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ON A SHARP MOSER–AUBIN–ONOFRI INEQUALITY
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We show that for $\alpha \geq \frac{1}{2}$, the following inequality holds:

$$\frac{\alpha}{2} \int_{-1}^1 (1 - x^2) |g'(x)|^2 dx + \int_{-1}^1 g(x) dx - \log \frac{1}{2} \int_{-1}^1 e^{2g(x)} dx \geq 0,$$

for every function g on $(-1, 1)$ satisfying $\|g\|^2 = \int_{-1}^1 (1 - x^2) |g'(x)|^2 dx < \infty$ and $\int_{-1}^1 e^{2g(x)} x dx = 0$. This improves a result of Feldman et al., 1998, and answers a question of Chang and Yang in the axially symmetric case.

1. Introduction.

On S^2 let J_α denote the functional on the Sobolev space $H^{1,2}(S^2)$ defined by

$$J_\alpha(g) = \alpha \int_{S^2} |\nabla g|^2 dw + 2 \int_{S^2} g dw - \log \int_{S^2} e^{2g} dw.$$

Here dw denotes the Lebesgue measure on the unit sphere, normalized to make $\int_{S^2} dw = 1$. The famous Moser-Trudinger inequality says that J_1 is bounded below by a non-positive constant C_1 . Later Onofri [6] showed that C_1 can be taken to be 0. (Another proof was also given by Osgood-Phillips-Sarnack [7].) On the other hand, if we restrict J_α to the class of \mathcal{G} of functions g for which e^{2g} has centre of mass equal to 0, that is $\int_{S^2} e^{2g} \vec{x} dw = 0$, then Aubin in [2] showed that for $\alpha \geq \frac{1}{2}$, the functional J_α is again bounded below by a non-positive constant C_α . In [3] and [4] A. Chang and P. Yang showed that $C_\alpha = 0$ for α close enough to 1. This led them to the following

Conjecture. Let \mathcal{G} denote the functions in $H^{1,2}(S^2)$ for which $\int_{S^2} e^{2g} \vec{x} dw = 0$. If $\alpha \geq \frac{1}{2}$, then $\inf_{g \in \mathcal{G}} J_\alpha(g) = 0$.

In this note, we prove this conjecture in the axially symmetric case. We note that Feldman, Froese, Ghoussoub and Gui [5] proved that the above conjecture holds for the axially symmetric case when $\alpha > \frac{16}{25} - \epsilon$ for some small ϵ . They also gave an example which says the inequality is not true if $\alpha < \frac{1}{2}$. It is also known that $J_\alpha(g) \geq 0$ if g is an even function, i.e., $g(\vec{x}) = g(-\vec{x})$ on S^2 . (See [7].)

Let θ and φ denote the usual angular coordinates on the sphere, and define $x = \cos(\theta)$. Axially symmetric functions depend on x only. For such functions, it is well-known (see [5]) that the functional J_α can be written as

$$I_\alpha(g) := \frac{\alpha}{2} \int_{-1}^1 (1 - x^2)|g'(x)|^2 dx + \int_{-1}^1 g(x) dx - \log \frac{1}{2} \int_{-1}^1 e^{2g(x)} dx.$$

The set \mathcal{G} is then replaced by

$$\mathcal{G}_r := \left\{ g \mid \int_{-1}^1 (1 - x^2)|g'(x)|^2 < \infty, \int_{-1}^1 e^{2g(x)} x dx = 0 \right\}.$$

It is proved in [5, Proposition 3.1] that any critical point g of I_α restricted to \mathcal{G}_r satisfies the following differential equation

$$(1.1) \quad \alpha((1 - x^2)g')' - 1 + \frac{2}{\lambda} e^{2g} = 0, \quad \lambda = \int_{-1}^1 e^{2g} dx.$$

The main result of this note is the following:

Theorem 1.1. *If $\alpha \geq \frac{1}{2}$, then the only critical points of the functional I_α restricted to \mathcal{G}_r are constant functions.*

As a consequence, the above theorem implies that the Conjecture of Chang and Yang is true in the axially symmetric case.

Theorem 1.2. *If $\alpha \geq \frac{1}{2}$, then $I_\alpha(g) \geq 0$ for $g \in \mathcal{G}_r$.*

The rest of the paper is devoted to the study of (1.1). To this end, we need some notations and some basic facts.

Let g be a solution of (1.1). Following [5], we set

$$G = (1 - x^2)g'.$$

Then G satisfies (see [5])

$$(1.2) \quad \alpha G' - 1 + \frac{2}{\lambda} e^{2g} = 0,$$

and

$$(1.3) \quad \begin{cases} (1 - x^2)G'' + \frac{2}{\alpha}G - 2GG' = 0 \\ G(-1) = G(1) = 0. \end{cases}$$

We also need some facts about the Legendre's polynomials.

