A GENERALIZATION OF CURVE GENUS FOR AMPLIFICATION VECTOR BUNDLES, II

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Let $X$ be a compact complex manifold of dimension $n \geq 2$ and $E$ an ample vector bundle of rank $r < n$ on $X$. As the continuation of Part I, we further study the properties of $g(X, E)$ that is an invariant for pairs $(X, E)$ and is equal to curve genus when $r = n - 1$. Main results are the classifications of $(X, E)$ with $g(X, E) = 2$ (resp. $3$) when $E$ has a regular section (resp. $E$ is ample and spanned) and $1 < r < n - 1$.

Introduction.

The present paper is a continuation of [I]. For a pair $(X, E)$ which consists of a compact complex manifold $X$ of dimension $n \geq 2$ and an ample vector bundle $E$ of rank $r < n$ on $X$, we defined in [I] an invariant $g(X, E)$ by the formula

$$2g(X, E) - 2 := (K_X + (n - r)c_1(E))c_1(E)^{n-r-1}c_r(E).$$

We note that $g(X, E)$ is a nonnegative integer, and $g(X, E)$ is equal to the curve genus of $(X, E)$ when $r = n - 1$. As in the case of curve genus, above $(X, E)$ with $g(X, E) \leq 1$ have been classified in [I]; moreover, it is shown that $g(X, E) \geq q(X)$ for spanned $E$ and its equality condition is given in [I]. ($q(X)$ is the irregularity of $X$.)

After we recall some preliminary results in Section 1, we consider the cases $g(X, E) = 2$ and $g(X, E) = 3$ when $1 < r < n - 1$ in Section 2. Corresponding results for $c_1$-sectional genus are given in [Fj2] and [BiLL] respectively. In Section 3 we consider the cases $g(X, E) = q(X) + 1$ and $g(X, E) = q(X) + 2$ when $1 < r < n - 1$. Related results for $c_1$-sectional genus are given in [R]. In Section 4 we give another relation between $g(X, E)$ and $q(X)$, namely $g(X, E) \geq 2q(X) - 1$ for $1 < r < n - 1$. When $r = 1$, this inequality is satisfied except one case. In Section 5 we show that $g(X, E) \geq g(C)$ when there exists a fibration $f : X \to C$ over a curve. We also give its equality condition. Finally in Appendix we give a classification of $(X, L)$ with $g(X, L) = q(X) + 2$ and $n = 2$ for ample and spanned line bundles $L$ on $X$. 
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1. Preliminaries.

We use a notation similar to that in [I]. For example, we denote by $H(E)$ the tautological line bundle on $\mathbb{P}_X(E)$, the projective space bundle associated to a vector bundle $E$ on a variety $X$. We say that a vector bundle $E$ is spanned if $H(E)$ is spanned. A polarized manifold $(X,L)$ is said to be a scroll over a variety $W$ if $(X,L) \cong (\mathbb{P}_W(F),H(F))$ for some ample vector bundle $F$ on $W$. We denote by $F_e$ the Hirzebruch surfaces $\mathbb{P}\mathbb{P}_1(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-e)) \ (e > 0)$, by $\sigma$ a minimal section, and by $f$ a fiber of the ruling $F_e \to \mathbb{P}_1$. Numerical equivalence is denoted by $\equiv$.

Definition 1.1. Let $X$ be a compact complex manifold of dimension $n \geq 2$ and $E$ an ample vector bundle of rank $r < n$ on $X$. We define a rational number $g(X,E)$ for the pair $(X,E)$ by the formula

$$2g(X,E) - 2 := (K_X + (n-r)c_1(E))c_1(E)^{n-r-1}c_r(E).$$

It turns out that $g(X,E)$ is a nonnegative integer (see [I]). When $r = 1$ (resp. $r = n-1$), $g(X,E)$ is nothing but the sectional genus (resp. curve genus) of $(X,E)$.

Remark 1.2. Let $(X,E)$ be as above. Suppose that $(X,E)$ satisfies the condition

(*) There exists a section $s \in H^0(X,E)$ whose zero locus $Z := (s)_0$ is a smooth submanifold of $X$ of the expected dimension $n-r$.

Then we have $g(X,E) = g(Z,\det E_Z)$ (see [I]). If $E$ is spanned, then $E$ satisfies (*) by Bertini’s theorem.

The following facts are used in the subsequent sections.

Proposition 1.3. Let $X$ be an $n$-dimensional compact complex manifold and $E$ an ample vector bundle of rank $r < n$ on $X$ with the property $(\ast)$ in (1.2). Let $\iota : Z \hookrightarrow X$ be the embedding. Then

1. $H^i(\iota) : H^i(X,Z) \to H^i(Z,Z)$ is an isomorphism for $i < n-r$.
2. $H^i(\iota)$ is injective and its cokernel is torsion free for $i = n-r$.
3. Pic($\iota$) : Pic($X$) $\to$ Pic($Z$) is an isomorphism for $n-r > 2$.
4. Pic($\iota$) is injective and its cokernel is torsion free for $n-r = 2$.

Proof. See Theorem 1.3 in [LM1] and see also Theorem 1.1 in [LM2].

Proposition 1.4. Let $X$ be an $n$-dimensional compact complex manifold and $E$ an ample vector bundle of rank $r \geq 2$ on $X$ with the property $(\ast)$. If $Z \cong \mathbb{P}^{n-r}(n-r \geq 1)$, then $(X,E)$ is one of the following:
(P1) \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})\);
(P2) \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (n-2)})\);
(P3) \((\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus (n-1)})\);
(P4) \(X \simeq \mathbb{P}^1(\mathcal{F})\) for some vector bundle \(\mathcal{F}\) of rank \(n\) on \(\mathbb{P}^1\) and \(\mathcal{E} = \bigoplus_{j=1}^{n-1}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))\), where \(\pi : X \to \mathbb{P}^1\) is the bundle projection.

If \(Z \simeq \mathbb{Q}^n(r - n \geq 2)\), then \((X, \mathcal{E})\) is one of the following:

(Q1) \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (r-1)})\);
(Q2) \((\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus r})\);
(Q3) \(X \simeq \mathbb{P}^1(\mathcal{F})\) and \(\mathcal{E} = \bigoplus_{j=1}^{n-2}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))\), where \(\mathcal{F}\) is the same as that in (P4).

Proof. See Theorem A and Theorem B in [LM1].

\(\square\)

**Proposition 1.5.** Let \(X\) be a complex projective manifold of dimension \(n\) and let \(\mathcal{E}\) be an ample vector bundle of rank \(n - 2 \geq 2\) on \(X\) satisfying \((\ast)\).

1. If \(Z\) is a geometrically ruled surface over a smooth curve \(B\) such that \(Z \neq \mathbb{F}_0, \mathbb{F}_1\), then \(X\) is a \(\mathbb{P}^{n-1}\)-bundle over \(B\) and \(\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus (n-2)}\) for every fiber \(F\) of the bundle map \(X \to B\).
2. If \(Z = \mathbb{F}_0\), then \((X, \mathcal{E})\) is either the type in (1) with \(B = \mathbb{P}^1\) or \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (n-3)})\) or \((\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus (n-2)})\).
3. If \(Z = \mathbb{F}_1\), then \((X, \mathcal{E})\) is either the type in (1) with \(B = \mathbb{P}^1\) or possibly \(X \simeq \mathbb{P}^2(\mathcal{F})\) for some ample vector bundle \(\mathcal{F}\) on \(\mathbb{P}^2\) with \(c_1(\mathcal{F}) = k(n-2) + 3\) for some positive integer \(k\) and \(\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus (n-2)}\) for every fiber \(F\) of the bundle map \(X \to \mathbb{P}^2\).

Proof. See [LM3].

