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Let $\mathfrak{g}$ be a noncompact real form of the simple complex Lie algebra $\mathfrak{g}^c$ of type E_8. We obtain a list of representatives of the adjoint orbits of triples $\{E, H, F\} \subset \mathfrak{g}$, spanning a subalgebra isomorphic to $\mathfrak{sl}_2(\mathrm{R})$, such that $[H, E] = 2E$, $[H, F] = -2F$, and $[F, E] = H$. They are chosen to be Cayley triples with respect to a fixed Cartan decomposition of $\mathfrak{g}$.

1. Introduction.

The nilpotent adjoint orbits in real forms $\mathfrak{g}$ of exceptional complex simple Lie algebras $\mathfrak{g}^c$ were enumerated in our papers [**4**, **5**] by using the Kostant-Sekiguchi bijection. No representatives for these orbits were listed there. For an adaptation of the Bala-Carter approach to this classification problem see the recent paper of A.G. Noël [**14**]. In the two recent papers [**7**, **8**] we have computed representatives for adjoint orbits of standard triples (E, H, F) in real forms of G_2, F_4, E_6, and E_7. The representatives (E, H, F) were chosen to be real Cayley triples with respect to a fixed Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The nilpositive elements E of these triples are representatives of the nonzero nilpotent orbits of $\mathfrak{g}$. We accomplish here the same objective for the noncompact real forms EVIII $= E_{8(8)}$ and EIX $= E_{8(-24)}$ of E_8. The present paper is a continuation of [**7**, **8**] and we refer the reader to them for the definitions of the technical terms that we use.

We now describe our method which is somewhat different from the one used in the previous two papers. Let σ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$, and θ the Cartan involution of $\mathfrak{g}$. We fix a maximally split θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{p}$ be the corresponding Cartan subspace. Let Φ be the root system of $(\mathfrak{g}^c, \mathfrak{h}^c)$ and Φ_0 the subsystem consisting of all $\alpha \in \Phi$ that vanish on $\mathfrak{a}$. Choose a base Π of Φ such that $\Pi_0 := \Pi \cap \Phi_0$ is a base of Φ_0. Let $\mathcal{O}^c \subset \mathfrak{g}^c$ be a complex nonzero nilpotent orbit and $H \in \mathfrak{h}^c$ its characteristic. One knows that $\mathcal{O}^c$ meets $\mathfrak{g}$ if and only if $H \in \mathfrak{a}$ (see [**6**]). We assume that this is the case. A subalgebra $\mathfrak{s}^c$ of $\mathfrak{g}^c$ is called *regular* if it is normalized by a Cartan subalgebra of $\mathfrak{g}^c$. Dynkin [**9**] has enumerated, up to conjugacy by the adjoint group G^c of $\mathfrak{g}^c$, all minimal semisimple regular subalgebras $\mathfrak{s}^c$ that meet $\mathcal{O}^c$. Since his list contains several errors, we use the corrected list from [**10**]. The intersection $\mathcal{O}_1^c := \mathcal{O}^c \cap \mathfrak{s}^c$ is a single nilpotent orbit of $\mathfrak{s}^c$. Now assume that $\mathfrak{s}^c$ is σ-stable and let $\mathfrak{s}$ be the corresponding real form of $\mathfrak{s}^c$. If $\mathcal{O}_1^c$ meets $\mathfrak{s}$ then $\mathcal{O}_1^c \cap \mathfrak{s}$ is a union of nilpotent adjoint orbits of $\mathfrak{s}$. We compute representatives of these orbits of $\mathfrak{s}$ and identify the nilpotent orbits of $\mathfrak{g}$ to which these representatives belong. In this way we were able to find representatives, under the action of the adjoint group G of $\mathfrak{g}$, for all nilpotent orbits of EVIII and EIX. As $\mathfrak{s}^c$ is not unique, we may obtain several representatives for the same nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$. Anticipating a possible application of our results to determine the closure ordering (see [**1**, **3**]) for nilpotent orbits of $\mathfrak{g}$, we decided to record one representative of $\mathcal{O}$ for each successful choice of $\mathfrak{s}^c$. In that sense the tables in this paper are more comprehensive than those in [**7**, **8**].

Let us say that a real Cayley triple (E, H, F) is *nice* if in the expression $E = \sum \lambda_k X_k$, $k > 0$, the square of each λ_k is an integer (negative integers are allowed). Almost all representative Cayley triples in this and the previous two papers are nice. The basic examples of representatives that are not nice are those for the orbits 17, 18 of EI, orbit 12 of EII, orbit 91 of EV, oribt 14 of EVI, and the orbit 114 of EVIII. For orbits 17 and 18 of EI we have the following nice representatives

$$E = \sqrt{6}(X_7 - X_{11} + X_{15}) + \sqrt{10}X_8 + X_{12} + X_{16},$$

$$E = 2\sqrt{3}(X_1 + X_6) + 4X_2 + \sqrt{7}(X_3 + X_5) + \sqrt{15}(X_9 - X_{10})$$

which should be used instead of those listed in [**7**, Table 7]. These two E's produce nice representatives for some other orbits. For instance the second of the two E's above was used to find nice representatives for the orbits 78, 88, and 91 of EVIII (see Table 4) and the orbit 35 of EIX (see Table 5). The two E's above are already incorporated in the tables of [**8**]. Unfortunately, one can check that the orbit 91 of EV has no nice representative Cayley triples.

All computations were performed on a SUN workstation using our own programs. We also used Maple [**13**] to compute the matrix of the operator ad $(H')|_{\mathfrak{k}^c}$ and the related invariants (see the next section). Once these tables are constructed, it is a routine but tedious job to verify the correctness of the representative triples. For an example of how such a verification can be performed see [**7**].

We point out that, due to a systemic error, all the entries in the last column of Tables 7 and 10 of [**7**] are wrong: One should subtract 2 from each of them to obtain the correct values for inv. These two tables are associated with the algebras EI and EIV which are the only ones of outer type. There are no repercussions from this error because the values of the invariant inv were used only to distinguish real orbits that are contained in the same complex orbit.

Due to typing errors, the representatives of orbits 15 and 16 of $E_{6(2)}$ given in Table 8 of [**7**] are incorrect. The correct representatives are:

Orbit 15 of $E_{6(2)}$: $E = \sqrt{3}(X_2 - X_{24}) + 2X_{23}$,

Orbit 16 of $E_{6(2)}$: $E = \sqrt{3}(X_2 + X_{24}) + 2X_{23}$.

2. Structure constants and the action of σ.

The positive roots are enumerated (see Table 1) as $\alpha_1, \alpha_2, \dots, \alpha_{120}$ so that $\mathrm{ht}(\alpha_i) \le \mathrm{ht}(\alpha_j)$ for $i < j$, and $\Pi = \{\alpha_1, \dots, \alpha_8\}$. The negative root $-\alpha_i$ is written also as α_{-i}. The extended Dynkin diagram of E_8 is given in Fig. 1.

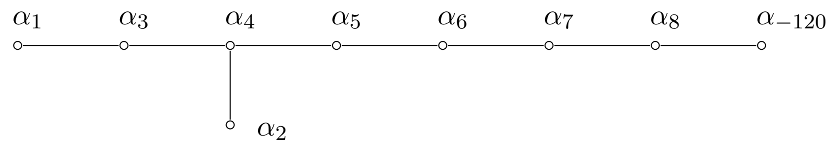

Figure 1.

Since E_8 is simply laced, if $\alpha_i = k_1\alpha_1 + \cdots + k_8\alpha_8$ is a positive root, then the corresponding coroot H_i is given by $H_i = k_1H_1 + \cdots + k_8H_8$.

As in [**8**], we use an algorithm of J. Kurtzke [**12**] to fix the choice of a Chevalley basis of $\mathfrak{g}^c$:

$$H_i, \quad 1 \le i \le 8; \quad X_i, X_{-i}, \quad 1 \le i \le 120.$$

For all i's we have $[X_{-i}, X_i] = H_i$. If $\alpha_i + \alpha_j = \alpha_k$, then $[X_i, X_j] = N(i,j)X_k$. As all roots of E_8 have the same length, the $N(i,j)$ are ± 1. We specify that $N(1,3) = +1$. Then, by Kurtzke's algorithm,

$$N(3,4) = N(5,6) = N(7,8) = -1, \; N(4,2) = N(4,5) = N(6,7) = +1,$$

and $N(i,j) = +1$ whenever $\alpha_i + \alpha_j$ is a root and $1 \le i \le 8 < j \le 120$. For the convenience of the reader, we list in the Appendix, the nonzero structure constants $N(i,j)$ for $0 < i < j \le 120$, and explain how the other $N(i,j)$'s can be computed.

The action of σ on $\mathfrak{h}^c$ induces naturally an action on the dual space of $\mathfrak{h}^c$ which preserves Φ. If $\sigma(\alpha_i) = \alpha_j$ we also write $\sigma(i) = j$. Note that $\sigma(i) = j$ implies that $\sigma(-i) = -j$. One can further assume that the Chevalley basis has been chosen so that, in addition to the properties mentioned above, the action of σ on the X_i's is given by

$$\sigma(X_i) = \xi_i X_{\sigma(i)}$$

where $\xi_i = \pm 1$. When $\mathfrak{g}$ is of type EVIII, i.e., $\mathfrak{g}$ is the split real form of $\mathfrak{g}^c$, then the action of σ on Φ is trivial, and $\sigma(X_i) = X_i$ for all i.

Table 1.
Positive roots of E_8.

i	α_i, H_i	i	α_i, H_i	i	α_i, H_i	i	α_i, H_i	i	α_i, H_i
1	10000000	25	01111000	49	01121110	73	11122211	97	22343210
2	01000000	26	01011100	50	01111111	74	01122221	98	12343211
3	00100000	27	00111100	51	11221100	75	12232110	99	12243221
4	00010000	28	00011110	52	11122100	76	11232210	100	12233321
5	00001000	29	00001111	53	11121110	77	11232111	101	22343211
6	00000100	30	11111000	54	11111111	78	11222211	102	12343221
7	00000010	31	10111100	55	01122110	79	11122221	103	12243321
8	00000001	32	01121000	56	01121111	80	12232210	104	22343221
9	10100000	33	01111100	57	11222100	81	12232111	105	12343321
10	01010000	34	01011110	58	11221110	82	11233210	106	12244321
11	00110000	35	00111110	59	11122110	83	11232211	107	22343321
12	00011000	36	00011111	60	11121111	84	11222221	108	12344321
13	00001100	37	11121000	61	01122210	85	12233210	109	22344321
14	00000110	38	11111100	62	01122111	86	12232211	110	12354321
15	00000011	39	10111110	63	11232100	87	11233211	111	22354321
16	10110000	40	01121100	64	11222110	88	11232221	112	13354321
17	01110000	41	01111110	65	11221111	89	12243210	113	23354321
18	01011000	42	01011111	66	11122210	90	12233211	114	22454321
19	00111000	43	00111111	67	11122111	91	12232221	115	23454321
20	00011100	44	11221000	68	01122211	92	11233221	116	23464321
21	00001110	45	11121100	69	12232100	93	12343210	117	23465321
22	00000111	46	11111110	70	11232110	94	12243211	118	23465421
23	11110000	47	10111111	71	11222210	95	12233221	119	23465431
24	10111000	48	01122100	72	11222111	96	11233321	120	23465432

Assume now that $\mathfrak{g}$ is of type EIX. We have $\xi_i = 1$ whenever $\alpha_i \in \Phi_0$, and $\xi_{-i} = \xi_i$ for all i. We may choose $\xi_i = 1$ for $1 \leq i \leq 8$. In Table 2 we list the vectors $\sigma(X_i)$ for $i > 0$. If $\mathfrak{g}_s$ is the split real form of $\mathfrak{g}^c$ which is the real span of the above Chevalley basis, then one can check that $\mathfrak{g} \cap \mathfrak{g}_s$ is isomorphic to $\mathrm{EVI} + \mathfrak{su}(2)$.

