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It is a classical result that a closed exceptional polar set is removable for subharmonic functions which are bounded above. Gardiner has shown that in the case of a compact exceptional set the above boundedness condition can be relaxed by imposing certain smoothness and Hausdorff measure conditions on the set. We give related results for a closed exceptional set, by replacing the smoothness and Hausdorff measure conditions with one sole condition on Minkowski upper content.

1. Introduction.

In the sequel Ω is always an open set in $\mathbb{R}^n$, $n \geq 2$, and $E \subset \Omega$ is closed in Ω. It is a classical result [**HK**, Theorem 5.18, p. 237] that if f is subharmonic in $\Omega \setminus E$ and bounded above and moreover E is polar, then f has a subharmonic extension to the whole of Ω. Imposing certain constraints on the geometry and size of the set E, Gardiner relaxed considerably the boundedness requirement of f [**Ga**, Theorems 1 and 3, pp. 71-74]. To state his results, let $\Phi : \Omega \to \mathbb{R}$ be a $\mathcal{C}^2$ function with nonvanishing gradient throughout Ω. Put $S = \{ x \in \Omega : \Phi(x) = 0 \}$. Write $d(x, S)$ for the distance from $x \in \mathbb{R}^n$ to S and let Λ_α be the α-dimensional Hausdorff (outer) measure in $\mathbb{R}^n$.

Theorem A. *Let $\alpha \in (0, n-2)$ and E be a compact subset of S such that $\Lambda_\alpha(E) = 0$. If f is subharmonic in $\Omega \setminus E$ and satisfies*

$$f(x) \leq C\, d(x, S)^{\alpha+2-n} \qquad (x \in \Omega \setminus S)$$

for some positive constant C, then f has a subharmonic extension to Ω.

Theorem B. *Let $\alpha \in (0, n-2)$ and E be a compact subset of S such that $\Lambda_\alpha(E) < \infty$. If f is subharmonic in $\Omega \setminus E$ and satisfies*

$$f(x) \leq u(d(x, S)) \qquad (x \in \Omega \setminus S)$$

where $t^{n-2-\alpha} u(t) \to 0$ $(t \to 0+)$, then f has a subharmonic extension to Ω.

Our notation is more or less standard or will be explained below. For example, $B(x, r)$ is the open ball in $\mathbb{R}^n$, with center x and radius r. The family of test functions on Ω is denoted by $\mathcal{D}(\Omega)$. The differential operator $(\mathcal{D}_1)^{\lambda_1} \cdots (\mathcal{D}_n)^{\lambda_n} = (\frac{\partial}{\partial x_1})^{\lambda_1} \cdots (\frac{\partial}{\partial x_n})^{\lambda_n}$ is denoted by $\mathcal{D}^\lambda$. Here $\lambda =$

$(\lambda_1, \dots, \lambda_n) \in (\mathbb{N} \cup \{0\})^n$ is a multi-index, and $|\lambda| = \lambda_1 + \cdots + \lambda_n$. The Laplacian is $\varDelta = \mathcal{D}_1^2 + \cdots + \mathcal{D}_n^2$. The notation $C(n, \alpha, \dots)$, say, means that C is a constant depending only on n, $\alpha, \dots$. As usual, constants may vary from line to line.

Gardiner also shows [**Ga**, Theorems 2 and 4, pp. 72-73] that his results are sharp in the following sense: If one drops the smoothness assumption $E \subset S$ then the exceptional set E is not any more necessarily removable. Our purpose is to point out that there exist, however, results which are in a certain sense parallel to Gardiner's results but where no smoothness conditions are necessary to impose on the exceptional set. As a matter of fact, we show below in Theorems 1 and 2 that results similar to Gardiner's hold when his conditions,

(i) $E \subset S$ where S is a $\mathcal{C}^2$ $(n-1)$-dimensional manifold in Ω,
(ii) $\Lambda_\alpha(E) = 0$ (resp. $\Lambda_\alpha(E) < \infty$),

are replaced by one geometric measure condition $M^\alpha(E) = 0$ (resp. $M^\alpha(E) < \infty$) where M^α is the upper Minkowski content. Our proofs are different and perhaps shorter than those of Gardiner. Moreover, our approach does not require the exceptional set E to be compact, unlike in Gardiner's results. On the other hand, as is shown in Examples 1 and 2 below, Gardiner's and our results are independent: Neither our nor Gardiner's results are included in the other's.

