connected sum and the Smale invariant actually gives a group isomorphism. [4] gives explicit generators of $\text{Imm}[S^3, R^4]$ and $\text{Imm}[S^3, R^5]$.

Hughes-Melvin [5] determine which classes of $\text{Imm}[S^n, R^{n+2}]$ are represented by an embedding, and prove that $\text{Emb}[S^n, R^{n+2}]$ is isomorphic to $\mathbb{Z}$ if $n \equiv 3 \mod 4$, and to 0 otherwise. Furthermore, [5] proves that the regular homotopy class of an embedding $S^n \hookrightarrow R^{n+2}(n \equiv 3 \mod 4)$ can be completely determined by the signature of its oriented “Seifert” manifold. For example, in the case $n = 3$, we have the following diagram:

$$
\begin{align*}
\text{Emb}[S^3, R^5] & \xrightarrow{s} 24\mathbb{Z} \\
\pi_3(V_{5,3}) & \cong \mathbb{Z} \\
\end{align*}
$$

where $V^4$ is an oriented Seifert manifold for $f$. 

249
This implies that there exist many n-knots which cannot be transformed to the standard embedding even through a smooth deformation admitting self-intersections \((n \equiv 3 \mod 4)\).

The purpose of this paper is to prove a similar statement for embeddings of \(\mathbb{Z}_2\)-homology 3-spheres in \(\mathbb{R}^5\). More precisely we prove that the regular homotopy class of an embedding of a \(\mathbb{Z}_2\)-homology 3-sphere in \(\mathbb{R}^5\) is completely determined by the signature of its oriented Seifert manifold.

Throughout this paper, manifolds and immersions are of class \(C^\infty\). The symbol \(\approx\) denotes an appropriate isomorphism between algebraic objects; \(\sim\) and \(\sim_r\) mean respectively “homotopic” and “regularly homotopic”. We often do not distinguish between an immersion \(f\) and its regular homotopy class, both of which we denote by \(f\).

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2. Preliminaries.

We recall some results of [9]. Let \(M^n\) be a parallelizable \(n\)-manifold, and \(f : M^n \to \mathbb{R}^N\) be an immersion. Fix a trivialization \(TM \cong M^n \times \mathbb{R}^n\); we can associate to \(f\) a map \(\overline{df} : M^n \to V_{N,n}\) from \(M^n\) to the Stiefel manifold \(V_{N,n}\), where \(V_{N,n}\) is identified with the set of all injective linear maps from \(\mathbb{R}^n\) to \(\mathbb{R}^N\). \(\overline{df}\) is essentially the differential of \(f\). By Hirsch’s theorem [2], the correspondence \(f \mapsto \overline{df}\) gives a bijection between \(\text{Imm}[M^n, \mathbb{R}^N]\) and the homotopy set \([M^n, V_{N,n}]\). Every oriented 3-manifold \(M^3\) is parallelizable, so \(\text{Imm}[M^3, \mathbb{R}^3] \approx [M^3, V_{3,3}]\).

We now study the set \([M^3, V_{5,3}]\). Since \(V_{5,3}\) is simply connected, we can make use of the results of Whitney [8]. Let \(\pi_1 = \pi_1(V_{5,3})\), then \(\pi_1 = 0\), \(\pi_2 \approx \pi_3 \approx \mathbb{Z}\). Therefore we must consider the secondary difference.

Identify \(\pi_2\) and \(\pi_3\) with \(\mathbb{Z}\) in the same way as [9, Proof of Theorem 2]. For a map \(\xi : M^3 \to V_{5,3}\) we can suppose \(\xi(M^{(1)}) = p \in V_{5,3}\) because \(\pi_1 = 0\), \((p\) is a point in \(V_{5,3}\) and \(M^{(q)}\) denotes the q-skeleton of \(M\). So we can consider the difference 2-cochain between \(\xi\) and the constant map to the point \(p\). Since \(\xi\) is defined over \(M^3\), this 2-cochain is actually a 2-cocycle. Let \(C^2_\xi\) denote its cohomology class in \(H^2(M^3; \mathbb{Z})\).

Next, for two maps \(\xi, \eta : M^3 \to V_{5,3}\) with \(\xi|M^{(2)} \sim \eta|M^{(2)}\), denote by \(\Delta^3_{\xi,\eta}\) the difference 3-cochain.

The following is an application of [8, Theorem 8A] to our special case of mappings of \(M^3\) in \(V_{5,3}\) (see also [9, proof of Theorem 2]).