Let (E, H, F) be a real Cayley triple and (E', H', F') its Cayley transform. In order to distinguish between various G-orbits in $\mathfrak{g}$ which are contained in the same nilpotent G^c-orbit in $\mathfrak{g}^c$, we use two invariants tr and inv defined by

$$\mathrm{tr} := \mathrm{trace}\ (\mathrm{ad}\ (H')^2|_{\mathfrak{k}^c}), \quad \mathrm{inv} := \dim Z_{\mathfrak{k}^c}(H').$$

Table 2.
EIX $= E_{8(-24)}$: Action of σ on the X_i's.

i	$\sigma(X_i)$	i	$\sigma(X_i)$	i	$\sigma(X_i)$	i	$\sigma(X_i)$	i	$\sigma(X_i)$	i	$\sigma(X_i)$
1	X_{44}	21	X_{49}	41	$-X_{28}$	61	X_{61}	81	X_{47}	101	X_{101}
2	X_{-2}	22	$-X_{62}$	42	X_{43}	62	$-X_{22}$	82	X_{80}	102	$-X_{79}$
3	X_{-3}	23	X_{24}	43	X_{42}	63	$-X_{38}$	83	$-X_{90}$	103	X_{108}
4	X_{-4}	24	X_{23}	44	X_1	64	X_{53}	84	X_{99}	104	X_{104}
5	X_{-5}	25	X_{-25}	45	X_{57}	65	$-X_{67}$	85	$-X_{76}$	105	$-X_{106}$
6	X_{48}	26	X_{27}	46	$-X_{70}$	66	$-X_{93}$	86	X_{87}	106	$-X_{105}$
7	X_7	27	X_{26}	47	X_{81}	67	$-X_{65}$	87	X_{86}	107	$-X_{117}$
8	X_8	28	$-X_{41}$	48	X_6	68	X_{68}	88	$-X_{95}$	108	X_{103}
9	X_{37}	29	X_{56}	49	X_{21}	69	X_{31}	89	X_{71}	109	X_{116}
10	X_{-10}	30	$-X_{16}$	50	$-X_{36}$	70	$-X_{46}$	90	$-X_{83}$	110	$-X_{100}$
11	X_{-11}	31	X_{69}	51	$-X_{52}$	71	X_{89}	91	X_{92}	111	$-X_{115}$
12	X_{-12}	32	X_{-32}	52	$-X_{51}$	72	X_{60}	92	X_{91}	112	X_{96}
13	X_{40}	33	$-X_{20}$	53	X_{64}	73	$-X_{98}$	93	$-X_{66}$	113	X_{114}
14	$-X_{55}$	34	X_{35}	54	$-X_{77}$	74	X_{74}	94	X_{78}	114	X_{113}
15	X_{15}	35	X_{34}	55	$-X_{14}$	75	X_{39}	95	$-X_{88}$	115	$-X_{111}$
16	$-X_{30}$	36	$-X_{50}$	56	X_{29}	76	$-X_{85}$	96	X_{112}	116	X_{109}
17	X_{-17}	37	X_9	57	X_{45}	77	$-X_{54}$	97	X_{97}	117	$-X_{107}$
18	X_{-18}	38	$-X_{63}$	58	$-X_{59}$	78	X_{94}	98	$-X_{73}$	118	X_{118}
19	X_{-19}	39	X_{75}	59	$-X_{58}$	79	$-X_{102}$	99	X_{84}	119	X_{119}
20	$-X_{33}$	40	X_{13}	60	X_{72}	80	X_{82}	100	$-X_{110}$	120	X_{120}

We use the following procedure to compute these invariants. It will be convenient to identify an element of $\mathfrak{g}^c$ with the column vector of its coordinates with respect to the above Chevalley basis. The matrices of the operators ad (X_k) are easy to compute from the known structure constants. As $H' = i(E+F)$ and E and F are expressed as linear combinations of the X_k's, we can compute the matrix, say X, of ad (H'). By applying the Maple function *colspan* to the matrix of the projector $(1+\theta)/2$ we obtain a matrix, say S, whose columns form a basis of $\mathfrak{k}^c$. The output, say Y, of the Maple call *linsolve*(S, XS) is the matrix of the restriction ad $(H')|_{\mathfrak{k}^c}$. Finally, tr is the trace of Y^2 and, since Y is semisimple, inv is equal to the corank of Y.

3. Complex nonzero nilpotent orbits.

In Table 3 we list the characteristics H of all 69 nonzero nilpotent orbits $\mathcal{O}^c$ in $\mathfrak{g}^c$. Column 2 contains the labels $\alpha_i(H)$ for $1 \leq i \leq 8$, while the next column lists the coefficients k_i in the linear combination $H = k_1H_1 + \cdots + k_8H_8$. Column 4 lists the G^c-conjugacy types of all minimal regular semisimple subalgebras $\mathfrak{s}^c$ that meet $\mathcal{O}^c$. More precisely, we identify there the intersection $\mathcal{O}_1^c := \mathcal{O}^c \cap \mathfrak{s}^c$. As mentioned earlier, this intersection is a single nilpotent orbit of $\mathfrak{s}^c$.

We now explain how to identify $\mathcal{O}_1^c$. Let $X + Y + \cdots + Z$ be the Cartan symbol of $\mathfrak{s}^c$. Thus each of $X, Y, \ldots, Z$ represents a simple ideal of $\mathfrak{s}^c$. The orbit $\mathcal{O}_1^c$ is uniquely determined by its projections $\mathcal{O}_X, \mathcal{O}_Y, \ldots, \mathcal{O}_Z$ in $X, Y, \ldots, Z$. The projection $\mathcal{O}_X$, for instance, is a nonzero nilpotent orbit of the ideal X. In fact the minimality assumption for $\mathfrak{s}^c$ implies that $\mathcal{O}_X$ does not meet any proper regular subalgebra of X, and similarly for other components. We use the Cartan symbol for X to denote the fact that $\mathcal{O}_X$ is the principal orbit of X. If not, we identify this nilpotent orbit of X by the symbol $X(a_k)$ instead of plain X. For the precise definition of this symbol see [**11**, p. 199].

The last two columns indicate which nilpotent orbits of the real Lie algebra $\mathfrak{g}$ of type EVIII or EIX constitute the intersection $\mathcal{O}^c \cap \mathfrak{g}$. The numbering of nonzero nilpotent orbits of $\mathfrak{g}$ is the same as in our paper [**5**, Tables XIV and XV].

Usually two isomorphic regular semisimple subalgebras of $\mathfrak{g}^c$ are G^c-conjugate, but there are five exceptions, namely:

$$4A_1,\ A_3 + 2A_1,\ 2A_3,\ A_5 + A_1,\ A_7.$$

In these five cases there are exactly two G^c-conjugacy classes of such algebras X, and they are denoted by $(X)'$ and $(X)''$. It is easy to distinguish these two conjugacy classes: the algebras $(X)'$ are Levi subalgebras of $\mathfrak{g}$ while the algebras $(X)''$ are not. Another way to distinguish them is to observe that all the nonzero labels $\alpha_i(H)$ of the characteristic H of the nilpotent orbit $\mathcal{O}^c$ of $\mathfrak{g}$ containing the principal orbit of $(X)'$ (resp. $(X)''$) are 1 (resp 2). Lastly, it follows from Table 3 that, in the case of $(X)''$, $\mathcal{O}^c$ meets the real form of type EIX, while this is not true for $(X)'$.

Table 3.
Nonzero nilpotent orbits in $\mathfrak{g}^c = E_8$.

No.	$\alpha_i(H)$	k_i	$\mathfrak{s}^c$	EVIII	EIX
1	00000001	2,3,4,6,5,4,3,2	A_1	1	1
2	10000000	4,5,7,10,8,6,4,2	$2A_1$	2	2,3
3	00000010	4,6,8,12,10,8,6,3	$3A_1$	3	4,5
4	00000002	4,6,8,12,10,8,6,4	$A_2, (4A_1)''$	4,5	6,7,8
5	01000000	5,8,10,15,12,9,6,3	$(4A_1)'$	6	
6	10000001	6,8,11,16,13,10,7,4	$A_2 + A_1, 5A_1$	7,8	9
7	00000100	6,9,12,18,15,12,8,4	$A_2 + 2A_1, 6A_1$	9,10	10,11
8	10000002	8,11,15,22,18,14,10,6	A_3	11	12,13
9	00100000	7,10,14,20,16,12,8,4	$A_2 + 3A_1, 7A_1$	12,13	
10	20000000	8,10,14,20,16,12,8,4	$A_2 + 4A_1, 2A_2, 8A_1$	14,15, 16	14
11	10000010	8,11,15,22,18,14,10,5	$2A_2 + A_1$	17	15
12	00000101	8,12,16,24,20,16,11,6	$A_3 + A_1$	18	16,17
13	00000020	8,12,16,24,20,16,12,6	$3A_2, D_4(a_1),$ $(A_3 + 2A_1)''$	19,20	18,19, 20
14	00000022	12,18,24,36,30,24,18,10	D_4	21	21,22
15	00001000	8,12,16,24,20,15,10,5	$2A_2 + 2A_1$	22	
16	00100001	9,13,18,26,21,16,11,6	$(A_3 + 2A_1)'$	23	
17	01000010	9,14,18,27,22,17,12,6	$A_3 + 3A_1,$ $3A_2 + A_1,$ $D_4(a_1) + A_1$	24,25	
18	10000100	10,14,19,28,23,18,12,6	$A_3 + 4A_1, A_3 + A_2,$ $D_4(a_1) + 2A_1$	26,27, 28	23
19	20000002	12,16,22,32,26,20,14,8	$A_4, (2A_3)''$	29,30	24,25
20	00010000	10,15,20,30,24,18,12,6	$A_3 + A_2 + A_1,$ $D_4(a_1) + 3A_1$	31,32	
21	01000012	13,20,26,39,32,25,18,10	$D_4 + A_1$	33	
22	02000000	10,16,20,30,24,18,12,6	$A_3 + A_2 + 2A_1,$ $4A_2, D_4(a_1) + A_2,$ $D_4(a_1) + 4A_1$	34,35, 36	
23	10000101	12,17,23,34,28,22,15,8	$A_4 + A_1, 2A_3 + A_1$	37,38	26
24	10001000	12,17,23,34,28,21,14,7	$(2A_3)'$	39	
25	10000102	14,20,27,40,33,26,18,10	$D_4 + 2A_1, D_5(a_1)$	40,41	27
26	00010001	12,18,24,36,29,22,15,8	$A_4 + 2A_1,$ $2A_3 + 2A_1,$ $D_4(a_1) + A_3$	42,43, 44	
27	00000200	12,18,24,36,30,24,16,8	$A_4 + A_2, 2D_4(a_1)$	45,46	28