\(\square\)

**Proposition 1.6.** Let \(X\) be a complex projective manifold of dimension \(n\) and let \(\mathcal{E}\) be an ample vector bundle of rank \(r \geq 2\) on \(X\). If \(g(X, \det \mathcal{E}) = 2\), then \(n = 2\) and \((X, \mathcal{E})\) is one of the following:

1. \(X\) is the Jacobian variety of a smooth curve \(B\) of genus \(2\) and \(\mathcal{E} \simeq \mathcal{E}_r(B, o) \oplus \mathcal{N}\) for some \(N \in \text{Pic} \ X\) with \(N \equiv 0\), where \(\mathcal{E}_r(B, o)\) is the Jacobian bundle for some point \(o\) on \(B\);
2. \(X \simeq \mathbb{P}^1(\mathcal{F})\) for some stable vector bundle \(\mathcal{F}\) of rank \(2\) on an elliptic curve \(B\) with \(c_1(\mathcal{F}) = 1\). There is an exact sequence

\[0 \to \mathcal{O}_X[2H(\mathcal{F}) + \rho^*G] \to \mathcal{E} \to \mathcal{O}_X[H(\mathcal{F}) + \rho^*T] \to 0,\]

where \(G, T \in \text{Pic} \ B\) and \(\rho\) is the projection \(X \to B\). We have \((\deg G, \deg T) = (-2, 1)\) or \((-1, 0)\);
3. \((2^2)\) \(X, \mathcal{F}, B\) and \(\rho\) are as in (2) and \(\mathcal{E} \simeq \rho^* \mathcal{G} \otimes H(\mathcal{F})\) for some stable vector bundle \(\mathcal{G}\) of rank \(3\) on \(B\) with \(c_1(\mathcal{G}) = -1\);
3. \((3)\) \(X \simeq \mathbb{P}^1(\mathcal{F})\) and \(\mathcal{E} \simeq \rho^* \mathcal{G} \otimes H(\mathcal{F})\) for some semistable vector bundles \(\mathcal{F}\) and \(\mathcal{G}\) of rank \(2\) on an elliptic curve \(B\) with \((c_1(\mathcal{F}), c_1(\mathcal{G})) = (1, 0)\) or \((0, 1)\);
Proof. See (1.10) Theorem in [Fj2]. □

Proposition 1.7. Let \( X \) be a complex projective manifold of dimension \( n \) and let \( E \) be an ample and spanned vector bundle of rank \( r \geq 2 \) on \( X \). If \( g(X, \det E) = 3 \), then \( n = 2 \) and \( (X, E) \) is one of the following:

1a) \( X = \mathbb{P}^2 \), \( E = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4} \);
1b) \( X = \mathbb{P}^2 \), and either \( E = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(2) \) or \( E = \mathcal{T}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \);
1c) \( X = \mathbb{P}^2 \), rank \( E = 2 \) and \( \det E = \mathcal{O}_{\mathbb{P}^2}(4) \);
2a) \( X = \mathbb{F}_0 \), and either \( E = [\sigma + f] \oplus [\sigma + 3f] \) or \( E = [\sigma + 2f]^{\oplus 2} \);
2b) \( X = \mathbb{F}_1 \), \( E = [\sigma + 2f] \oplus [\sigma + 3f] \);
2c) \( X = \mathbb{F}_2 \), \( E = [\sigma + 3f]^{\oplus 2} \);
3) \( X \) is a Del Pezzo surface with \( K_X^2 = 2 \) and either \( E = [-K_X]^{\oplus 2} \), or \( E = \psi^*(\mathcal{Q}_Y) \), where \( \psi \) is a birational morphism from \( X \) to a surface \( Y \) of bidegree \( (4,4) \) in the Grassmannian of lines of \( \mathbb{P}^3 \), and \( \mathcal{Q} \) is the universal rank 2 quotient bundle;
4) \( X = \mathbb{P}(\mathcal{F}) \), where \( \mathcal{F} \) is a rank 2 vector bundle on an elliptic curve \( B \) with \( c_1(\mathcal{F}) = 1 \) and \( E = H(\mathcal{F}) \otimes \rho^*\mathcal{G} \), where \( \rho : X \to B \) is the bundle projection and \( \mathcal{G} \) is any rank 2 vector bundle on \( B \) defined by a nonsplitting exact sequence \( 0 \to \mathcal{O}_B \to \mathcal{G} \to \mathcal{O}_B(x) \to 0 \), where \( x \in B \).

Proof. See (1.10) Theorem in [BiLL]. □

2. The cases \( g(X, E) = 2 \) and \( g(X, E) = 3 \).

Theorem 2.1. Let \( X \) be a compact complex manifold of dimension \( n \) and \( E \) an ample vector bundle of rank \( r \) on \( X \) with \( 1 < r < n - 1 \) and the property (*) in (1.2). If \( g(X, E) = 2 \), then \( (X, E) \) is one of the following:

i) There exists an ample line bundle \( A \) on \( X \) such that \( (X, A) \) is a Del Pezzo 4-fold of degree 1 and \( E = A^{\oplus 2} \) (see also (2.2.1));
ii) \( X \cong \mathbb{P}_B(\mathcal{F}) \) and \( E = H(\mathcal{F}) \otimes \pi^*\mathcal{G} \), where \( \mathcal{F} \) and \( \mathcal{G} \) are vector bundles on an elliptic curve \( B \) such that rank \( \mathcal{F} = 4 \), rank \( \mathcal{G} = 2 \), \( c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 1 \), and \( \pi : X \to B \) is the bundle projection;
iii) \( X \cong \mathbb{P}_B(\mathcal{F}) \) and \( E = H(\mathcal{F}) \otimes \pi^*\mathcal{G} \), where \( \mathcal{F} \) and \( \mathcal{G} \) are vector bundles on an elliptic curve \( B \) such that rank \( \mathcal{F} = 5 \), rank \( \mathcal{G} = 3 \), \( 3c_1(\mathcal{F}) \) and \( 5c_1(\mathcal{G}) = 1 \), and \( \pi : X \to B \) is the bundle projection.

Proof. Suppose that \( g(X, E) = 2 \). Since \( E \) satisfies (*), there exists a nonzero section \( s \in H^0(X, E) \) whose zero locus \( Z := \{s\}_0 \) is a smooth submanifold of \( X \) of dimension \( n - r \) and \( 2 = g(X, E) = g(Z, \det E|_Z) \). From (1.6) we see
that $n - r = 2$ and $(Z, E_Z)$ is one of the cases in (1.6). We make a case by case analysis in the following.

(2.1.1) If $(Z, E_Z)$ is in case (1.6;1), then $K_Z = O_Z$. We have $K_X + \det E = O_X$ since $[K_X + \det E]_Z = K_Z$ and Pic$(a) : \text{Pic}(X) \to \text{Pic}(Z)$ is injective by (1.3). We get also that $H^1(i) : H^1(X, Z) \to H^1(Z, Z)$ is an isomorphism by (1.3), but this is impossible since $X$ is a Fano manifold and $Z$ is an abelian surface.

(2.1.2) If $(Z, E_Z)$ is in case (1.6;5), we have $r = 2$ and $n = 4$. By (1.4), $(X, E)$ is one of the cases (Q1),(Q2) and (Q3). We easily see that $g(X, E) \neq 2$ in cases (Q1) and (Q2). In case (Q3), we can write $F$ is ample, $E$ a $X$, $a \neq 0$ with $Z = \sum \alpha_1 a_i$. Hence we have $B$ with $Z = a_iH$, $H = \pi \alpha_1 a_i$. From these two equalities we get (2.1.4) If $(Z, E_Z)$ is in case (1.6;4), then $r = 2$ and $n = 4$. We have $2K_X + 3 \det E = O_X$ since, by adjunction, $[2K_X + 3 \det E]_Z = 2K_Z + \det E_Z = O_Z$ and the restriction map Pic$(X) \to \text{Pic}(Z)$ is injective. By setting $A := K_X + 2 \det E$, we get det $E = 2A$ and $K_X + 3A = O_X$, hence $(X, A)$ is a
Del Pezzo 4-fold. Then we set $\mathcal{E} := \mathcal{E} \oplus A$; we get $K_X + \det \mathcal{E}' = \mathcal{O}_X$ and $\mathcal{E}' \simeq A^{\oplus 3}$ by using Proposition 7.4 in [PSW]. It follows that $\mathcal{E} \simeq A^{\oplus 2}$ and

$$2 = 2g(X, \mathcal{E}) - 2 = (K_X + 2c_1(\mathcal{E}))[c_1(\mathcal{E})c_2(\mathcal{E})] = 2A^4,$$

hence $A^4 = 1$. Thus we obtain that $(X, \mathcal{E})$ is the case (i) of our theorem.

