COMMUTATORS WITH POWER CENTRAL VALUES ON
A LIE IDEAL

LUISA CARINI AND VINCENZO DE FILIPPIS
COMMUTATORS WITH POWER CENTRAL VALUES ON A LIE IDEAL

LUISA CARINI AND VINCENZO DE FILIPPIS

Let $R$ be a prime ring of characteristic $\neq 2$ with a derivation $d \neq 0$, $L$ a noncentral Lie ideal of $R$ such that $[d(u), u]^n$ is central, for all $u \in L$. We prove that $R$ must satisfy $s_4$ the standard identity in 4 variables. We also examine the case $R$ is a 2-torsion free semiprime ring and $[d([x, y]), [x, y]]^n$ is central, for all $x, y \in R$.

Let $R$ be a prime ring and $d$ a nonzero derivation of $R$. A well known result of Posner [14] states that if the commutator $[d(x), x] \in Z(R)$, the center of $R$, for any $x \in R$, then $R$ is commutative.

In [11] C. Lanski generalizes the result of Posner to a Lie ideal. To be more specific, the statement of Lanski’s theorem is the following:

**Theorem ([11], Theorem 2, page 282]).** Let $R$ be a prime ring, $L$ a noncommutative Lie ideal of $R$ and $d \neq 0$ a derivation of $R$. If $[d(x), x] \in Z(R)$, for all $x \in L$, then either $R$ is commutative, or $\text{char}(R) = 2$ and $R$ satisfies $s_4$, the standard identity in 4 variables.

Here we will examine what happens in case $[d(x), x]^n \in Z(R)$, for any $x \in L$, a noncommutative Lie ideal of $R$ and $n \geq 1$ a fixed integer.

One cannot expect the same conclusion of Lanski’s theorem as the following example shows:

**Example 1.** Let $R = M_2(F)$, the $2 \times 2$ matrices over a field $F$, and take $L = R$ as a noncommutative Lie ideal of $R$. Since $[x, y]^2 \in Z(R)$, for all $x, y \in R$, then also $[d(x), x]^2 \in Z(R)$, for all $x \in R$.

We will prove that:

**Theorem 1.1.** Let $R$ be a prime ring of characteristic different from $2$, $L$ a noncentral Lie ideal of $R$, $d$ a nonzero derivation of $R$ such that $[d(u), u]^n \in Z(R)$, for any $u \in L$. Then $R$ satisfies $s_4$.

We will proceed by first proving that:

**Lemma 1.1.** Let $R$ be a prime ring of characteristic different from $2$, $L$ a noncentral Lie ideal of $R$, $d$ a nonzero derivation of $R$, $n \geq 1$. If $d$ satisfies $[d(u), u]^n = 0$, for any $u \in L$, then $R$ is commutative.
We then examine the case $R$ is a 2-torsion free semiprime ring. The results we obtain are:

**Theorem 2.1.** Let $R$ be a 2-torsion free semiprime ring, $d$ a nonzero derivation of $R$, $n$ a fixed positive integer, $U$ the left Utumi quotient ring of $R$ and $[d([x,y]), [x,y]]^n = 0$, for any $x, y \in R$. Then there exists a central idempotent element $e$ of $U$ such that on the direct sum decomposition $eU \oplus (1-e)U$, $d$ vanishes identically on $eU$ and the ring $(1-e)U$ is commutative.

**Theorem 2.2.** Let $R$ be a 2-torsion free semiprime ring, $d$ a nonzero derivation of $R$, $n$ a fixed positive integer, $U$ the left Utumi quotient ring of $R$ and $[d([x,y]), [x,y]]^n \in Z(R)$, for any $x, y \in R$. Then there exists a central idempotent $e$ of $U$ such that, on the direct sum decomposition $U = eU \oplus (1-e)U$, the derivation $d$ vanishes identically on $eU$ and the ring $(1-e)U$ satisfies $s_4$.