Table 3.
(continued)

No.	$\alpha_i(H)$	k_i	$\mathfrak{s}^c$	EVIII	EIX
28	20000101	16,22,30,44,36,28,19,10	A_5	47	29
29	00010002	14,21,28,42,34,26,18,10	D_4+3A_1, $D_5(a_1)+A_1$	48,49	
30	00100100	13,19,26,38,31,24,16,8	$A_4+A_2+A_1$	50	
31	02000002	14,22,28,42,34,26,18,10	D_4+A_2, D_4+4A_1, $D_5(a_1)+2A_1$	51,52,53	
32	20000020	16,22,30,44,36,28,20,10	$(A_5+A_1)''$	54,55	30,31
33	20000022	20,28,38,56,46,36,26,14	D_5	56	32,33
34	00010010	14,21,28,42,34,26,18,9	A_4+A_3	57	
35	10010001	16,23,31,46,37,28,19,10	$(A_5+A_1)'$	58	
36	00100101	15,22,30,44,36,28,19,10	$D_5(a_1)+A_2$	59	
37	01100010	16,24,32,47,38,29,20,10	$D_4+A_3, D_6(a_2)$	60,61	
38	10001010	16,23,31,46,38,29,20,10	A_5+2A_1	62,63	
39	00010100	16,24,32,48,39,30,20,10	A_5+A_2, $D_6(a_2)+A_1$	64,65	
40	10001012	20,29,39,58,48,37,26,14	D_5+A_1	66	
41	00002000	16,24,32,48,40,30,20,10	$A_5+A_2+A_1$, $2A_4$, $D_6(a_2)+2A_1$, $D_5(a_1)+A_3$, $D_4+D_4(a_1)$	67,68,69	
42	20000200	20,28,38,56,46,36,24,12	$A_6, 2D_4$	70,71	34
43	01100012	20,30,40,59,48,37,26,14	$D_5+2A_1, D_6(a_1)$	72,73	
44	10010100	20,29,39,58,47,36,24,12	A_6+A_1	74	
45	00010102	20,30,40,60,49,38,26,14	$D_6(a_1)+A_1$	75,76	
46	20000202	24,34,46,68,56,44,30,16	$(A_7)'', E_6(a_1)$	77,78	35
47	00002002	20,30,40,60,50,38,26,14	D_5+A_2, $D_6(a_1)+2A_1$	79,80,81	
48	20000222	32,46,62,92,76,60,42,22	E_6	82	36
49	21100012	28,40,54,79,64,49,34,18	D_6	83	
50	10010101	22,32,43,64,52,40,27,14	$D_5+A_3, D_7(a_2)$	84,85	
51	10010110	24,35,47,70,57,44,30,15	$(A_7)'$	86	
52	10010102	24,35,47,70,57,44,30,16	A_7+A_1, $E_6(a_1)+A_1$	87,88	
53	20010102	28,40,54,80,65,50,34,18	D_6+A_1	89,90	
54	00020002	24,36,48,72,58,44,30,16	$D_8(a_3)$, $E_6(a_1)+A_2$	91,92	
55	20002002	28,40,54,80,66,50,34,18	$D_6+2A_1, D_7(a_1)$	93,94,95	

Table 3.
(continued)

No.	$\alpha_i(H)$	k_i	$\mathfrak{s}^c$	EVIII	EIX
56	10010122	$32, 47, 63, 94, 77, 60, 42, 22$	$E_6 + A_1$	96	
57	01101022	$32, 48, 64, 95, 78, 60, 42, 22$	$E_7(a_2)$	97	
58	00020020	$28, 42, 56, 84, 68, 52, 36, 18$	$A_8, D_8(a_2)$	$98, 99$	
59	21101101	$36, 52, 70, 103, 84, 64, 43, 22$	D_7	100	
60	00020022	$32, 48, 64, 96, 78, 60, 42, 22$	$E_6 + A_2$, $E_7(a_2) + A_1$	$101, 102$	
61	21101022	$40, 58, 78, 115, 94, 72, 50, 26$	$E_7(a_1)$	103	
62	20020020	$36, 52, 70, 104, 84, 64, 44, 22$	$D_8(a_1)$	$104, 105$	
63	20020022	$40, 58, 78, 116, 94, 72, 50, 26$	$E_7(a_1) + A_1$	$106, 107$	
64	21101222	$52, 76, 102, 151, 124, 96, 66, 34$	E_7	108	
65	20020202	$44, 64, 86, 128, 104, 80, 54, 28$	D_8	$109, 110$	
66	20020222	$52, 76, 102, 152, 124, 96, 66, 34$	$E_7 + A_1$	$111, 112$	
67	22202022	$60, 88, 118, 174, 142, 108, 74, 38$	$E_8(a_2)$	113	
68	22202222	$72, 106, 142, 210, 172, 132, 90, 46$	$E_8(a_1)$	114	
69	22222222	$92, 136, 182, 270, 220, 168, 114, 58$	E_8	115	

We point out that the coordinates k_i are also listed in Dynkin's paper [**9**] but his k_6 is incorrect for orbits 62 and 63.

4. Real Cayley triples in EVIII.

There are 115 nonzero nilpotent orbits in $\mathfrak{g}$ = EVIII. Consequently, there are also 115 K-orbits of real Cayley triples (E, H, F) in $\mathfrak{g}$ (see [**8**, Proposition 1]), where K is the connected subgroup of G with Lie algebra $\mathfrak{k}$.

We list in Table 4 the representatives (E, H, F) for K-orbits of real Cayley triples in $\mathfrak{g}$. The elements E are the representatives of the nonzero nilpotent G-orbits in $\mathfrak{g}$. We record only the elements E because H is given in Table 3 and F can be easily computed since

$$F = \theta(E) = \theta\sigma(E) = \sigma_u(E)$$

and $\sigma_u(X_i) = X_{-i}$ (for all i). In all cases the restriction of σ to $\mathfrak{s}^c$ is such that $\mathfrak{s}$ is the split real form of $\mathfrak{s}^c$. If $\mathfrak{s}^c = X + Y + \cdots + Z$, as in the previous section, then E is written as

$$E = [E_X] + [E_Y] + \cdots + [E_Z],$$

where $E_X \in X,\ E_Y \in Y, \ldots, E_Z \in Z$, and each of these components belongs to $\mathfrak{s}$. If, say, $Z = A_1$ then we omit the brackets around E_Z. We also omit the brackets around E_X if $\mathfrak{s}^c = X$ is simple.

As an example let us consider the orbit 19. In that case Table 4 provides two representatives (E, H, F) for the same K-orbit of real Cayley triples. At the same time they are two distinct representatives of the corresponding G-orbit of standard triples. The first representative arises from the regular subalgebra $\mathfrak{s}^c = (A_3 + 2A_1)''$. In this case we have $X = A_3$, $Y = A_1$, and $Z = A_1$. The nilpositive element E, from Table 4, is

$$E = \left[\sqrt{3}(X_{22} + X_{62}) + 2X_{97}\right] + X_7 - X_{61},$$

and its components are $E_X = \sqrt{3}(X_{22} + X_{62}) + 2X_{97}$, $E_Y = X_7$, and $E_Z = -X_{61}$. From row 13 of Table 3 we find that the neutral element H is given by

$$H = 8H_1 + 12H_2 + 16H_3 + 24H_4 + 20H_5 + 16H_6 + 12H_7 + 6H_8.$$

By applying the conjugation σ_u to E, we find that

$$F = \left[\sqrt{3}(X_{-22} + X_{-62}) + 2X_{-97}\right] + X_{-7} - X_{-61}.$$

The second representative arises from the subalgebra $\mathfrak{s}^c = 3A_2$. Hence in this case $X = A_2$, $Y = A_2$, and $Z = A_2$. We have

$$E_X = \sqrt{2}(X_{14} + X_{101}), \quad E_Y = \sqrt{2}(X_{15} + X_{93}), \quad E_Z = \sqrt{2}(X_{59} + X_{68}),$$

and $E = E_X + E_Y + E_Z$. The neutral element H is the same as above and for F we obtain

$$F = \left[\sqrt{2}(X_{-14} + X_{-101})\right] + \left[\sqrt{2}(X_{-15} + X_{-93})\right] + \left[\sqrt{2}(X_{-59} + X_{-68})\right].$$

The nilpotent G-orbits in $\mathfrak{g}$ which are contained in the same nilpotent G^c-orbit in $\mathfrak{g}^c$ are distinguished by the invariant inv. There are only 3 cases where this is not so, namely the orbit pairs $(52, 53)$, $(79, 81)$, and $(93, 95)$. In these three cases the orbits can be distinguished by comparing the multiplicities of various eigenvalues of the linear operator ad $(H')|_{\mathfrak{k}^c}$. For instance, the largest eigenvalue $\lambda = 8$ of that operator has multiplicity 3 for the orbit 52, and multiplicity 4 for the orbit 53.

Table 4.
Cayley triples in EVIII = $E_{8(8)}$.

