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REMOVABLE SETS FOR SUBHARMONIC FUNCTIONS

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It is a classical result that a closed exceptional polar set is removable for subharmonic functions which are bounded above. Gardiner has shown that in the case of a compact exceptional set the above boundedness condition can be relaxed by imposing certain smoothness and Hausdorff measure conditions on the set. We give related results for a closed exceptional set, by replacing the smoothness and Hausdorff measure conditions with one sole condition on Minkowski upper content.

1. Introduction.

In the sequel $\Omega$ is always an open set in $\mathbb{R}^n$, $n \geq 2$, and $E \subset \Omega$ is closed in $\Omega$. It is a classical result [HK, Theorem 5.18, p. 237] that if $f$ is subharmonic in $\Omega \setminus E$ and bounded above and moreover $E$ is polar, then $f$ has a subharmonic extension to the whole of $\Omega$. Imposing certain constraints on the geometry and size of the set $E$, Gardiner relaxed considerably the boundedness requirement of $f$ [Ga, Theorems 1 and 3, pp. 71-74]. To state his results, let $\Phi : \Omega \to \mathbb{R}$ be a $C^2$ function with nonvanishing gradient throughout $\Omega$. Put $S = \{ x \in \Omega : \Phi(x) = 0 \}$. Write $d(x, S)$ for the distance from $x \in \mathbb{R}^n$ to $S$ and let $\Lambda_\alpha$ be the $\alpha$-dimensional Hausdorff (outer) measure in $\mathbb{R}^n$.

**Theorem A.** Let $\alpha \in (0, n-2)$ and $E$ be a compact subset of $S$ such that $\Lambda_\alpha(E) = 0$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies
\[
f(x) \leq C \, d(x, S)^{\alpha+2-n} \quad (x \in \Omega \setminus S)
\]
for some positive constant $C$, then $f$ has a subharmonic extension to $\Omega$.

**Theorem B.** Let $\alpha \in (0, n-2)$ and $E$ be a compact subset of $S$ such that $\Lambda_\alpha(E) < \infty$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies
\[
f(x) \leq u(d(x, S)) \quad (x \in \Omega \setminus S)
\]
where $t^{n-2-\alpha}u(t) \to 0$ ($t \to 0^+$), then $f$ has a subharmonic extension to $\Omega$.

Our notation is more or less standard or will be explained below. For example, $B(x, r)$ is the open ball in $\mathbb{R}^n$, with center $x$ and radius $r$. The family of test functions on $\Omega$ is denoted by $D(\Omega)$. The differential operator $(\partial_{x_1})^\lambda_1 \cdots (\partial_{x_n})^\lambda_n = (\frac{\partial}{\partial x_1})^\lambda_1 \cdots (\frac{\partial}{\partial x_n})^\lambda_n$ is denoted by $D^\lambda$. Here $\lambda =$
($\lambda_1, \ldots, \lambda_n) \in (\mathbb{N} \cup \{0\})^n$ is a multi-index, and $|\lambda| = \lambda_1 + \cdots + \lambda_n$. The Laplacian is $\Delta = D^2_1 + \cdots + D^2_n$. The notation $C(n, \alpha, \ldots)$, say, means that $C$ is a constant depending only on $n, \alpha, \ldots$. As usual, constants may vary from line to line.

Gardiner also shows [Ga, Theorems 2 and 4, pp. 72-73] that his results are sharp in the following sense: If one drops the smoothness assumption $E \subset S$ then the exceptional set $E$ is not any more necessarily removable. Our purpose is to point out that there exist, however, results which are in a certain sense parallel to Gardiner’s results but where no smoothness conditions are necessary to impose on the exceptional set. As a matter of fact, we show below in Theorems 1 and 2 that results similar to Gardiner’s hold when his conditions,

(i) $E \subset S$ where $S$ is a $C^2$ $(n-1)$-dimensional manifold in $\Omega$,
(ii) $\Lambda_\alpha(E) = 0$ (resp. $\Lambda_\alpha(E) < \infty$),

are replaced by one geometric measure condition $M^\alpha(E) = 0$ (resp. $M^\alpha(E) < \infty$) where $M^\alpha$ is the upper Minkowski content. Our proofs are different and perhaps shorter than those of Gardiner. Moreover, our approach does not require the exceptional set $E$ to be compact, unlike in Gardiner’s results. On the other hand, as is shown in Examples 1 and 2 below, Gardiner’s and our results are independent: Neither our nor Gardiner’s results are included in the other’s.