(2.1.5) If $(Z, \mathcal{E}_Z)$ is in case (1.6; 2), then $r = 2$ and $n = 4$. Since $Z$ is a geometrically ruled surface over an elliptic curve $B$, by (1.5), $X$ is a $\mathbb{P}^3$-bundle over $B$ and $\mathcal{E}|_F = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ for every fiber of the ruling $\pi : X \to B$. On the other hand, we have $\mathcal{E}_Z|_F = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ for every fiber $F$ of the ruling $\rho : Z \to B$. This is a contradiction since $\pi|_Z = \rho$. If $(Z, \mathcal{E}_Z)$ is in case (1.6; 2) or (1.6; 3), by using (1.5), we obtain that $(X, \mathcal{E})$ is the case (ii) or (iii) of our theorem respectively. This completes the proof.

\textbf{Remark 2.2.} We make some comments on (2.1).

(2.2.1) In case (2.1; i), Del Pezzo 4-folds of degree 1 have been classified in [Fj1], Part III. In particular, they are weighted hypersurfaces of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, 1, 1, 1)$.

(2.2.2) We give an example of $(X, \mathcal{E})$ in case (2.1; ii) in the following. Let $L_1$ and $L_2$ be line bundles on an elliptic curve $B$ such that $\deg L_1 = \deg L_2$ and $L_1 \neq L_2$ in $\text{Pic} B$. Let $\mathcal{F}$ be an indecomposable vector bundle of rank 4 on $B$ with $c_1(\mathcal{F}) = 1 - 2 \deg L_1 - 2 \deg L_2$. We set $X := \mathbb{P}_B(\mathcal{F})$, $\mathcal{G} := L_1 \oplus L_2$, and $\mathcal{E} := H(\mathcal{F}) \otimes \pi^* \mathcal{G} = \oplus_{i=0}^{2}[H(\mathcal{F}) + \pi^* L_i]$, where $\pi : X \to B$ is the bundle projection. Since $c_1(\mathcal{F} \otimes L_i) = 1$, $\mathcal{F} \otimes L_i$ is ample and $h^0(B, \mathcal{F} \otimes L_i) = 1$. Then there exists an exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{F} \otimes L_i \to Q_i \to 0,$$

where $Q_i$ is a locally free sheaf of rank 3 on $B$. Since $Q_i$ is ample and $c_1(Q_i) = 1$, we see that $Q_i$ is indecomposable. We set $D_i := \mathbb{P}_B(Q_i)$ and $Z := D_1 \cap D_2$. Since $c_1(Q_2 \otimes [L_1 - L_2]) = 1$, there exists an exact sequence

$$0 \to \mathcal{O}_B \to Q_2 \otimes [L_1 - L_2] \to Q \to 0,$$

where $Q$ is a locally free sheaf of rank 2 on $B$. Then we have $Z = \mathbb{P}_B(Q)$ in $[H(Q_2) + (\pi|_{D_2})^*(L_1 - L_2)]$. Thus we see that $(X, \mathcal{E})$ satisfies the condition (*) and $(X, \mathcal{E})$ is an example of (2.1; ii).

(2.2.3) The authors have no example for case (2.1; iii). We note that without the condition (*) we have examples for the case. Indeed, we can take semistable vector bundles $\mathcal{F}$ and $\mathcal{G}$ on an elliptic curve $B$ with the property that rank $\mathcal{F} = 5$, rank $\mathcal{G} = 3$, and $3c_1(\mathcal{F}) + 5c_1(\mathcal{G}) = 1$. Let $\pi : \mathbb{P}(\mathcal{F}) \to B$ and $\pi' : \mathbb{P}(\mathcal{G}) \to B$ be the bundle projections. Then $5H(\mathcal{F}) - \pi^* \det \mathcal{F}$ is nef on $\mathbb{P}(\mathcal{F})$ and $3H(\mathcal{G}) - (\pi')^* \det \mathcal{G}$ is nef on $\mathbb{P}(\mathcal{G})$ by Theorem 3.1 in [Mi]. We set $\mathcal{E} := H(\mathcal{F}) \otimes \pi^* \mathcal{G}$ and let $p : \mathbb{P}(\mathcal{E}) \to B$ be the composition of the projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{F})$ and $\pi$. Then $15H(\mathcal{E}) - F$ is nef on $\mathbb{P}(\mathcal{E})$ for a fiber $F$ of $p$, hence $\mathcal{E}$ is ample. But it is uncertain that such $\mathcal{E}$ satisfies (*).
(2.2.4) We see that every vector bundle \( E \) appeared in (2.1) is not spanned. Indeed, it is clear for case (2.1; i). For cases (2.1; ii) and (2.1; iii), we use the following:

**Lemma 2.2.5.** Let \( F \) be a vector bundle of rank \( r \) on an elliptic curve. Then there exists a line sub-bundle \( L \) of \( F \) such that \( \deg L \geq \lceil c_1(F)/r \rceil \), where \( \lceil c_1(F)/r \rceil \) is the largest integer that is not greater than \( c_1(F)/r \).

This is a consequence of the Mukai-Sakai theorem [MuS], hence proof is omitted.

Suppose that \( E \) is spanned in case (2.1; ii). Applying the lemma to \( F^\vee \) and \( G^\vee \), we get quotient line bundles \( L_1 \) and \( L_2 \) of \( F \) and \( G \) respectively, with the property that \( \deg L_1 \leq -\lceil c_1(F)/4 \rceil \) and \( \deg L_2 \leq -\lceil c_1(G)/2 \rceil \). The surjection \( F \to L_1 \) gives a section \( C := \mathbb{P}(L_1) \) of the projection \( \pi : \mathbb{P}_B(F) \to B \). Since \( H(F)|_C = (\pi|_C)^*L_1 \), we see that \((\pi|_C)^*(L_1 \otimes L_2)\) is a quotient line bundle of \( E_C \), hence \( L_1 \otimes L_2 \) is ample and spanned. From \( c_1(F) + 2c_1(G) = 1 \) we get \( \deg L_1 + \deg L_2 \leq \left( \left\lceil 2c_1(G) - 1 \right\rceil / 4 \right) - \lceil c_1(G)/2 \rceil - 1 \); this leads to a contradiction since \( B \) is an elliptic curve. Similarly we can show that \( E \) is not spanned in case (2.1; iii).

**Theorem 2.3.** Let \( X \) be a compact complex manifold of dimension \( n \) and \( E \) an ample and spanned vector bundle of rank \( r \) on \( X \) with \( 1 < r < n-1 \). If \( g(X,E) = 3 \), then \((X,E)\) is one of the following:

(i) \( (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)^{\oplus 4}) \);
(ii) \( (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,1)^{\oplus 2}) \);
(iii) There exists a double covering \( f : X \to \mathbb{P}^4 \) with branch locus \( B \in |\mathcal{O}_{\mathbb{P}^4}(1)| \) and \( E = f^*\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2} \).