Gardiner also [**Ga**, Theorem 5, p. 74] proves the following result:

Theorem C. *Let $\alpha \in (0, n-2)$ and E be a compact subset of S such that $\Lambda_\alpha(E) = 0$. If f is subharmonic in $\Omega \setminus E$ and satisfies*

$$\mathcal{A}(f^+, x, r) \le C\, r^{\alpha+2-n} \qquad \left(\overline{B(x,r)} \subset \Omega\right)$$

then f has a subharmonic extension to Ω.

Here $f^+ = \max\{f, 0\}$ and $\mathcal{A}(f^+, x, r)$ is the mean value of f^+ over the ball $B(x,r)$, with respect to the Lebesgue measure m in $\mathbb{R}^n$.

Below in Theorem 3 we improve this result by dropping the condition that E is compact. Again our approach is essentially different than that of Gardiner.

2. Net measure and Minkowski content.

For readers' convenience we first recall certain basic facts concerning net measure and Minkowski content and their relationship with the standard Hausdorff measure. For a more thorough discussion see e.g., [**HP**, pp. 41-44] and [**Fa**, pp. 33, 42].

Let $A \subset \mathbb{R}^n$ and $\alpha \in [0, n]$. For each $\epsilon > 0$ define

$$\mathcal{L}_\alpha^\epsilon(A) = \inf \sum_{i=1}^{\infty} s_i^\alpha$$

where the infimum is over all coverings of A by countable disjoint collection of dyadic cubes Q_i with (side)length $s_i \le \epsilon$. Define the *α-dimensional net measure* of A by

$$\mathcal{L}_\alpha(A) = \lim_{\epsilon \to 0+} \mathcal{L}_\alpha^\epsilon(A).$$

It is well-known that the standard Hausdorff measure Λ_α and the net measure $\mathcal{L}_\alpha$ are comparable: There are positive constants $C_1 = C_1(n)$ and $C_2 = C_2(n)$ such that

$$C_1 \mathcal{L}_\alpha(A) \le \Lambda_\alpha(A) \le C_2 \mathcal{L}_\alpha(A) \tag{1}$$

for all $A \subset \mathbb{R}^n$.

To define the Minkowski content, let $A \subset \mathbb{R}^n$, $\alpha \in [0, n]$ and $\epsilon > 0$. Write

$$A_\epsilon = \{\, x \in \mathbb{R}^n \,:\, d(x, A) < \epsilon \,\}.$$

The *α-dimensional upper Minkowski content* of A is defined by

$$\mathcal{M}^\alpha(A) = \limsup_{\epsilon \to 0+} \frac{m(A_\epsilon)}{\epsilon^{n-\alpha}}.$$

It is well-known that there is a positive constant $C_3 = C_3(n, \alpha)$ such that

$$C_3 \Lambda_\alpha(A) \le \mathcal{M}^\alpha(A)$$

for all $A \subset \mathbb{R}^n$. The reverse inequality does not hold in general, but is true for certain smooth sets, even for α rectifiable closed subsets of $\mathbb{R}^n$ (here α is a positive integer). See [**HP**, p. 41] and [**Fe**, 3.2.39, p. 275].

Our argument will essentially be based on the following type of partition of unity, see [**HP**, Lemma 3.1, p. 43]:

Lemma. *Let $\{Q_i : i = 1, \dots, N\}$ be a finite disjoint collection of dyadic cubes of length $s(Q_i) = s_i$. For each i, there is a function $\varphi_i \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with support $spt\, \varphi_i \subset \frac{3}{2} Q_i$ such that $\sum_{i=1}^{\infty} \varphi_i(x) = 1$ for all $x \in \cup_{i=1}^N Q_i$. Furthermore, for each multi-index λ, there is a constant $C_\lambda = C_\lambda(\lambda, n)$ for which $|\mathcal{D}^\lambda \varphi_i(x)| \le C_\lambda s_i^{-|\lambda|}$ for all $x \in \mathbb{R}^n$ and $i = 1, \dots, N$.*

3. The results.

Our first result is parallel to Gardiner's Theorem A:

Theorem 1. *Suppose that $\alpha \in [0, n-2]$ and $\mathcal{M}^\alpha(E) = 0$. If f is subharmonic in $\Omega \setminus E$ and satisfies*

$$f(x) \le C^* \, d(x, E)^{\alpha + 2 - n} \qquad (x \in \Omega \setminus E)$$

for some positive constant C^*, then f has a subharmonic extension to Ω.