Gardiner also [Ga, Theorem 5, p. 74] proves the following result:

**Theorem C.** Let $\alpha \in (0, n-2)$ and $E$ be a compact subset of $S$ such that $\Lambda_\alpha(E) = 0$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies

$$\mathcal{A}(f^+, x, r) \leq Cr^{\alpha+2-n} \quad \left( B(x, r) \subset \Omega \right)$$

then $f$ has a subharmonic extension to $\Omega$.

Here $f^+ = \max\{f, 0\}$ and $\mathcal{A}(f^+, x, r)$ is the mean value of $f^+$ over the ball $B(x, r)$, with respect to the Lebesgue measure $m$ in $\mathbb{R}^n$.

Below in Theorem 3 we improve this result by dropping the condition that $E$ is compact. Again our approach is essentially different than that of Gardiner.

## 2. Net measure and Minkowski content.

For readers’ convenience we first recall certain basic facts concerning net measure and Minkowski content and their relationship with the standard Hausdorff measure. For a more thorough discussion see e.g., [HP, pp. 41-44] and [Fa, pp. 33, 42].
Let $A \subset \mathbb{R}^n$ and $\alpha \in [0, n]$. For each $\epsilon > 0$ define

$$L^\epsilon_\alpha(A) = \inf \sum_{i=1}^{\infty} s_i^\alpha$$

where the infimum is over all coverings of $A$ by countable disjoint collection of dyadic cubes $Q_i$ with (side)length $s_i \leq \epsilon$. Define the $\alpha$-dimensional net measure of $A$ by

$$L_\alpha(A) = \lim_{\epsilon \to 0^+} L^\epsilon_\alpha(A).$$

It is well-known that the standard Hausdorff measure $\Lambda_\alpha$ and the net measure $L_\alpha$ are comparable: There are positive constants $C_1 = C_1(n)$ and $C_2 = C_2(n)$ such that

$$C_1 L_\alpha(A) \leq \Lambda_\alpha(A) \leq C_2 L_\alpha(A)$$

for all $A \subset \mathbb{R}^n$.

To define the Minkowski content, let $A \subset \mathbb{R}^n$, $\alpha \in [0, n]$ and $\epsilon > 0$. Write

$$A_\epsilon = \{ x \in \mathbb{R}^n : d(x, A) < \epsilon \}.$$

The $\alpha$-dimensional upper Minkowski content of $A$ is defined by

$$\mathcal{M}^\alpha(A) = \limsup_{\epsilon \to 0^+} \frac{m(A_\epsilon)}{\epsilon^{n-\alpha}}.$$

It is well-known that there is a positive constant $C_3 = C_3(n, \alpha)$ such that

$$C_3 \Lambda_\alpha(A) \leq \mathcal{M}^\alpha(A)$$

for all $A \subset \mathbb{R}^n$. The reverse inequality does not hold in general, but is true for certain smooth sets, even for $\alpha$ rectifiable closed subsets of $\mathbb{R}^n$ (here $\alpha$ is a positive integer). See [HP, p. 41] and [Fe, 3.2.39, p. 275].

Our argument will essentially be based on the following type of partition of unity, see [HP, Lemma 3.1, p. 43]:

**Lemma.** Let $\{Q_i : i = 1, \ldots, N\}$ be a finite disjoint collection of dyadic cubes of length $s(Q_i) = s_i$. For each $i$, there is a function $\varphi_i \in C_0^\infty(\mathbb{R}^n)$ with support $\text{spt} \varphi_i \subset \frac{3}{2}Q_i$ such that $\sum_{i=1}^{\infty} \varphi_i(x) = 1$ for all $x \in \cup_{i=1}^{N}Q_i$. Furthermore, for each multi-index $\lambda$, there is a constant $C_{\lambda} = C_{\lambda}(\lambda, n)$ for which $|D^\lambda \varphi_i(x)| \leq C_{\lambda}s_i^{-|\lambda|}$ for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, N$.