**Proof.** Suppose that \( g(X,E) = 3 \). We argue as in the proof of (2.1). Since \( E \) is spanned, there exists a nonzero section \( s \in H^0(X,E) \) whose zero locus \( Z := \{s\}_0 \) is a smooth submanifold of \( X \) of dimension \( n-r \) and \( 3 = g(X,E) = g(Z, \det E_Z) \). From (1.7) we see that \( n-r = 2 \) and \((Z,E_Z)\) is one of the cases in (1.7).

(2.3.1) If \((Z,E_Z)\) is in case (1a), (1b), or (1c) of (1.7), then \( Z = \mathbb{P}^2 \) and \((X,E)\) is the case (P1) of (1.4) since \( n-r = 2 \). We obtain that \((X,E)\) is the case (i) of our theorem by \( g(X,E) = 3 \).

(2.3.2) If \((Z,E_Z)\) is in case (3) of (1.7), then \( r = 2 \) and \( n = 4 \). By setting \( A := K_X + 2\det E \), we infer that \((X,A)\) is a Del Pezzo manifold and \( E = A^{\oplus 2} \) from the same argument as that in (2.1.4). Then we find that \( A^4 = 2 \) since \( g(X,E) = 3 \). Hence we obtain that \((X,E)\) is the case (iii) of our theorem by [Fj1], Part I.

(2.3.3) If \((Z,E_Z)\) is in case (2a), (2b), (2c), or (4) of (1.7), then \( r = 2 \) and \( n = 4 \). Since \( Z \) is a geometrically ruled surface, by (1.5), \((X,E)\) is one of the following:
(R1) \((\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2))\);

(R2) \((\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})\);

(R3) \(X\) is a \(\mathbb{P}^3\)-bundle over a smooth curve \(B\) and \(\mathcal{E}_\tilde{F} = \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}\) for every fiber \(\tilde{F}\) of the bundle map \(\pi : X \to B\);

(R4) \(Z = \mathbb{F}_1, X \cong \mathbb{P}^2(\mathcal{F})\) for some ample vector bundle \(\mathcal{F}\) on \(\mathbb{P}^2\) with \(c_1(\mathcal{F}) = 2k + 3\) \((k > 0)\), and \(\mathcal{E}_\tilde{F} = \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}\) for every fiber \(\tilde{F}\) of the bundle map \(\pi : X \to \mathbb{P}^2\).

Cases (R1) and (R2) are ruled out by \(g(X, \mathcal{E}) = 3\). Case (R4) comes from (2b) of (1.7), hence \(E\) is a blowing-up \(\mathbb{F}_1 \to \mathbb{P}^2\) and \(E_Z = [\sigma + 2f] \oplus [\sigma + 3f]\). We can write \(E = H(\mathcal{F}) \otimes \pi^*\mathcal{G}\) for some vector bundle \(\mathcal{G}\) of rank 2 on \(\mathbb{P}^2\) and \(H(\mathcal{F})_Z = a\sigma + bf\) for some \(a, b \in \mathbb{Z}\). Then

\[
2\sigma + 5f = \det E_Z = 2H(\mathcal{F})_Z + (\pi_\ast E)^* E_G = (2a + c_1(\mathcal{G}))\sigma + (2b + c_1(\mathcal{G}))f,
\]

hence \(2a - 2b = -3\), a contradiction. In case (R3), we have \(X \cong \mathbb{P}_B(\mathcal{F})\) and \(E = H(\mathcal{F}) \otimes \pi^*\mathcal{G}\) for some vector bundles \(\mathcal{F}\) and \(\mathcal{G}\) on \(B\) such that rank \(\mathcal{F} = 4\) and rank \(\mathcal{G} = 2\). Then

\[
4 = 2g(X, \mathcal{E}) - 2 = (K_X + 2c_1(\mathcal{E}))c_1(\mathcal{E})c_2(\mathcal{E}) = 2(2g(B) - 2 + c_1(\mathcal{F}) + 2c_1(\mathcal{G})),
\]

where \(g(B)\) is the genus of \(B\). Since \(E\) is ample, we find that \(c_1(\mathcal{F}) + 2c_1(\mathcal{G}) > 0\) from \((\det E)^4 > 0\). It follows that \(g(B) \leq 1\). In case \(g(B) = 0\), we have \(B \cong \mathbb{P}^1\) and \(c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 4\). Then we can write \(\mathcal{F} = \sum_{i=1}^4 \mathcal{O}(a_i)\) and \(\mathcal{G} = \sum_{j=1}^2 \mathcal{O}(b_j)\). By the same argument as that in (2.1.2), we infer that \(a_i + b_j = 1\) for every \(i\) and \(j\). It follows that \(a_1 = \cdots = a_4\) and \(b_1 = b_2\), hence \(\mathbb{P}_B(\mathcal{F}) \cong \mathbb{P}^4 \times \mathbb{P}^3\) and \(E = \mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}^3}(1,1)^{\oplus 2}\), which is the case (ii) of our theorem. In case \(g(B) = 1\), we have \(c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 2\). Then we get a contradiction by the same argument as that in (2.2.4). We have thus completed the proof.

\[\square\]

### 3. The cases \(g(X, \mathcal{E}) = g(X) + 1\) and \(g(X, \mathcal{E}) = g(X) + 2\).

**Theorem 3.1.** Let \(X\) be a compact complex manifold of dimension \(n\) and let \(\mathcal{E}\) be an ample and spanned vector bundle of rank \(r\) with \(1 < r < n - 1\). Then \(g(X, \mathcal{E}) = g(X) + 1\) if and only if \((X, \mathcal{E})\) is one of the following:

1. \((\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2})\);
2. \((\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 3})\);
3. \((\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})\).

**Proof.** First we note that if \((X, \mathcal{E})\) is one of the cases (1),(2) and (3) of our theorem, then we easily see that \(g(X, \mathcal{E}) = 1 = g(X) + 1\). Suppose that \(g(X, \mathcal{E}) = g(X) + 1\) on the contrary. Let \(Z\) be a smooth submanifold of \(X\) with \(\dim Z = n - r\) defined as the zero locus of some \(s \in H^0(X, \mathcal{E})\).
Then \( g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z) \). We put \( A := \det \mathcal{E}_Z \); then \( A \) is ample and spanned. If \( n - r \geq 3 \), we take general members \( D_1, \ldots, D_{n-r-2} \in |A| \) with the property that \( S := D_1 \cap \cdots \cap D_{n-r-2} \) is a smooth surface. If \( n - r = 2 \), we set \( S = Z \). Hence there exists a polarized surface \((S, A_S)\) such that \( g(Z, A) = g(S, A_S) \). We get \( q(X) = q(Z) = q(S) \) by using (1.3). Thus we get \( g(S, A_S) = q(S) + 1 \).

We show that \( h^0(K_S) = 0 \). Indeed, it is obvious if \( \kappa(S) = -\infty \), where \( \kappa(S) \) is the Kodaira dimension of \( S \). When \( \kappa(S) \geq 0 \), by Riemann-Roch Theorem and Vanishing Theorem, we get

\[
h^0(K_S + A_S) - h^0(K_S) = g(S, A_S) - q(S) = 1.
\]

If \( h^0(K_S) > 0 \), then

\[
h^0(K_S + A_S) \geq h^0(K_S) + h^0(A_S) - 1.
\]

But this is impossible since \( h^0(A_S) \geq 3 \). Hence \( h^0(K_S) = 0 \). Thus we get \( g(S, A_S) \geq 2q(S) \) by Lemma 1.4 in [Ma1] since \((S, A_S)\) is not a scroll over a smooth curve. Then \( q(S) \leq 1 \) and \( g(X, \mathcal{E}) \leq 2 \) by the above argument. So we obtain that \((X, \mathcal{E})\) is the case (1), (2), or (3) of our theorem by using (2.1), (2.2.4) and [I]. □

**Remark 3.2.** Let \( L \) be an ample and spanned line bundle on a compact complex manifold \( X \) of dimension \( n \geq 2 \). When \( n \geq 3 \), we have \( g(X, L) = q(X) + 1 \) if and only if \((X, L)\) is a Del Pezzo manifold (see [Fk3]). When \( n = 2 \), we have \( g(X, L) = q(X) + 1 \) if and only if \((X, L)\) is a Del Pezzo surface (i.e., \( L = -K_X \)) or \( X \simeq \mathbb{P}_B(\mathcal{F}) \) and \( L \equiv 2H(\mathcal{F}) \) for some ample vector bundle \( \mathcal{F} \) of rank 2 on an elliptic curve \( B \) with \( c_1(\mathcal{F}) = 1 \). We can prove this by the argument in (3.1) and Theorem 3.1 in [LP].