1. The case: $R$ prime ring.

In all that follows, unless stated otherwise, $R$ will be a prime ring of characteristic $\neq 2$, $L$ a Lie ideal of $R$, $d \neq 0$ a derivation of $R$ and $n \geq 1$ a fixed integer such that $[d(x), x]^n \in Z(R)$, for all $x \in L$.

For any ring $S$, $Z(S)$ will denote its center, and $[a, b] = ab - ba$, $[a, b]_2 = [[a, b], b], a, b \in S$. In addition $s_4$ will denote the standard identity in 4 variables.

We will also make frequent use of the following result due to Kharchenko [8] (see also [12]):

Let $R$ be a prime ring, $d$ a nonzero derivation of $R$ and $I$ a nonzero two-sided ideal of $R$. Let $f(x_1, \ldots, x_n, d(x_1, \ldots, x_n))$ a differential identity in $I$, that is

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \quad \forall r_1, \ldots, r_n \in I.$$ 

One of the following holds:

1) Either $d$ is an inner derivation in $Q$, the Martindale quotient ring of $R$, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q) (x) = [q, x]$, for all $x \in R$, and $I$ satisfies the generalized polynomial identity

$$f(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]) = 0;$$

2) or $I$ satisfies the generalized polynomial identity

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.$$

**Lemma 1.1.** Let $R$ be a prime ring of characteristic different from 2, $U$ a noncentral Lie ideal of $R$, $d$ a nonzero derivation of $R$ and $n \geq 1$. If $([d(u), u])^n = 0$, for any $u \in L$, then $R$ is commutative.
Proof. Since we assume that \( \text{char} (R) \neq 2 \), by a result of Herstein [6], \( L \supseteq [I, R] \), for some \( I \neq 0 \), an ideal of \( R \), and also \( L \) is not commutative. Therefore we will assume throughout that \( L \supseteq [I, R] \). Without loss of generality we can assume \( L = [I, I] \).

Hence \( d([x, y]), [x, y]^n = 0 \), for any \( x, y \in I \), then \( I \) satisfies the differential identity
\[
f(x, y, d(x), d(y)) = [[d(x), y] + [x, d(y)], [x, y]]^n = 0.
\]
If the derivation \( d \) is not inner, by Kharchenko’s theorem [8], \( I \) satisfies the polynomial identity
\[
f(x, y, t, z) = [[[z, y] + [x, t], [x, y]]^n = 0
\]
and in particular, for \( z = 0 \),
\[
[[x, t], [x, y]]^n = 0.
\]
Since the latter is a polynomial identity for \( I \), and so for \( R \) too, it is well known that there exists a field \( F \) such that \( R \) and \( F \) satisfy the same polynomial identities (see [7, page 57, page 89]). Let \( e_{ij} \) the matrix unit with 1 in \((i,j)\)-entry and zero elsewhere. Suppose \( m \geq 2 \). If we choose \( x = e_{11}, y = e_{21}, t = e_{12}, \) then we get the contradiction
\[
0 = [[e_{11}, e_{12}], [e_{11}, e_{21}]^n = [e_{12}, -e_{21}]^n = (-1)^n e_{11} + e_{22} \neq 0.
\]
Therefore \( m = 1 \) and so \( R \) is commutative.

Let now \( d \) be an inner derivation induced by an element \( A \in Q \), the Martindale quotient ring of \( R \). Then, for any \( x, y \in I \), \( ([A, [x, y]]_2)^n = 0 \). Since by [2] \( I \) and \( Q \) satisfy the same generalized polynomial identities, we have \( ([A, [x, y]]_2)^n = 0 \), for any \( x, y \in Q \). Moreover, since \( Q \) remains prime by the primeness of \( R \), replacing \( R \) by \( Q \) we may assume that \( A \in R \) and \( C \) is just the center of \( R \). Note that \( R \) is a centrally closed prime \( C \)-algebra in the present situation [4], i.e., \( RC = R \). By Martindale’s theorem in [13], \( RC \) (and so \( R \)) is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space \( V \) over a division ring \( D \). Since \( R \) is primitive then there exist a vector space \( V \) and the division ring \( D \) such that \( R \) is dense of \( D \)-linear transformation over \( V \).