3. The results.

Our first result is parallel to Gardiner’s Theorem A:

**Theorem 1.** Suppose that $\alpha \in [0, n-2]$ and $\mathcal{M}^\alpha(E) = 0$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies

$$f(x) \leq C^* d(x, E)^{\alpha+2-n} \quad (x \in \Omega \setminus E)$$

for some positive constant $C^*$, then $f$ has a subharmonic extension to $\Omega$. 
Proof. If \( \alpha = 0 \) then \( E = \emptyset \). If \( \alpha = n - 2 \), then \( E \) is polar e.g., by [HK, Theorem 5.14, p. 288]. Since \( f \) is then also bounded above, the claim follows from the classical result [HK, Theorem 5.18, p. 237].

It remains to consider the case \( \alpha \in (0, n - 2) \). Since \( f^+ \) is subharmonic, and also

\[
f^+(x) \leq C^* d(x, E)^{\alpha+2-n} \quad (x \in \Omega \setminus E),
\]

we may suppose that \( f \geq 0 \).

We first show that \( f \in L^1_{\text{loc}}(\Omega) \), cf. [HP, p. 42] and [Ri, pp. 730-731]. It is sufficient to show that for some \( r > 0 \),

\[
\int_{E_r} f \, dm < \infty.
\]

Take \( \epsilon > 0 \) arbitrarily. Since \( M^\alpha(E) = 0 \), there is \( r_0, 0 < r_0 < 1 \), such that \( m(E_r) \leq \epsilon r^{n-\alpha} \) for all \( r, 0 \leq r \leq r_0 \). Take any such \( r \), and write for each \( j = 0, 1, \ldots \),

\[
K_j = \{ x \in \mathbb{R}^n : d(x, E) < r 2^{-j} \}.
\]

Then

\[
E_r = \bigcup_{j=0}^{\infty} (K_j \setminus K_{j+1}),
\]

and

\[
\int_{E_r} f(x) \, dm(x) \leq C^* \int_{E_r} d(x, E)^{\alpha+2-n} \, dm(x)
\]

\[
= C^* \sum_{j=0}^{\infty} \int_{K_j \setminus K_{j+1}} d(x, E)^{\alpha+2-n} \, dm(x)
\]

\[
\leq C^* \sum_{j=0}^{\infty} \left[ r 2^{-(j+1)} \right]^{\alpha+2-n} m(K_j)
\]

\[
\leq C^* 2^{n-2-\alpha} r^{\alpha+2-n} \sum_{j=0}^{\infty} 2^{(n-2-\alpha)j} \epsilon (r 2^{-j})^{n-\alpha}
\]

\[
\leq C^* 2^{n-2-\alpha} r^2 \epsilon \sum_{j=0}^{\infty} 2^{-2j} < \infty.
\]

Thus \( f \in L^1_{\text{loc}}(\Omega) \). For later use we observe that we also got

\[
\int_{E_r} d(x, E)^{\alpha+2-n} \, dm(x) \leq C r^2 \epsilon
\]

for all \( r, 0 \leq r \leq r_0 \), where \( C = C(n, \alpha, C^*) \).
To complete the proof, it remains to show that for any nonnegative test function $\varphi \in D(\Omega)$,
\[
\int f \Delta \varphi \, dm \geq 0.
\]
We may suppose that $0 \leq \varphi \leq 1$ and $|D^\lambda \varphi| \leq 1$ for each multi-index $\lambda$, $|\lambda| \leq 2$. Compare [KW, p. 113].

Let $K = spt \varphi$. We may suppose that $K_{r_0} \subset \Omega$. Choose $s = 2^{-k}$ so small that $3s\sqrt{n} \leq r_0$. Cover $K$ by a finite, disjoint collection of dyadic cubes $Q_i$ with length $s(Q_i) = s$, $i = 1, \ldots, N$. We may suppose that
\[
\frac{3}{2} \text{ } Q_i \cap E \neq \emptyset \text{ for } i = 1, \ldots, N^*;
\]
and
\[
\frac{3}{2} \text{ } Q_i \cap E = \emptyset \text{ for } i = N^* + 1, \ldots, N,
\]
for some $N^* \in \mathbb{N}$, $1 \leq N^* \leq N$. Let $\varphi_i$, $i = 1, \ldots, N$, be the test functions related to the collection $Q_i$, $i = 1, \ldots, N$, and possessing the properties described in the above Lemma.

