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A SCHLÄFLI DIFFERENTIAL FORMULA FOR SIMPLICES
IN SEMI-RIEMANNIAN HYPERQUADRICS,
GAUSS-BONNET FORMULAS FOR SIMPLICES IN THE
DE SITTER SPHERE AND THE DUAL VOLUME OF A
HYPERBOLIC SIMPLEX

EVA SUÁREZ-PEIRÓ

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In this paper we prove a Schläfli differential formula for the volume of simplices in central unit hyperquadrics of semi-Euclidean space \mathbb{R}_q^{n+1} . Then we apply this result to obtain Gauss-Bonnet formulas for simplices with riemannian faces in the de Sitter sphere, and to generalize a formula of L. Santaló relating the volume of a hyperbolic simplex with the measure of the set of hyperbolic hyperplanes intersecting it.

Introduction.

Schläfli's differential formula plays a central role in the computation of the volume of hyperbolic and spherical polyhedra of any dimension. Given a family of n -simplices Δ which vary in a differentiable manner in a non-Euclidean space of dimension n and constant curvature $\kappa = 1$ or $\kappa = -1$, the Schläfli formula expresses the differential of the volume of Δ in terms of the volumes of its faces of codimension 2 and the dihedral angles at these faces. Namely,

$$dV_n(\Delta) = \frac{\kappa}{n-1} \sum_F V_{n-2}(F) d\alpha_F$$

where the sum is taken over all faces F of codimension 2 of Δ , $V_{n-2}(F)$ denotes the $((n-2)$ -dimensional) volume of face F and α_F denotes the dihedral angle at face F .

Around 1852, L. Schläfli proved this formula for spherical simplices of any dimension ([Schlä]). In 1936 H. Kneser gave a different proof, which he could easily generalize to the hyperbolic case ([Kne]). A more recent, very recommended reference is J. Milnor's paper ([Mil]), where some remarks on the history of the Schläfli differential formula can be found, as well as a new proof which is particularly transparent. In the last years great use has been made of this formula to compute the volume of hyperbolic 3-manifolds, possibly with cone-type singularities (cf. [HLM₁], [HLM₂], [HLM₃]), following an idea of C. Hodgson ([Ho]). It has also been used to study the

volume of hyperbolic spaceforms in general (cf. [Kel]). Recently F. Bonahon has found a Schläfli type formula for the volume of the convex core of hyperbolic 3-manifolds (cf. [Bo]).

On the other hand, it is also interesting to compute the volume of simplices contained in complete semi-riemannian manifolds of constant curvature; some work has been done in this direction by J.-M. Schlenker ([Schle₁], [Schle₂]). An important example is the n -dimensional de Sitter sphere \mathbb{S}_1^n , which is a complete Lorentz manifold of constant curvature 1. There is a duality relationship between \mathbb{S}_1^n and n -dimensional hyperbolic space \mathbb{H}^n (when both are considered as submanifolds of Minkowski space \mathbb{R}_1^{n+1}), associating to each point \mathbf{v} in \mathbb{S}_1^n the hyperplane in \mathbb{H}^n with unit normal \mathbf{v} (cf. [HR]). Therefore, a Schläfli formula for the volume of simplices in \mathbb{S}_1^n can be used to relate the volume of a hyperbolic simplex with the measure of the set of hyperplanes intersecting it. (We will call the measure of this set of hyperplanes, the *dual volume* of the given hyperbolic simplex.) For $n = 3$ such a relationship was found by L. Santaló ([Sa₂]), using methods of integral geometry. (An analogous equation, relating the volume of a spherical tetrahedron with the measure of the set of planes intersecting it, was proved by Milnor using the spherical Schläfli formula, cf. [Mil].) My PhD advisor, José María Montesinos ([Mo]), obtained a Schläfli differential equality for tetrahedra in \mathbb{S}_1^3 by differentiating the above mentioned formula of Santaló and then applying Schläfli's equation for hyperbolic tetrahedra. He suggested to me the possibility of obtaining a Schläfli formula for simplices in \mathbb{S}_1^n following Kneser's proof, and of generalizing with it Santaló's equality to higher dimensions. This is, in a somewhat wider context, the aim of this paper, which is a part of my PhD thesis ([Su]).

In Section 1 we adapt and generalize Kneser's proof to obtain a Schläfli differential formula for a large class of simplices in any central unit hyperquadric of semi-Euclidean space \mathbb{R}_q^{n+1} (cf. [O'N]). For this, a suitable definition of dihedral angle in a semi-riemannian geometry is necessary. This is a point that requires some care and it is tackled in subsection 1.4, where the (partial) standard definition is generalized (cf. [O'N]). (A different definition of dihedral angle (with complex values) in the semi-riemannian case, giving rise to a different version of the Schläfli formula, has been used by J.-M. Schlenker, cf. [Schle₁], [Schle₂].)

Then we give three applications of the Schläfli differential formula for simplices in the de Sitter sphere. In Section 2 we give a new proof of Santaló's formula in dimension 3 (relating the volume of a hyperbolic tetrahedron with the measure of the set of hyperplanes intersecting it), using the Schläfli formulas in \mathbb{H}^3 y \mathbb{S}_1^3 . In this section we also introduce the notions of polar dual and complementary dual of a hyperbolic simplex of arbitrary dimension, which will be necessary to generalize Santaló's formula to higher dimensions.

(The polar dual of a convex hyperbolic polyhedron has been extensively studied in [Ri] and [HR].)

In Section 3 we obtain Gauss-Bonnet formulas for simplices in the n -dimensional de Sitter sphere S_1^n with riemannian faces. These formulas are analogous to the generalized Gauss-Bonnet formulas for spherical and hyperbolic simplices, which relate the volumes of all even-dimensional faces of the simplex and the dihedral angles at those faces (cf. [Sa₁], [AVS]).

Finally Section 4 is a consequence of the previous two, and provides a generalization to higher dimensions of Santaló’s formula relating the volume of a hyperbolic simplex with the measure of the set of hyperplanes intersecting it.

I thank José María Montesinos for suggesting the idea of this paper and for all his helpful advice and encouragement. I also thank Francis Bonahon for giving me the opportunity to talk about this result in a students’ seminar at the I.H.P. in Paris, and I. Rivin and J.-M. Schlenker for their suggestions and for drawing my attention to their results.

Added in the revised version. After submission of this paper, I. Rivin and J.-M. Schlenker have obtained a smooth analogue of the Schläfli formula for the volume bounded by a hypersurface in a general Einstein manifold, using methods of differential geometry (cf. [RS]). The Schläfli formula for polyhedra in the de Sitter sphere proved in this paper follows as a corollary from their result. J.-M. Schlenker and R. Souam have also obtained higher dimensional analogues of the Schläfli formula ([S-S]), which generalize the formulas given in Proposition 3.1. of this paper.

1. A Schläfli differential formula for simplices in semi-riemannian hyperquadrics.

1.1. Background and definitions.

To fix notations we recall now some standard definitions in semi-riemannian geometry (see [O’N]).

Semi-Euclidean space \mathbb{R}_q^{n+1} is \mathbb{R}^{n+1} with the semi-riemannian metric defined by the bilinear form of index q

$$\langle \mathbf{x}, \mathbf{y} \rangle = - \sum_{i=0}^{q-1} x_i y_i + \sum_{j=q}^n x_j y_j$$

where $\mathbf{x} = (x_0, \dots, x_n)$ and $\mathbf{y} = (y_0, \dots, y_n)$. The *norm* of a vector $\mathbf{x} \in \mathbb{R}_q^{n+1}$ is $|\mathbf{x}| = |\langle \mathbf{x}, \mathbf{x} \rangle|^{1/2}$. A vector \mathbf{x} is *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, *spacelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ and *null* if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$.

Given $\epsilon \in \{\pm 1\}$, the *central unit hyperquadric of \mathbb{R}_q^{n+1} with sign ϵ* is the submanifold

$$Q_q^n(\epsilon) = \{ \mathbf{x} \in \mathbb{R}_q^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = \epsilon \}.$$

It is a complete semi-riemannian n -dimensional manifold (not necessarily connected) with constant curvature ϵ . The totally geodesic submanifolds of $Q_q^n(\epsilon)$ are the connected components of the intersection of $Q_q^n(\epsilon)$ with linear subspaces of \mathbb{R}^{n+1} . The isometries of $Q_q^n(\epsilon)$ are the restriction to $Q_q^n(\epsilon)$ of the linear automorphisms of \mathbb{R}_q^{n+1} leaving $Q_q^n(\epsilon)$ invariant. Important examples are the n -dimensional sphere $\mathbb{S}^n = Q_0^n(1)$, the n -dimensional hyperbolic space \mathbb{H}^n , which is one of the two components of $Q_1^n(-1)$, the n -dimensional de Sitter sphere $\mathbb{S}_1^n = Q_1^n(1)$, and the 3-dimensional hyperquadric $Q_2^3(1)$, which can be identified with $SL(2, \mathbb{R})$ with the metric given by its canonical Killing form. From now on we will write $S_q^n(\epsilon)$ to denote any connected component of $Q_q^n(\epsilon)$, and we will also call $S_q^n(\epsilon)$ an n -dimensional hyperquadric.

Let us now define an n -simplex in the hyperquadric $S_q^n(\epsilon)$. We define a *linear halfspace* in \mathbb{R}_q^{n+1} as a closed halfspace whose boundary is a linear hyperplane of \mathbb{R}_q^{n+1} . A *simplicial cone* in \mathbb{R}_q^{n+1} is the intersection of $(n + 1)$ linear halfspaces in general position (i.e., such that the $(n + 1)$ boundary hyperplanes intersect only at the origin). A *face* of codimension k of a simplicial cone C ($k = 0, \dots, n$) is the intersection of C with k of the hyperplanes forming its boundary. Observe that the intersection of an arbitrary simplicial cone C with the hyperquadric $S_q^n(\epsilon)$ can be empty, and it can also be non-compact.

Definition 1.1. An n -simplex in the n -dimensional hyperquadric $S_q^n(\epsilon)$ is the intersection of a simplicial cone with $S_q^n(\epsilon)$, when this intersection is non-empty and compact. A *face* of codimension k of an n -simplex $\Delta \subset S_q^n(\epsilon)$ ($k = 0, \dots, n$) is the intersection with $S_q^n(\epsilon)$ of a face of codimension k of the corresponding simplicial cone.