No.	inv	$\mathfrak{s}^c$	E
1	64	A_1	X_{120}
2	44	$2A_1$	$X_{97} + X_{120}$
3	40	$3A_1$	$X_{74} + X_{104} + X_{118}$
4	70	$(4A_1)''$	$X_8 + X_{74} + X_{104} + X_{118}$
5	64	$(4A_1)''$	$X_8 + X_{74} + X_{104} - X_{118}$
		A_2	$\sqrt{2}(X_8 + X_{119})$
6	32	$(4A_1)'$	$X_{69} + X_{91} + X_{106} + X_{114}$
7	38	$5A_1$	$X_{47} - X_{81} + X_{97} + X_{100} + X_{110}$
8	32	$5A_1$	$X_{47} + X_{81} + X_{97} + X_{100} + X_{110}$
		$A_2 + A_1$	$[\sqrt{2}(X_{47} + X_{112})] + X_{97}$
9	38	$6A_1$	$X_{61} - X_{73} + X_{84} + X_{97} + X_{98} + X_{99}$
10	26	$6A_1$	$X_{61} + X_{73} + X_{84} + X_{97} + X_{98} + X_{99}$
		$A_2 + 2A_1$	$[\sqrt{2}(X_{73} + X_{102})] + X_{61} + X_{97}$
11	32	A_3	$\sqrt{3}(X_8 + X_{74}) + 2X_{97}$
12	50	$7A_1$	$X_{44} + X_{71} - X_{83} + X_{89} + X_{90} + X_{91} + X_{92}$
13	26	$7A_1$	$X_{44} + X_{71} + X_{83} + X_{89} + X_{90} + X_{91} + X_{92}$
		$A_2 + 3A_1$	$[\sqrt{2}(X_{83} + X_{95})] + X_{44} + X_{71} + X_{89}$
14	92	$8A_1$	$X_1 + X_{44} + X_{71} - X_{83} + X_{89} + X_{90} + X_{91} + X_{92}$
15	50	$8A_1$	$X_1 + X_{44} + X_{71} + X_{83} + X_{89} + X_{90} + X_{91} + X_{92}$
		$A_2 + 4A_1$	$[\sqrt{2}(X_{83} + X_{95})] + X_1 + X_{44} + X_{71} + X_{89}$
16	44	$8A_1$	$X_1 - X_{44} + X_{71} + X_{83} + X_{89} + X_{90} + X_{91} + X_{92}$
		$A_2 + 4A_1$	$[\sqrt{2}(X_{83} + X_{95})] + X_1 - X_{44} + X_{71} + X_{89}$
		$2A_2$	$[\sqrt{2}(X_1 + X_{112})] + [\sqrt{2}(X_{44} + X_{96})]$
17	24	$2A_2 + A_1$	$[\sqrt{2}(X_{39} + X_{98})] + [\sqrt{2}(X_{54} + X_{89})] + X_{74}$
18	24	$A_3 + A_1$	$[\sqrt{3}(X_{22} + X_{62}) + 2X_{97}] + X_{61}$
19	40	$(A_3 + 2A_1)''$	$[\sqrt{3}(X_{22} + X_{62}) + 2X_{97}] + X_7 - X_{61}$
		$3A_2$	$[\sqrt{2}(X_{14} + X_{101})] + [\sqrt{2}(X_{15} + X_{93})]$
			$+[\sqrt{2}(X_{59} + X_{68})]$
20	42	$(A_3 + 2A_1)''$	$[\sqrt{3}(X_{22} + X_{62}) + 2X_{97}] + X_7 + X_{61}$
		$D_4(a_1)$	$2X_{34} + X_{58} + X_{82} + \sqrt{3}(X_{65} - X_{87})$
21	40	D_4	$\sqrt{6}(X_7 + X_{61} + X_{97}) + \sqrt{10}X_8$
22	20	$2A_2 + 2A_1$	$[\sqrt{2}(X_{48} + X_{91})] + [\sqrt{2}(X_{59} + X_{83})] + X_{71} + X_{72}$
23	20	$(A_3 + 2A_1)'$	$[\sqrt{3}(X_{43} + X_{54}) + 2X_{89}] + X_{44} + X_{71}$
24	20	$A_3 + 3A_1$	$[\sqrt{3}(X_{50} + X_{60}) + 2X_{82}] + X_{34} - X_{58} + X_{69}$
		$3A_2 + A_1$	$[\sqrt{2}(X_{34} + X_{87})] + [\sqrt{2}(X_{54} + X_{76})]$
			$+[\sqrt{2}(X_{56} + X_{64})] + X_{69}$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
25	22	A_3+3A_1	$[\sqrt{3}(X_{50}+X_{60})+2X_{82}]+X_{34}+X_{58}+X_{69}$
		$D_4(a_1)+A_1$	$[2X_{34}+X_{58}+X_{82}+\sqrt{3}(X_{65}-X_{87})]+X_{69}$
26	22	A_3+4A_1	$[\sqrt{3}(X_{60}+X_{72})+2X_{61}]+X_{31}-X_{46}+X_{69}+X_{70}$
27	26	A_3+4A_1	$[\sqrt{3}(X_{60}+X_{72})+2X_{61}]+X_{31}+X_{46}-X_{69}+X_{70}$
		$D_4(a_1)+2A_1$	$[X_{51}+X_{52}+2X_{61}+\sqrt{3}(X_{65}-X_{67})]+X_{46}+X_{70}$
28	18	A_3+4A_1	$[\sqrt{3}(X_{60}+X_{72})+2X_{61}]+X_{31}+X_{46}+X_{69}+X_{70}$
		$D_4(a_1)+2A_1$	$[X_{51}+X_{52}+2X_{61}+\sqrt{3}(X_{65}-X_{67})]+X_{46}-X_{70}$
		A_3+A_2	$[\sqrt{3}(X_{31}+X_{69})+2X_{74}]+[\sqrt{2}(X_{46}+X_{77})]$
29	36	$(2A_3)''$	$[\sqrt{3}(X_1+X_{44})+2X_{74}]+[\sqrt{3}(X_{71}+X_{89})+2X_8]$
30	32	$(2A_3)''$	$[\sqrt{3}(X_1-X_{44})+2X_{74}]+[\sqrt{3}(X_{71}+X_{89})+2X_8]$
		A_4	$\sqrt{6}(X_1+X_{93})+2(X_8+X_{74})$
31	34	$D_4(a_1)+3A_1$	$[2X_{58}+X_{59}+X_{61}+\sqrt{3}(X_{67}-X_{68})]$ $-X_{32}+X_{45}+X_{57}$
32	18	$D_4(a_1)+3A_1$	$[2X_{58}+X_{59}+X_{61}+\sqrt{3}(X_{67}-X_{68})]$ $+X_{32}+X_{45}+X_{57}$
		$A_3+A_2+A_1$	$[\sqrt{3}(X_{45}+X_{57})+2X_{74}]+[\sqrt{2}(X_{58}+X_{67})]+X_{32}$
33	20	D_4+A_1	$[\sqrt{6}(X_{34}+X_{58}+X_{82})+\sqrt{10}X_8]+X_{69}$
34	64	$D_4(a_1)+4A_1$	$[2X_{34}+X_{58}+X_{82}+\sqrt{3}(X_{65}-X_{87})]$ $+X_{25}+X_{37}+X_{38}-X_{40}$
35	32	$D_4(a_1)+4A_1$	$[2X_{34}+X_{58}+X_{82}+\sqrt{3}(X_{65}-X_{87})]$ $+X_{25}+X_{37}-X_{38}+X_{40}$
		$A_3+A_2+2A_1$	$[\sqrt{3}(X_{45}+X_{57})+2X_{74}]$ $+[\sqrt{2}(X_{58}+X_{67})]+X_2-X_{32}$
		$4A_2$	$[\sqrt{2}(X_2+X_{76})]+[\sqrt{2}(X_{32}+X_{73})]$ $+[\sqrt{2}(X_{57}+X_{74})]+[\sqrt{2}(X_{59}+X_{65})]$
36	34	$D_4(a_1)+4A_1$	$[2X_{34}+X_{58}+X_{82}+\sqrt{3}(X_{65}-X_{87})]$ $+X_{25}+X_{37}+X_{38}+X_{40}$
		$A_3+A_2+2A_1$	$[\sqrt{3}(X_{45}+X_{57})+2X_{74}]$ $+[\sqrt{2}(X_{58}+X_{67})]+X_2+X_{32}$
		$D_4(a_1)+A_2$	$[2X_{32}+X_{57}+X_{59}+\sqrt{3}(X_{78}-X_{79})]$ $+[\sqrt{2}(X_2+X_{76})]$
37	20	$2A_3+A_1$	$[\sqrt{3}(X_{31}+X_{46})+2X_{62}]$ $+[\sqrt{3}(X_{69}-X_{70})+2X_{22}]+X_{61}$
38	16	$2A_3+A_1$	$[\sqrt{3}(X_{31}+X_{46})+2X_{62}]$ $+[\sqrt{3}(X_{69}+X_{70})+2X_{22}]+X_{61}$
		A_4+A_1	$[2(X_{22}+X_{62})+\sqrt{6}(X_{31}+X_{75})]+X_{61}$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
39	16	$(2A_3)'$	$[\sqrt{3}(X_{24}+X_{38})+2X_{74}]$ $+[\sqrt{3}(X_{53}+X_{65})+2X_{48}]$
40	22	D_4+2A_1	$[\sqrt{6}(X_{46}+X_{61}+X_{70})+\sqrt{10}X_8]+X_{31}-X_{69}$
41	16	D_4+2A_1	$[\sqrt{6}(X_{46}+X_{61}+X_{70})+\sqrt{10}X_8]+X_{31}+X_{69}$
		$D_5(a_1)$	$X_{58}+X_{59}+\sqrt{10}X_{15}+\sqrt{6}(X_{51}-X_{52}+X_{61})$
42	16	$2A_3+2A_1$	$[\sqrt{3}(X_{45}-X_{59})+2X_{50}]$ $+[\sqrt{3}(X_{57}+X_{58})+2X_{36}]+X_{32}+X_{61}$
43	20	$2A_3+2A_1$	$[\sqrt{3}(X_{45}+X_{59})+2X_{50}]$ $+[\sqrt{3}(X_{57}-X_{58})+2X_{36}]+X_{32}+X_{61}$
		$D_4(a_1)+A_3$	$[X_{42}+2X_{52}+X_{53}+\sqrt{3}(X_{50}-X_{58})]$ $+[\sqrt{3}(X_{40}-X_{55})+2X_{47}]$
44	14	$2A_3+2A_1$	$[\sqrt{3}(X_{45}+X_{59})+2X_{50}]$ $+[\sqrt{3}(X_{57}+X_{58})+2X_{36}]+X_{32}+X_{61}$
		$D_4(a_1)+A_3$	$[X_{42}+2X_{52}+X_{53}+\sqrt{3}(X_{50}-X_{58})]$ $+[\sqrt{3}(X_{40}+X_{55})+2X_{47}]$
		A_4+2A_1	$[2(X_{36}+X_{54})+\sqrt{6}(X_{32}+X_{71})]+X_{45}+X_{59}$
45	54	$2D_4(a_1)$	$[X_{13}+2X_{34}+X_{62}+\sqrt{3}(X_{31}-X_{77})]$ $+[-X_{22}+2X_{35}+X_{40}+\sqrt{3}(X_{54}+X_{69})]$
46	26	$2D_4(a_1)$	$[X_{13}-2X_{34}+X_{62}+\sqrt{3}(X_{31}-X_{77})]$ $+[-X_{22}+2X_{35}+X_{40}+\sqrt{3}(X_{54}+X_{69})]$
		A_4+A_2	$[2(X_6+X_{57})+\sqrt{6}(X_{36}+X_{75})]$ $+[\sqrt{2}(X_{21}+X_{65})]$
47	16	A_5	$3X_{61}+2\sqrt{2}(X_1+X_{44})+\sqrt{5}(X_{22}+X_{62})$
48	26	D_4+3A_1	$[\sqrt{6}(X_{58}+X_{59}+X_{61})+\sqrt{10}X_8]$ $+X_{32}-X_{45}+X_{57}$
49	14	D_4+3A_1	$[\sqrt{6}(X_{58}+X_{59}+X_{61})+\sqrt{10}X_8]$ $+X_{32}+X_{45}+X_{57}$
		$D_5(a_1)+A_1$	$[X_{64}+X_{66}+\sqrt{10}X_{29}+\sqrt{6}(X_{44}-X_{45}+X_{49})]$ $+X_{48}$
50	14	$A_4+A_2+A_1$	$[2(X_{27}+X_{38})+\sqrt{6}(X_{49}+X_{67})]$ $+[\sqrt{2}(X_{39}+X_{50})]+X_{44}$
51	50	D_4+4A_1	$[\sqrt{6}(X_{58}+X_{59}+X_{61})+\sqrt{10}X_8]$ $+X_2+X_{32}-X_{45}+X_{57}$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
52	26	D_4+4A_1	$[\sqrt{6}(X_{58}+X_{59}+X_{61})+\sqrt{10}X_8]$ $+X_2+X_{32}+X_{45}-X_{57}$
		$D_5(a_1)+2A_1$	$[X_{64}+X_{76}+\sqrt{6}(X_{30}+X_{34}-X_{45})+\sqrt{10}X_{43}]$ $+X_{17}-X_{48}$
53	26	D_4+4A_1	$[\sqrt{6}(X_{58}+X_{59}+X_{61})+\sqrt{10}X_8]$ $+X_2+X_{32}+X_{45}+X_{57}$
		$D_5(a_1)+2A_1$	$[X_{64}+X_{76}+\sqrt{6}(X_{30}+X_{34}-X_{45})+\sqrt{10}X_{43}]$ $+X_{17}+X_{48}$
		D_4+A_2	$[\sqrt{6}(X_{32}+X_{45}+X_{58})+\sqrt{10}X_{29}]$ $+[\sqrt{2}(X_2+X_{82})]$
54	24	$(A_5+A_1)''$	$[-3X_{61}+2\sqrt{2}(X_1+X_{44})+\sqrt{5}(X_{22}+X_{62})]+X_7$
55	26	$(A_5+A_1)''$	$[3X_{61}+2\sqrt{2}(X_1+X_{44})+\sqrt{5}(X_{22}+X_{62})]+X_7$
56	24	D_5	$\sqrt{10}(X_1+X_{44})+\sqrt{2}(2X_7+3X_{61})+\sqrt{14}X_8$
57	12	A_4+A_3	$[2(X_{32}+X_{45})+\sqrt{6}(X_{41}+X_{47})]$ $+[\sqrt{3}(X_{28}+X_{42})+2X_{57}]$
58	12	$(A_5+A_1)'$	$[3X_{61}+2\sqrt{2}(X_{16}+X_{30})+\sqrt{5}(X_{36}+X_{50})]+X_{32}$