Proof. If $\alpha = 0$ then $E = \emptyset$. If $\alpha = n-2$, then E is polar e.g., by [**HK**, Theorem 5.14, p. 288]. Since f is then also bounded above, the claim follows from the classical result [**HK**, Theorem 5.18, p. 237].

It remains to consider the case $\alpha \in (0, n-2)$. Since f^+ is subharmonic, and also

$$f^+(x) \leq C^* \, d(x,E)^{\alpha+2-n} \qquad (x \in \Omega \setminus E),$$

we may suppose that $f \geq 0$.

We first show that $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$, cf. [**HP**, p. 42] and [**Ri**, pp. 730-731]. It is sufficient to show that for some $r > 0$,

$$\int_{E_r} f \, dm < \infty.$$

Take $\epsilon > 0$ arbitrarily. Since $\mathcal{M}^\alpha(E) = 0$, there is r_o, $0 < r_o < 1$, such that $m(E_r) \leq \epsilon \, r^{n-\alpha}$ for all r, $0 \leq r \leq r_o$. Take any such r, and write for each $j = 0, 1, \ldots,$

$$K_j = \{ \, x \in \mathbb{R}^n \, : \, d(x,E) < r \, 2^{-j} \, \}.$$

Then

$$E_r = \bigcup_{j=0}^{\infty} (K_j \setminus K_{j+1}),$$

and

$$\begin{aligned}
\int_{E_r} f(x) \, dm(x) &\leq C^* \int_{E_r} d(x,E)^{\alpha+2-n} dm(x) \\
&= C^* \sum_{j=0}^{\infty} \int_{K_j \setminus K_{j+1}} d(x,E)^{\alpha+2-n} dm(x) \\
&\leq C^* \sum_{j=0}^{\infty} \left[r \, 2^{-(j+1)} \right]^{\alpha+2-n} \, m(K_j) \\
&\leq C^* \, 2^{n-2-\alpha} \, r^{\alpha+2-n} \sum_{j=0}^{\infty} 2^{(n-2-\alpha)j} \epsilon \, (r \, 2^{-j})^{n-\alpha} \\
&\leq C^* \, 2^{n-2-\alpha} r^2 \, \epsilon \sum_{j=0}^{\infty} 2^{-2j} < \infty.
\end{aligned}$$

Thus $f \in \mathcal{L}^1_{loc}(\Omega)$. For later use we observe that we also got

$$\int_{E_r} d(x,E)^{\alpha+2-n} dm(x) \leq C \, r^2 \epsilon \tag{2}$$

for all r, $0 \leq r \leq r_o$, where $C = C(n, \alpha, C^*)$.

To complete the proof, it remains to show that for any nonnegative test function $\varphi \in \mathcal{D}(\Omega)$,

$$\int f \Delta\varphi \, dm \geq 0.$$

We may suppose that $0 \leq \varphi \leq 1$ and $|\mathcal{D}^\lambda \varphi| \leq 1$ for each multi-index λ, $|\lambda| \leq 2$. Compare [**KW**, p. 113].

Let $K = spt\, \varphi$. We may suppose that $K_{r_o} \subset \Omega$. Choose $s = 2^{-k}$ so small that $3s\sqrt{n} \leq r_o$. Cover K by a finite, disjoint collection of dyadic cubes Q_i with length $s(Q_i) = s$, $i = 1, \ldots, N$. We may suppose that

$$\frac{3}{2} Q_i \cap E \neq \emptyset \ \text{ for } \ i = 1, \ldots, N^*,$$

and

$$\frac{3}{2} Q_i \cap E = \emptyset \ \text{ for } \ i = N^* + 1, \ldots, N,$$

for some $N^* \in \mathbb{N}$, $1 \leq N^* \leq N$. Let φ_i, $i = 1, \ldots, N$, be the test functions related to the collection Q_i, $i = 1, \ldots, N$, and possessing the properties described in the above Lemma.