Remark 1.1. Every simplicial cone C in \mathbb{R}_q^{n+1} can be written as

$$C = \{x_0\mathbf{v}_0 + \dots + x_n\mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\}$$

for some basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ of \mathbb{R}_q^{n+1} . We will use the following notation for the faces of codimension k of a simplicial cone:

$$C_{i_1 \dots i_k} = \{x_0\mathbf{v}_0 + \dots + x_n\mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0; x_{i_1} = \dots = x_{i_k} = 0\}.$$

A subset $\Delta \subset S_q^n(\epsilon)$ is an n -simplex if and only if it can be written as

$$\Delta = \{x_0\mathbf{v}_0 + \dots + x_n\mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap S_q^n(\epsilon)$$

for some basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ of \mathbb{R}_q^{n+1} such that $\epsilon \cdot \langle x_0\mathbf{v}_0 + \dots + x_n\mathbf{v}_n, x_0\mathbf{v}_0 + \dots + x_n\mathbf{v}_n \rangle > 0$ if $x_0 \geq 0, \dots, x_n \geq 0$ and they are not all zero. We will denote the codimension k faces of an n -simplex Δ as follows:

$$F_{i_1 \dots i_k} = \{x_0\mathbf{v}_0 + \dots + x_n\mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0; x_{i_1} = \dots = x_{i_k} = 0\} \cap S_q^n(\epsilon).$$

Remark 1.2. Let C be a simplicial cone in \mathbb{R}_q^{n+1} such that $\Delta = C \cap S_q^n(\epsilon)$ is an n -simplex. Each codimension k face $C_{i_1 \dots i_k}$ of C spans an $(n+1-k)$ -dimensional linear subspace $\langle C_{i_1 \dots i_k} \rangle$ of \mathbb{R}_q^{n+1} . If the induced metric in this subspace is non-degenerate of a certain index ν , then $\langle C_{i_1 \dots i_k} \rangle \cap S_q^n(\epsilon)$ is an $(n-k)$ -dimensional hyperquadric of type $S_\nu^{n-k}(\epsilon)$, and the corresponding face of Δ , $F_{i_1 \dots i_k} = C_{i_1 \dots i_k} \cap S_q^n(\epsilon)$, is an $(n-k)$ -simplex inside this hyperquadric. If the induced metric in $\langle C_{i_1 \dots i_k} \rangle$ is degenerate, then the face $F_{i_1 \dots i_k}$ of Δ also has a degenerate metric as a submanifold of $S_q^n(\epsilon)$.

In what follows we will restrict our attention to the class of n -simplices Δ in $S_q^n(\epsilon)$ satisfying the following additional condition (*), in order to avoid dividing by zero in the coming calculations:

(*) *All faces of codimension 1 and 2 of Δ have a non-degenerate metric.*

1.2. The volume function.

Let Δ be an n -simplex in $S_q^n(\epsilon)$, and let C be the simplicial cone of \mathbb{R}_q^{n+1} such that $\Delta = C \cap S_q^n(\epsilon)$. To compute the volume of Δ , it is very useful to consider the following idea of Kneser ([Kne]). Define the function $r(\mathbf{x}) = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, for $\mathbf{x} \in \mathbb{R}_q^{n+1}$. Then the following relation holds inside the simplicial cone C :

$$d\mathbb{R}_q^{n+1} = r^n dr \wedge dS_q^n(\epsilon)$$

where $d\mathbb{R}_q^{n+1}$ denotes the volume form of \mathbb{R}_q^{n+1} (with the standard orientation) and $dS_q^n(\epsilon)$ denotes the volume form of $S_q^n(\epsilon)$ (oriented in such a way that the normal vector $\frac{\partial}{\partial r}$ followed by a positive basis of tangent vectors to $S_q^n(\epsilon)$, form a positive basis of \mathbb{R}_q^{n+1}).

Hence the volume $V_n(\Delta)$ of Δ can be obtained from the following equality:

$$\begin{aligned} (1) \quad \int_C e^{-\frac{r^2}{2}} d\mathbb{R}_q^{n+1} &= \int_C r^n e^{-\frac{r^2}{2}} dr \wedge dS_q^n(\epsilon) \\ &= \int_\Delta dS_q^n(\epsilon) \cdot \int_0^{+\infty} r^n e^{-\frac{r^2}{2}} dr \\ &= 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) V_n(\Delta). \end{aligned}$$

Let $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be a positive basis of \mathbb{R}_q^{n+1} such that $C = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\}$. Denote by (x_0, \dots, x_n) the coordinates in this basis and let $\Phi(x_0, \dots, x_n) = \langle x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n, x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \rangle$ be the quadratic form of \mathbb{R}_q^{n+1} . Then inside the simplicial cone C we have $\Phi = \epsilon \cdot r^2$. On the other hand, consider the parallelepiped determined in \mathbb{R}_q^{n+1} by the vectors $\mathbf{v}_0, \dots, \mathbf{v}_n$, and denote by $V(\mathbf{v}_0, \dots, \mathbf{v}_n)$ its volume in

\mathbb{R}_q^{n+1} (which coincides with its euclidean volume). Then the volume form of \mathbb{R}_q^{n+1} is

$$d\mathbb{R}_q^{n+1} = V(\mathbf{v}_0, \dots, \mathbf{v}_n) dx_0 \wedge \dots \wedge dx_n.$$

Therefore, Equation (1) can be written in the coordinates (x_0, \dots, x_n) as

$$(2) \quad 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) V_n(\Delta) = V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \int_C e^{-\epsilon \Phi/2} dx_0 \dots dx_n.$$

1.3. The volume differential.

The space \mathcal{S} of all n -simplices in $S_q^n(\epsilon)$ has a natural structure of $n(n+1)$ -dimensional manifold, with charts consisting, for instance, of products of sufficiently small neighborhoods of the $n+1$ vertices inside $S_q^n(\epsilon)$. The volume function $V_n : \mathcal{S} \rightarrow \mathbb{R}$, associating to each simplex Δ its volume $V_n(\Delta)$, is a differentiable map, and we are interested in finding a formula for its differential.

To do this, we will look at an arbitrary point $\Delta \in \mathcal{S}$ and consider a special basis formed by $n(n+1)$ linearly independent tangent vectors of \mathcal{S} at Δ , and we will compute the value of the form dV_n applied to each of these tangent vectors. This special basis is defined as follows: for each of the $n+1$ vertices of Δ , we take n linearly independent tangent vectors of \mathcal{S} at Δ that represent n ways of moving the chosen vertex in different directions, keeping all the other vertices fixed. More explicitly, the chosen vertex will move along the line joining it to each of the n remaining vertices, as described below.

Let $C = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\}$ be the simplicial cone such that $\Delta = C \cap S_q^n(\epsilon)$. Since the ordering of the vertices is arbitrary, it is enough to study the case where the vertex \mathbf{v}_1 moves towards the vertex \mathbf{v}_0 and all other vertices remain fixed. For every $t \in \mathbb{R}$ consider the basis formed

by the vectors $\begin{cases} \mathbf{v}_0(t) = \mathbf{v}_0 \\ \mathbf{v}_1(t) = \mathbf{v}_1 - t \mathbf{v}_0 \\ \mathbf{v}_i(t) = \mathbf{v}_i \text{ if } 2 \leq i \leq n \end{cases}$. When t is sufficiently close to

0, the simplicial cone $C_t = \{\lambda_0 \mathbf{v}_0(t) + \dots + \lambda_n \mathbf{v}_n(t) \mid \lambda_0 \geq 0, \dots, \lambda_n \geq 0\}$ is such that $\Delta_t = C_t \cap S_q^n(\epsilon)$ is an n -simplex. Then we will take as one of the tangent vectors of \mathcal{S} at Δ , the tangent vector to the path Δ_t at time $t = 0$.

The value of the form dV_n applied to this tangent vector is $\left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0}$.

Using formula (2),

$$2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) V_n(\Delta_t) = V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \int_{C_t} e^{-\epsilon \Phi/2} dx_0 \dots dx_n.$$

Since

$$\begin{aligned} C_t &= \{\lambda_0 \mathbf{v}_0(t) + \dots + \lambda_n \mathbf{v}_n(t) \mid \lambda_0 \geq 0, \dots, \lambda_n \geq 0\} \\ &= \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 + x_1 t \geq 0, x_1 \geq 0, \dots, x_n \geq 0\} \end{aligned}$$

we have

$$2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) V_n(\Delta_t) = V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \int_{x_1=0}^{+\infty} \int_{x_0=-x_1 t}^{+\infty} \int_{x_2=0}^{+\infty} \dots \int_{x_n=0}^{+\infty} e^{-\epsilon \Phi/2} dx_0 \dots dx_n$$

and differentiating we obtain

$$(3) \quad 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0} = V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \int_{x_1=0}^{+\infty} \int_{x_2=0}^{+\infty} \dots \int_{x_n=0}^{+\infty} x_1 e^{-\epsilon \Phi/2} dx_1 \dots dx_n \Big|_{x_0=0}.$$

Let us now consider the dual basis $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$ of $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$, which satisfies the relation $\langle \mathbf{v}_i, \mathbf{w}_j \rangle = \delta_{ij}$. Taking multiples of the vectors \mathbf{v}_i if necessary, we can always assume that $|\mathbf{w}_j| = 1$ for $j = 0, \dots, n$. Denote by (x_0, \dots, x_n) the coordinates in the basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$, and by (y_0, \dots, y_n) the coordinates in the dual basis $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$. Then the following relation holds:

$$(4) \quad x_i = \sum_{j=0}^n \langle \mathbf{w}_i, \mathbf{w}_j \rangle y_j \quad y_i = \sum_{j=0}^n \langle \mathbf{v}_i, \mathbf{v}_j \rangle x_j.$$

Now the quadratic form of \mathbb{R}_q^{n+1} is $\Phi = \sum_{i,j=0}^n \langle \mathbf{v}_i, \mathbf{v}_j \rangle x_i x_j$. Therefore,

$$\frac{\partial \Phi}{\partial x_i} = 2 \sum_{j=0}^n \langle \mathbf{v}_i, \mathbf{v}_j \rangle x_j = 2 y_i.$$