Assume first that \( \text{dim}_D V \geq 3 \).

Step 1.
We want to show that, for any \( v \in V \), \( v \) and \( Av \) are linearly \( D \)-dependent.
Since if \( Av = 0 \) then \( \{v, Av\} \) is \( D \)-dependent, suppose that \( Av \neq 0 \). If \( v \) and \( Av \) are \( D \)-dependent, since \( \text{dim}_D V \geq 3 \), then there exists \( w \in V \) such that \( v, Av, w \) are also linearly independent. By the density of \( I \), there exist \( x, y \in I \) such that
\[
xv = 0, \ Axv = w, \ xw = v
\]
\[
yv = 0, \ yAv = 0, \ yw = w.
\]
These imply that

\[ [A, [x, y]]_2 v = -v \quad \text{and} \quad 0 = ([A, [x, y]]_2)^n v = (-1)^n v, \]

which is a contradiction.

So we can conclude that \( v \) are \( Av \) are linearly D-dependent, for all \( v \in V \).

**Step 2.**

We show here that there exists \( b \in D \) such that \( Av = vb \), for any \( v \in V \).

Now choose \( v, w \in V \) linearly independent. Since \( \dim_D V \geq 3 \), there exists \( u \in V \) such that \( v, w, u \) are linearly independent. By Step 1, there exist \( a_v, a_w, a_u \in D \) such that

\[ Av = va_v, \quad Aw = wa_w, \quad Au = ua_u \quad \text{that is} \quad A(v + w + u) = va_v + wa_w + ua_u. \]

Moreover \( A(v + w + u) = (v + w + u)a_{v+w+u} \), for a suitable \( a_{v+w+u} \in D \). Then \( 0 = v(a_{v+w+u} - a_v) + w(a_{v+w+u} - a_w) + u(a_{v+w+u} - a_u) \) and, because \( v, w, u \) are linearly independent, \( a_u = a_w = a_v = a_{v+w+u} \). This completes the proof of Step 2.

Let now \( r \in R \) and \( v \in V \). By Step 2, \( Av = vb, r(Av) = r(vb) \), and also \( A(rv) = (rv)b \). Thus \( 0 = [A, r]v \), for any \( v \in V \), that is \( [A, r]V = 0 \).

Since \( V \) is a left faithful irreducible R-module, \( [A, r] = 0 \), for all \( r \in R \), i.e., \( A \in Z(R) \) and \( d = 0 \), which contradicts our hypothesis.

Therefore \( \dim_D V \) must be \( \leq 2 \). In this case \( R \) is a simple GPI ring with 1, and so it is a central simple algebra finite dimensional over its center. From Lemma 2 in [10] it follows that there exists a suitable field \( F \) such that \( R \subseteq M_k(F) \), the ring of all \( k \times k \) matrices over \( F \), and moreover \( M_k(F) \) satisfies the same generalized polynomial identity of \( R \).

If we assume \( k \geq 3 \), by the same argument as in Steps 1 and 2, we get a contradiction.

Obviously if \( k = 1 \) then \( R \) is commutative. Thus we may assume \( R \subseteq M_2(F) \), where \( M_2(F) \) satisfies \( ([A, [x, y]]_2)^n = 0 \).