Since $f$ is subharmonic in $\Omega \setminus E$ and all $\varphi \varphi_i$, $i = N^* + 1, \ldots, N$, are nonnegative test functions in $D(\Omega \setminus E)$, we have
\[
\int f \Delta (\varphi \varphi_i) \, dm \geq 0 \text{ for } i = N^* + 1, \ldots, N.
\]
In view of these inequalities, we get
\[
\int f \Delta \varphi \, dm = \int f \Delta \left[ \varphi \left( \sum_{j=1}^{N} \varphi_i \right) \right] \, dm = \sum_{i=1}^{N} \int_{\frac{3}{2} Q_i} f \Delta (\varphi \varphi_i) \, dm
\]
(3)
\[
\geq \sum_{i=1}^{N^*} \int_{\frac{3}{2} Q_i} f \Delta (\varphi \varphi_i) \, dm.
\]

An easy computation shows that
\[
\Delta (\varphi \varphi_i) = (\Delta \varphi) \varphi_i + \varphi (\Delta \varphi_i) + 2 \sum_{j=1}^{n} D_j \varphi D_j \varphi_i.
\]
By the properties of the test functions $\varphi_i$ and $\varphi$, we have for all $i = 1, \ldots, N^*$ and $x \in \mathbb{R}^n$,
\[
|\Delta (\varphi \varphi_i)(x)| \leq |\Delta \varphi(x)||\varphi_i(x)| + |\varphi(x)||\Delta \varphi_i(x)| + 2 \sum_{j=1}^{n} |D_j \varphi(x)||D_j \varphi_i(x)|
\]
(4)
\[
\leq 1 + \frac{C_2}{s^2} + \frac{C_1}{s} \leq \frac{C}{s^2},
\]
where $C = C(n, C_1, C_2)$. The last inequality here follows from the fact that, since $0 < r_0 < 1$, also $0 < s < 1$.

For each cube $Q_i$, $i = 1, \ldots, N^*$, there are clearly at most $3^n$ cubes $Q_j$, $s(Q_j) = s$, $j = 1, \ldots, N_i \leq 3^n$ (just the adjacent cubes to $Q_i$ with equal length), such that

$$3^n Q_i \cap \frac{3}{2} Q_j \neq \emptyset.$$  

Using this, the fact that $\frac{3}{2} Q_i \subset E_{3s\sqrt{n}}$, $i = 1, \ldots, N^*$, (3) and (4), we get

$$\int f \Delta \varphi \, dm \geq -\frac{C}{s^2} \sum_{i=1}^{N^*} \int_{\frac{3}{2} Q_i} f \, dm = -\frac{C}{s^2} \sum_{i=1}^{N^*} \int_{E_{3s\sqrt{n}}} f \chi_{\frac{3}{2} Q_i} \, dm$$

$$= -\frac{C}{s^2} \int_{E_{3s\sqrt{n}}} f \left( \sum_{i=1}^{N^*} \chi_{\frac{3}{2} Q_i} \right) \, dm$$

$$\geq -\frac{3^n C}{s^2} \int_{E_{3s\sqrt{n}}} f \, dm.$$  

Here $\chi_{\frac{3}{2} Q_i}$ is the characteristic function of $\frac{3}{2} Q_i$, $i = 1, \ldots, N^*$. Above we have used the fact that $\sum_{i=1}^{N^*} \chi_{\frac{3}{2} Q_i}(x) \leq 3^n$ for all $x \in E_{3s\sqrt{n}}$. Indeed, if $x \notin \frac{3}{2} Q_i$ for $i = 1, \ldots, N^*$, then $\sum_{i=1}^{N^*} \chi_{\frac{3}{2} Q_i}(x) = 0$. If $x \in \frac{3}{2} Q_{i_0}$ for some $i_0$, $1 \leq i_0 \leq N^*$, then by (5) we see that among the cubes $\frac{3}{2} Q_i$, $i = 1, \ldots, N^*$, there are at most $N_{i_0}$ such for which $x \in \frac{3}{2} Q_i$. Since $N_{i_0} \leq 3^n$ (see (5) above), also $\sum_{i=1}^{N^*} \chi_{\frac{3}{2} Q_i}(x) \leq 3^n$. Proceeding further then, and using also (2), we get