59	12	$D_5(a_1)+A_2$	$[X_{31}-X_{35}+\sqrt{6}(X_{44}+X_{52}+X_{55})+\sqrt{10}X_{22}]$ $+[\sqrt{2}(X_{40}+X_{46})]$
60	12	D_4+A_3	$[\sqrt{6}(X_{17}+X_{34}+X_{48})+\sqrt{10}X_{47}]$ $+[2X_{35}+\sqrt{3}(X_{30}-X_{45})]$
61	16	D_4+A_3	$[\sqrt{6}(X_{17}+X_{34}+X_{48})+\sqrt{10}X_{47}]$ $+[2X_{35}+\sqrt{3}(X_{30}+X_{45})]$
		$D_6(a_2)$	$2X_{43}+\sqrt{3}(X_{37}-X_{38})+\sqrt{6}(X_{32}+X_{33}+X_{42})$ $+\sqrt{10}X_{39}$
62	12	A_5+2A_1	$[3X_{49}+2\sqrt{2}(X_{24}+X_{38})+\sqrt{5}(X_{36}-X_{50})]$ $+X_{21}+X_{48}$
63	14	A_5+2A_1	$[3X_{49}+2\sqrt{2}(X_{24}+X_{38})+\sqrt{5}(X_{36}+X_{50})]$ $+X_{21}+X_{48}$
64	22	$D_6(a_2)+A_1$	$[2X_{43}+\sqrt{10}X_{39}+\sqrt{3}(X_{37}-X_{38})$ $+\sqrt{6}(X_{32}+X_{33}+X_{42})]+X_{20}$
65	12	$D_6(a_2)+A_1$	$[2X_{43}+\sqrt{10}X_{39}+\sqrt{3}(X_{37}-X_{38})$ $+\sqrt{6}(X_{32}+X_{33}+X_{42})]-X_{20}$
		A_5+A_2	$[3X_{32}+2\sqrt{2}(X_{39}+X_{54})+\sqrt{5}(X_{20}+X_{33})]$ $+[\sqrt{2}(X_{34}+X_{43})]$
66	12	D_5+A_1	$[\sqrt{14}X_8+\sqrt{2}(2X_{21}+3X_{49})+\sqrt{10}(X_{24}+X_{38})]$ $+X_{48}$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
67	20	$D_6(a_2) + 2A_1$	$[2X_{43} + \sqrt{10}X_{39} + \sqrt{3}(X_{37} - X_{38})$ $+\sqrt{6}(X_{32} + X_{33} + X_{42})] + X_5 - X_{20}$
		$D_5(a_1) + A_3$	$[X_{51} + X_{53} + \sqrt{10}X_{47} + \sqrt{6}(X_{32} + X_{33} - X_{34})]$ $+[2X_{30} + \sqrt{3}(X_{20} + X_{35})]$
		$D_4 + D_4(a_1)$	$[\sqrt{6}(X_{13} + X_{28} + X_{42}) + \sqrt{10}X_{44}]$ $+[2X_{40} + X_{41} + X_{43} + \sqrt{3}(X_{46} - X_{47})]$
		$A_5 + A_2 + A_1$	$[3X_{32} + 2\sqrt{2}(X_{39} + X_{54}) + \sqrt{5}(X_{20} + X_{33})]$ $+[\sqrt{2}(X_{34} + X_{43})] - X_5$
		$2A_4$	$[2(X_{12} + X_{31}) + \sqrt{6}(X_{34} + X_{65})]$ $+[2(X_{29} + X_{35}) + \sqrt{6}(X_{30} + X_{40})]$
68	40	$D_6(a_2) + 2A_1$	$[2X_{43} + \sqrt{10}X_{39} + \sqrt{3}(X_{37} - X_{38})$ $+\sqrt{6}(X_{32} + X_{33} + X_{42})] - X_5 + X_{20}$
		$D_4 + D_4(a_1)$	$[\sqrt{6}(X_{13} + X_{28} - X_{42}) + \sqrt{10}X_{44}]$ $+[2X_{40} + X_{41} + X_{43} + \sqrt{3}(X_{46} - X_{47})]$
69	22	$D_6(a_2) + 2A_1$	$[2X_{43} + \sqrt{10}X_{39} + \sqrt{3}(X_{37} - X_{38})$ $+\sqrt{6}(X_{32} + X_{33} + X_{42})] + X_5 + X_{20}$
		$D_5(a_1) + A_3$	$[X_{51} + X_{53} + \sqrt{10}X_{47} + \sqrt{6}(X_{32} + X_{33} - X_{34})]$ $+[2X_{30} + \sqrt{3}(X_{20} - X_{35})]$
		$A_5 + A_2 + A_1$	$[3X_{32} + 2\sqrt{2}(X_{39} + X_{54}) + \sqrt{5}(X_{20} + X_{33})]$ $+[\sqrt{2}(X_{34} + X_{43})] + X_5$
70	38	$2D_4$	$[\sqrt{6}(X_{48} + X_{49} - X_{50}) + \sqrt{10}X_1]$ $+[\sqrt{6}(X_6 + X_{21} + X_{36}) + \sqrt{10}X_{44}]$
71	18	$2D_4$	$[\sqrt{6}(X_{48} + X_{49} + X_{50}) + \sqrt{10}X_1]$ $+[\sqrt{6}(X_6 + X_{21} + X_{36}) + \sqrt{10}X_{44}]$
		A_6	$\sqrt{10}(X_1 + X_{44}) + 2\sqrt{3}(X_{21} + X_{56}) + \sqrt{6}(X_6 + X_{48})$
72	12	$D_5 + 2A_1$	$[\sqrt{14}X_8 + \sqrt{2}(2X_{34} + 3X_{35}) + \sqrt{10}(X_{30} + X_{45})]$ $+X_{17} - X_{48}$
73	14	$D_5 + 2A_1$	$[\sqrt{14}X_8 + \sqrt{2}(2X_{34} + 3X_{35})$ $+\sqrt{10}(X_{30} + X_{45})] + X_{17} + X_{48}$
		$D_6(a_1)$	$\sqrt{14}X_8 + \sqrt{2}(2X_{34} + 3X_{39}) + X_{37} + X_{38}$ $+\sqrt{10}(X_{32} - X_{33})$
74	10	$A_6 + A_1$	$[\sqrt{10}(X_{16} + X_{30}) + 2\sqrt{3}(X_{34} + X_{43})$ $+\sqrt{6}(X_{20} + X_{33})] + X_{32}$
75	10	$D_6(a_1) + A_1$	$[\sqrt{2}(2X_{34} + 3X_{35}) + \sqrt{14}X_8 + X_{32}$ $-X_{33} + \sqrt{10}(X_{37} + X_{38})] - X_{20}$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
76	18	$D_6(a_1)+A_1$	$[\sqrt{2}(2X_{34}+3X_{35})+\sqrt{14}X_8+X_{32}$ $-X_{33}+\sqrt{10}(X_{37}+X_{38})]+X_{20}$
77	18	$(A_7)''$	$4X_8+2\sqrt{3}(X_1+X_{44})+\sqrt{7}(X_6+X_{48})$ $+\sqrt{15}(X_{21}+X_{49})$
78	16	$(A_7)''$	$4X_8+2\sqrt{3}(X_1+X_{44})+\sqrt{7}(X_6+X_{48})$ $+\sqrt{15}(X_{21}-X_{49})$
		$E_6(a_1)$	$4X_8+2\sqrt{3}(X_1+X_{44})+\sqrt{7}(X_{26}+X_{27})$ $+\sqrt{15}(X_{34}-X_{35})$
79	18	$D_6(a_1)+2A_1$	$[X_{28}+X_{38}+\sqrt{10}(X_{32}-X_{33})+\sqrt{14}X_8$ $+\sqrt{2}(2X_{34}+3X_{39})]-X_5+X_{20}$
80	34	$D_6(a_1)+2A_1$	$[X_{28}+X_{38}+\sqrt{10}(X_{32}-X_{33})+\sqrt{14}X_8$ $+\sqrt{2}(2X_{34}+3X_{39})]+X_5-X_{20}$
81	18	$D_6(a_1)+2A_1$	$[X_{28}+X_{38}+\sqrt{10}(X_{32}-X_{33})+\sqrt{14}X_8$ $+\sqrt{2}(2X_{34}+3X_{39})]+X_5+X_{20}$
		D_5+A_2	$[\sqrt{14}X_8+\sqrt{10}(X_5+X_{20})+\sqrt{2}(2X_{34}+3X_{58})]$ $+[\sqrt{2}(X_{32}+X_{38})]$
82	16	E_6	$4(X_1+X_{44})+\sqrt{30}(X_6+X_{48})+\sqrt{42}X_7+\sqrt{22}X_8$
83	12	D_6	$2\sqrt{7}X_1+3\sqrt{2}X_8+\sqrt{10}X_{34}+2\sqrt{6}X_{35}$ $+\sqrt{15}(X_{17}+X_{48})$
84	12	D_5+A_3	$[\sqrt{14}X_{22}+\sqrt{2}(2X_{32}+3X_{30})+\sqrt{10}(X_{20}-X_{35})]$ $+[2X_{16}+\sqrt{3}(X_{33}+X_{34})]$
85	8	D_5+A_3	$[\sqrt{14}X_{22}+\sqrt{2}(2X_{32}+3X_{30})+\sqrt{10}(X_{20}+X_{35})]$ $+[2X_{16}+\sqrt{3}(X_{33}+X_{34})]$
		$D_7(a_2)$	$\sqrt{14}X_{22}+\sqrt{2}(2X_{18}+3X_{24})+2X_{23}$ $+\sqrt{3}(X_{13}-X_{28})+\sqrt{10}(X_{33}+X_{49})$
86	8	$(A_7)'$	$4X_{32}+2\sqrt{3}(X_{16}+X_{30})+\sqrt{7}(X_{20}+X_{33})$ $+\sqrt{15}(X_{14}+X_{29})$
87	10	A_7+A_1	$[4X_8+2\sqrt{3}(X_{16}+X_{30})+\sqrt{7}(X_{20}+X_{33})$ $+\sqrt{15}(X_{34}-X_{35})]+X_{32}$
88	8	A_7+A_1	$[4X_8+2\sqrt{3}(X_{16}+X_{30})+\sqrt{7}(X_{20}+X_{33})$ $+\sqrt{15}(X_{34}+X_{35})]+X_{32}$
		$E_6(a_1)+A_1$	$[2\sqrt{3}(X_{16}+X_{30})+4X_8+\sqrt{7}(X_{26}+X_{27})$ $+\sqrt{15}(X_{34}-X_{35})]+X_{32}$
89	8	D_6+A_1	$[2\sqrt{7}X_1+3\sqrt{2}X_8+\sqrt{10}X_{34}+2\sqrt{6}X_{35}$ $+\sqrt{15}(X_{32}+X_{33})]+X_{20}$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
90	14	D_6+A_1	$[2\sqrt{7}X_1+3\sqrt{2}X_8+\sqrt{10}X_{34}+2\sqrt{6}X_{35}$ $+\sqrt{15}(X_{32}-X_{33})]+X_{20}$
91	14	$D_8(a_3)$	$\sqrt{14}X_{17}+\sqrt{2}(2X_{20}+3X_{39})$ $+\sqrt{6}(X_{38}-X_{22}+X_{34})+\sqrt{10}(X_8+X_{18}+X_{19})$
		$E_6(a_1)+A_2$	$[4X_8+2\sqrt{3}(X_{17}+X_{19})+\sqrt{7}(X_{26}+X_{31})$ $+\sqrt{15}(X_{34}-X_{39})]+[\sqrt{2}(X_4+X_{30})]$
92	28	$D_8(a_3)$	$\sqrt{14}X_{17}+\sqrt{2}(2X_{20}+3X_{39})$ $+\sqrt{6}(X_{38}+X_{22}-X_{34})+\sqrt{10}(X_8+X_{18}+X_{19})$
93	14	D_6+2A_1	$[2\sqrt{7}X_1+3\sqrt{2}X_8+2\sqrt{6}X_{35}+\sqrt{10}X_{34}$ $+\sqrt{15}(X_{33}-X_{32})]+X_5+X_{20}$
94	26	D_6+2A_1	$[2\sqrt{7}X_1+3\sqrt{2}X_8+2\sqrt{6}X_{35}+\sqrt{10}X_{34}$ $+\sqrt{15}(X_{32}-X_{33})]+X_5+X_{20}$
95	14	D_6+2A_1	$[2\sqrt{7}X_1+3\sqrt{2}X_8+2\sqrt{6}X_{35}+\sqrt{10}X_{34}$ $+\sqrt{15}(X_{32}+X_{33})]+X_5+X_{20}$
		$D_7(a_1)$	$\sqrt{10}X_{18}+3\sqrt{2}X_{22}+2\sqrt{6}X_{19}+2\sqrt{7}X_1$ $+X_{20}-X_{21}+\sqrt{15}(X_{40}+X_{41})$
96	8	E_6+A_1	$[4(X_{16}+X_{30})+\sqrt{30}(X_{20}+X_{33})$ $+\sqrt{42}X_7+\sqrt{22}X_8]+X_{32}$
97	12	$E_7(a_2)$	$\sqrt{22}X_8-X_{18}+\sqrt{42}X_7+4(X_{17}-X_{19})$ $+\sqrt{30}(X_{26}+X_{31})+X_{23}+X_{24}$
98	24	$D_8(a_2)$	$\sqrt{10}X_4+3\sqrt{2}X_{21}+2\sqrt{6}X_{17}+2X_{19}$ $+\sqrt{15}(X_{15}-X_{18})+\sqrt{3}(X_{22}+X_{26})+2\sqrt{7}X_{31}$
99	12	$D_8(a_2)$	$\sqrt{10}X_4+3\sqrt{2}X_{21}+2\sqrt{6}X_{17}+2X_{19}$ $+\sqrt{15}(X_{18}-X_{15})+\sqrt{3}(X_{22}+X_{26})+2\sqrt{7}X_{31}$
		A_8	$\sqrt{2}(2X_4+2X_{17}+3X_7+3X_{22})$ $+2\sqrt{5}(X_{18}+X_{27})+\sqrt{14}(X_{24}+X_{38})$
100	8	D_7	$6X_1+2\sqrt{3}X_{18}+2\sqrt{10}X_{17}$ $+\sqrt{21}(X_{13}+X_{28})+\sqrt{22}X_{22}+\sqrt{30}X_{19}$
101	12	$E_7(a_2)+A_1$	$[\sqrt{22}X_8-X_{18}+\sqrt{42}X_7+4(X_{17}-X_{19})$ $+\sqrt{30}(X_{26}+X_{31})+X_{23}+X_{24}]+X_4$
		E_6+A_2	$[4(X_4+X_{18})+\sqrt{30}(X_{27}+X_{38})+\sqrt{42}X_7$ $+\sqrt{22}X_8]+[\sqrt{2}(X_{17}+X_{24})]$
102	22	$E_7(a_2)+A_1$	$[\sqrt{22}X_8-X_{18}+\sqrt{42}X_7+4(X_{17}-X_{19})$ $+\sqrt{30}(X_{26}+X_{31})+X_{23}+X_{24}]-X_4$