From (4) we deduce that, inside the hyperplane $x_0 = 0$,

$$\begin{cases} x_1 = \sum_{j=0}^n \langle \mathbf{w}_1, \mathbf{w}_j \rangle y_j \\ x_0 = \langle \mathbf{w}_0, \mathbf{w}_0 \rangle y_0 + \sum_{i=1}^n \langle \mathbf{w}_0, \mathbf{w}_i \rangle y_i = 0. \end{cases}$$

Hence

$$\begin{aligned} x_1 &= \sum_{i=1}^n \left(\langle \mathbf{w}_1, \mathbf{w}_i \rangle - \frac{\langle \mathbf{w}_0, \mathbf{w}_1 \rangle \langle \mathbf{w}_0, \mathbf{w}_i \rangle}{\langle \mathbf{w}_0, \mathbf{w}_0 \rangle} \right) y_i \\ &= \sum_{i=1}^n \left(\langle \mathbf{w}_1, \mathbf{w}_i \rangle - \frac{\langle \mathbf{w}_0, \mathbf{w}_1 \rangle \langle \mathbf{w}_0, \mathbf{w}_i \rangle}{\langle \mathbf{w}_0, \mathbf{w}_0 \rangle} \right) \frac{\partial(\Phi/2)}{\partial x_i}. \end{aligned}$$

For every $t \in \mathbb{R}$ let $\{\mathbf{w}_0(t), \dots, \mathbf{w}_n(t)\}$ be the dual basis of $\{\mathbf{v}_0(t), \dots, \mathbf{v}_n(t)\}$. It is easy to check that $\begin{cases} \mathbf{w}_0(t) = \mathbf{w}_0 + t \mathbf{w}_1 \\ \mathbf{w}_i(t) = \mathbf{w}_i \text{ for } 1 \leq i \leq n \end{cases}$. Now define for every $i \in \{1, \dots, n\}$ the function

$$f_{0i}(t) = \frac{\langle \mathbf{w}_0(t), \mathbf{w}_i(t) \rangle}{|\mathbf{w}_0(t)| \cdot |\mathbf{w}_i(t)|}.$$

Considering that $|\mathbf{w}_j| = 1$ for all j , we obtain on differentiating that

$$\left. \frac{df_{0i}(t)}{dt} \right|_{t=0} = \langle \mathbf{w}_1, \mathbf{w}_i \rangle - \frac{\langle \mathbf{w}_0, \mathbf{w}_1 \rangle \langle \mathbf{w}_0, \mathbf{w}_i \rangle}{\langle \mathbf{w}_0, \mathbf{w}_0 \rangle}.$$

Therefore,

$$x_1 = \sum_{i=1}^n \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \cdot \frac{\partial(\Phi/2)}{\partial x_i}$$

and plugging this into (3) it follows that

$$\begin{aligned} (5) \quad & 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0} \\ &= V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \int_0^{+\infty} \dots \int_0^{+\infty} \sum_{i=1}^n \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \\ & \quad \cdot \left. \frac{\partial(\Phi/2)}{\partial x_i} e^{-\epsilon \Phi/2} dx_1 \dots dx_n \right|_{x_0=0} \\ &= -\epsilon \cdot V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \sum_{i=1}^n \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \\ & \quad \cdot \int_0^{+\infty} \dots \int_0^{+\infty} \left. \frac{\partial(e^{-\epsilon \Phi/2})}{\partial x_i} dx_1 \dots dx_n \right|_{x_0=0} \\ &= \epsilon \cdot V(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot \sum_{i=1}^n \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \\ & \quad \cdot \int_0^{+\infty} \dots \int_0^{+\infty} \left. e^{-\epsilon \Phi/2} dx_1 \dots dx_n \right|_{x_0=x_i=0}. \end{aligned}$$

Now let us observe that the volume of the codimension 2 face of Δ , $F_{0i} = \{x_1 \mathbf{v}_1 + \dots + x_n \mathbf{v}_n \mid x_1 \geq 0, \dots, x_n \geq 0; x_i = 0\} \cap S_q^n(\epsilon)$, can also be computed using the relation (2) in the $(n-1)$ -dimensional subspace of \mathbb{R}_q^{n+1}

spanned by $\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n$. In other words,

$$2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right) V_{n-2}(F_{0i}) = V(\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n) \cdot \int_0^{+\infty} \dots \int_0^{+\infty} e^{-\epsilon \Phi/2} dx_1 \dots d\widehat{x}_i \dots dx_n \Big|_{x_0 = x_i = 0}.$$

Substituting this last expression in (5), we get

$$\begin{aligned} & 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \frac{dV_n(\Delta_t)}{dt} \Big|_{t=0} \\ &= \epsilon \sum_{i=1}^n \frac{V(\mathbf{v}_0, \dots, \mathbf{v}_n)}{V(\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n)} \cdot \frac{df_{0i}(t)}{dt} \Big|_{t=0} 2^{\frac{n-3}{2}} \cdot \Gamma\left(\frac{n-1}{2}\right) V_{n-2}(F_{0i}) \end{aligned}$$

and since the gamma function has the property that $\Gamma\left(\frac{n+1}{2}\right) = \frac{n-1}{2} \Gamma\left(\frac{n-1}{2}\right)$, it follows that

$$\begin{aligned} (6) \quad & \frac{dV_n(\Delta_t)}{dt} \Big|_{t=0} \\ &= \frac{\epsilon}{n-1} \sum_{i=1}^n \frac{V(\mathbf{v}_0, \dots, \mathbf{v}_n)}{V(\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n)} \cdot \frac{df_{0i}(t)}{dt} \Big|_{t=0} \cdot V_{n-2}(F_{0i}). \end{aligned}$$

Let us now make some elementary observations about the volume of a parallelepiped in \mathbb{R}_q^{n+1} , which will allow us to simplify the factor $\frac{V(\mathbf{v}_0, \dots, \mathbf{v}_n)}{V(\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n)}$ in the last expression.

- (i) The volume $V(\mathbf{v}_0, \dots, \mathbf{v}_n)$ is the same as the euclidean volume of the parallelepiped determined in \mathbb{R}^{n+1} by the vectors $\mathbf{v}_0, \dots, \mathbf{v}_n$ (because the matrix J_q of the quadratic form of \mathbb{R}_q^{n+1} has determinant ± 1). Hence $V(\mathbf{v}_0, \dots, \mathbf{v}_n)$ is the absolute value of the determinant of the matrix having as columns the coordinates of the vectors $\mathbf{v}_0, \dots, \mathbf{v}_n$ in the standard basis of \mathbb{R}_q^{n+1} . It also equals $\sqrt{|\det(\langle \mathbf{v}_i, \mathbf{v}_j \rangle)_{i,j}|}$.
- (ii) Let $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$ be the dual basis of $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$, and let M (resp. N) be the matrix having as columns the coordinates of $\mathbf{v}_0, \dots, \mathbf{v}_n$ (resp. $\mathbf{w}_0, \dots, \mathbf{w}_n$). Then $M^t \cdot J_q \cdot N$ is the identity matrix, so $|\det(N)| = |\det(M)|^{-1}$ and

$$V(\mathbf{w}_0, \dots, \mathbf{w}_n) = \frac{1}{V(\mathbf{v}_0, \dots, \mathbf{v}_n)}$$

- (iii) For any given k between 0 and $n-1$, let $\mathbf{v}'_0, \dots, \mathbf{v}'_k$ be the orthogonal projections of the vectors $\mathbf{v}_0, \dots, \mathbf{v}_k$ onto the subspace spanned

by $\mathbf{w}_0, \dots, \mathbf{w}_k$ (which is itself orthogonal to the remaining vectors $\mathbf{v}_{k+1}, \dots, \mathbf{v}_n$). Then for every $i \in \{0, \dots, k\}$, $\mathbf{v}'_i - \mathbf{v}_i$ is a linear combination of the vectors $\mathbf{v}_{k+1}, \dots, \mathbf{v}_n$, and therefore

$$\begin{aligned} V(\mathbf{v}_0, \dots, \mathbf{v}_n) &= V(\mathbf{v}'_0, \dots, \mathbf{v}'_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n) \\ &= V(\mathbf{v}'_0, \dots, \mathbf{v}'_k) \cdot V(\mathbf{v}_{k+1}, \dots, \mathbf{v}_n) \end{aligned}$$

(the last equality being a consequence of the orthogonality). On the other hand, it is immediately checked that $\{\mathbf{w}_0, \dots, \mathbf{w}_k\}$ is the dual basis of $\{\mathbf{v}'_0, \dots, \mathbf{v}'_k\}$ inside the subspace spanned by $\mathbf{w}_0, \dots, \mathbf{w}_k$. Hence from (ii) follows that

$$V(\mathbf{v}_0, \dots, \mathbf{v}_n) = \frac{V(\mathbf{v}_{k+1}, \dots, \mathbf{v}_n)}{V(\mathbf{w}_0, \dots, \mathbf{w}_k)}.$$

Using this last property in the particular case of expression (6) we get

$$\begin{aligned} (7) \quad & \left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0} \\ &= \frac{\epsilon}{n-1} \sum_{i=1}^n \frac{1}{V(\mathbf{w}_0, \mathbf{w}_i)} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \cdot V_{n-2}(F_{0i}) \\ &= \frac{\epsilon}{n-1} \sum_{i=1}^n \frac{1}{\sqrt{|\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2|}} \\ & \quad \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \cdot V_{n-2}(F_{0i}). \end{aligned}$$

Before we can proceed further to arrive at a Schläfli formula, we need a definition of dihedral angle in semi-riemannian geometry. A natural definition has been given by J-M. Schlenker (cf. [Schle₁], [Schle₂]), in which the dihedral angle can take complex values. This definition, together with a suitable definition of semi-riemannian volume (also complex-valued), provides a Schläfli formula and Gauss-Bonnet formulas in the semi-riemannian case which are identical to the corresponding spherical formulas (cf. [Schle₁]).

However, here we are interested in defining the dihedral angle $\alpha_{0i}(t)$ at the face $F_{0i}(t)$ of Δ_t as a *real* number, in such a way that

$$\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{|\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2|}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}$$

where $f_{0i}(t) = \frac{\langle \mathbf{w}_0(t), \mathbf{w}_i(t) \rangle}{|\mathbf{w}_0(t)| \cdot |\mathbf{w}_i(t)|}$.