Since for any \( a, b \in M_2(F) \), \( [a, b]^2 \in Z(R) \) then it follows easily that \( ([A, [x, y]]^2 = 0 \), for any \( x, y \in M_2(F) \). Let \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \). If we choose \( x = e_{12}, y = e_{21} \) then we get:

\[
[A, e_{11} - e_{22}]_2 = \begin{bmatrix}
0 & 4a_{12} \\
4a_{21} & 0
\end{bmatrix}
\]

\[ 0 = ([A, e_{11} - e_{22}]_2)^2 = \begin{bmatrix}
16(a_{12}a_{21}) & 0 \\
0 & 16(a_{12}a_{21})
\end{bmatrix}. \]

Therefore either \( a_{12} = 0 \) or \( a_{21} = 0 \). Without loss of generality we can pick \( a_{12} = 0 \).
Now let \([x, y] = [e_{11}, e_{12} + e_{21}] = e_{12} - e_{21}\). In this case we have:

\[\begin{bmatrix}
2(a_{22} - a_{11}) & -2a_{21} \\
-2a_{21} & 2(a_{11} - a_{22})
\end{bmatrix}
\]

that is

\[4(a_{21})^2 = 4(a_{11} - a_{22})^2 = 0\]

On the other hand if \([x, y] = [e_{11}, e_{12} - e_{21}] = e_{12} + e_{21}\) then

\[\begin{bmatrix}
2(a_{11} - a_{22}) & -2a_{21} \\
2a_{21} & 2(a_{22} - a_{11})
\end{bmatrix}
\]

that is

\[4(a_{22} - a_{11})^2 = 4(a_{21})^2 = 0\]

(1) and (2) imply that \(a_{21} = 0\) and \(a_{11} = a_{22}\) which means that \(A\) is a central matrix in \(M_2(F)\), \(A \in F\) and \(d = 0\), a contradiction. Therefore \(k = 1\), i.e., \(R\) is commutative.

\[\square\]

**Lemma 1.2.** Let \(R = M_k(F)\), the ring of \(k \times k\) matrices over a field \(F\) of characteristic \(\neq 2\). If \(q \neq 0\) is a noncentral element of \(R\) such that \(([q, [x, y]]_2)^n \in F\), for any \(x, y \in R\), then \(k \leq 2\).

**Proof.** Suppose \(k \geq 3\). Let \(i, j, r\) be distinct indices and \(q = \sum a_{mn} e_{mn}\), with \(a_{mn} \in F\). For simplicity we assume that \(i = 1, j = 2, r = 3\). If we choose \([x, y] = [e_{12}, e_{23} - e_{31}] = e_{13} + e_{32}\), then

\[([q, [x, y]]_2)_n = a_{21}e_{11} + a_{21}e_{22} - 2a_{21}e_{33} + \sum_{n \neq 1} \gamma_n e_{1n} + \sum_{m \neq 2} \delta_m e_{m2}\]

with \(\gamma_n, \delta_m \in F\), and

\[([q, [x, y]]_2)^n = (a_{21})^n e_{11} + (a_{21})^n e_{22} + (-2a_{21})^n e_{33} + \sum_{n \neq 1} \alpha_n e_{1n} + \sum_{m \neq 2} \beta_m e_{m2}\]

with \(\alpha_n, \beta_m \in F\). Since by assumption \(([q, [x, y]]_2)^n \in F\), then \(a_{21} = \beta_m = 0\), for all \(m, n\), and \((a_{21})^n = (-2a_{21})^n = 0\), i.e., \(a_{21} = 0\). In a similar way we may conclude that \(a_{ij} = 0\), for any \(i \neq j\). Therefore if \(k \geq 3\), \(q\) is a diagonal matrix, \(q = \sum a_{tt} e_{tt}\), with \(a_t \in F\).

If we show that \(q\) is a central matrix, then we get a contradiction to our assumption and so \(k\) must be less or equal than 2.
Let \([x, y] = [e_{ij} - e_{ji}, e_{jj}] = e_{ij} + e_{ji}\). Therefore
\[
[q, [x, y]]_2 = 2(a_{ii} - a_{jj})e_{ii} + 2(a_{jj} - a_{ii})e_{jj}
\]
and
\[
([q, [x, y]]_2)^n = 2^n(a_{ii} - a_{jj})^n e_{ii} + 2^n(a_{jj} - a_{ii})^n e_{jj}.
\]
Since \(([q, [x, y]]_2)^n \in F\) and \(k \geq 3\), it follows that \(a_{ii} = a_{jj}\). Thus \(q\) is a central matrix.