$$\int f \Delta \varphi \, dm \geq -\frac{C}{s^2} \int_{E_{3s\sqrt{n}}} f \, dm$$

$$\geq -\frac{C}{s^2} \int_{E_{3s\sqrt{n}}} d(x, E)^{\alpha+2-n} \, dm(x)$$

$$\geq -\frac{C}{s^2} (3s\sqrt{n})^2 \epsilon = -C \epsilon.$$  

Since $\epsilon > 0$ was arbitrary and $C = C(n, \alpha, C^*)$, it follows that

$$\int f \Delta \varphi \, dm \geq 0,$$

concluding the proof. \hfill \Box

As Gardiner points out [Ga, p. 73], a slight modification of his proof of Theorem A yields Theorem B. In our frame the situation is similar:
Theorem 2. Suppose that $\alpha \in [0, n - 2]$ and $\mathcal{M}^\alpha(E) < \infty$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies

$$f(x) \leq u(d(x, E)) \quad (x \in \Omega \setminus E)$$

where $u(t)$ is a Borel measurable function such that $t^{n-2-\alpha}u(t) \to 0$ ($t \to 0^+$), then $f$ has a subharmonic extension to $\Omega$.

The proof goes along the same lines as above with only minor changes. In fact, take $\epsilon > 0$ arbitrarily. Choose then $r_0 > 0$, $0 < r < 1$, such that

$$u(t) < \epsilon t^{\alpha+2-n}$$

whenever $0 < t < r_0$. Since $\mathcal{M}^\alpha(E) < \infty$, we may suppose that $m(E_r) < M r^{n-\alpha}$ for all $r$, $0 < r \leq r_0$. Proceeding then as in the proof of Theorem 1 (see (2) above), one sees that for all $r$, $0 < r \leq r_0$,

$$\int_{E_r} f(x) \, dm(x) \leq \int_{E_r} u(d(x, E)) \, dm(x) \leq \epsilon \int_{E_r} d(x, E)^{\alpha+2-n} \, dm(x)$$

$$< \epsilon C r^2 M < \infty.$$  

The rest of the proof goes as in the proof of Theorem 1.

Example 1. Let $0 < \alpha < 1$ be arbitrarily given. By [Fa, Example 4.5, p. 58] there is a uniform Cantor set $F \subset [0, 1]$ such that $\mathcal{M}^\alpha(F) = 0$. Set $E = F \times \cdots \times F$. Then $E$ is closed and by [Fa, Example 7.6, p. 95], $\mathcal{M}^{\alpha n}(E) = 0$. Clearly $E$ is not contained in any $C^2 (n - 1)$-dimensional manifold. Thus our results, Theorems 1 and 2 above, can be applied in situations where Gardiner’s Theorems A and B cannot be used.

Example 2. By [Ko, 2.3, p. 462] there is for each $\alpha$, $0 < \alpha < 2$, a countable, compact subset $F$ of the complex plane $\mathbb{C}$ with $\mathcal{M}^\alpha(F) > 0$. Let $E = F \times \{0\} \subset \mathbb{R}^3$. One sees easily that $\mathcal{M}^\alpha(E) > 0$. Since $E$ is countable, $\Lambda_\alpha(E) = 0$. Thus we have an example where Gardiner’s theorems can be used whereas our results are not applicable.

Our last theorem improves Gardiner’s Theorem C by allowing the exceptional set to be noncompact. The proof we present is different from that of Gardiner.

Theorem 3. Suppose that $\alpha \in [0, n - 2]$ and $\Lambda_\alpha(E) = 0$. If $f$ is subharmonic in $\Omega \setminus E$ and satisfies

$$\mathcal{A}(f^+, x, r) \leq C^* r^{\alpha+2-n} \quad \left( B(x, r) \subset \Omega \right)$$

for some positive constant $C^*$, then $f$ has a subharmonic extension to $\Omega$.