Table 4.
(continued)

No.	inv	$\mathfrak{s}^c$	E
103	10	$E_7(a_1)$	$\sqrt{26}X_8 + 5\sqrt{2}X_7 + 2\sqrt{10}X_1 + \sqrt{21}X_{17}$ $+\frac{1}{\sqrt{5}}(\sqrt{33}X_{19} - \sqrt{77}X_{18} - 6\sqrt{3}X_{26} - 6\sqrt{7}X_{27})$
104	10	$D_8(a_1)$	$2\sqrt{3}X_4 + \sqrt{22}X_{21} + \sqrt{30}X_{17} + 6X_1$ $+2\sqrt{10}X_{19} - X_{15} + X_{18} + \sqrt{21}(X_{22} + X_{26})$
105	20	$D_8(a_1)$	$2\sqrt{3}X_4 + \sqrt{22}X_{21} + \sqrt{30}X_{17} + 6X_1$ $+2\sqrt{10}X_{19} + X_{15} - X_{18} + \sqrt{21}(X_{22} + X_{26})$
106	10	$E_7(a_1) + A_1$	$[\sqrt{26}X_8 + 5\sqrt{2}X_7 + 2\sqrt{10}X_1 + \sqrt{21}X_{17}$ $+\frac{1}{\sqrt{5}}(\sqrt{33}X_{19} - \sqrt{77}X_{18} - 6\sqrt{3}X_{26} - 6\sqrt{7}X_{27})] + X_4$
107	18	$E_7(a_1) + A_1$	$[\sqrt{26}X_8 + 5\sqrt{2}X_7 + 2\sqrt{10}X_1 + \sqrt{21}X_{17}$ $+\frac{1}{\sqrt{5}}(\sqrt{33}X_{19} - \sqrt{77}X_{18} - 6\sqrt{3}X_{26} - 6\sqrt{7}X_{27})] - X_4$
108	8	E_7	$7X_{18} + \sqrt{3}(3X_{17} + 5X_{19}) + 4\sqrt{6}X_6 + 2\sqrt{13}X_1$ $+\sqrt{66}X_7 + \sqrt{34}X_8$
109	8	D_8	$6X_{19} + 5\sqrt{2}X_{17} + 3\sqrt{6}X_{21} + 2\sqrt{7}(X_4 + X_8)$ $+2\sqrt{11}X_1 - \sqrt{14}X_{18} + \sqrt{26}X_6$
110	16	D_8	$6X_{19} + 5\sqrt{2}X_{17} + 3\sqrt{6}X_{21} + 2\sqrt{7}(X_4 + X_8)$ $+2\sqrt{11}X_1 + \sqrt{14}X_{18} + \sqrt{26}X_6$
111	8	$E_7 + A_1$	$[7X_{18} + \sqrt{3}(3X_{17} + 5X_{19}) + 4\sqrt{6}X_6$ $+2\sqrt{13}X_1 + \sqrt{34}X_8 + \sqrt{66}X_7] - X_4$
112	14	$E_7 + A_1$	$[7X_{18} + \sqrt{3}(3X_{17} + 5X_{19}) + 4\sqrt{6}X_6$ $+2\sqrt{13}X_1 + \sqrt{34}X_8 + \sqrt{66}X_7] + X_4$
113	12	$E_8(a_2)$	$2\sqrt{15}X_1 + \sqrt{88}X_2 + 2\sqrt{13}X_5 + \sqrt{74}X_7$ $+\sqrt{38}X_8 + \sqrt{118}X_{11} + 2\sqrt{14}X_{20}$
114	10	$E_8(a_1)$	$6\sqrt{2}X_1 + \sqrt{106}X_2 + 2\sqrt{33}X_6 + 3\sqrt{10}X_7 + \sqrt{46}X_8$ $+\frac{1}{\sqrt{53}}(2\sqrt{13}(\sqrt{38}X_5 - 2\sqrt{17}X_3)$ $+\sqrt{105}(\sqrt{38}X_{11} + 2\sqrt{17}X_{12}))$
115	8	E_8	$2\sqrt{23}X_1 + 2\sqrt{34}X_2 + \sqrt{182}X_3 + 3\sqrt{30}X_4$ $+2\sqrt{55}X_5 + 2\sqrt{42}X_6 + \sqrt{114}X_7 + \sqrt{58}X_8$

5. Real Cayley triples in EIX.

There are 36 nonzero nilpotent orbits in $\mathfrak{g}$ = EIX. We list in Table 5 the representatives (E, H, F) for K-orbits of real Cayley triples in $\mathfrak{g}$. In the fourth column we record the isomorphism type of the real form $\mathfrak{s}$ of $\mathfrak{s}^c$ determined by the restriction of σ. We remark that $\mathfrak{s}$ is quasi-split in all cases.

Table 5.
Cayley triples in EIX = $E_{8(-24)}$.