**Lemma 2.1** ([8, Theorem 8A], [9, Theorem 2]). Two maps \(\xi, \eta : M^3 \to V_{5,3}\) are homotopic if and only if
\[
\text{(a) } C^2_\xi = C^2_\eta \in H^2(M^3; \mathbb{Z}).
\]
(b) There is a 1-cocycle $X^1$ and a 2-cochain $Y^2$ such that $\Delta^3_{\xi,\eta} = 4X^1 \cup C^2_\xi + \delta Y^2$.

3. Main results.

Let $M^3$ be a closed oriented 3-manifold. Let $D^3$ be the 3-disk, which from now on we will often identify with the northern hemisphere of the 3-sphere $S^3$. Fix an inclusion $D^3 \subset M^3$, and put $M_0 = M^3 - \text{int} D^3$. Suppose $F_0 : M^3 \hookrightarrow R^5$ is an embedding such that $F_0|D^3$ coincides with the northern part of the standard embedding $S^3 \subset R^5$. For an immersion $f : S^3 \hookrightarrow R^5$, we can assume $f|(the\ southern\ hemisphere)$ is standard, so define the map

\[ \sharp F_0 : \text{Imm}[S^3, R^5] \longrightarrow \text{Imm}[M^3, R^5] \]

\[ f \longrightarrow F_0 \sharp f \]

where $(F_0 \sharp f)|M_0 = F_0|M_0$, and $(F_0 \sharp f)|D^3 = f|D^3$. The normal bundle of $F_0$ is trivial and if $F_0$ is altered on $D^3$ its normal bundle does not change. So we can in fact define the map

\[ \sharp F_0 : \text{Imm}[S^3, R^5] \longrightarrow \text{Imm}[M^3, R^5]_0 \]

where $\text{Imm}[M^3, R^5]_0$ is the subset of $\text{Imm}[M^3, R^5]$ consisting of all regular homotopy classes of immersions with trivial normal bundle. Note that $\text{Emb}[M^3, R^5] \subset \text{Imm}[M^3, R^5]_0$.

**Proposition 3.1.** If $H^2(M^3; Z)$ has no elements of even order, then

\[ \sharp F_0 : \text{Imm}[S^3, R^5] \longrightarrow \text{Imm}[M^3, R^5]_0 \]

is bijective.

**Proof.** Let $\nu_F$ be the normal bundle of an immersion $F : M^3 \hookrightarrow R^5$. Since there is the bundle map

\[ \begin{array}{ccc}
\nu_F & \longrightarrow & V_{5,5} \\
\downarrow & & \downarrow \\
M^3 & \xrightarrow{dF} & V_{5,3}
\end{array} \]

and since the Euler class of the $S^1$-bundle $V_{5,5} \rightarrow V_{5,3}$ is equal to $2\Sigma^2$ for a generator $\Sigma^2 \in H^2(V_{5,3}; Z)$, we have

\[ \nu_F \text{ is trivial}, \]

\[ \Leftrightarrow \text{the normal Euler class of } F \ (\text{denoted by } \chi_F) \text{ is zero}, \]

\[ \Leftrightarrow dF^*(2\chi_F) = 2dF^*(\chi_F) = 0, \]

\[ \Leftrightarrow 2C_2^2\frac{dF}{dF} = 0, \]

\[ \Leftrightarrow C_2^2\frac{dF}{dF} = 0. \]

Therefore, $\text{Imm}[M_0^3, R^5]_0 \approx H^3(M_0; Z) = 0$ by [9, Theorem 2]. This means that $\sharp F_0$ is surjective from the covering homotopy property for immersion spaces (see [7]).
We next prove the injectivity. For two immersions \( f, g : S^3 \hookrightarrow \mathbb{R}^5 \), by Lemma 2.1,
\[
F_0^\sharp f \simeq_r F_0^\sharp g,
\]
\[
\Leftrightarrow d(F_0^\sharp f) \sim d(F_0^\sharp g),
\]
\[
\Leftrightarrow \Delta^3_{d(F_0^\sharp f),d(F_0^\sharp g)} \text{ is a coboundary}.
\]

If we consider \( D^3 \) as a 3-cell, \( \Delta^3 d(F_0^\sharp f),d(F_0^\sharp g) \) is a 3-cochain which assigns \( s(f) - s(g) \in \pi_3(V^5_5, 3) \) to \( D^3 \), and \( 0 \in \pi_3(V^5_5, 3) \) to other 3-cells by definition. So clearly
\[
\Delta^3_{d(F_0^\sharp f),d(F_0^\sharp g)} \text{ is a coboundary},
\]
\[
\Leftrightarrow s(f) = s(g) \in \pi_3(V^5_5, 3),
\]
\[
\Leftrightarrow f \simeq_r g : S^3 \rightarrow \mathbb{R}^5.
\]
This completes the proof. \( \square \)