1.4. A definition of dihedral angle in semi-riemannian geometry.

Consider the 2-dimensional vector subspace of \mathbb{R}_q^{n+1} spanned by the vectors \mathbf{w}_0 and \mathbf{w}_i , which is orthogonal to the codimension 2 subspace $\langle C_{0i} \rangle$ spanned by $\mathbf{v}_1, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n$. The restriction of the quadratic form of \mathbb{R}_q^{n+1} to

this plane is non-degenerate, because of the condition (*) imposed upon the simplex Δ (cf. Remark 1.2). If it is a (positive or negative) definite quadratic form, then the dihedral angle at the face F_{0i} can be defined in the usual manner, as

$$\alpha_{0i} = \pi - \arccos \left(\frac{\langle \mathbf{w}_0, \mathbf{w}_i \rangle}{|\mathbf{w}_0| \cdot |\mathbf{w}_i|} \right) = \pi - \arccos(f_{0i}).$$

However, this orthogonal plane can also have a Lorentz metric, so we need to define the angle between two non-null vectors in the Lorentz-Minkowski plane \mathbb{R}_1^2 . The difficulty lies in the fact that the set of unit vectors in \mathbb{R}_1^2 is not bounded. In the Euclidean plane \mathbb{R}^2 , the angle between two unit vectors $\mathbf{w}_1, \mathbf{w}_2$ can be defined as the length of the arc they subtend on the unit sphere \mathbb{S}^1 . If we try to define the angle between two unit vectors in \mathbb{R}_1^2 in an analogous manner, it can happen that the arc they subtend on the set of all unit vectors has infinite branches.

A natural way to solve the problem is the following. Since the hyperbola \mathcal{H}^- corresponding to timelike unit vectors in \mathbb{R}_1^2 has a positive definite metric, while the hyperbola \mathcal{H}^+ corresponding to spacelike unit vectors has a negative definite metric, we will assign positive length to all arcs contained in \mathcal{H}^- and negative length to all arcs contained in \mathcal{H}^+ . Given two non-null unit vectors $\mathbf{w}_1, \mathbf{w}_2$ in \mathbb{R}_1^2 , consider the arc (possibly with infinite branches) subtended by them in $\mathcal{H}^- \cup \mathcal{H}^+$, and let l_r be the sum of the lengths (with the above sign convention) of the portions of this arc contained inside the Euclidean ball of radius r in \mathbb{R}^2 . Then define the angle between \mathbf{w}_1 and \mathbf{w}_2 as the limit of l_r as r tends to infinity. It is easy to check that this is equivalent to the following definition.

Definition 1.2. Given two non-null vectors $\mathbf{w}_1, \mathbf{w}_2$ in \mathbb{R}_1^2 , the *angle* between \mathbf{w}_1 and \mathbf{w}_2 is defined by

$$\left\{ \begin{array}{ll} \text{ang}(\mathbf{w}_1, \mathbf{w}_2) = \operatorname{arccosh} \left(\frac{-\langle \mathbf{w}_1, \mathbf{w}_2 \rangle}{|\mathbf{w}_1| \cdot |\mathbf{w}_2|} \right) & \text{if } \langle \mathbf{w}_1, \mathbf{w}_1 \rangle \cdot \langle \mathbf{w}_2, \mathbf{w}_2 \rangle > 0 \\ & \text{and } \langle \mathbf{w}_1, \mathbf{w}_2 \rangle < 0; \\ \text{ang}(\mathbf{w}_1, \mathbf{w}_2) = -\operatorname{arccosh} \left(\frac{\langle \mathbf{w}_1, \mathbf{w}_2 \rangle}{|\mathbf{w}_1| \cdot |\mathbf{w}_2|} \right) & \text{if } \langle \mathbf{w}_1, \mathbf{w}_1 \rangle \cdot \langle \mathbf{w}_2, \mathbf{w}_2 \rangle > 0 \\ & \text{and } \langle \mathbf{w}_1, \mathbf{w}_2 \rangle > 0; \\ \text{ang}(\mathbf{w}_1, \mathbf{w}_2) = -\operatorname{arcsinh} \left(\frac{\langle \mathbf{w}_1, \mathbf{w}_2 \rangle}{|\mathbf{w}_1| \cdot |\mathbf{w}_2|} \right) & \text{if } \langle \mathbf{w}_1, \mathbf{w}_1 \rangle \cdot \langle \mathbf{w}_2, \mathbf{w}_2 \rangle < 0. \end{array} \right.$$

This definition is consistent with the standard one for the hyperbolic angle between two timelike vectors lying in the same timecone of a Lorentz vector space (cf. [O’N]), and it is also the adequate one for our purpose.

Since with our sign convention we have assigned zero length to the set of all unit vectors in \mathbb{R}_1^2 , the following definition of dihedral angle is also natural.

Definition 1.3. Given two non-null vectors $\mathbf{w}_1, \mathbf{w}_2$ in \mathbb{R}_q^{n+1} which span a Lorentz plane, the *dihedral angle* α at the edge of the dihedron $\{\mathbf{v} \in \mathbb{R}_q^{n+1} : \langle \mathbf{v}, \mathbf{w}_1 \rangle \geq 0, \langle \mathbf{v}, \mathbf{w}_2 \rangle \geq 0\}$ is defined as $\alpha = -\text{ang}(\mathbf{w}_1, \mathbf{w}_2)$.

1.5. A Schläfli differential formula.

Theorem 1.1. Let $S_q^n(\epsilon)$ be a connected component of the central unit hyperquadric of sign ϵ of \mathbb{R}_q^{n+1} , and let Δ be a family of n -simplices in $S_q^n(\epsilon)$ varying differentiably with respect to one or more parameters, and such that all their faces of codimension 1 and 2 have a non-degenerate induced metric. Then the differential of their volume $V_n(\Delta)$ satisfies the following equality:

$$dV_n(\Delta) = \frac{\epsilon}{n-1} \sum_F V_{n-2}(F) d\alpha_F$$

where the sum is taken over all codimension 2 faces F of Δ , $V_{n-2}(F)$ is the $((n-2)$ -dimensional) volume of the face F and α_F is the dihedral angle at the face F . (When $n-2=0$, we set $V_0(F) = 1$.)

Proof. Let us continue the proof at the point where we had arrived in subsection 1.3. Our starting point was an n -simplex

$$\Delta = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap S_q^n(\epsilon)$$

(where the basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ is chosen so that its dual basis is formed by unit vectors \mathbf{w}_i), and we deformed it by moving one of its vertices, \mathbf{v}_1 , towards another one, \mathbf{v}_0 , keeping all the others fixed. More precisely, for every t sufficiently close to 0 we defined the n -simplex

$$\Delta_t = \{\lambda_0 \mathbf{v}_0 + \lambda_1 (\mathbf{v}_1 - t \mathbf{v}_0) + \lambda_2 \mathbf{v}_2 + \dots + \lambda_n \mathbf{v}_n \mid \lambda_0 \geq 0, \dots, \lambda_n \geq 0\} \cap S_q^n(\epsilon).$$

The only dihedral angles that change are the angles at the edges of the face opposite to the vertex \mathbf{v}_0 , that is, the dihedral angles $\alpha_{0i}(t)$ at the codimension 2 faces of Δ_t of the form $F_{0i}(t)$ ($i = 1, \dots, n$). Therefore, what we want to prove is that

$$\left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0} = \frac{\epsilon}{n-1} \sum_{i=1}^n V_{n-2}(F_{0i}) \cdot \left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0}.$$

In subsection 1.3 we showed that

$$\begin{aligned} & \left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0} \\ &= \frac{\epsilon}{n-1} \sum_{i=1}^n V_{n-2}(F_{0i}) \cdot \frac{1}{\sqrt{|\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2|}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0} \end{aligned}$$

where $f_{0i}(t) = \frac{\langle \mathbf{w}_0(t), \mathbf{w}_i(t) \rangle}{|\mathbf{w}_0(t)| \cdot |\mathbf{w}_i(t)|}$. Hence it only remains to prove that

$$\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{|\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}.$$

We need to distinguish several cases:

- **Case 1.** If $\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2 > 0$, then for t close to 0, the plane spanned by $\mathbf{w}_0(t)$ and $\mathbf{w}_i(t)$ has a definite metric, and the dihedral angle at the face $F_{0i}(t)$ is $\alpha_{0i}(t) = \pi - \arccos(f_{0i}(t))$. Hence $\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{1 - f_{0i}^2(0)}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}$

- **Case 2.** If $\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2 < 0$, then for t close to 0, the plane spanned by $\mathbf{w}_0(t)$ and $\mathbf{w}_i(t)$ has a Lorentz metric. There are three possibilities:

- (i) $\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle > 0$ and $\langle \mathbf{w}_0, \mathbf{w}_i \rangle < 0$.

Then $\alpha_{0i}(t) = -\operatorname{arccosh}(-f_{0i}(t))$ and

$$\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{f_{0i}^2(0) - 1}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}.$$

- (ii) $\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle > 0$ and $\langle \mathbf{w}_0, \mathbf{w}_i \rangle > 0$.

Now $\alpha_{0i}(t) = \operatorname{arccosh}(f_{0i}(t))$ and

$$\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{f_{0i}^2(0) - 1}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}.$$

- (iii) $\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle < 0$.