Since f is subharmonic in $\Omega \setminus E$ and all $\varphi\varphi_i$, $i = N^* + 1, \ldots, N$, are nonnegative test functions in $\mathcal{D}(\Omega \setminus E)$, we have

$$\int f \Delta(\varphi\varphi_i) \, dm \geq 0 \ \text{ for } \ i = N^* + 1, \ldots, N.$$

In view of these inequalities, we get

$$(3) \qquad \int f \, \Delta\varphi \, dm = \int f \, \Delta \left[\varphi \left(\sum_{j=1}^{N} \varphi_i \right) \right] dm = \sum_{i=1}^{N} \int_{\frac{3}{2}Q_i} f \, \Delta(\varphi\varphi_i) \, dm$$
$$\geq \sum_{i=1}^{N^*} \int_{\frac{3}{2}Q_i} f \, \Delta(\varphi\varphi_i) \, dm.$$

An easy computation shows that

$$\Delta(\varphi\varphi_i) = (\Delta\varphi)\varphi_i + \varphi(\Delta\varphi_i) + 2\sum_{j=1}^{n} \mathcal{D}_j\varphi \, \mathcal{D}_j\varphi_i.$$

By the properties of the test functions φ_i and φ, we have for all $i = 1, \ldots, N^*$ and $x \in \mathbb{R}^n$,

(4)

$$|\Delta(\varphi\varphi_i)(x)| \leq |\Delta\varphi(x)||\varphi_i(x)| + |\varphi(x)||\Delta\varphi_i(x)| + 2\sum_{j=1}^{n} |\mathcal{D}_j\varphi(x)||\mathcal{D}_j\varphi_i(x)|$$
$$\leq 1 + \frac{C_2}{s^2} + \frac{C_1}{s} \leq \frac{C}{s^2},$$

where $C = C(n, C_1, C_2)$. The last inequality here follows from the fact that, since $0 < r_o < 1$, also $0 < s < 1$.

For each cube Q_i, $i = 1, \dots, N^*$, there are clearly at most 3^n cubes Q_j, $s(Q_j) = s$, $j = 1, \dots, N_i \le 3^n$ (just the adjacent cubes to Q_i with equal length), such that

$$\frac{3}{2}Q_i \cap \frac{3}{2}Q_j \neq \emptyset. \tag{5}$$

Using this, the fact that $\frac{3}{2}Q_i \subset E_{3s\sqrt{n}}$, $i = 1, \dots, N^*$, (3) and (4), we get

$$\begin{aligned}\int f\, \Delta\varphi\, dm &\ge -\frac{C}{s^2}\sum_{i=1}^{N^*}\int_{\frac{3}{2}Q_i} f\, dm = -\frac{C}{s^2}\sum_{i=1}^{N^*}\int_{E_{3s\sqrt{n}}} f\, \chi_{\frac{3}{2}Q_i}\, dm \\ &= -\frac{C}{s^2}\int_{E_{3s\sqrt{n}}} f\left(\sum_{i=1}^{N^*}\chi_{\frac{3}{2}Q_i}\right) dm \\ &\ge -\frac{3^n C}{s^2}\int_{E_{3s\sqrt{n}}} f\, dm.\end{aligned}$$

Here $\chi_{\frac{3}{2}Q_i}$ is the characteristic function of $\frac{3}{2}Q_i$, $i = 1, \dots, N^*$. Above we have used the fact that $\sum_{i=1}^{N^*}\chi_{\frac{3}{2}Q_i}(x) \le 3^n$ for all $x \in E_{3s\sqrt{n}}$. Indeed, if $x \notin \frac{3}{2}Q_i$ for $i = 1, \dots, N^*$, then $\sum_{i=1}^{N^*}\chi_{\frac{3}{2}Q_i}(x) = 0$. If $x \in \frac{3}{2}Q_{i_o}$ for some i_o, $1 \le i_o \le N^*$, then by (5) we see that among the cubes $\frac{3}{2}Q_i$, $i = 1, \dots, N^*$, there are at most N_{i_o} such for which $x \in \frac{3}{2}Q_i$. Since $N_{i_o} \le 3^n$ (see (5) above), also $\sum_{i=1}^{N^*}\chi_{\frac{3}{2}Q_i}(x) \le 3^n$. Proceeding further then, and using also (2), we get