No.	tr	$\mathfrak{s}^c$	$\mathfrak{s}$	E
1	56	A_1	$\mathfrak{sl}_2(\mathbf{R})$	X_{120}
2	80	$2A_1$	$2\mathfrak{sl}_2(\mathbf{R})$	$X_{97} - X_{120}$
3	144	$2A_1$	$2\mathfrak{sl}_2(\mathbf{R})$	$X_{97} + X_{120}$
		$2A_1$	$\mathfrak{sl}_2(\mathbf{C})$	$X_{113} + X_{114}$
4	72	$3A_1$	$3\mathfrak{sl}_2(\mathbf{R})$	$X_{74} - X_{104} + X_{118}$
5	200	$3A_1$	$3\mathfrak{sl}_2(\mathbf{R})$	$X_{74} + X_{104} + X_{118}$
		$3A_1$	$\mathfrak{sl}_2(\mathbf{C}) + \mathfrak{sl}_2(\mathbf{R})$	$[X_{91} + X_{92}] + X_{118}$
6	32	$(4A_1)''$	$4\mathfrak{sl}_2(\mathbf{R})$	$X_8 - X_{74} + X_{104} - X_{118}$
7	224	$(4A_1)''$	$4\mathfrak{sl}_2(\mathbf{R})$	$X_8 + X_{74} + X_{104} - X_{118}$
		$(4A_1)''$	$\mathfrak{sl}_2(\mathbf{C}) + 2\mathfrak{sl}_2(\mathbf{R})$	$[X_{105} - X_{106}] + X_{104} + X_8$
		A_2	$\mathfrak{sl}_3(\mathbf{R})$	$\sqrt{2}(X_8 + X_{119})$
8	288	$(4A_1)''$	$4\mathfrak{sl}_2(\mathbf{R})$	$X_8 + X_{74} + X_{104} + X_{118}$
		$(4A_1)''$	$\mathfrak{sl}_2(\mathbf{C}) + 2\mathfrak{sl}_2(\mathbf{R})$	$[X_{105} - X_{106}] + X_{104} - X_8$
		$(4A_1)''$	$2\mathfrak{sl}_2(\mathbf{C})$	$[X_{42} + X_{43}] + [X_{107} - X_{117}]$
9	344	$5A_1$	$2\mathfrak{sl}_2(\mathbf{C}) + \mathfrak{sl}_2(\mathbf{R})$	$[X_{47} + X_{81}] + [X_{100} - X_{110}] + X_{97}$
10	368	$6A_1$	$2\mathfrak{sl}_2(\mathbf{C}) + 2\mathfrak{sl}_2(\mathbf{R})$	$[X_{73} - X_{98}] + [X_{84} + X_{99}]$ $-X_{61} + X_{97}$
		$A_2 + 2A_1$	$\mathfrak{sl}_3(\mathbf{R}) + \mathfrak{sl}_2(\mathbf{C})$	$[\sqrt{2}(X_{61} + X_{104})] + [X_{73} - X_{98}]$
11	432	$6A_1$	$2\mathfrak{sl}_2(\mathbf{C}) + 2\mathfrak{sl}_2(\mathbf{R})$	$[X_{73} - X_{98}] + [X_{84} + X_{99}]$ $+X_{61} + X_{97}$
		$6A_1$	$3\mathfrak{sl}_2(\mathbf{C})$	$[X_{66} - X_{93}] + [X_{78} + X_{94}]$ $+[X_{91} + X_{92}]$
12	464	A_3	$\mathfrak{sl}_4(\mathbf{R})$	$\sqrt{3}(X_8 - X_{74}) + 2X_{97}$
13	656	A_3	$\mathfrak{sl}_4(\mathbf{R})$	$\sqrt{3}(X_8 + X_{74}) + 2X_{97}$
		A_3	$\mathfrak{su}(2,2)$	$\sqrt{3}(X_{42} + X_{43}) + 2X_{97}$
14	576	$2A_2$	$\mathfrak{sl}_3(\mathbf{C})$	$\sqrt{2}(X_1 + X_{44} + X_{96} + X_{112})$
		$8A_1$	$4\mathfrak{sl}_2(\mathbf{C})$	$[X_1 + X_{44}] + [X_{71} + X_{89}]$ $+[X_{83} - X_{90}] + [X_{91} + X_{92}]$
15	632	$2A_2 + A_1$	$\mathfrak{sl}_3(\mathbf{C}) + \mathfrak{sl}_2(\mathbf{R})$	$[\sqrt{2}(X_{39} + X_{73} + X_{75} - X_{98})] + X_{74}$
16	648	$A_3 + A_1$	$\mathfrak{su}(2,2) + \mathfrak{sl}_2(\mathbf{R})$	$[2X_{97} + \sqrt{3}(X_{22} - X_{62})] - X_{61}$
17	776	$A_3 + A_1$	$\mathfrak{su}(2,2) + \mathfrak{sl}_2(\mathbf{R})$	$[2X_{97} + \sqrt{3}(X_{22} - X_{62})] + X_{61}$
18	608	$(A_3 + 2A_1)''$	$\mathfrak{su}(2,2) + 2\mathfrak{sl}_2(\mathbf{R})$	$[2X_{97} + \sqrt{3}(X_{22} - X_{62})] + X_7 - X_{61}$
		$(A_3 + 2A_1)''$	$\mathfrak{sl}_4(\mathbf{R}) + \mathfrak{sl}_2(\mathbf{C})$	$[2X_{68} + \sqrt{3}(X_7 + X_{97})]$ $+[X_{54} - X_{77}]$
		$D_4(a_1)$	$\mathfrak{so}(4,4)$	$2X_7 + \sqrt{3}(X_{68} + X_{101}) - X_{61} + X_{97}$
19	864	$(A_3 + 2A_1)''$	$\mathfrak{su}(2,2) + 2\mathfrak{sl}_2(\mathbf{R})$	$[2X_{97} + \sqrt{3}(X_{22} - X_{62})] - X_7 + X_{61}$
		$D_4(a_1)$	$\mathfrak{so}(4,4)$	$2X_7 + \sqrt{3}(X_{68} + X_{101}) + X_{61} - X_{97}$

Table 5.
(continued)

No.	tr	$\mathfrak{s}^c$	$\mathfrak{s}$	E
20	800	$(A_3+2A_1)''$	$\mathfrak{su}(2,2)+2\mathfrak{sl}_2(\mathbf{R})$	$[2X_{97}+\sqrt{3}(X_{22}-X_{62})]+X_7+X_{61}$
		$(A_3+2A_1)''$	$\mathfrak{sl}_4(\mathbf{R})+\mathfrak{sl}_2(\mathbf{C})$	$[2X_{68}+\sqrt{3}(X_7-X_{97})]+[X_{54}-X_{77}]$
		$(A_3+2A_1)''$	$\mathfrak{su}(2,2)+\mathfrak{sl}_2(\mathbf{C})$	$[2X_7+\sqrt{3}(X_{78}+X_{94})]+[X_{66}-X_{93}]$
		$3A_2$	$\mathfrak{sl}_3(\mathbf{C})+\mathfrak{sl}_3(\mathbf{R})$	$[\sqrt{2}(X_{22}-X_{62}+X_{66}-X_{93})]$ $+[\sqrt{2}(X_7+X_{11})]$
		$D_4(a_1)$	$\mathfrak{so}(5,3)$	$2X_{61}+X_{46}-X_{70}+i\sqrt{3}(X_{54}+X_{77})$
21	992	D_4	$\mathfrak{so}(4,4)$	$\sqrt{10}X_8+\sqrt{6}(X_7-X_{61}+X_{97})$
22	1760	D_4	$\mathfrak{so}(4,4)$	$\sqrt{10}X_8+\sqrt{6}(X_7+X_{61}+X_{97})$
		D_4	$\mathfrak{so}(5,3)$	$\sqrt{10}X_8+\sqrt{6}(X_{58}-X_{59}+X_{61})$
23	944	A_3+4A_1	$\mathfrak{su}(2,2)+2\mathfrak{sl}_2(\mathbf{C})$	$[2X_{61}+\sqrt{3}(X_{60}+X_{72})]$ $+[X_{31}+X_{69}]+[X_{46}-X_{70}]$
		$D_4(a_1)+2A_1$	$\mathfrak{so}(5,3)+\mathfrak{sl}_2(\mathbf{C})$	$[2X_{61}+X_{46}-X_{70}+i\sqrt{3}(X_{54}+X_{77})]$ $+[X_{31}+X_{69}]$
24	1184	$(2A_3)''$	$2\mathfrak{su}(2,2)$	$[2X_8+\sqrt{3}(X_{71}+X_{89})]$ $+[-2X_{74}+\sqrt{3}(X_1+X_{44})]$
25	1440	$(2A_3)''$	$2\mathfrak{su}(2,2)$	$[2X_8+\sqrt{3}(X_{71}+X_{89})]$ $+[2X_{74}+\sqrt{3}(X_1+X_{44})]$
		$(2A_3)''$	$\mathfrak{sl}_4(\mathbf{C})$	$\sqrt{3}(X_{16}-X_{30}+X_{66}-X_{93})$ $+2(X_{42}+X_{43})$
26	1496	$2A_3+A_1$	$\mathfrak{sl}_4(\mathbf{C})+\mathfrak{sl}_2(\mathbf{R})$	$[\sqrt{3}(X_{31}+X_{46}+X_{69}-X_{70})$ $+2(X_{22}-X_{62})]+X_{61}$
27	1904	D_4+2A_1	$\mathfrak{so}(5,3)+\mathfrak{sl}_2(\mathbf{C})$	$[\sqrt{10}X_8+\sqrt{6}(X_{46}+X_{61}-X_{70})]$ $+[X_{31}+X_{69}]$
		$D_5(a_1)$	$\mathfrak{so}(6,4)$	$\sqrt{6}(X_{61}+X_{31}+X_{69})+\sqrt{10}X_{15}$ $+i(X_{39}-X_{75})$
28	1728	$2D_4(a_1)$	$\mathfrak{so}_8(\mathbf{C})$	$X_{13}+2X_{34}+X_{62}+\sqrt{3}(X_{31}-X_{77})$ $+X_{40}+2X_{35}-X_{22}+\sqrt{3}(X_{54}+X_{69})$
29	2376	A_5	$\mathfrak{su}(3,3)$	$3X_{61}+2\sqrt{2}(X_1+X_{44})$ $+\sqrt{5}(X_{22}-X_{62})$
30	2528	$(A_5+A_1)''$	$\mathfrak{su}(3,3)+\mathfrak{sl}_2(\mathbf{R})$	$[3X_{61}+2\sqrt{2}(X_1+X_{44})$ $+\sqrt{5}(X_{22}-X_{62})]+X_7$
31	2336	$(A_5+A_1)''$	$\mathfrak{su}(3,3)+\mathfrak{sl}_2(\mathbf{R})$	$[3X_{61}+2\sqrt{2}(X_1+X_{44})$ $+\sqrt{5}(X_{22}-X_{62})]-X_7$

Table 5.
(continued)

No.	tr	$\mathfrak{s}^c$	$\mathfrak{s}$	E
32	4064	D_5	$\mathfrak{so}(6,4)$	$\sqrt{14}X_8 + \sqrt{2}(2X_7 + 3X_{61})$ $+\sqrt{10}(X_1 + X_{44})$
33	3296	D_5	$\mathfrak{so}(6,4)$	$\sqrt{14}X_8 + \sqrt{2}(-2X_7 + 3X_{61})$ $+\sqrt{10}(X_1 + X_{44})$
34	4032	$2D_4$	$\mathfrak{so}_8(\mathbf{C})$	$\sqrt{6}(X_6 + X_{21} + X_{36} + X_{48}$ $+X_{49} - X_{50}) + \sqrt{10}(X_1 + X_{44})$
35	5792	$(A_7)''$	$\mathfrak{su}(4,4)$	$4X_8 + 2\sqrt{3}(X_1 + X_{44})$ $+\sqrt{7}(X_6 + X_{48}) + \sqrt{15}(X_{21} + X_{49})$
		$E_6(a_1)$	$E_{6(2)}$	$4X_8 + 2\sqrt{3}(X_1 + X_{44})$ $+\sqrt{7}(X_6 + X_{48}) + i\sqrt{15}(X_{14} + X_{55})$
36	10208	E_6	$E_{6(2)}$	$4(X_1 + X_{44}) + \sqrt{22}X_8$ $+\sqrt{30}(X_6 + X_{48}) + \sqrt{42}X_7$

As in the previous section, we record only the elements E. The brackets $[\cdot]$ in the expressions for E have a slightly different meaning: They indicate the contributions of the various simple ideals of the real form $\mathfrak{s}$, not of $\mathfrak{s}^c$ itself. We omit them if the ideal is isomorphic to $\mathfrak{sl}_2(\mathbf{R})$ or if $\mathfrak{s}$ itself is simple. The nilpotent G-orbits of $\mathfrak{g}$ which are contained in the same nilpotent G^c-orbit of $\mathfrak{g}^c$ are now distinguished by the invariant tr, which is recorded in the second column.