Now $\alpha_{0i}(t) = \operatorname{arcsinh}(f_{0i}(t))$ and

$$\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{1 + f_{0i}^2(0)}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}.$$

Taking into account the fact that all the vectors \mathbf{w}_i have norm 1, we see that in all these cases

$$\left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0} = \frac{1}{\sqrt{|\langle \mathbf{w}_0, \mathbf{w}_0 \rangle \cdot \langle \mathbf{w}_i, \mathbf{w}_i \rangle - \langle \mathbf{w}_0, \mathbf{w}_i \rangle^2}} \cdot \left. \frac{df_{0i}(t)}{dt} \right|_{t=0}.$$

Therefore,

$$\left. \frac{dV_n(\Delta_t)}{dt} \right|_{t=0} = \frac{\epsilon}{n-1} \sum_{i=1}^n V_{n-2}(F_{0i}) \cdot \left. \frac{d\alpha_{0i}(t)}{dt} \right|_{t=0}.$$

□

2. Dual volume of a tetrahedron in 3-dimensional hyperbolic space. (A formula of Santaló, revisited.)

We define the *dual volume* of a hyperbolic n -simplex Δ in \mathbb{H}^n as the “measure” of the set of hyperplanes of \mathbb{H}^n intersecting it. There are two natural ways to define a measure on the set \mathcal{P}^n of hyperplanes of \mathbb{H}^n . On the one hand, \mathcal{P}^n can be identified with the Lie group $O(1, n)/\mathcal{J}$, where \mathcal{J} is the subgroup of $O(1, n)$ formed by all transformations leaving a given hyperplane of \mathbb{H}^n invariant. This Lie group has a left-invariant volume form, which defines a measure on \mathcal{P}^n (cf. [Sa₂]). On the other hand, the de Sitter sphere $\mathbb{S}_1^n(1)$ (which we will simply denote by \mathbb{S}_1^n from now on) is a double cover of \mathcal{P}^n , via the map associating to each point $\mathbf{v} \in \mathbb{S}_1^n$ the hyperplane of \mathbb{H}^n orthogonal to \mathbf{v} . Since the volume form of \mathbb{S}_1^n is invariant under the antipodal map, it also induces a measure on \mathcal{P}^n . It can be checked that both definitions coincide. Hence the dual volume of a hyperbolic n -simplex can be computed as the volume of a certain subset of the de Sitter sphere.

2.1. Polar dual and complementary dual of a hyperbolic simplex.

Let Δ be an n -simplex in hyperbolic space \mathbb{H}^n . Let $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be a basis of \mathbb{R}_1^{n+1} such that $\Delta = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap \mathbb{H}^n$, and let $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$ be the dual basis, which is formed by spacelike vectors. We will associate to Δ the following two subsets of the de Sitter sphere \mathbb{S}_1^n :

- (i) the *polar dual*

$$\Delta^* = \{x_0 \mathbf{w}_0 + \dots + x_n \mathbf{w}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap \mathbb{S}_1^n$$

which is the intersection of \mathbb{S}_1^n with a simplicial cone of \mathbb{R}_1^{n+1} but is not bounded; and

- (ii) the *complementary dual*

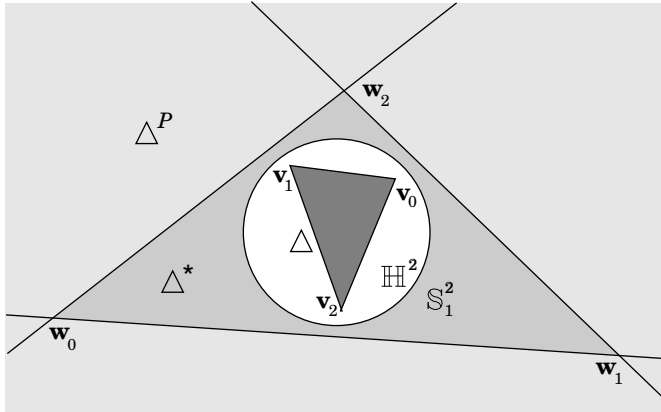
$$\Delta^P = \{x_0 \mathbf{w}_0 + \dots + x_n \mathbf{w}_n \mid x_0 \geq 0 \text{ and } x_i \leq 0 \text{ for some } i \in \{1, \dots, n\}\} \cap \mathbb{S}_1^n$$

which is a compact subset of \mathbb{S}_1^n but is not a simplex.

Remark 2.1. The following properties are easily checked. The complementary dual Δ^P is a compact polyhedron in \mathbb{S}_1^n , whose faces of codimension ≥ 1 are riemannian (i.e., they are spherical simplices). The polar dual Δ^* is not bounded, but all its faces of codimension ≥ 1 are also spherical simplices. Moreover, every face of Δ^* of codimension ≥ 2 is also a face of Δ^P . The dihedral angle of Δ^* at a codimension 2 face is the opposite of the dihedral angle of Δ^P at the same face. In fact, if $F_{ij}^* = \{x_0 \mathbf{w}_0 + \dots + x_n \mathbf{w}_n \mid x_0 \geq 0, \dots, x_n \geq 0; x_i = x_j = 0\} \cap \mathbb{S}_1^n$ is a codimension 2 face of Δ^* , then the dihedral angle of Δ^* at the face F_{ij}^*

is $-\text{ang}(\mathbf{v}_i, \mathbf{v}_j) = -\text{arccosh} \left(\frac{-\langle \mathbf{v}_i, \mathbf{v}_j \rangle}{|\mathbf{v}_i| \cdot |\mathbf{v}_j|} \right)$, while the dihedral angle of Δ^P at the same face is $\text{ang}(\mathbf{v}_i, \mathbf{v}_j)$.

In the following figure we show an example of Δ^* and Δ^P in dimension 2, seen in the projective Klein model.



What makes the complementary dual Δ^P interesting is its geometric interpretation. Consider the map associating to each point \mathbf{w} of Δ^P the hyperplane of \mathbb{H}^n orthogonal to \mathbf{w} . This is a surjective map from Δ^P onto the set of all hyperbolic hyperplanes intersecting Δ . It is not one-to-one, because there are pairs of antipodal points inside Δ^P . However, these pairs of points are contained in the boundary of Δ^P and they form a subset of measure zero. Hence *there is a bijection between Δ^P minus a subset of measure zero, and the set of all hyperbolic hyperplanes intersecting Δ .* Therefore the dual volume of a hyperbolic n -simplex is equal to the volume of its complementary dual Δ^P . To compute this volume we use the Schläfli differential formula in the de Sitter sphere, obtained in the previous section. (Although Δ^P is not a simplex, we can just apply the Schläfli formula to a triangulation of Δ^P .)

Lemma 2.1. *Let Δ be a family of hyperbolic n -simplices varying differentiably with respect to one or more parameters. Then their complementary duals also vary in a differentiable manner, and the differential of their volume $V_n(\Delta^P)$ satisfies the following equality:*

$$dV_n(\Delta^P) = \frac{1}{n-1} \sum_{F^*} V_{n-2}(F^*) d\alpha_{F^*}$$

where the sum is taken over all codimension 2 faces F^* of the polar dual Δ^* (which are also codimension 2 faces of Δ^P), $V_{n-2}(F^*)$ is the $((n-2)$ -dimensional) volume of the face F^* and α_{F^*} is the dihedral angle of Δ^P at the face F^* . (When $n-2=0$, we set $V_0(F^*)=1$.)

2.2. Dual volume of a hyperbolic tetrahedron in dimension 3. The following proposition can be found in [Sa₂, §IV.17.5, Note 1]. Here we prove it using the Schläfli formula (compare with [Mil]).

Proposition 2.1 (Santaló). *Let Δ be a hyperbolic tetrahedron in \mathbb{H}^3 , and let Δ^P be its complementary dual. Then the volume of Δ and its dual volume are related as follows:*

$$V_3(\Delta) + V_3(\Delta^P) = \frac{1}{2} \sum_F (\pi - \alpha_F) V_1(F)$$

where the sum runs over all edges F of the tetrahedron Δ , $V_1(F)$ is the length of the edge F and α_F is the dihedral angle at the edge F .

Proof. Let $\Delta = \{x_0 \mathbf{v}_0 + \dots + x_3 \mathbf{v}_3 \mid x_0 \geq 0, \dots, x_3 \geq 0\} \cap \mathbb{H}^3$, and let $\{\mathbf{w}_0, \dots, \mathbf{w}_3\}$ be the dual basis of $\{\mathbf{v}_0, \dots, \mathbf{v}_3\}$. The hyperbolic Schläfli formula says that

$$dV_3(\Delta) = -\frac{1}{2} \sum_{0 \leq i < j \leq 3} V_1(F_{ij}) d\alpha_{ij}$$

where $F_{ij} = \{x_0 \mathbf{v}_0 + \dots + x_3 \mathbf{v}_3 \mid x_0 \geq 0, \dots, x_3 \geq 0; x_i = x_j = 0\} \cap \mathbb{H}^3$ and α_{ij} is the dihedral angle of Δ at the edge F_{ij} .

On the other hand, we know from Lemma 2.1 above that

$$dV_3(\Delta^P) = \frac{1}{2} \sum_{0 \leq k < l \leq 3} V_1(F_{kl}^*) d\alpha_{kl}^*$$

where $F_{kl}^* = \{x_0 \mathbf{w}_0 + \dots + x_3 \mathbf{w}_3 \mid x_0 \geq 0, \dots, x_3 \geq 0; x_k = x_l = 0\} \cap \mathbb{S}_1^3$ and α_{kl}^* is the dihedral angle of Δ^P at the edge F_{kl}^* . By Remark 2.1, $\alpha_{kl}^* = \text{ang}(\mathbf{v}_k, \mathbf{v}_l)$.

Then the following relations hold. Given $0 \leq i < j \leq 3$, denote by $0 \leq k < l \leq 3$ the other two elements of $\{0, 1, 2, 3\}$ different from i, j . Then:

$$V_1(F_{kl}^*) = \text{ang}(\mathbf{w}_k, \mathbf{w}_l) = \pi - \alpha_{ij} \quad \text{and} \quad \alpha_{kl}^* = \text{ang}(\mathbf{v}_k, \mathbf{v}_l) = V_1(F_{ij}).$$

Hence

$$\begin{aligned} dV_3(\Delta^P) &= \frac{1}{2} \sum_{0 \leq i < j \leq 3} (\pi - \alpha_{ij}) dV_1(F_{ij}) \\ &= \frac{1}{2} \sum_{0 \leq i < j \leq 3} (\pi - \alpha_{ij}) dV_1(F_{ij}) - \frac{1}{2} \sum_{0 \leq i < j \leq 3} V_1(F_{ij}) d\alpha_{ij} \\ &\quad + \frac{1}{2} \sum_{0 \leq i < j \leq 3} V_1(F_{ij}) d\alpha_{ij} \\ &= \frac{1}{2} d \left(\sum_{0 \leq i < j \leq 3} (\pi - \alpha_{ij}) \cdot V_1(F_{ij}) \right) - dV_3(\Delta) \end{aligned}$$

and therefore

$$d(V_3(\Delta) + V_3(\Delta^P)) = d \left(\frac{1}{2} \sum_F (\pi - \alpha_F) \cdot V_1(F) \right)$$

where the sum extends over all edges F of the tetrahedron Δ . We obtain on integrating that

$$V_3(\Delta) + V_3(\Delta^P) = \frac{1}{2} \sum_F (\pi - \alpha_F) \cdot V_1(F) + \text{constant}.$$

To determine the constant, consider the limiting case when Δ degenerates to a point, so that the expression $\frac{1}{2} \sum_F (\pi - \alpha_F) V_1(F)$ tends to zero. In this case, the complementary dual Δ^P degenerates to a plane and $V_3(\Delta) + V_3(\Delta^P)$ also tends to zero. Hence the constant is zero and the formula is proved. \square

In higher dimensions the relation between the volume and the dual volume of a hyperbolic n -simplex Δ is not so straightforward. The reason is that for $n > 3$, the volume of a codimension 2 face of Δ is equal to the dihedral angle of Δ^P at a face of codimension $n - 1 \neq 2$, which does not appear in the Schläfli formula for the volume of Δ^P . To generalize Santaló’s formula to higher dimensions, we will need some equalities that will be proved in the next section (see Proposition 3.1 below). We will also benefit from them to obtain Gauss-Bonnet formulas for n -simplices with riemannian faces in the de Sitter sphere \mathbb{S}_1^n .