**Remark 3.2.** We already know, by a result of Wu ([9, Theorem 2]), that for a closed oriented 3-manifold \( M^3 \), \( \text{Imm}[M^3, \mathbb{R}^5]_0 \approx \mathbb{Z} \sqcup \cdots \sqcup \mathbb{Z} \), i.e., the disjoint union of as many copies of \( \mathbb{Z} \) as the number of elements \( c \in H^2(M^3; \mathbb{Z}) \) with \( 2c = 0 \). In particular, in our case where \( H^2(M^3; \mathbb{Z}) \) has no elements of even order, this implies that \( \text{Imm}[M^3, \mathbb{R}^5]_0 \approx \mathbb{Z} \).

We now investigate \( \sharp F_0 \) restricted to \( \text{Emb}[M^3, \mathbb{R}^5] \). We want to show that \( \sharp F_0 \) gives a bijection between \( \text{Emb}[S^3, \mathbb{R}^5] \) and \( \text{Emb}[M^3, \mathbb{R}^5] \).

**Theorem 3.3.** If \( H^1(M^3; \mathbb{Z}_2) = 0 \), then
\[
\sharp F_0 : \text{Emb}[S^3, \mathbb{R}^5] \longrightarrow \text{Emb}[M^3, \mathbb{R}^5]
\]
is bijective.

Furthermore, under the identification \( \text{Imm}[M^3, \mathbb{R}^5]_0 \approx \text{Smale inv.} \mathbb{Z} \),
\[
\text{Emb}[M^3, \mathbb{R}^5] \approx 24\mathbb{Z},
\]
where \( W^4_F \) stands for an oriented Seifert manifold for \( F \), and \( \sigma(W^4_F) \) is its signature.

**Proof.** Extend the embedding \( F_0 : M^3 \hookrightarrow \mathbb{R}^5 \) to an embedding \( \overline{F}_0 : W^4_{F_0} \hookrightarrow \mathbb{R}^5 \). Take a suitable neighbourhood of \( M^3 \) in \( W^4_{F_0} \) diffeomorphic to \( M^3 \times [0, 1) \), and further extend \( \overline{F}_0 \) to an embedding (denoted again by \( \overline{F}_0 \))
\[
\overline{F}_0 : W^4_{F_0} \cup_{M \times \{0\}} M^3 \times (-1, 0] \hookrightarrow \mathbb{R}^5.
\]

Let \( F : M^3 \hookrightarrow \mathbb{R}^5 \) be an embedding, and extend \( F \) to
\[
\overline{F} : W^4_F \cup_{M \times \{0\}} M^3 \times (-1, 0] \hookrightarrow \mathbb{R}^5.
\]
in the same way as above.

Take a neighbourhood $M'_0$ of $M_0$ in $M^3$. Since $M'_0 \times (-1,1)$ is parallelizable,

$$\text{Imm}[M'_0 \times (-1,1), \mathbb{R}^5] \approx [M'_0 \times (-1,1), V_{5,4}] \approx [M_0, SO(5)].$$

And it follows by obstruction theory that $\text{Imm}[M'_0 \times (-1,1), \mathbb{R}^5] \approx [M_0, SO(5)]$ consists of a unique element, because $\pi_2(SO(5)) = 0$, $H^3(M_0; \pi_3(SO(5))) = 0$, and $H^1(M_0; \pi_1(SO(5))) \approx H^1(M^3; \mathbb{Z}_2) = 0$. Therefore we can alter $F$ by a regular homotopy (we use again the letter $F$ to represent the resulting immersion) so that

$$F|(M'_0 \times (-1,1))(x,t) = F_0|(M'_0 \times (-1,1))(x,-t), \quad (x,t) \in M'_0 \times (-1,1).$$

Consider the manifold $V_F^4 = W_{F_0}^4 \cup_{M_0 \times \{0\}} W_F^4$ (the orientation of $V_F^4$ is taken to be in accord with the one of $W_{F_0}^4$), whose boundary is $S^3$. Using $F$ and $F_0$, construct a map from $V_F^4$ to $\mathbb{R}^5$. This map is an immersion except on $S^2 = \partial M_0 \subset \partial V_F^4$. Pushing a neighbourhood of $S^2$ into $V_F^4$, we have an immersion $G$ of the whole $V_F^4$ in $\mathbb{R}^5$ (Figure 1).

![Figure 1](image-url)
Figure 2.