**Proposition 3.3.** Let \( X \) be a compact complex manifold of dimension \( n \) and let \( \mathcal{E} \) be an ample and spanned vector bundle of rank \( r \) with \( 1 < r < n-1 \). Then we have \( g(X, \mathcal{E}) \neq q(X) + 2 \).

**Proof.** The following argument is similar to the proof of (3.1). Suppose that \( g(X, \mathcal{E}) = q(X) + 2 \). Let \( Z \) be a smooth submanifold of \( X \) with \( \dim Z = n - r \) defined as the zero locus of some \( s \in H^0(X, \mathcal{E}) \). Then \( g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z) \) and \( \det \mathcal{E}_Z \) is ample and spanned. As in the proof of (3.1), we get a smooth surface \( S \) such that \( g(Z, \det \mathcal{E}_Z) = g(S, \det \mathcal{E}_S) \). We have \( q(X) = q(Z) = q(S) \), thus we get \( g(S, \det \mathcal{E}_S) = q(S) + 2 \). Then we find that \( q(S) \leq 1 \) by Theorem 3.4 in [R]. It follows that \( g(X, \mathcal{E}) \leq 3 \) and we infer that \((X, \mathcal{E})\) does not exist from (2.1), (2.2.4) and (2.3). This completes the proof. □

**Remark 3.4.** We see that the pairs \((X, \mathcal{E})\) in (2.3) satisfy \( g(X, \mathcal{E}) = q(X) + 3 \). In Appendix we give a classification of polarized surfaces \((X, L)\) such that \( g(X, L) = q(X) + 2 \) and \( L \) is spanned.
4. Another Lower bound for $g(X, \mathcal{E})$.

**Proposition 4.1.** Let $L$ be an ample and spanned line bundle on a compact complex manifold $X$ with $\dim X = n \geq 2$. Then $g(X, L) \geq 2q(X) - 1$ unless $(X, L)$ is a scroll over a smooth curve $B$ of genus $g(B) \geq 2$.

**Proof.** Since $L$ is ample and spanned, if $n \geq 3$, we can take general members $D_1, \ldots, D_{n-2} \in |L|$ such that $S := D_1 \cap \cdots \cap D_{n-2}$ is a smooth surface. If $n = 2$, we set $S = X$. Then we get $g(X, L) = g(S, L_S)$ and $q(X) = q(S)$.

If $\kappa(S) \geq 0$, then $g(X, L) = g(S, L_S) \geq 2q(S) - 1 = 2q(X) - 1$ by Corollary 3.2 in [Fk1].

If $\kappa(S) = -\infty$ and $(S, L_S)$ is not a scroll over a smooth curve, then $g(X, L) = g(S, L_S) \geq 2q(S) = 2q(X)$ by Lemma 1.4 in [Ma1].

If $\kappa(S) = -\infty$ and $(S, L_S)$ is a scroll over a smooth curve, then $g(X, L) = g(S, L_S) = g(S) = g(X)$. Hence we get $g(X, L) \geq 2q(X) - 1$ if $q(S) \leq 1$. So we may assume that $q(S) \geq 2$. Then we obtain that $(X, L)$ is a scroll over a smooth curve $B$ of genus $g(B) \geq 2$ by using Theorem 3 in [Bà]. \hfill $\square$

**Theorem 4.2.** Let $X$ be a compact complex manifold with $\dim X = n$ and let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r$ with $1 < r < n - 1$. Then $g(X, \mathcal{E}) \geq 2q(X) - 1$.

**Proof.** Let $Z$ be the zero locus of some $s \in H^0(X, \mathcal{E})$ such that $Z$ is a smooth submanifold of $X$ with $\dim Z = n - r$. Then $g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z)$ and $q(X) = q(Z)$. We put $A := \det \mathcal{E}_Z$; then $A$ is ample and spanned. Since $(Z, A)$ is not a scroll, by (4.1), we obtain that $g(X, \mathcal{E}) = g(Z, A) \geq 2q(Z) - 1 = 2q(X) - 1$. \hfill $\square$

5. The case of a fiber space over a curve.

**Definition 5.1.** Here we say that a quartet $(f, X, C, \mathcal{E})$ is a generalized polarized fiber space over a curve if:

1. $X$ and $C$ are compact complex manifolds with $1 = \dim C < \dim X = n$,
2. $f : X \rightarrow C$ is a surjective morphism with connected fibers, and
3. $\mathcal{E}$ is an ample vector bundle of rank $r$ on $X$.

**Theorem 5.2.** Let $(f, X, C, \mathcal{E})$ be a generalized polarized fiber space over a curve with $r \leq n - 1$. Then $g(X, \mathcal{E}) \geq g(C)$.

**Proof.** First we remark that the following equality holds:

$$g(X, \mathcal{E}) = g(C) + \frac{1}{2}(K_{X/C} + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E})$$

$$+ (g(C) - 1)(c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E})F - 1),$$

where $K_{X/C} := K_X - f^*(K_C)$ and $F$ is a general fiber of $f$. 

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If \( g(C) = 0 \), then Theorem 5.2 is true by [I]. So we may assume that \( g(C) \geq 1 \).

(I) The case in which \( K_X/C + (n-r)c_1(\mathcal{E}) \) is \( f \)-nef.

Then there exists a surjective map

\[
f^* \circ f_* (\mathcal{O}(m(K_X/C + (n-r)c_1(\mathcal{E})))) \rightarrow \mathcal{O}(m(K_X/C + (n-r)c_1(\mathcal{E})))
\]
for any large \( m \) by base point free theorem.

By Theorem A in Appendix in [Fk2], \( f_* (\mathcal{O}(m(K_X/C + (n-r)c_1(\mathcal{E})))) \) is semipositive. Hence \( K_X/C + (n-r)c_1(\mathcal{E}) \) is nef. So we get

\[
(K_X/C + (n-r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 0.
\]
Hence we obtain \( g(X, \mathcal{E}) \geq g(C) \) because of (5.2.1) and \( c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 1 \).

(II) The case in which \( K_X/C + (n-r)c_1(\mathcal{E}) \) is not \( f \)-nef.

Then \( K_X + (n-r)c_1(\mathcal{E}) \) is not nef. So by Mori Theory, there exists an extremal rational curve \( l \) such that \( (K_X + (n-r)c_1(\mathcal{E}))l < 0 \). Hence

\[
n + 1 \geq -K_Xl > (n-r)c_1(\mathcal{E})l \geq (n-r)r \geq n - 1.
\]
If \( (n-r)r = n \), then \( (n,r) = (4,2) \).
If \( (n-r)r = n - 1 \), then \( r = 1 \) or \( r = n - 1 \).

(II-1) The case where \( (n,r) = (4,2) \).

Then \(-K_Xl = 5 = n + 1 \). So we have \( \text{Pic} X \cong \mathbb{Z} \) by [W]. But this is impossible because \( X \) has a nontrivial fibration.

(II-2) The case in which \( r = 1 \).

Then Theorem 5.2 is true by Theorem 1.2.1 in [Fk2].

(II-3) The case in which \( r = n - 1 \).