Proof. As in the proof of Theorem 1, we may suppose that $\alpha \in (0, n - 2)$ and $f \geq 0$. Since $f \in \mathcal{L}^1_{loc}(\Omega)$, it is sufficient to show that

$$(6) \quad \int f \Delta \varphi \, dm \geq 0$$
for any nonnegative test function $\varphi \in \mathcal{D}(\Omega)$. Take such a $\varphi$ arbitrarily. As in the proof of Theorem 1, we may suppose that $0 \leq \varphi \leq 1$ and $|\mathcal{D}^\lambda \varphi| \leq 1$ for each multi-index $\lambda$, $|\lambda| \leq 2$. Let $K = \text{spt} \varphi$. Choose $r_o$, $0 < r_o < 1$, such that $\hat{K} = K_{r_o} \subset K_{2r_o} \subset K_{2r_o} \subset \Omega$. Let $\epsilon > 0$ be arbitrarily given. We will cover $K$ by a suitable collection of mutually disjoint dyadic cubes. This will be done in three steps.

First, using the assumption $\Lambda_\alpha(E) = 0$ and (1), we find a sequence of mutually disjoint dyadic cubes $Q_i$, $s(Q_i) = s_i$, $i = 1, 2, \ldots$, such that

\[(7) \sum_{i=1}^{\infty} s_i^i \epsilon < \epsilon.\]

We may suppose that $3s_i \sqrt{n} < r_o$, $i = 1, 2, \ldots$. Since $E \cap \hat{K}$ is compact, there is $N_1 \in \mathbb{N}$ such that

\[(8) E \cap \hat{K} \subset \bigcup_{i=1}^{N_1} Q_i.\]

Second, we attach to each cube $Q_i$, $s(Q_i) = s_i$, $i = 1, \ldots, N_1$, all adjacent dyadic cubes with the same length $s_i$. Since two dyadic cubes are either mutually disjoint or one is contained in the other, we may drop extra cubes away. Proceeding in this way we get a collection of mutually disjoint cubes $Q_i^j$, $j = 0, \ldots, n_i$, $i = 1, \ldots, N_1$, such that

\[(9) s(Q_i^j) = s(Q_i) = s_i, \quad j = 0, \ldots, n_i \leq 3^n - 1, \quad i = 1, \ldots, N_1.\]

(That indeed $n_i \leq 3^n - 1$ for all $i = 1, \ldots, N_1$, follows just from the fact that we are considering adjacent cubes of the same length.)

Third, cover the remaining bounded set $K \setminus ((\bigcup_{i=1}^{N_1} Q_i) \cup (\bigcup_{i=1}^{n_i} (\bigcup_{j=0}^{n_i} Q_i^j)))$ by mutually disjoint, dyadic cubes $\tilde{Q}_k$, all with the same length $s(\tilde{Q}_k) = s$, $k = 0, \ldots, N_2$, where $s = \min\{s_i : i = 1, \ldots, N_1\}$. Using then the facts that $Q_i$ and $Q_i^j$ are adjacent, that $s(Q_i) = s(Q_i^j) = s_i$, $j = 0, \ldots, n_i$, and $s(\tilde{Q}_k) = s \leq s_i$, $i = 1, \ldots, N_1$, $k = 0, \ldots, N_2$, one sees easily that

\[(10) \frac{3}{2} \tilde{Q}_k \cap E = \emptyset \quad \text{for} \quad k = 0, \ldots, N_2.\]

In order to show that (6) holds, we next choose nonnegative test functions $\varphi_i$, $\varphi_i^j$, $j = 0, \ldots, n_i$, $i = 1, \ldots, N_1$, and $\tilde{\varphi}_k$, $k = 0, \ldots, N_2$, from $\mathcal{D}(\Omega)$ with the aid of the above Lemma, and thus with the following properties:

\[(11) \text{spt} \varphi_i \subset \frac{3}{2} Q_i, \quad |\mathcal{D}^\lambda \varphi_i| \leq \frac{C_\lambda}{s_i^{|\lambda|}} \quad \text{for} \quad \lambda, |\lambda| \leq 2, \quad i = 1, \ldots, N_1;\]
(12) \( \text{spt } \varphi_i^j \subset \frac{3}{2} Q_i^j, \ |D^\lambda \varphi_i^j| \leq \frac{C}{s_i^{\lambda}} \) for \( \lambda, |\lambda| \leq 2, \ j_i = 0, \ldots, n_i; \ i = 1, \ldots, N_1; \)