Notice that if \(n = 1\) then by using the same argument and choosing \([x, y] = e_{12}\), we get \(N = [q, [x, y]]_2 = -2e_{12}q e_{12}\), which has rank 1 and so it cannot be central in \(M_k(F)\), with \(k \geq 2\). This implies that if \(n = 1\) then \(k = 1\), and \(R\) must be a commutative field. The proof of Lemma 1.2 is now complete.

\[\Box\]

**Theorem 1.1.** Let \(R\) be a prime ring of characteristic different from 2, \(L\) a noncentral Lie ideal of \(R\), \(d\) a nonzero derivation of \(R\) such that \([d(u), u]^n \in Z(R)\), for any \(u \in L\). Then \(R\) satisfies \(s_4\).

**Proof.** Let \(I\) be the nonzero two-sided ideal of \(R\) such that \(0 \neq [I, R] \subseteq L\) and \(J\) be any nonzero two-sided ideal of \(R\). Then \(V = [I, J]^2 \subseteq L\) is a Lie ideal of \(R\). If, for every \(v \in V\), \([d(v), v]^n = 0\), by Lemma 1.1, \(R\) is commutative. Otherwise, by our assumptions, \(J \cap Z(R) \neq 0\). Let now \(K\) be a nonzero two-sided ideal of \(R_Z\), the ring of the central quotients of \(R\). Since \(K \cap R\) is an ideal of \(R\) then \(K \cap R \cap Z(R) \neq 0\), that is \(K\) contains an invertible element in \(R_Z\), and so \(R_Z\) is simple with 1.

Moreover we may assume \(L = [I, I]\). For any \(x, y \in I\), \([d([x, y]), [x, y]]^n \in Z(R)\), i.e.,
\[
[[d([x, y]), [x, y]]^n, r] = 0 \quad \text{for any } x \in R.
\]
Thus \(I\) satisfies the differential identity
\[
f(x, y, r, d(x), d(y)) = [[[d(x), y] + [x, d(y)], [x, y]]^n, r] = 0.
\]
If the derivation is not inner, by [8], \(I\) satisfies the polynomial identity
\[
f(x, y, r, z, t) = [[[t, y] + [x, z], [x, y]]^n, r] = 0
\]
and in particular, for \(z = 0\),
\[
[[[t, y], [x, y]]^n, r] = 0.
\]
In this case we know that there exists a field \(F\) such that \(R\) and \(F_m\) satisfy the same polynomial identities. Thus \([[t, y], [x, y]]^n\) is central in \(F_m\). Suppose \(m \geq 3\) and choose \(x = e_{32}, y = e_{33}, t = e_{23}\).
\[
[t, y] = e_{23}, \quad [x, y] = -e_{32}
\]
\[
[[t, y], [x, y]] = -e_{22} + e_{33}
\]
\[
[[t, y], [x, y]]^n = (-1)^n e_{22} + e_{33} \notin Z(R)
\]
contrary to our assumptions. This forces \(m \leq 2\), i.e., \(R\) satisfies \(s_4\).
Notice that in the case $n = 1$, $[[t, y], [x, y]]$ must be central in $F_m$. But if $m \geq 2$ and $t = e_{11}, y = e_{12}, x = e_{21}$, we get the contradiction $[[t, y], [x, y]] = 2e_{12} \notin Z(R)$. Therefore $m$ must be equal to 1 and $R$ is commutative.

Now let $d$ be an inner derivation induced by an element $A \in Q$. By localizing $R$ at $Z(R)$ it follows that $([A, [x, y]]_2)^n \in Z(R_Z)$, for all $x, y \in R_Z$.