If \( n = 2 \), then \( r = 1 \) and so we may assume that \( n \geq 3 \). Since \( X \) has a nontrivial fibration, \((X,\mathcal{E})\) is the following type by [YZ]: There exists a surjective morphism \( \pi : X \rightarrow B \) such that any fiber of \( \pi \) is \( \mathbb{P}^{n-1} \) and \( \mathcal{E}|_{F_x} \cong \mathcal{O}(1)^{\oplus n-1} \), where \( B \) is a smooth curve and \( F_x \) is a fiber of \( \pi \).

Since any fiber of \( \pi \) is \( \mathbb{P}^{n-1} \), there exists a morphism \( \delta : B \rightarrow C \) such that \( f = \delta \circ \pi \). Because \( f \) has connected fibers, \( \delta \) is an isomorphism. In particular, \( g(B) = g(C) \). On the other hand, by [Ma2], \( g(X, \mathcal{E}) = g(B) \). Hence \( g(X, \mathcal{E}) = g(C) \). This completes the proof of Theorem 5.2. \( \square \)

**Theorem 5.3.** Let \((f,X,C,\mathcal{E})\) be a generalized polarized fiber space over a curve with \( 2 \leq r \leq n - 1 \). If \( g(X, \mathcal{E}) = g(C) \), then \( r = n - 1 \), any fiber \( F \) of \( f \) is isomorphic to \( \mathbb{P}^{n-1} \) and \( \mathcal{E}|_{F} \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^{n-1}}(1) \).

**Proof.** (I) The case in which \( g(C) \leq 1 \).

Then \( g(X, \mathcal{E}) = g(C) \leq 1 \), and by the classification results of [I] and [Ma2], we get the following: \( X \) is a \( \mathbb{P}^{n-1} \)-bundle over \( \mathbb{P}^1 \) or a smooth elliptic curve and \( \mathcal{E}|_{F_x} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n-1} \), where \( F_x \) is a fiber of its bundle map.
\(\pi : X \to B\) and \(B\) is \(\mathbb{P}^1\) or a smooth elliptic curve. Since any fiber of \(\pi\) is \(\mathbb{P}^{n-1}\), there exists a morphism \(\delta : B \to C\) such that \(f = \delta \circ \pi\). Because \(f\) has connected fibers, \(\delta\) is an isomorphism. Therefore we get the assertion.

(II) The case in which \(g(C) \geq 2\).

(II-1) \(n - r \geq 2\) case.

If \(K_{X/C} + (n - r - 1)c_1(\mathcal{E})\) is \(f\)-nef, then by the same argument as in the proof of Theorem 5.2 we get

\[
(K_{X/C} + (n - r - 1)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 0
\]

and

\[
(K_{X/C} + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 1.
\]

Hence we obtain that \(g(X, \mathcal{E}) > g(C)\) by (5.2.1). So we may assume that \(K_{X/C} + (n - r - 1)c_1(\mathcal{E})\) is not \(f\)-nef. Then by Mori Theory, there exists an extremal rational curve \(l\) such that \((K_X + (n - r - 1)c_1(\mathcal{E}))l < 0\). Hence we get

\[
n + 1 \geq -K_Xl > (n - r - 1)c_1(\mathcal{E})l \geq (n - r - 1)r \geq n - 2.
\]

If \((n - r - 1)r = n\), then \(-K_X l = n + 1\) and \(\text{Pic} \, X \cong \mathbb{Z}\) by [W]. But this is impossible.

If \((n - r - 1)r = n - 1\), then \(n = 5\) and \(r = 2\).

Here we prove the following Lemma.

**Lemma 5.4.** Let \((f, X, C, \mathcal{E})\) be a generalized polarized fiber space over a curve with \(2 \leq r \leq n - 1\) and \(g(C) \geq 1\). If \(\kappa(K_F + xc_1(\mathcal{E}_F)) \geq 0\) for a rational number \(x\) with \(x < n - r\) and a general fiber \(F\) of \(f\), then \(g(X, \mathcal{E}) \geq g(C) + 1\).

**Proof.** By assumption, there exists a natural number \(N\) such that

\[
f_*(\mathcal{O}(N(K_{X/C} + xc_1(\mathcal{E})))) \neq 0.
\]

By Remark 1.3.2 in [Fk2], \(N(K_{X/C} + xc_1(\mathcal{E}))\) is pseudo effective. Therefore

\[
(K_{X/C} + xc_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 0
\]

and we get

\[
(K_{X/C} + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 1.
\]

Since \(g(C) \geq 1\), we get that \(g(X, \mathcal{E}) \geq g(C) + 1\) by (5.2.1). \(\Box\)

We continue the proof of Theorem 5.3. If \(K_F + xc_1(\mathcal{E}_F)\) is nef for a rational number \(x\) with \(x < 3\), then we can prove that \(g(X, \mathcal{E}) > g(C)\) by Lemma 5.4.

Assume that \(K_F + xc_1(\mathcal{E}_F)\) is not nef for a rational number \(x\) with \(x < 3\). Then there exists an extremal rational curve \(l\) on \(F\) such that \(n \geq -K_Fl > xc_1(\mathcal{E}_F)l \geq rx\). Since \(n = 5\) and \(r = 2\), we have \(x < 5/2\). Therefore there exists a rational number \(y < 3\) such that \(K_F + yc_1(\mathcal{E}_F)\) is nef, and we get \(g(X, \mathcal{E}) > g(C)\).
If \((n-r-1)r = n-2\), then \(r = n-2\) by assumption. Assume that 
\(K_F + xc_1(\mathcal{E}_F)\) is not nef for a rational number \(x\) with \(x < 2\). Then we 
get \(n > rx\) by the same argument as above. Since \(r = n-2\), we get 
\(x < n/(n-2) = 1 + 2/(n-2)\). By assumption, we get \(n \geq 4\). So we 
have \(x < 2\). Therefore there exists a rational number \(y < 2\) such that 
\(K_F + yc_1(\mathcal{E}_F)\) is nef. Hence we have \(g(X, \mathcal{E}) > g(C)\).

\((\Pi-2)\ n-r = 1\) case.

First we assume that \(K_F + c_1(\mathcal{E}_F)\) is nef for a general fiber \(F\) of \(f\). If 
\(K_F + c_1(\mathcal{E}_F)\) is ample, then there exists a rational number \(t > 0\) such that 
\(\kappa(K_F + (1-t)c_1(\mathcal{E}_F)) \geq 0\) by Kodaira’s Lemma. So we get that \(g(X, \mathcal{E}) > g(C)\) by the same argument as above. Assume that \(K_F + c_1(\mathcal{E}_F)\) is not 
ample. Since \(\dim F = \text{rank} \mathcal{E}_F\), by [Fj3], we get that \((F, \mathcal{E}_F)\) is one of the following:

(A) \((\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n-2})\),
(B) \((\mathbb{P}^{n-1}, T_{\mathbb{P}^{n-1}})\),
(C) \((\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus n-1})\),
(D) \(F\) is a \(\mathbb{P}^{n-2}\)-bundle over a smooth curve \(B\) and \(\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{1}}(1)^{\oplus n-1}\) for 
every fiber \(F_x\) of the projection \(\pi : F \to B\).

If \((F, \mathcal{E}_F)\) is one of the type (A), (B), or (C), then \(h^0(K_F + c_1(\mathcal{E}_F)) > 0\) by 
easy calculation. Here we prove the following Lemma.

**Lemma 5.5.** Let \((f, X, C, \mathcal{E})\) be a generalized polarized fiber space over a 
curve with \(2 \leq r \leq n-1\). If \(h^0(K_F + c_1(\mathcal{E}_F)) > 0\) for a general fiber \(F\) of 
\(f\), then \((K_{X/C} + c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) > 0\).