Now clearly $F \sim_r F_0 \sharp (G|\partial V^4_F) : M^3 \inj R^5$ (Figure 2). By Proposition 3.1, the regular homotopy class of $F$ depends only on the regular homotopy class of $G|\partial V^4_F : S^3 \inj R^5$. Since [5, Proof of Theorem and Corollary 2] actually proves that if an immersion $f : S^3 \inj R^5$ bounds an immersion of an oriented 4-manifold $V^4$ then $s(f)$ is equal to $-\frac{3}{2}\sigma(V^4)$, we can see $s\left(G|\partial V^4_F\right) = -\frac{3}{2}\sigma(V^4) \in 24\mathbb{Z}$, and $G|\partial V^4_F \in \text{Emb}[S^3, R^5]$. Thus, the map $\sharp F_0$ gives a bijection from $\text{Emb}[S^3, R^5]$ to $\text{Emb}[M^3, R^5]$. Therefore, identifying $\text{Imm}[M^3, R^5] \approx \text{Z}$, $F \in \text{Emb}[M^3, R^5]$ corresponds to $\frac{3}{2}\sigma(V^4_F) = -\frac{3}{2}(\sigma(W^4_F) - \sigma(W^4_{F_0}))$ by Novikov additivity. This completes the proof. 

Remark 3.4. We actually proved here that if an immersion $F : M^3 \inj R^5$ bounds an immersion of an oriented 4-manifold $W^4$ then $F$ corresponds to $\frac{3}{2}(\sigma(W^4_F) - \sigma(W^4_{F_0}))$ under the above identification $\text{Imm}[M^3, R^5] \approx \mathbb{Z}$. We realize $h$-cobordisms in $R^5$.

In this section, we study the following problem. Suppose $M_1$, $M_2$ are two $\mathbb{Z}_2$-homology 3-spheres which are mutually h-cobordant and let $S_i : \text{Imm}[M_i, R^5] \rightarrow \mathbb{Z}$ $(i = 1, 2)$ denote the bijections as in Remark 3.5. Is it possible to relate $S_1$ to $S_2$?

Let $M_1$, $M_2$ be as above, and $V$ be an h-cobordism between $M_1$ and $M_2$. Let $F_i : M_i \inj R^5$ be embeddings and $W_i$ be oriented Seifert manifolds
EMBEDDINGS OF $\mathbb{Z}_2$-HOMOLOGY 3-SPHERES

255

for them ($i = 1, 2$). Abstractly each $M_i$ bounds a simply connected spin 4-manifold $W'_i$ of signature $\sigma(W'_i) = \sigma(W_i)$ (taking a connected sum with some copies of the $\pm K3$-surface if necessary) ($i = 1, 2$)(see [3]). Consider the closed manifold

$$Y = W'_1 \cup_{M_1} V \cup_{M_2} W'_2.$$  

$Y$ is a simply connected spin 4-manifold of signature $\pm(\sigma(W'_1) - \sigma(W'_2))$, since $W'_1 \cup_{M_1} V$ is homotopy equivalent to $W'_1$ and since each $M_i$ admits a unique spin structure. By Cochran [1], $Y$ can embed in $\mathbb{R}^5$ if $\sigma(W'_1) = \sigma(W'_2)$. Clearly this embedding restricted to each $M_i$ is regularly homotopic to $F_i$ ($i = 1, 2$), using Theorem 3.3.

Conversely, suppose $H : V \hookrightarrow \mathbb{R}^5$ is an embedding. $H$ can extend to an immersion of $W_1 \cup V$ in $\mathbb{R}^5$ for a Seifert manifold $W_1$ for $H|_{M_1}$, if the trivialization of the normal bundle of $H|_{M_1}$ (for the construction of $W_1$) is suitably chosen. This, together with Theorem 3.3, implies that $S_1(H|_{M_1}) = S_2(H|_{M_2}) \in \mathbb{Z}$ because $\sigma(W_1) = \sigma(W_1 \cup V)$.

Thus, we have:

**Proposition 4.1.** Let $M_i$, $S_i$ ($i = 1, 2$) and $V$ be as above. For embeddings $F_i : M_i \hookrightarrow \mathbb{R}^5$ ($i = 1, 2$), $S_1(F_1) = S_2(F_2) \in \mathbb{Z}$ if and only if there is an embedding $H : V \hookrightarrow \mathbb{R}^5$ with $H|_{M_i} \sim_r F_i$ ($i = 1, 2$) (or equivalently, there is an immersion $H : V \looparrowright \mathbb{R}^5$ with $H|_{M_i} = F_i$ ($i = 1, 2$)).

**References**


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