3. Gauss-Bonnet formulas in the de Sitter sphere, for simplices with riemannian faces.

Let Δ be an n -simplex in the de Sitter sphere \mathbb{S}_1^n , such that all its faces have a riemannian induced metric. Let $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be a basis of \mathbb{R}_1^{n+1} such that $\Delta = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap \mathbb{S}_1^n$, and let $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$ be the dual basis, which is formed by timelike vectors. Then we can find signs $\epsilon_0, \dots, \epsilon_n \in \{\pm 1\}$ such that $\epsilon_0 \mathbf{w}_0, \dots, \epsilon_n \mathbf{w}_n$ are contained in the upper timecone. We will define the *polar hyperbolic simplex* of Δ as

$$\Delta^* = \{x_0 \cdot \epsilon_0 \mathbf{w}_0 + \dots + x_n \cdot \epsilon_n \mathbf{w}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap \mathbb{H}^n.$$

Definition 3.1. Given $0 \leq r \leq n - 1$ and a codimension $r + 1$ face of the simplex Δ ,

$$F_{i_0 \dots i_r} = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0; \\ x_{i_0} = \dots = x_{i_r} = 0\} \cap \mathbb{S}_1^n$$

we define the *polar angle* of Δ at the face $F_{i_0 \dots i_r}$, as the r -dimensional hyperbolic simplex

$$\Theta_{i_0 \dots i_r} = \{x_{i_0} \cdot \epsilon_{i_0} \mathbf{w}_{i_0} + \dots + x_{i_r} \cdot \epsilon_{i_r} \mathbf{w}_{i_r} \mid x_{i_0} \geq 0, \dots, x_{i_r} \geq 0\} \cap \mathbb{H}^n$$

(which is a face of the polar hyperbolic simplex Δ^*). We define the *algebraic measure of the polar angle* $\Theta_{i_0 \dots i_r}$ as:

$$\theta_{i_0 \dots i_r} = \epsilon_{i_0} \cdots \epsilon_{i_r} \cdot V_r(\Theta_{i_0 \dots i_r})$$

where $V_r(\Theta_{i_0 \dots i_r})$ is the (r -dimensional) volume of the hyperbolic simplex $\Theta_{i_0 \dots i_r}$. When $r = 0$, we set $\theta_i = \epsilon_i = \pm 1$ for $i = 0, \dots, n$.

Proposition 3.1. *Let Δ be a family of n -simplices in \mathbb{S}_1^n such that all their faces are riemannian, and such that they vary differentiably with respect to one or more parameters. For every $1 \leq r \leq n - 2$ the following equality holds:*

$$\begin{aligned} (8) \quad & \Gamma\left(\frac{r+2}{2}\right) \cdot \Gamma\left(\frac{n-r-1}{2}\right) \cdot \sum_{\dim(F)=r+1} \theta_F \cdot dV_{r+1}(F) \\ & = \Gamma\left(\frac{r}{2}\right) \cdot \Gamma\left(\frac{n-r+1}{2}\right) \cdot \sum_{\dim(F)=r-1} V_{r-1}(F) \cdot d\theta_F \end{aligned}$$

where the sum runs over all faces F of the simplex Δ of the given dimension, θ_F is the algebraic measure of the polar angle at the face F and $V_k(F)$ is the (k -dimensional) volume of a face F of dimension k of Δ .

Proof. Let F be a codimension $n - r - 1$ face of Δ . Then F is an $(r + 1)$ -dimensional spherical simplex, and the differential of its volume is given by the spherical Schläfli formula

$$(9) \quad dV_{r+1}(F) = \frac{1}{r} \sum_{\substack{L \subset F \\ \dim(L)=r-1}} V_{r-1}(L) \cdot d\alpha(L, F)$$

where the sum runs over all faces L of dimension $r - 1$ of F and $\alpha(L, F)$ is the dihedral angle of the spherical simplex F at the face L .

On the other hand, let L be a codimension $n - r + 1$ face of Δ , and let Θ_L be the polar angle of Δ at the face L . Then Θ_L is an $(n - r)$ -dimensional hyperbolic simplex, and the differential of its volume is given by the hyperbolic Schläfli formula. Since the faces of Θ_L coincide with the polar angles of Δ at the faces that contain L , we can write

$$dV_r(\Theta_L) = \frac{-1}{n - r - 1} \sum_{\substack{L \subset F \\ \dim(F)=r+1}} V_{n-r-2}(\Theta_F) \cdot d\beta(\Theta_F, \Theta_L)$$

where the sum runs over all $(r + 1)$ -dimensional faces F of Δ containing L , and $\beta(\Theta_F, \Theta_L)$ is the dihedral angle of the hyperbolic simplex Θ_L at the face Θ_F .

Now the algebraic measures of Θ_F and Θ_L are in fact $\theta_F = \pm V_{n-r-2}(\Theta_F)$ and $\theta_L = \pm V_{n-r}(\Theta_L)$, so we can write the last expression as follows:

$$(10) \quad d\theta_L = \frac{-1}{n-r-1} \sum_{\substack{L \subset F \\ \dim(F)=r+1}} \theta_F \cdot \tau(L, F) d\beta(\Theta_F, \Theta_L)$$

where $\tau(L, F) = \frac{\theta_L \cdot V_{n-r-2}(\Theta_F)}{V_{n-r}(\Theta_L) \cdot \theta_F} \in \{\pm 1\}$ is a sign depending on the faces L and F .

For the sake of clarity in the notation, let us assume that $F = \{x_0 \mathbf{v}_0 + \dots + x_{r+1} \mathbf{v}_{r+1} \mid x_0 \geq 0, \dots, x_{r+1} \geq 0\} \cap \mathbb{S}_1^n$, and $L = \{x_0 \mathbf{v}_0 + \dots + x_{r+1} \mathbf{v}_{r+1} \mid x_0 \geq 0, \dots, x_{r+1} \geq 0; x_0 = x_1 = 0\} \cap \mathbb{S}_1^n$. The angle of F at its $(r-1)$ -dimensional face L is

$$\alpha(L, F) = \pi - \text{ang}(\mathbf{w}'_0, \mathbf{w}'_1)$$

where $\mathbf{w}'_0, \mathbf{w}'_1$ are two vectors belonging to the subspace of \mathbb{R}_1^{n+1} spanned by $\{\mathbf{v}_0, \dots, \mathbf{v}_{r+1}\}$, such that $\langle \mathbf{w}'_i, \mathbf{v}_j \rangle = \delta_{ij}$ for $i = 0, 1$ and $j = 0, \dots, r+1$. Now the subspace spanned by $\{\mathbf{v}_0, \dots, \mathbf{v}_{r+1}\}$ is orthogonal to the subspace spanned by $\{\mathbf{w}_{r+2}, \dots, \mathbf{w}_n\}$. Hence the vectors $\mathbf{w}'_0, \mathbf{w}'_1$ are uniquely determined by the following conditions:

$$(11) \quad \begin{cases} \langle \mathbf{w}'_i, \mathbf{v}_j \rangle = \delta_{ij} \\ \langle \mathbf{w}'_i, \mathbf{w}_k \rangle = 0 \end{cases} \quad \text{for } i = 0, 1; \quad j = 0, \dots, r+1; \quad k = r+2, \dots, n.$$

On the other hand, the polar angles of Δ at the faces F and L are $\Theta_F = \{x_{r+2} \cdot \epsilon_{r+2} \mathbf{w}_{r+2} + \dots + x_n \cdot \epsilon_n \mathbf{w}_n \mid x_{r+2} \geq 0, \dots, x_n \geq 0\} \cap \mathbb{H}^n$, and $\Theta_L = \{x_0 \cdot \epsilon_0 \mathbf{w}_0 + x_1 \cdot \epsilon_1 \mathbf{w}_1 + (x_{r+2} \cdot \epsilon_{r+2} \mathbf{w}_{r+2} + \dots + x_n \cdot \epsilon_n \mathbf{w}_n) \mid x_0 \geq 0, x_1 \geq 0, x_{r+2} \geq 0, \dots, x_n \geq 0\} \cap \mathbb{H}^n$, respectively. Their algebraic measures are $\theta_F = \epsilon_{r+2} \cdot \dots \cdot \epsilon_n \cdot V_{n-r-2}(\Theta_F)$ and $\theta_L = \epsilon_0 \cdot \epsilon_1 \cdot \epsilon_{r+2} \cdot \dots \cdot \epsilon_n \cdot V_{n-r}(\Theta_L)$. Hence the sign $\tau(L, F)$ appearing in the Schläfli formula (10) is $\tau(L, F) = \epsilon_0 \cdot \epsilon_1$.