**Proof.** By hypothesis, \(f_*\mathcal{O}(K_{X/C} + c_1(\mathcal{E})) \neq 0\). By Theorem 2.4 and Corol-
ary 2.5 in [EV], we get that \(f_*\mathcal{O}(K_{X/C} + c_1(\mathcal{E}))\) is ample. By the proof of 
Lemma 1.4.1 in [Fk2], we get that \(m(K_{X/C} + c_1(\mathcal{E})) - f^*A\) is an effective 
divisor for a large number \(m\) and an ample divisor \(A\) on \(C\). Hence we obtain 
\((K_{X/C} + c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) > 0\). \(\square\)

By Lemma 5.5, we get that \(g(X, \mathcal{E}) > g(C)\) if \((F, \mathcal{E}_F)\) is one of the type 
(A), (B), or (C).

Assume that \((F, \mathcal{E}_F)\) is the type (D). Then there exist vector bundles 
\(\mathcal{F}\) and \(\mathcal{G}\) on \(B\) with \(\text{rank} \mathcal{F} = \text{rank} \mathcal{G} = n-1\) such that 
\(\mathcal{E}_F \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})\), where \(H(\mathcal{F})\) is the tautological line bundle of \(\mathbb{P}(\mathcal{F})\). Then 
\(K_F + c_1(\mathcal{E}_F) = \pi^*(K_B + \det \mathcal{F} + \det \mathcal{G})\). Since \(K_F + c_1(\mathcal{E}_F)\) is nef, we get 
\((K_{X/C} + c_1(\mathcal{E}))c_r(\mathcal{E}) \geq 0\) by the proof of Lemma 5.4. We have \(g(X, \mathcal{E}) = g(C)\), then
Because \(K_F + c_1(\mathcal{E}_F)\) is nef, we obtain that \(\deg(K_F + \det \mathcal{F} + \det \mathcal{G}) \geq 0\). Hence \(g(B) \geq 1\). Therefore \(h^0(K_F + c_1(\mathcal{E}_F)) \geq 1\). By Lemma 5.5 we obtain that \(g(X, \mathcal{E}) > g(C)\) and this is a contradiction.

Next we assume that \(K_F + c_1(\mathcal{E}_F)\) is not nef. Then \(K_X + c_1(\mathcal{E})\) is not nef and the same argument as in the proof of Theorem 5.2, case (II-3), shows that \((f, X, C, \mathcal{E})\) is as required. This completes the proof of Theorem 5.3. \(\square\)

**Remark 5.6.** Let \((f, X, C, \mathcal{E})\) be as in Theorem 5.2. Suppose that \(g(X, \mathcal{E}) = g(C)\) and \(r = 1\). Then by Theorem 1.4.2 and Proposition 1.4.3 in [Fk2], \((f, X, C, \mathcal{E})\) is a scroll (in the sense of [Fk2], §0) unless \(n = 2\) and \((f, X, C, \mathcal{E}) \cong (\pi, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1, L)\), where \(\pi\) is one projection such that \(\mathcal{L}_\pi \geq 2\) for a fiber \(F_\pi\) of \(\pi\). By the other projection \(\rho\), however, \((\rho, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1, L)\) becomes a scroll.

### Appendix.

**Proposition A.** Let \((X, L)\) be a quasi-polarized surface (i.e., \(L\) is a nef and big line bundle on a smooth surface \(X\)) such that \(\kappa(X) = 2\) and \(h^0(L) \geq 2\). Then \(K_X L \geq 2g(X) - 2\). If equality holds and \((X, L)\) is \(L\)-minimal (i.e., \(LE > 0\) for any \((-1)\)-curve \(E\) on \(X\)), then \((X, L)\) is the following:

\[X \cong F \times C\] and \(L \equiv C + 2F\), where \(F\) and \(C\) are smooth curves with \(g(F) = 2\) and \(g(C) \geq 2\).

*Proof.* See [Fk4]. \(\square\)

**Proposition B.** Let \((X, L)\) be a polarized surface with \(\kappa(X) = 0\) or 1. Assume that \(L\) is spanned. Then \(g(L) := g(X, L) \geq 2g(X)\). Furthermore if \(g(L) = 2g(X)\), then \((X, L)\) is one of the following:

1. \((X, L)\) is a polarized abelian surface with \(L^2 = 6\) such that \((X, L) \not\cong (E_1 \times E_2, \pi_1^*(D_1) + \pi_2^*(D_2))\), where \(E_i\) is a smooth elliptic curve, \(\pi_i\) is the \(i\)-th projection, and \(D_i \in \text{Pic}(E_i)\) for \(i = 1, 2\) with \(\deg D_1 = 1\) and \(\deg D_2 = 3\).
2. \(X\) is a one point blowing up of \(S\), and \(L = \mu^* A - 2E\), where \(S\) is an abelian surface, \(A\) is an ample line bundle with \(A^2 = 8\), \(\mu : X \rightarrow S\) is its blowing up, and \(E\) is a \((-1)\)-curve of \(\mu\).
3. \(\kappa(X) = 1\), \(L^2 = 4\), \(g(X) = 3\), \(X\) has a locally trivial elliptic fibration \(f : X \rightarrow C\), and \(L F = 3\) for a fiber \(F\) of \(f\), where \(C\) is a smooth curve with \(g(C) = 2\).
Proof. See [Fk5]. □

**Theorem.** Let $X$ be a smooth projective surface and let $L$ be an ample and spanned line bundle on $X$. If $g(L) = q(X) + 2$, then $(X, L)$ is one of the following:

1. $(X, L)$ is a relatively minimal conic bundle over a smooth curve $B$ of genus two (i.e., $X$ is a $\mathbb{P}^1$-bundle over $B$ and $L_F = \mathcal{O}_{\mathbb{P}^1}(2)$ for every fiber $F$ of the ruling).
2. $X$ is a $\mathbb{P}^1$-bundle $X_0$ blown-up at $s$ ($0 \leq s \leq 4$) points $p_1, \ldots, p_s$ on distinct fibers and $L = \pi^*L_0 - E_1 - \cdots - E_s$, where $\pi : X \to X_0$ is the blowing up, $E_i = \pi^{-1}(p_i)$. $X_0$ is an elliptic $\mathbb{P}^1$-bundle of invariant $e < 0$, and $L_0 \equiv 2\sigma + (e + 2)f$ ($\sigma$ is a minimal section with $\sigma^2 = -e$ and $f$ is a fiber).
3. $X$ is an $\mathbb{F}_e$ ($e \leq 2$) blown-up at $s$ ($0 \leq s \leq 9$) points $p_1, \ldots, p_s$ on distinct fibers and $L = \pi^*L_0 - E_1 - \cdots - E_s$, where $\pi : X \to \mathbb{F}_e$ is the blowing up, $E_i = \pi^{-1}(p_i)$, and $L_0 \equiv 2\sigma + (e + 3)f$.
4. $X$ is a Del Pezzo surface of degree one and there exists a double covering $\pi : X \to Q \subset \mathbb{P}^3$ of a quadric cone $Q$ branched at the vertex and along the transverse intersection of $Q$ with a cubic surface and $L = \pi^*(\mathcal{O}_Q(1))$.
5. $(X, L)$ is a polarized abelian surface with $L^2 = 6$ such that $(X, L) \not\equiv (E_1 \times E_2, p_1^*(D_1) + p_2^*(D_2))$, where $E_i$ is a smooth elliptic curve, $p_i$ is the $i$-th projection, and $D_i \in \text{Pic}(E_i)$ for $i = 1, 2$ with $\deg D_1 = 1$ and $\deg D_2 = 3$.
6. $X$ is a blowing up of an abelian surface $S$ at one point $p$ and $L = \pi^*A - 2E$, where $\pi : X \to S$ is the blowing up, $E = \pi^{-1}(p)$, and $A$ is an ample line bundle on $S$ with $A^2 = 8$.
7. $X$ is a $K3$ surface which is a double covering of $\mathbb{P}^2$ branched along a smooth curve of degree six and $L$ is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$.