6. Appendix.

We list in Table 6 the nonzero structure constants $N(i,j)$ of E_8 for $0 < i < j \leq 120$. The second (resp., third) column lists those $j > i$ for which $N(i,j)$ is $+1$ (resp., -1). Note that one or both of these sets can be empty. They are both empty if $i > 88$ and so we omit these cases.

There is a simple algorithm to compute $N(i,j)$'s for arbitrary i and j from those that we have tabulated. First of all, since $N(-i,-j) = N(i,j)$, we may assume that $i > 0$. If $\alpha_i + \alpha_j$ is not a root, then $[X_i, X_j] = 0$. Assume that it is a root, say $\alpha_i + \alpha_j = \alpha_k$. One can compute k by using Table 1. Since $\alpha_i + \alpha_j + \alpha_{-k} = 0$, it follows from [**2**, Chapitre 8, §2, Exercice 4] that $N(i,j) = N(-j,k) = N(-k,i)$. Since one of the pairs $(-j,k)$, $(-k,i)$ consists of positive integers, this shows that it suffices to know the $N(i,j)$'s for $i,j > 0$. As $N(j,i) = -N(i,j)$ we may also impose the restriction $i < j$.

Table 6.
Structure constants $N(i, j)$, $i < j$, of E_8.

i	$j:\ N(i,j) = +1$	$j:\ N(i,j) = -1$
1	3, 11, 17, 19, 25, 27, 32, 33, 35, 40, 41, 43, 48, 49, 50, 55, 56, 61, 62, 68, 74, 93, 98, 102, 105, 108, 110, 112	
2	11, 12, 16, 19, 20, 24, 27, 28, 31, 35, 36, 39, 43, 47, 63, 70, 76, 77, 82, 83, 87, 88, 92, 96, 110, 111, 114	4
3	10, 12, 18, 20, 26, 28, 34, 36, 37, 42, 45, 52, 53, 59, 60, 66, 67, 73, 79, 89, 94, 99, 103, 106, 111, 113	4
4	5, 9, 13, 21, 25, 29, 30, 33, 38, 41, 46, 50, 54, 57, 64, 71, 72, 78, 84, 85, 90, 95, 100, 108, 109, 115	
5	10, 11, 14, 16, 17, 22, 23, 40, 45, 49, 51, 53, 56, 58, 60, 65, 76, 80, 83, 86, 88, 91, 103, 105, 107, 116	6
6	7, 12, 15, 18, 19, 24, 25, 30, 32, 37, 44, 55, 59, 62, 64, 67, 70, 72, 75, 77, 81, 92, 95, 99, 102, 104, 117	
7	13, 20, 26, 27, 31, 33, 38, 40, 45, 48, 51, 52, 57, 63, 68, 69, 73, 78, 83, 86, 87, 90, 94, 98, 101, 118	8
8	14, 21, 28, 34, 35, 39, 41, 46, 49, 53, 55, 58, 59, 61, 64, 66, 70, 71, 75, 76, 80, 82, 85, 89, 93, 97, 119	
9	10, 12, 18, 20, 26, 28, 34, 36, 42, 89, 94, 99, 103, 106	32, 40, 48, 49, 55, 56, 61, 62, 68, 74, 110, 112
10	19, 24, 27, 31, 35, 39, 43, 47, 82, 87, 92, 96, 114	13, 21, 29, 57, 64, 71, 72, 78, 84, 108, 109
11	18, 26, 34, 42, 52, 59, 66, 67, 73, 79, 106, 113	13, 21, 29, 30, 38, 46, 54, 85, 90, 95, 100, 109
12	14, 17, 22, 23, 51, 58, 65, 80, 86, 91, 105, 107	33, 38, 41, 46, 50, 54, 71, 78, 84, 100, 115
13	16, 17, 23, 49, 53, 56, 58, 60, 65, 88, 91, 116	15, 32, 37, 44, 70, 75, 77, 77, 81, 99, 102, 104
14	48, 52, 57, 63, 69, 87, 90, 94, 98, 101	18, 19, 24, 25, 30, 32, 37, 44, 62, 67, 72, 77, 81, 117

Table 6.
(continued)

i	$j:\ N(i,j)=+1$	$j:\ N(i,j)=-1$
15	20, 26, 27, 31, 33, 38, 40, 45, 48, 51, 52, 57, 63, 69, 118	61, 66, 71, 76, 80, 82, 85, 89, 93, 97
16	18, 25, 26, 33, 34, 41, 42, 50, 106, 108	21, 29, 48, 55, 61, 62, 68, 74, 85, 90, 95, 100, 112
17	24, 31, 39, 47, 52, 59, 66, 67, 73, 79, 82, 87, 92, 96, 106	20, 21, 28, 29, 36, 109, 111
18	22, 27, 31, 35, 39, 43, 47, 51, 58, 65, 96, 105, 107, 114	71, 76, 78, 83, 84, 88
19	22, 23, 26, 34, 42, 66, 73, 79, 80, 86, 91, 107, 113	38, 45, 46, 53, 54, 60, 100, 103
20	23, 25, 30, 58, 64, 65, 72, 91, 95	41, 44, 46, 50, 54, 75, 81, 84, 102, 104, 115
21	23, 56, 60, 63, 65, 69, 94, 98, 101, 116	32, 37, 40, 44, 45, 51, 77, 81, 83, 86
22	48, 52, 55, 57, 59, 63, 64, 69, 70, 75	24, 25, 30, 32, 37, 44, 82, 85, 89, 93, 97, 117
23	82, 87, 92, 96, 106, 108, 110	27, 28, 29, 35, 36, 43, 48, 55, 61, 62, 68, 74
24	26, 33, 34, 40, 41, 42, 49, 50, 56, 80, 86, 91	61, 68, 74, 100, 103, 105, 112
25	31, 39, 47, 66, 73, 79, 96, 107	28, 36, 45, 53, 60, 76, 83, 88, 103, 111
26	35, 39, 43, 47, 58, 64, 65, 70, 72, 77, 114	44, 84, 88, 92, 102, 104
27	30, 34, 37, 42, 79, 91, 95, 99, 113	46, 53, 54, 59, 60, 67, 75, 81, 104
28	30, 33, 38, 65, 69, 72, 78, 98, 101	44, 50, 51, 54, 57, 81, 86, 90, 115
29	63, 69, 70, 75, 76, 80, 116	32, 37, 40, 44, 45, 49, 51, 53, 58, 89, 93, 97
30	40, 49, 56, 96, 110	35, 36, 43, 61, 68, 74, 76, 83, 88, 103, 105
31	34, 41, 42, 49, 50, 55, 56, 62, 91, 95, 99, 102	32, 74, 75, 81, 112
32	38, 39, 46, 47, 54, 66, 71, 73, 78, 79, 84, 96, 100, 107, 109	
33	37, 39, 47, 70, 77, 79, 99	36, 53, 59, 60, 67, 88, 92, 104, 111
34	43, 47, 65, 72, 77, 78, 83, 87, 98, 101, 114	44, 51, 57, 63
35	37, 38, 42, 45, 52, 69, 101, 113	54, 60, 67, 73, 81, 86, 90, 94

Table 6.
(continued)

i	$j:\ N(i,j)=+1$	$j:\ N(i,j)=-1$
36	38, 41, 46, 69, 75, 80, 85	44, 51, 57, 58, 64, 71, 93, 97, 115
37	71, 78, 84, 96, 100	41, 43, 50, 61, 68, 74, 105, 108
38	49, 55, 56, 62, 70, 77, 99, 102, 110	43, 74, 88, 92
39	42, 50, 56, 62, 68, 69	40, 48, 81, 86, 90, 94, 98, 112
40	46, 47, 54, 79, 84, 109	59, 64, 67, 72, 92, 95, 104
41	45, 47, 52, 77, 83, 87, 101	60, 63, 67, 73, 94, 111
42	114	44, 51, 57, 58, 63, 64, 70, 71, 76, 82, 93, 97
43	45, 46, 52, 53, 59, 66, 69, 75, 80, 85, 89, 113	97
44	96, 100, 103, 106	61, 66, 68, 73, 74, 79
45	55, 62, 84, 102	50, 64, 72, 74, 92, 95, 108
46	56, 62, 68, 77, 83, 87, 110	48, 63, 94, 98
47	69, 75, 80, 85, 89, 93	48, 49, 55, 61, 112
48	53, 54, 58, 60, 65, 79, 84, 88, 91	104, 107
49	52, 54, 57, 87, 90, 101, 109	67, 72, 73, 78
50	52, 53, 59, 66, 89	63, 70, 76, 82, 97, 111
51	55, 59, 62, 67, 106	74, 79, 92, 95, 99
52	58, 65, 84, 88, 91, 102, 105	56, 74
53	57, 62, 68, 87, 90	72, 78, 98, 108
54	89, 93, 110	55, 61, 63, 70, 76, 82
55	60, 65, 101	73, 78, 83, 86, 107
56	57, 59, 64, 66, 71, 109	82, 85, 97
57	88, 91	60, 74, 79, 99, 103
58	62, 67, 68, 73, 87, 90, 94, 106	
59	65, 68, 105	78, 83, 86, 98
60	64, 71, 93	61, 82, 85, 108
61	65, 67, 72, 77, 81, 101, 104	
62	66, 71, 76, 80	97, 107
63	91, 95, 100	74, 79, 84
64	68, 73, 94	83, 86, 103
65	106	66, 82, 85, 89
66	72, 77, 81	98, 102
67	71, 76, 80, 93, 105	
68	104	70, 75, 97
69		74, 79, 84, 88, 92, 96
70	73, 78, 100	86, 90

Table 6.
(continued)

i	$j:\ N(i,j)=+1$	$j:\ N(i,j)=-1$
71	$77, 81, 94, 99$	
72	$76, 80$	$89, 103$
73	93	$75, 102$
74		$97, 101$
75	$78, 83, 87$	96
76	81	$90, 95$
77	$80, 85, 100$	
78	99	89
79	$93, 98$	
80	$87, 92$	
81		$82, 96$
82	$86, 91$	
83	85	95
84		$89, 94$
85		88
86	92	
87	91	
88	90	

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