The dihedral angle of the hyperbolic simplex Θ_L at the face Θ_F is

$$\beta(\Theta_F, \Theta_L) = \pi - \text{ang}(\mathbf{v}'_0, \mathbf{v}'_1)$$

where $\mathbf{v}'_0, \mathbf{v}'_1$ are two vectors belonging to the subspace of \mathbb{R}_1^{n+1} spanned by $\{\epsilon_0 \mathbf{w}_0, \epsilon_1 \mathbf{w}_1, \epsilon_{r+2} \mathbf{w}_{r+2}, \dots, \epsilon_n \mathbf{w}_n\}$, such that $\langle \mathbf{v}'_i, \epsilon_j \mathbf{w}_j \rangle = \delta_{ij}$ for $i = 0, 1$ and $j = 0, 1, r+2, \dots, n$. Again the subspace spanned by $\{\epsilon_0 \mathbf{w}_0, \epsilon_1 \mathbf{w}_1, \epsilon_{r+2} \mathbf{w}_{r+2}, \dots, \epsilon_n \mathbf{w}_n\}$ is orthogonal to the subspace spanned by $\{\mathbf{v}_2, \dots, \mathbf{v}_{r+1}\}$. Hence the vectors $\mathbf{v}'_0, \mathbf{v}'_1$ are uniquely determined by the following conditions:

$$(12) \quad \begin{cases} \langle \epsilon_i \mathbf{v}'_i, \mathbf{w}_j \rangle = \delta_{ij} \\ \langle \epsilon_i \mathbf{v}'_i, \mathbf{v}_k \rangle = 0 \end{cases} \quad \text{for } \begin{matrix} i = 0, 1; & j = 0, 1, r+2, \dots, n; \\ & k = 2, \dots, r+1 \end{matrix}.$$

Consider the subspace Π of \mathbb{R}_1^{n+1} spanned by the vectors $\{\mathbf{v}_2, \dots, \mathbf{v}_{r+1}, \mathbf{w}_{r+2}, \dots, \mathbf{w}_n\}$. It has codimension 2 in \mathbb{R}_1^{n+1} , because it is the direct sum of the two mutually orthogonal subspaces spanned by $\{\mathbf{v}_2, \dots, \mathbf{v}_{r+1}\}$ and

$\{\mathbf{w}_{r+2}, \dots, \mathbf{w}_n\}$, respectively. From conditions (11) and (12) we deduce that $\{\mathbf{w}'_0, \mathbf{w}'_1\}$ and $\{\epsilon_0 \mathbf{v}'_0, \epsilon_1 \mathbf{v}'_1\}$ are in fact *dual basis* of the plane Π^\perp , orthogonal to Π in \mathbb{R}_1^{n+1} . Hence

$$\text{ang}(\mathbf{w}'_0, \mathbf{w}'_1) = \pi - \text{ang}(\epsilon_0 \mathbf{v}'_0, \epsilon_1 \mathbf{v}'_1)$$

and therefore

$$\begin{aligned} d\alpha(L, F) &= -d(\text{ang}(\mathbf{w}'_0, \mathbf{w}'_1)) \\ &= d(\text{ang}(\epsilon_0 \mathbf{v}'_0, \epsilon_1 \mathbf{v}'_1)) \\ &= \epsilon_0 \epsilon_1 \cdot d(\text{ang}(\mathbf{v}'_0, \mathbf{v}'_1)) \\ &= -\tau(L, F) \cdot d\beta(\Theta_F, \Theta_L). \end{aligned}$$

Applying now the Schläfli formulas (9) and (10), we get

$$\begin{aligned} &\sum_{\dim(F)=r+1} \theta_F \cdot dV_{r+1}(F) \\ &= \frac{1}{r} \sum_{\dim(F)=r+1} \sum_{\substack{L \subset F \\ \dim(L)=r-1}} \theta_F \cdot V_{r-1}(L) \cdot d\alpha(L, F) \\ &= \frac{-1}{r} \sum_{\dim(L)=r-1} \sum_{\substack{L \subset F \\ \dim(F)=r+1}} V_{r-1}(L) \cdot \theta_F \cdot \tau(L, F) \cdot d\beta(\Theta_F, \Theta_L) \\ &= \frac{n-r-1}{r} \sum_{\dim(L)=r-1} V_{r-1}(L) \\ &\quad \cdot \left(\frac{-1}{n-r-1} \sum_{\substack{L \subset F \\ \dim(F)=r+1}} \theta_F \cdot \tau(L, F) \cdot d\beta(\Theta_F, \Theta_L) \right) \\ &= \frac{n-r-1}{r} \sum_{\dim(L)=r-1} V_{r-1}(L) \cdot d\theta_L. \end{aligned}$$

Using the properties of the gamma function we finally arrive at the desired formula. □

Let us define the following constants (cf. [Sa₁]) for $0 \leq i \leq n-1$:

$$c_i = \frac{\Gamma(\frac{i+1}{2}) \cdot \Gamma(\frac{n-i}{2})}{2 \Gamma(\frac{n+1}{2})} = \frac{\text{Vol}(\mathbb{S}^n)}{\text{Vol}(\mathbb{S}^i) \cdot \text{Vol}(\mathbb{S}^{n-1-i})}.$$

Then for $1 \leq r \leq n-2$, the equality (8) can also be written as follows:

$$(13) \quad c_{r+1} \cdot \sum_{\dim(F)=r+1} \theta_F \cdot dV_{r+1}(F) - c_{r-1} \cdot \sum_{\dim(F)=r-1} V_{r-1}(F) \cdot d\theta_F = 0$$

Remark 3.1. In the proof of Proposition 3.1 we have never used the fact that the simplex Δ is compact. Therefore, formula (13) holds for any non-empty subset of the de Sitter sphere \mathbb{S}_1^n which is the intersection of \mathbb{S}_1^n with a simplicial cone of \mathbb{R}_1^{n+1} , and such that all its faces of codimension ≥ 1 are riemannian. (For example, the polar dual Δ^* of a hyperbolic n -simplex Δ , cf. §2.1.)

A Gauss-Bonnet formula.

Proposition 3.2. *Let Δ be an n -simplex in the de Sitter sphere \mathbb{S}_1^n such that all its faces are riemannian. Then*

$$(14) \quad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \cdot c_{2k} \cdot \sum_{\dim(F)=2k} V_{2k}(F) \cdot \theta_F = 0$$

where the sum runs over all faces F of Δ of even dimension, $V_{2k}(F)$ is the volume of a face F of dimension $2k$, θ_F is the algebraic measure of the polar angle of Δ at the face F , and c_i is the constant defined by

$$c_i = \frac{\text{Vol}(\mathbb{S}^n)}{\text{Vol}(\mathbb{S}^i) \cdot \text{Vol}(\mathbb{S}^{n-1-i})} = \frac{\Gamma(\frac{i+1}{2}) \cdot \Gamma(\frac{n-i}{2})}{2 \Gamma(\frac{n+1}{2})}$$

if $0 \leq i \leq n - 1$ and $c_n = 1$.

(We are using here the following conventions: if $k = \frac{n}{2}$, then the polar angle θ_Δ at the whole simplex is taken to be equal to -1 ; if $k = \frac{n-1}{2}$, then the polar angles at the codimension 1 faces are ± 1 (see Definition 3.1), and if $k = 0$, then the volume of a vertex is always 1.)

Remark 3.2. For even dimension n , the Gauss-Bonnet formula (14) gives an expression of the volume of the simplex Δ in terms of the volumes of simplices of lesser dimension:

$$(-1)^{\frac{n}{2}} \cdot V_n(\Delta) = \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \cdot c_{2k} \cdot \sum_{\dim(F)=2k} V_{2k}(F) \cdot \theta_F \quad \text{if } n \text{ is even.}$$

When n is odd, the Gauss-Bonnet formula does not involve the volume of the simplex itself. Therefore, it cannot be used to reduce $V_n(\Delta)$ to volumes in smaller dimensions:

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \cdot c_{2k} \cdot \sum_{\dim(F)=2k} V_{2k}(F) \cdot \theta_F = 0 \quad \text{if } n \text{ is odd.}$$

Proof. We will only give the proof in the case when n is even. The other case is analogous.

Let Δ be an n -simplex in S_1^n such that all its faces are riemannian, where n is even. The Schläfli formula in the de Sitter sphere (Theorem 1.1) says that

$$dV_n(\Delta) = \frac{1}{n-1} \sum_F V_{n-2}(F) d\alpha_F$$

where the sum is taken over all codimension 2 faces F of Δ and α_F is the dihedral angle of Δ at the face F . Now from the definitions of dihedral angle and polar angle it follows easily that the polar angle of Δ at the face F is $\theta_F = -\alpha_F$. Considering also that $c_{n-2} = \frac{\Gamma(\frac{n-1}{2})}{2 \Gamma(\frac{n+1}{2})} = \frac{1}{n-1}$, we can write

$$(-1)^{\frac{n}{2}} \cdot dV_n(\Delta) = (-1)^{\frac{(n-2)}{2}} \cdot c_{n-2} \cdot \sum_{\dim(F)=n-2} V_{n-2}(F) d\theta_F.$$

On the other hand, from formula (13) we get

$$\left\{ \begin{array}{l} (-1)^{\frac{(n-2)}{2}} \cdot c_{n-2} \cdot \sum_{\dim(F)=n-2} V_{n-2}(F) d\theta_F = (-1)^{\frac{n}{2}} \cdot dV_n(\Delta) \\ (-1)^{\frac{(n-2)}{2}} \cdot c_{n-2} \cdot \sum_{\dim(F)=n-2} \theta_F dV_{n-2}(F) + (-1)^{\frac{(n-4)}{2}} \cdot c_{n-4} \\ \cdot \sum_{\dim(F)=n-4} V_{n-4}(F) d\theta_F = 0 \\ \vdots \\ c_4 \cdot \sum_{\dim(F)=4} \theta_F dV_4(F) - c_2 \cdot \sum_{\dim(F)=2} V_2(F) d\theta_F = 0 \\ -c_2 \cdot \sum_{\dim(F)=2} \theta_F dV_2(F) + c_0 \cdot \sum_{\dim(F)=0} d\theta_F = 0. \end{array} \right.$$

Adding up all these equalities and integrating, we obtain

$$(-1)^{\frac{n}{2}} \cdot V_n(\Delta) = \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \cdot c_{2k} \cdot \sum_{\dim(F)=2k} V_{2k}(F) \cdot \theta_F + \text{constant}.$$

The constant can be found to be zero by considering the limiting case as the polar hyperbolic simplex Δ^* degenerates to a point. Then all polar angles $\theta_F \rightarrow 0$, and the simplex Δ degenerates into a subset of a hyperplane, so $V_n(\Delta) \rightarrow 0$. □

Remark 3.3. I have called formula (14) a *Gauss-Bonnet formula* because of its analogy to the generalized Gauss-Bonnet formulas for convex polyhedra in non-Euclidean space of constant curvature $\kappa = \pm 1$ (cf. [Sa₁], [AVS]).