$$\begin{aligned}\int f\Delta\varphi\, dm &\ge -\frac{C}{s^2}\int_{E_{3s\sqrt{n}}} f\, dm \\ &\ge -\frac{C}{s^2}\int_{E_{3s\sqrt{n}}} d(x, E)^{\alpha+2-n}\, dm(x) \\ &\ge -\frac{C}{s^2}\left(3s\sqrt{n}\right)^2 \epsilon = -C\,\epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary and $C = C(n, \alpha, C^*)$, it follows that

$$\int f\, \Delta\varphi\, dm \ge 0,$$

concluding the proof. □

As Gardiner points out [**Ga**, p. 73], a slight modification of his proof of Theorem A yields Theorem B. In our frame the situation is similar:

Theorem 2. *Suppose that $\alpha \in [0, n-2]$ and $\mathcal{M}^\alpha(E) < \infty$. If f is subharmonic in $\Omega \setminus E$ and satisfies*

$$f(x) \le u(d(x,E)) \qquad (x \in \Omega \setminus E)$$

where $u(t)$ is a Borel measurable function such that $t^{n-2-\alpha}u(t) \to 0$ $(t \to 0+)$, then f has a subharmonic extension to Ω.

The proof goes along the same lines as above with only minor changes. In fact, take $\epsilon > 0$ arbitrarily. Choose then r_o, $0 < r_o < 1$, such that

$$u(t) < \epsilon\, t^{\alpha+2-n}$$

whenever $0 < t < r_o$. Since $\mathcal{M}^\alpha(E) < \infty$, we may suppose that $m(E_r) < M\, r^{n-\alpha}$ for all r, $0 < r \le r_o$. Proceeding then as in the proof of Theorem 1 (see (2) above), one sees that for all r, $0 < r \le r_o$,

$$\int_{E_r} f(x)\, dm(x) \le \int_{E_r} u(d(x,E))\, dm(x) \le \epsilon \int_{E_r} d(x,E)^{\alpha+2-n} dm(x) < \epsilon\, C\, r^2\, M < \infty.$$

The rest of the proof goes as in the proof of Theorem 1.

Example 1. Let $0 < \alpha < 1$ be arbitrarily given. By [**Fa**, Example 4.5, p. 58] there is a uniform Cantor set $F \subset [0,1]$ such that $\mathcal{M}^\alpha(F) = 0$. Set $E = F \times \cdots \times F$. Then E is closed and by [**Fa**, Example 7.6, p. 95], $\mathcal{M}^{\alpha n}(E) = 0$. Clearly E is not contained in any $\mathcal{C}^2$ $(n-1)$-dimensional manifold. Thus our results, Theorems 1 and 2 above, can be applied in situations where Gardiner's Theorems A and B cannot be used.

Example 2. By [**Ko**, 2.3, p. 462] there is for each α, $0 < \alpha < 2$, a countable, compact subset F of the complex plane $\mathbb{C}$ with $\mathcal{M}^\alpha(F) > 0$. Let $E = F \times \{0\} \subset \mathbb{R}^3$. One sees easily that $\mathcal{M}^\alpha(E) > 0$. Since E is countable, $\Lambda_\alpha(E) = 0$. Thus we have an example where Gardiner's theorems can be used whereas our results are not applicable.

Our last theorem improves Gardiner's Theorem C by allowing the exceptional set to be noncompact. The proof we present is different from that of Gardiner.

Theorem 3. *Suppose that $\alpha \in [0, n-2]$ and $\Lambda_\alpha(E) = 0$. If f is subharmonic in $\Omega \setminus E$ and satisfies*

$$\mathcal{A}(f^+, x, r) \le C^*\, r^{\alpha+2-n} \qquad \left(\overline{B(x,r)} \subset \Omega\right)$$

for some positive constant C^*, then f has a subharmonic extension to Ω.