Since $R$ and $R_Z$ satisfy the same polynomial identities, in order to prove that $R$ satisfies $S_4(x_1, x_2, x_3, x_4)$, we may assume that $R$ is simple with 1 and $[R, R] \subseteq L$.

In this case, $([A, [x, y]]_2)^n \in Z(R)$, for all $x, y \in R$. Therefore $R$ satisfies a generalized polynomial identity and it is simple with 1, which implies that $Q = RC = R$ and $R$ has a minimal right ideal. Thus $A \in R = Q$ and $R$ is simple artinian that is $R = D_k$, where $D$ is a division ring finite dimensional over $Z(R)$ [13]. From Lemma 2 in [10] it follows that there exists a suitable field $F$ such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over $F$, and moreover $M_k(F)$ satisfies the generalized polynomial identity $([A, [x, y]]_2)^n, z] = 0$. By Lemma 1.2, if $n \geq 2$ then $k \leq 2$ and $R$ satisfies $s_4$, also if $n = 1$ then $k = 1$ and $R$ must be commutative. \[\square\]

2. The case: $R$ semiprime ring.

In all that follows $R$ will be a 2-torsion free semiprime ring. We cannot expect the same conclusion of previous section to hold, as the following example shows:

**Example 2.** Let $R_1$ be any prime ring not satisfying $s_4$ and $R_2 = M_2(F)$, the ring of $2 \times 2$ matrices over the field $F$. Let $R = R_1 \oplus R_2$, $d$ a nonzero derivation of $R$ such that $d = 0$ in $R_1$. Consider $L = [R, R]$. It is a non-central Lie ideal of $R$. Let $r_1, s_1 \in R_1, r_2, s_2 \in R_2, u = [(r_1, r_2), (s_1, s_2)]$. Therefore $d(u) = (0, d([r_2, s_2]))$ and $[d(u), u] = (0, [d([r_2, s_2]), [r_2, s_2]])$. Since $[d([r_2, s_2]), [r_2, s_2]]^2 \in Z(R_2)$, then

$$[d(u), u]^2 = 0, [d([r_2, s_2]), [r_2, s_2]]^2 = 0, [d([r_2, s_2]), [r_2, s_2]]^2 \in Z(R)$$

but $R$ does not satisfy $s_4$.

The related object we need to mention is the left Utumi quotient ring $U$ of $R$. For basic definitions and preliminary results we refer the reader to [1], [5], [9].

In order to prove the main result of this section we will make use of the following facts:

**Claim 1** ([1, Proposition 2.5.1]). Any derivation of a semiprime ring $R$ can be uniquely extended to a derivation of its left Utumi quotient ring $U$, and so any derivation of $R$ can be defined on the whole $U$. 
Claim 2 ([3, p. 38]). If $R$ is semiprime then so is its left Utumi quotient ring. The extended centroid $C$ of a semiprime ring coincides with the center of its left Utumi quotient ring.

Claim 3 ([3, p. 42]). Let $B$ be the set of all the idempotents in $C$, the extended centroid of $R$. Assume $R$ is a $B$-algebra orthogonal complete. For any maximal ideal $P$ of $B$, $PR$ forms a minimal prime ideal of $R$, which is invariant under any derivation of $R$.

We will prove the following:

**Theorem 2.1.** Let $R$ be a 2-torsion free semiprime ring, $d$ a nonzero derivation of $R$, $n$ a fixed positive integer, $U$ the left Utumi quotient ring of $R$ and $[d([x, y]), [x, y]]^n = 0$, for any $x, y \in R$. Then there exists a central idempotent element $e$ of $U$ such that on the direct sum decomposition $eU \oplus (1-e)U$, $d$ vanishes identically on $eU$ and the ring $(1-e)U$ is commutative.