There is, however, an essential difference in that the right-hand side of Equation (14) is zero. Therefore, if we tried to extend the formula (14) to polyhedra in \mathbb{S}_1^n which admit a triangulation by simplices with riemannian faces, the Euler characteristic of the polyhedron would never appear.

Remark 3.4. The same procedure used in the proofs of Propositions 3.1 and 3.2 can also be applied in the spherical and hyperbolic case to prove the above mentioned Gauss-Bonnet formulas for spherical and hyperbolic simplices.

4. Dual volume of a hyperbolic n -simplex. (A generalization of Santaló’s formula.)

Let $\Delta = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap \mathbb{H}^n$ be a hyperbolic n -simplex, and let $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$ be the dual basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$. Consider the polar dual of Δ , $\Delta^* = \{x_0 \mathbf{w}_0 + \dots + x_n \mathbf{w}_n \mid x_0 \geq 0, \dots, x_n \geq 0\} \cap \mathbb{S}_1^n$, and the complementary dual, $\Delta^P = \{x_0 \mathbf{w}_0 + \dots + x_n \mathbf{w}_n \mid x_0 \geq 0 \text{ and } x_i \leq 0 \text{ for some } i \in \{1, \dots, n\}\} \cap \mathbb{S}_1^n$. We saw in Section 2.1 that the dual volume of Δ (i.e., the measure of the set of hyperbolic hyperplanes intersecting it) is equal to the volume of its complementary dual Δ^P . Now we are going to find a relation between this dual volume $V_n(\Delta^P)$ and the volumes of all odd-dimensional faces of the hyperbolic simplex Δ and the polar angles at those faces.

Definition 4.1. For $0 \leq k \leq n - 1$, we will associate to each face of codimension $k + 1$ of the simplex Δ ,

$F = \{x_0 \mathbf{v}_0 + \dots + x_n \mathbf{v}_n \mid x_0 \geq 0, \dots, x_n \geq 0; x_{i_1} = \dots = x_{i_k} = 0\} \cap \mathbb{H}^n$
 the following k -dimensional face of the polar dual Δ^* :

$$F^* = \{x_{i_1} \mathbf{v}_{i_1} + \dots + x_{i_k} \mathbf{v}_{i_k} \mid x_{i_1} \geq 0, \dots, x_{i_k} \geq 0\} \cap \mathbb{S}_1^n.$$

We will call F and F^* *polar faces*. We define the *polar angle* θ_F of the hyperbolic simplex Δ at the face F as the volume of its polar face F^* (which is a k -dimensional spherical simplex).

Remark 4.1. If F is a codimension 2 face of Δ and α_F is the dihedral angle of Δ at F , then the polar angle at F is $\theta_F = \pi - \alpha_F$.

Proposition 4.1. *Let Δ be a hyperbolic n -simplex. Then its dual volume is*

$$(15) \quad V_n(\Delta^P) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot c_{2k+1} \cdot \sum_{\dim(F)=2k+1} V_{2k+1}(F) \cdot \theta_F$$

where the sum extends over all odd-dimensional faces F of Δ , $V_{2k+1}(F)$ is the volume of a face F of dimension $2k + 1$, θ_F is the polar angle of Δ at

the face F , and c_i is the constant defined by

$$c_i = \frac{\text{Vol}(\mathbb{S}^n)}{\text{Vol}(\mathbb{S}^i) \cdot \text{Vol}(\mathbb{S}^{n-1-i})}$$

$$= \frac{\Gamma(\frac{i+1}{2}) \cdot \Gamma(\frac{n-i}{2})}{2 \Gamma(\frac{n+1}{2})} \quad \text{if } 0 \leq i \leq n-1 \quad \text{and} \quad c_n = 1.$$

(We are using here the following conventions: if $k = \frac{n-1}{2}$, then the polar angle θ_Δ at the whole simplex is taken to be equal to 1 ; if $k = \frac{n-2}{2}$, then the polar angles at the codimension 1 faces are 1, and if $k = 0$, then the volume of a vertex is always 1.)

Remark 4.2. For odd dimension n , formula (15) relates the volume of the n - simplex Δ with its dual volume:

$$V_n(\Delta^P) + (-1)^{\frac{n+1}{2}} \cdot V_n(\Delta)$$

$$= \sum_{k=0}^{\frac{n-3}{2}} (-1)^k \cdot c_{2k+1} \cdot \sum_{\dim(F)=2k+1} V_{2k+1}(F) \cdot \theta_F \quad \text{if } n \text{ is odd.}$$

In particular, in dimension 3 we obtain again Santaló’s formula for hyperbolic tetrahedra.

When n is even, formula (15) relates the dual volume of the n -simplex Δ with its “surface area” $S_{n-1}(\Delta)$, i.e., with the sum of the volumes of all its faces of codimension 1:

$$V_n(\Delta^P) + (-1)^{\frac{n}{2}} \cdot c_{n-1} \cdot S_{n-1}(\Delta)$$

$$= \sum_{k=0}^{\frac{n-4}{2}} (-1)^k \cdot c_{2k+1} \cdot \sum_{\dim(F)=2k+1} V_{2k+1}(F) \cdot \theta_F \quad \text{if } n \text{ is even.}$$

In particular, in dimension 2 this says that the measure of the set of all lines intersecting a planar hyperbolic triangle, is equal to the perimeter of the triangle. In dimension 4, the expression is a little more complicated:

$$3 V_4(\Delta^P) + 2 S_3(\Delta) = \sum_{\dim(F)=1} V_1(F) \cdot \theta_F$$

Proof. We will do it only when n is odd, since the other case is analogous.

Let us apply the formulas (13) of Section 3 to the polar dual Δ^* of Δ (cf. Remark 3.1). Since the constants c_i satisfy the relation $c_i = c_{n-i-1}$, we

have that for $2 \leq r \leq n - 1$,

$$(16) \quad \begin{aligned} &c_{r-2} \cdot \sum_{\dim(F^*)=n-r+1} \theta_{F^*} \cdot dV_{n-r+1}(F^*) \\ &- c_r \cdot \sum_{\dim(F^*)=n-r-1} V_{n-r-1}(F^*) \cdot d\theta_{F^*} = 0 \end{aligned}$$

where the sums extend to all faces F^* of the polar dual Δ^* of the given dimension, θ_{F^*} is the algebraic measure of the polar angle of Δ^* at the face F^* , and $V_k(F^*)$ is the (k -dimensional) volume of a k -dimensional face F^* of Δ^* .

Now the dual basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ of $\{\mathbf{w}_0, \dots, \mathbf{w}_n\}$ is formed by timelike vectors contained in the upper timecone. Therefore, the algebraic measure of the polar angle of Δ^* at the face F^* coincides with the volume of the polar face F of Δ (cf. Definition 3.1). Hence if $F \subset \Delta$ and $F^* \subset \Delta^*$ are polar faces of dimensions r and $n - r - 1$, respectively, then

$$V_r(F) = \theta_{F^*} \quad \text{and} \quad V_{n-r-1}(F^*) = \theta_F.$$

Therefore, for $1 \leq r \leq n - 2$ the equality (16) can be written in terms of quantities associated only to the hyperbolic simplex Δ itself, as follows:

$$(17) \quad c_{r-2} \cdot \sum_{\dim(F)=r-2} V_{r-2}(F) \cdot d\theta_F - c_r \cdot \sum_{\dim(F)=r} \theta_F \cdot dV_r(F) = 0$$

where the sums run over all faces F of Δ of the given dimension, θ_F is the polar angle of Δ at the face F , and $V_k(F)$ is the (k -dimensional) volume of a face F of dimension k of Δ .

On the other hand, considering the Remark 4.1 and the fact that $c_1 = c_{n-2} = \frac{1}{n-1}$, we know from the hyperbolic Schläfli formula that

$$(18) \quad dV_n(\Delta) = c_{n-2} \cdot \sum_{\dim(F)=n-2} V_{n-2}(F) d\theta_F$$

and from the Schläfli formula in the de Sitter sphere, that

$$dV_n(\Delta^P) = c_1 \cdot \sum_{\dim(F^*)=n-2} V_{n-2}(F^*) d\alpha_{F^*}$$

where we sum over all codimension 2 faces F^* of Δ^* , and α_{F^*} is the dihedral angle of Δ^P at the face F^* . But $\alpha_{F^*} = V_1(F)$ and $V_{n-2}(F^*) = \theta_F$, where F is the polar face of F^* . Hence

$$(19) \quad dV_n(\Delta^P) = c_1 \cdot \sum_{\dim(F)=1} \theta_F dV_1(F).$$

Let us combine formulas (17), (18) and (19):

$$\left\{ \begin{array}{l} c_1 \cdot \sum_{\dim(F)=1} \theta_F dV_1(F) = dV_n(\Delta^P) \\ c_1 \cdot \sum_{\dim(F)=1} V_1(F) d\theta_F - c_3 \cdot \sum_{\dim(F)=3} \theta_F dV_3(F) = 0 \\ \vdots \\ (-1)^{\frac{(n-5)}{2}} \cdot c_{n-4} \cdot \sum_{\dim(F)=n-4} V_{n-4}(F) d\theta_F + (-1)^{\frac{(n-3)}{2}} \cdot c_{n-2} \\ \cdot \sum_{\dim(F)=n-2} \theta_F dV_{n-2}(F) = 0 \\ (-1)^{\frac{(n-3)}{2}} \cdot c_{n-2} \cdot \sum_{\dim(F)=n-2} V_{n-2}(F) d\theta_F = (-1)^{\frac{(n-3)}{2}} \cdot dV_n(\Delta). \end{array} \right.$$

Adding up all these equalities and integrating we obtain

$$\begin{aligned} &V_n(\Delta^P) + (-1)^{\frac{(n-3)}{2}} \cdot V_n(\Delta) \\ &= \sum_{k=0}^{\frac{n-3}{2}} (-1)^k \cdot c_{2k+1} \cdot \sum_{\dim(F)=2k+1} V_{2k+1}(F) \cdot \theta_F + \text{constant}. \end{aligned}$$

Again we see that the constant is zero by considering the limiting case as Δ degenerates to a point (and Δ^P degenerates to a hyperplane in \mathbb{S}_1^n). \square

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