Proof. As in the proof of Theorem 1, we may suppose that $\alpha \in (0, n-2)$ and $f \ge 0$. Since $f \in \mathcal{L}^1_{loc}(\Omega)$, it is sufficient to show that

$$\int f\, \Delta\varphi\, dm \ge 0 \tag{6}$$

for any nonnegative test function $\varphi \in \mathcal{D}(\Omega)$. Take such a φ arbitrarily. As in the proof of Theorem 1, we may suppose that $0 \le \varphi \le 1$ and $|\mathcal{D}^\lambda \varphi| \le 1$ for each multi-index λ, $|\lambda| \le 2$. Let $K = spt\,\varphi$. Choose r_o, $0 < r_o < 1$, such that $\widehat{K} = \overline{K}_{r_o} \subset K_{2r_o} \subset \overline{K}_{2r_o} \subset \Omega$. Let $\epsilon > 0$ be arbitrarily given. We will cover K by a suitable collection of mutually disjoint dyadic cubes. This will be done in three steps.

First, using the assumption $\Lambda_\alpha(E) = 0$ and (1), we find a sequence of mutually disjoint dyadic cubes Q_i, $s(Q_i) = s_i$, $i = 1, 2, \ldots$, such that

$$\sum_{i=1}^{\infty} s_i^\alpha < \epsilon. \tag{7}$$

We may suppose that $3s_i\sqrt{n} < r_o$, $i = 1, 2, \ldots$. Since $E \cap \widehat{K}$ is compact, there is $N_1 \in \mathbb{N}$ such that

$$E \cap \widehat{K} \subset \bigcup_{i=1}^{N_1} Q_i. \tag{8}$$

Second, we attach to each cube Q_i, $s(Q_i) = s_i$, $i = 1, \ldots, N_1$, all adjacent dyadic cubes with the same length s_i. Since two dyadic cubes are either mutually disjoint or one is contained in the other, we may drop extra cubes away. Proceeding in this way we get a collection of mutually disjoint cubes $Q_i^{j_i}$, $j_i = 0, \ldots, n_i$, $i = 1, \ldots, N_1$, such that

$$s(Q_i^{j_i}) = s(Q_i) = s_i, \quad j_i = 0, \ldots, n_i \le 3^n - 1,\ i = 1, \ldots, N_1. \tag{9}$$

(That indeed $n_i \le 3^n - 1$ for all $i = 1, \ldots, N_1$, follows just from the fact that we are considering adjacent cubes of the same length.)

Third, cover the remaining bounded set $K \setminus ((\cup_{i=1}^{N_1} Q_i) \cup (\cup_{i=1}^{N_1} (\cup_{j_i=0}^{n_i} Q_i^{j_i})))$ by mutually disjoint, dyadic cubes $\widetilde{Q}_k$, all with the same length $s(\widetilde{Q}_k) = s$, $k = 0, \ldots, N_2$, where $s = \min\{ s_i : i = 1, \ldots, N_1 \}$. Using then the facts that Q_i and $Q_i^{j_i}$ are adjacent, that $s(Q_i) = s(Q_i^{j_i}) = s_i$, $j_i = 0, \ldots, n_i$, and $s(\widetilde{Q}_k) = s \le s_i$, $i = 1, \ldots, N_1$, $k = 0, \ldots, N_2$, one sees easily that

$$\frac{3}{2}\widetilde{Q}_k \cap E = \emptyset \ \text{ for } \ k = 0, \ldots, N_2. \tag{10}$$

In order to show that (6) holds, we next choose nonnegative test functions φ_i, $\varphi_i^{j_i}$, $j_i = 0, \ldots, n_i$, $i = 1, \ldots, N_1$, and $\widetilde{\varphi}_k$, $k = 0, \ldots, N_2$, from $\mathcal{D}(\Omega)$ with the aid of the above Lemma, and thus with the following properties:

$$spt\,\varphi_i \subset \frac{3}{2}Q_i, \ \ |\mathcal{D}^\lambda \varphi_i| \le \frac{C_\lambda}{s_i^{|\lambda|}} \ \text{ for } \ \lambda, |\lambda| \le 2, i = 1, \ldots, N_1; \tag{11}$$

$$(12)\quad spt\,\varphi_i^{j_i} \subset \frac{3}{2}Q_i^{j_i},\ \ |\mathcal{D}^\lambda \varphi_i^{j_i}| \le \frac{C_\lambda}{s_i^{|\lambda|}} \quad \text{for } \ \lambda, |\lambda| \le 2,\ j_i = 0,\dots,n_i;\ i = 1,\dots,N_1;$$