**Proof.** Since $R$ is semiprime, by Claim 2, $Z(U) = C$, the extended centroid of $R$, and, by Claim 1, the derivation $d$ can be uniquely extended on $U$. Since $U$ and $R$ satisfy the same differential identities (see [12]), then $[d([x, y]), [x, y]]^n = 0$, for all $x, y \in U$. Let $B$ be the complete boolean algebra of idempotents in $C$ and $M$ be any maximal ideal of $B$.

Since $U$ is a $B$-algebra orthogonal complete (see [3, p. 42, (2) of Fact 1]), by Claim 3, $MU$ is a prime ideal of $U$, which is $d$-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by $d$ on $\overline{U}$. For any $\overline{x}, \overline{y} \in \overline{U}$, $[\overline{d}(\overline{[x, y]}), \overline{[x, y]}]^n = 0$. In particular $\overline{U}$ is a prime ring and so, by Lemma 1.1, $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ is commutative. This implies that, for any maximal ideal $M$ of $B$, $d(U) \subseteq MU$ or $[U, U] \subseteq MU$. In any case $d(U)[U, U] \subseteq MU$, for all $M$. Therefore $d(U)[U, U] \subseteq \bigcap_M MU = 0$.

By using the theory of orthogonal completion for semiprime rings (see [1, Chapter 3]), it follows that there exists a central idempotent element $e$ in $U$ such that on the direct sum decomposition $eU \oplus (1-e)U$, $d$ vanishes identically on $eU$ and the ring $(1-e)U$ is commutative. $\square$

We come now to our last result:

**Theorem 2.2.** Let $R$ be a 2-torsion free semiprime ring, $d$ a nonzero derivation of $R$, $n$ a fixed positive integer, $U$ the left Utumi quotient ring of $R$ and $[d([x, y]), [x, y]]^n \in Z(R)$, for any $x, y \in R$. Then there exists a central idempotent $e$ of $U$ such that, on the direct sum decomposition $U = eU \oplus (1-e)U$, the derivation $d$ vanishes identically on $eU$ and the ring $(1-e)U$ satisfies $s_4$.  


Proof. By Claim 2, $Z(U) = C$, and by Claim 1 $d$ can be uniquely defined on the whole $U$. Since $U$ and $R$ satisfy the same differential identities, then $[d([x, y]), [x, y]]^n \in C$, for all $x, y \in U$. Let $B$ be the complete boolean algebra of idempotents in $C$ and $M$ any maximal ideal of $B$. As already pointed out in the proof of Theorem 2.1, $U$ is a B-algebra orthogonal complete and by Claim 3, $MU$ is a prime ideal of $U$, which is $d$-invariant. Let $\overline{d}$ the derivation induced by $d$ on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C + MU)/MU = C/MU$, then $[\overline{d}(x, y), [x, y]]^n \in (C + MU)/MU$, for any $x, y \in \overline{U}$. Moreover $\overline{U}$ is a prime ring, hence we may conclude, by Theorem 1.1, that $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ satisfies $s_4$. This implies that, for any maximal ideal $M$ of $B$, $d(U) \subseteq MU$ or $s_4(x_1, x_2, x_3, x_4) \subseteq MU$, for all $x_1, x_2, x_3, x_4 \in U$. In any case $d(U)s_4(x_1, x_2, x_3, x_4) \subseteq \bigcap_M MU = 0$. From [1, Chapter 3], there exists a central idempotent element $e$ of $U$, the left Utumi quotient ring of $R$, such that there exists a central idempotent $e$ of $U$ such that $d(eU) = 0$ and $(1 - e)U$ satisfies $s_4$. □

References


Received August 19, 1998 and revised November 20, 1998.

Dipartimento di Matematica ed Applicazioni
Università di Palermo
90123 Palermo
Italy
E-mail address: lcarini@dipmat.unime.it

Dipartimento di Matematica
Università di Messina
98166 Messina
Italy
E-mail address: enzo@dipmat.unime.it