(13) \( \text{spt } \tilde{\varphi}_k \subset \frac{3}{2} \tilde{Q}_k, \ |D^\lambda \tilde{\varphi}_k| \leq \frac{C}{s_i^{\lambda}} \) for \( \lambda, |\lambda| \leq 2, \ k = 0, \ldots, N_2; \)

(14) \[
\sum_{i=1}^{N_1} \varphi_i(x) + \sum_{i=1}^{N_1} \sum_{j_i=0}^{n_i} \varphi_i^{j_i}(x) + \sum_{k=0}^{N_2} \tilde{\varphi}_k(x) = 1 \quad \text{for } x \in K.
\]

Using then (10), (13) and the fact that \( f \) is subharmonic in \( \Omega \setminus E \), one gets

\[
\int_{\frac{3}{2} \tilde{Q}_k} f \Delta (\varphi \tilde{\varphi}_k) \, dm \geq 0 \quad \text{for } k = 0, \ldots, N_2.
\]

From this, (14), (11) and (12), it follows that

\[
\int f \Delta \varphi \, dm = \int f \Delta \left[ \varphi \left( \sum_{i=1}^{N_1} \varphi_i + \sum_{i=1}^{N_1} \sum_{j_i=0}^{n_i} \varphi_i^{j_i} + \sum_{k=0}^{N_2} \tilde{\varphi}_k \right) \right] \, dm \\
\geq \sum_{i=1}^{N_1} \int_{\frac{3}{2} Q_i} f \Delta (\varphi \varphi_i) \, dm + \sum_{i=1}^{N_1} \sum_{j_i=0}^{n_i} \int_{\frac{3}{2} Q_i^j} f \Delta (\varphi \varphi_i^{j_i}) \, dm.
\]

Using then (11) and (12) and proceeding as in the proof of Theorem 1, we get similar estimates as in (4),

\[
|\Delta (\varphi \varphi_i)(x)| \leq \frac{C}{s_i^{\lambda}} \quad \text{for } x \in \frac{3}{2} Q_i, \ i = 1, \ldots, N_1;
\]

and

\[
|\Delta (\varphi \varphi_i^{j_i})(x)| \leq \frac{C}{s_i^{\lambda}} \quad \text{for } x \in \frac{3}{2} Q_i^j, \ j_i = 0, \ldots, n_i, \ i = 1, \ldots, N_1.
\]

In view of these inequalities, and of (8), (9) and (7), we get (in the sequel \( x_i \) and \( x_i^{j_i} \) are the centers of the cubes \( Q_i, Q_i^j, j_i = 0, \ldots, n_i, i = 1, \ldots, N_1, \)
respectively, and \( \nu_n = m(B(0, 1)) \)
\[
\int f \Delta \varphi \, dm \\
\geq -C \left( \sum_{i=1}^{N_1} \frac{1}{s_i^2} \int_{B(x_i, s_i \sqrt{n})} f \, dm + \sum_{i=1}^{N_1} \sum_{j_i=0}^{n_i} \frac{1}{s_i^2} \int_{B(x_i^{j_i}, \frac{3}{2} s_i \sqrt{n})} f \, dm \right) \\
\geq -C \left( \sum_{i=1}^{N_1} \frac{1}{s_i^2} \int_{B(x_i, s_i \sqrt{n})} f \, dm + \sum_{i=1}^{N_1} \sum_{j_i=0}^{n_i} \frac{1}{s_i^2} \int_{B(x_i^{j_i}, \frac{3}{2} s_i \sqrt{n})} f \, dm \right) \\
\geq -\left( \frac{3}{4} \sqrt{n} \right)^n \nu_n \frac{C}{n} \left( \sum_{i=1}^{N_1} s_i^\alpha + 3^n \sum_{i=1}^{N_1} s_i^\alpha \right) \\
\geq -C \sum_{i=1}^{N_1} s_i^\alpha \geq -C \epsilon.
\]
Since \( C = C(n, \alpha, C^*) \) and \( \epsilon \) was arbitrarily given, (6) follows and the proof is complete. \( \square \)

References


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