$$(13)\qquad spt\,\widetilde{\varphi}_k \subset \frac{3}{2}\widetilde{Q}_k,\ \ |\mathcal{D}^\lambda \widetilde{\varphi}_k| \le \frac{C_\lambda}{s^{|\lambda|}} \ \text{ for } \ \lambda, |\lambda| \le 2, k = 0,\dots,N_2;$$

$$(14)\qquad \sum_{i=1}^{N_1} \varphi_i(x) + \sum_{i=1}^{N_1}\sum_{j_i=0}^{n_i} \varphi_i^{j_i}(x) + \sum_{k=0}^{N_2} \widetilde{\varphi}_k(x) = 1 \ \text{ for } \ x \in K.$$

Using then (10), (13) and the fact that f is subharmonic in $\Omega \setminus E$, one gets

$$\int_{\frac{3}{2}\widetilde{Q}_k} f\, \Delta(\varphi\, \widetilde{\varphi}_k)\, dm \ge 0 \ \text{ for } \ k = 0,\dots,N_2.$$

From this, (14), (11) and (12), it follows that

$$\begin{aligned}\int f\,\Delta\varphi\, dm &= \int f\,\Delta\left[\varphi\left(\sum_{i=1}^{N_1}\varphi_i + \sum_{i=1}^{N_1}\sum_{j_i=0}^{n_i}\varphi_i^{j_i} + \sum_{k=0}^{N_2}\widetilde{\varphi}_k\right)\right] dm \\ &\ge \sum_{i=1}^{N_1}\int_{\frac{3}{2}Q_i} f\,\Delta(\varphi\varphi_i)\, dm + \sum_{i=1}^{N_1}\sum_{j_i=0}^{n_i}\int_{\frac{3}{2}Q_i^{j_i}} f\,\Delta(\varphi\varphi_i^{j_i})\, dm.\end{aligned}$$

Using then (11) and (12) and proceeding as in the proof of Theorem 1, we get similar estimates as in (4),

$$|\Delta(\varphi\,\varphi_i)(x)| \le \frac{C}{s_i^2} \ \text{ for } \ x \in \frac{3}{2}Q_i,\ i = 1,\dots,N_1;$$

and

$$|\Delta(\varphi\,\varphi_i^{j_i})(x)| \le \frac{C}{s_i^2} \ \text{ for } \ x \in \frac{3}{2}Q_i^{j_i},\ j_i = 0,\dots,n_i,\ i = 1,\dots,N_1.$$

In view of these inequalities, and of (8), (9) and (7), we get (in the sequel x_i and $x_i^{j_i}$ are the centers of the cubes Q_i, $Q_i^{j_i}$, $j_i = 0,\dots,n_i$, $i = 1,\dots,N_1$, respectively, and $\nu_n = m(B(0,1)))$

$$\begin{aligned}
&\int f\,\Delta\varphi\,dm \\
&\geq -C\left(\sum_{i=1}^{N_1}\frac{1}{s_i^2}\int_{\frac{3}{2}Q_i} f\,dm + \sum_{i=1}^{N_1}\sum_{j_i=0}^{n_i}\frac{1}{s_i^2}\int_{\frac{3}{2}Q_i^{j_i}} f\,dm\right) \\
&\geq -C\left(\sum_{i=1}^{N_1}\frac{1}{s_i^2}\int_{B(x_i,\frac{3}{4}s_i\sqrt{n})} f\,dm + \sum_{i=1}^{N_1}\sum_{j_i=0}^{n_i}\frac{1}{s_i^2}\int_{B(x_i^{j_i},\frac{3}{4}s_i\sqrt{n})} f\,dm\right) \\
&\geq -\left(\frac{3}{4}\sqrt{n}\right)^n \nu_n\,C\left(\sum_{i=1}^{N_1}s_i^\alpha + 3^n\sum_{i=1}^{N_1}s_i^\alpha\right) \\
&\geq -C\sum_{i=1}^{N_1}s_i^\alpha \geq -C\,\epsilon.
\end{aligned}$$

Since $C = C(n, \alpha, C^*)$ and ϵ was arbitrarily given, (6) follows and the proof is complete. □

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