Proof. (I) The case in which $\kappa(X) = 0$ or 1.

Then by Proposition B, we get that $g(L) \geq 2q(X)$. So we obtain $q(X) \leq 2$ by assumption.

(I-1) If $q(X) = 2$, then $g(L) = q(X) + 2 = 2q(X)$ and by Proposition B we get the type (5) and (6) in Theorem.

(I-2) If $q(X) \leq 1$, then $g(L) \leq 3$ and $L^2 \leq 4$ by $K_X L \geq 0$. So by Kodaira vanishing Theorem and Riemann-Roch Theorem, we get the equality: $h^0(L) = L^2/2 + \chi(\mathcal{O}_X) = 2 + \chi(\mathcal{O}_X)$. Because $L$ is ample and spanned, we obtain $h^0(L) \geq 3$ and $\chi(\mathcal{O}_X) \geq 1$. But then $q(X) = 0$ by the classification theory of surfaces and this is impossible.

(I-2-2) If $L^2 = 3$, then $g(L) = 3, K_X L = 1$, and $q(X) = 1$. We have $h^0(L) \geq 3$ since $L$ is ample spanned.
If $h^0(L) \geq 4$, then $g(L) > \Delta(L)$ and $L^2 \geq 2\Delta(L) + 1$, where $\Delta(L) := 2 + L^2 - h^0(L)$ is the $\Delta$-genus of $L$. But then $q(X) = 0$ (see e.g. (1.3.5) in \[\text{Be}\]).

If $h^0(L) = 3$, then there is a triple covering $\varphi|_L : X \to \mathbb{P}^2$ which is defined by $|L|$. Let $\mathcal{E}$ be a vector bundle of rank two on $\mathbb{P}^2$ such that $\pi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$.

By Lemma 3.2 in \[\text{(II-1)}\] The case of

Claim. The above three cases cannot occur.

Proof. (II-1) The case $\gamma$.

In this case $X$ is minimal because $K_X L = 1$. But then this is impossible by Hodge index Theorem.
(II-2) The case (β).
If $X$ is minimal, then $K_X^2 \geq 2g(X) = 4$ by Théorème 6.1 in [D]. On the other hand, $K_X^2 \leq 3$ by Hodge index Theorem and this is a contradiction.

So we get that $X$ is not minimal. Let $\mu := \mu_r \circ \cdots \circ \mu_1 : X := X_0 \to X_1 \to \cdots \to X_r =: X'$ be an admissible minimalization of $X$ and let $m = (m_r, \ldots, m_1)$ be the weight sequence of this minimalization (see (II.14.4) in [Fj4]). We remark that $m_r \geq \cdots \geq m_1$.

If $m_1 = 1$, then $g(L_1) = q(X_1) + 1$ and $h^0(L_1) \geq 2$, where $L_1 := (\mu_1)_*(L)$ in the sense of cycle theory. But then this is impossible by Proposition A because $2 = K_XL > K_XL_1$. So we get $m_1 \geq 2$. Then $L_1^2 \geq 7$ and $K_X, L_1 \leq 1$. Hence $X_1$ is minimal and this is a contradiction by Hodge index Theorem.

(II-3) The case (α).
If $X$ is minimal, then $\chi(\mathcal{O}_X) \geq 4$ because $3\chi(\mathcal{O}_X) = K_X^2 + 10$. Furthermore $p_g(X) \geq 6$ since $q(X) = 3$. Hence $K_X^2 \geq 2p_g(X) \geq 12$ by Théorème 6.1 in [D]. But this is impossible by Hodge index Theorem. So we get that $X$ is not minimal. By the same argument as in the case (II-2) we get a contradiction. \hfill \Box

We continue the proof of Theorem.

If $L^2 = 2$, then there exists a double covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Let $\mathcal{O}_{\mathbb{P}^2}(a)$ be a line bundle on $\mathbb{P}^2$ such that $B \in |\mathcal{O}_{\mathbb{P}^2}(2a)|$, where $B$ is the branch locus. Then $(\varphi_{|L|})_*(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-a)$. Hence

$$h^1(\mathcal{O}_X) = h^1((\varphi_{|L|})_*(\mathcal{O}_X)) = h^1(\mathcal{O}_{\mathbb{P}^2}) + h^1(\mathcal{O}_{\mathbb{P}^2}(-a)) = 0.$$  

So we get $g(L) = 2$. But since $K_XL > 0$ and $L^2 = 2$, this is impossible.

(III) The case in which $\kappa(X) = -\infty$.
Since $(X, L)$ is not a scroll over a smooth curve, we get $g(L) \geq 2q(X)$ by Lemma 1.4 in [Ma1]. So $q(X) \leq 2$.

(III-1) The case in which $q(X) = 2$.
In this case, $g(L) = q(X) + 2 = 2q(X)$. Since $K_X + L$ is nef, we get

$$0 \leq (K_X + L)^2 = (K_X)^2 + 2(K_X + L)L - L^2$$
$$\leq 8(1 - q(X)) + 4(g(L) - 1) - L^2$$
$$= 4(g(L) - 2q(X) + 1) - L^2.$$  

Hence $L^2 \leq 4$ in this case.

If $L^2 = 4$, then $X$ is relatively minimal and $(K_X + L)^2 = 0$, that is, $(X, L)$ is a relatively minimal conic bundle over a smooth curve. This is the type (1) in Theorem.

If $L^2 \leq 3$ and $h^0(L) \geq 4$, then we get a contradiction as in (I-2-2). So we may assume that $L^2 \leq 3$ and $h^0(L) = 3$.  

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If $L^2 = 3$, then $K_X L = 3$ and there is a triple covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Since $\chi(O_X) = -1$, we get that $K_X^2 = -12$ by Lemma 3.2 in [Be]. Here we calculate $(K_X + L)^2$:

$$(K_X + L)^2 = K_X^2 + 2K_X L + L^2 = -12 + 6 + 3 < 0.$$  

But this is a contradiction because $K_X + L$ is nef.

If $L^2 = 2$, then there is a double covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. But then $q(X) = 0$ and this is a contradiction.

(III-2) The case in which $q(X) = 1$.

Then $g(L) = 3$. Here we use the classification of polarized surfaces with sectional genus three by [LL].

**Claim.** *The case in which $L^2 = 3$ cannot occur.*

**Proof.** If $L^2 = 3$ and $h^0(L) \geq 4$, then $g(L) > 1 \geq \Delta(L)$ and $L^2 \geq 2\Delta(L) + 1$. But this is impossible because $q(X) = 1$. So we may assume that $h^0(L) = 3$. Then there is a triple covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Since $\chi(O_X) = 0$, we get $K_X^2 = -7$ by Lemma 3.2 in [Be]. But in the table II of [LL], the case in which $L^2 = 3$ cannot occur. □

Next we prove that the following case cannot occur (see (2.6) in [LL]):

* $X$ is an elliptic $\mathbb{P}^1$-bundle $X_2$ of invariant $e = 0$, blown up at a single point $p$ not lying on a curve $D \in |m\sigma|, m \leq 2$ and $L = \eta^*[4\sigma + (2e + 1)f] \otimes [E]^{-2}.$ (Here we use the same notations as in [LL].)

Let $\sigma'$ be the strict transform of $\sigma$ under $\eta$. Since

$$0 < Lo' = (4\sigma + f)\sigma - 2E\sigma' = 1 - 2E\sigma',$$

we see that $E\sigma' = 0$ and $Lo' = 1$. It follows that $\sigma \cong \sigma' \cong \mathbb{P}^1$ since $L$ is spanned. This is a contradiction.

By the above argument, we obtain the type (2) in Theorem by the classification of polarized surfaces with sectional genus three (see [LL]).

(III-3) The case in which $q(X) = 0$.

Then $g(L) = 2$. So by Theorem 3.1 in [LP] we get the type (3) and (4) in Theorem. □

**References**


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