Let $P_n(x)$ be the n -th Legendre polynomial, i.e., P_n satisfies

$$((1 - x^2)P'_n)' + \lambda_n P_n = 0, \lambda_n = n(n + 1), n = 0, 1, \dots$$

Note that

$$P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1), \dots$$

Moreover (see [1])

$$(1.4) \quad |P'_n(x)| \leq \frac{1}{2}\lambda_n, \int_{-1}^1 P_n^2 = \frac{2}{2n+1}.$$

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2. Proof of Theorem 1.1.

In this section, we shall prove Theorem 1.1.

Let

$$G(x) = \beta x + a_2 \frac{1}{2}(3x^2 - 1) + \sum_{k=3}^{\infty} a_k P_k(x),$$

$$G_2 = \sum_{k=3}^{\infty} a_k P_k(x)$$

and

$$b_k^2 = a_k^2 \int_{-1}^1 P_k^2, k \geq 2.$$

We first derive some equalities:

$$(2.1) \quad \int_{-1}^1 (1-x^2)(G')^2 = \left(\frac{2}{\alpha} - 1\right) \int_{-1}^1 G^2,$$

$$(2.2) \quad \int_{-1}^1 P_1 G = \frac{2}{3}\beta,$$

$$(2.3) \quad \int_{-1}^1 (1-x^2) \frac{e^{2g}}{\lambda} = \frac{2}{3}(1-\alpha\beta),$$

$$(2.4) \quad \int_{-1}^1 P_k G = -\frac{2}{\alpha\lambda_k} \int_{-1}^1 (1-x^2) P'_k \frac{e^{2g}}{\lambda}, k \geq 2,$$

$$(2.5) \quad \int_{-1}^1 G^2 = \left(6 - \frac{2}{\alpha}\right) \frac{2}{3}\beta,$$

$$(2.6) \quad \frac{2}{3}\beta \left(4\beta + \left(7 - \frac{2}{\alpha}\right) \left(\frac{2}{\alpha} - 6\right)\right) = \int_{-1}^1 (1-x^2)(G'_2)^2 - 6 \int_{-1}^1 G_2^2,$$

$$(2.7) \quad \int_{-1}^1 (1-x^2)(G_2')^2 - 6 \int_{-1}^1 G_2^2 = \sum_{k=3}^{\infty} (\lambda_k - 6)b_k^2.$$

Proofs of (2.1)-(2.7). Multiplying (1.3) by G and integrating over $[-1, 1]$, we obtain (2.1). The relation (2.2) follows by definition. Multiplying (1.2) by $\int_{-1}^x P_k(s)ds, k \geq 1$ and integrating over $[-1, 1]$ we obtain (2.3) and (2.4). Multiplying (1.3) by x and integrating from -1 to 1 we obtain (2.5). To show (2.6), we just need to use (2.1), (2.5) and the definition of G_2 . The equality (2.7) follows from definition. \square

We will show $\beta = 0$, which implies $G = 0$ by (2.5). Our basic strategy is to show that if $\beta \neq 0$, then

$$\beta = \frac{1}{\alpha},$$

which will lead to a contradiction.

Below we assume that $\beta \neq 0$.

Next we obtain some inequalities.

From (2.3) we have

$$(2.8) \quad \frac{1}{\alpha} - \beta > 0.$$

By definition we have

$$\begin{aligned} b_k^2 &= a_k^2 \int_{-1}^1 P_k^2 = \frac{(\int_{-1}^1 GP_k)^2}{\int_{-1}^1 P_k^2} \\ &\leq \frac{2k+1}{2} \left(\frac{2}{\alpha\lambda_k} \int_{-1}^1 (1-x^2)|P_k'| \frac{e^{2g}}{\lambda} \right)^2 \\ &\leq \frac{2k+1}{2} \left(\frac{2}{\alpha\lambda_k} \frac{\lambda_k}{2} \frac{2}{3}(1-\alpha\beta) \right)^2. \end{aligned}$$

Hence we obtain

$$(2.9) \quad b_k^2 \leq \frac{2(2k+1)}{9} \left(\frac{1}{\alpha} - \beta \right)^2, k \geq 2.$$

Similarly we obtain

$$(2.10) \quad \frac{3}{5}|a_2| \leq \frac{1}{\alpha} - \beta.$$

From (2.6) (since $\beta > 0$),

$$4\beta + \left(7 - \frac{2}{\alpha} \right) \left(\frac{2}{\alpha} - 6 \right) \geq 0.$$

Since $\alpha \geq 0.5$, we have

$$(2.11) \quad \beta \geq \frac{1}{4} \left(7 - \frac{2}{\alpha}\right) \left(6 - \frac{2}{\alpha}\right) \geq 1.5.$$

From (2.6) and (2.8), we have

$$\frac{4}{\alpha} + \left(7 - \frac{2}{\alpha}\right) \left(\frac{2}{\alpha} - 6\right) \geq 0$$

which implies that

$$\alpha \leq 0.537.$$

From (2.6) we have

$$\begin{aligned} & \frac{2}{3}\beta \left(4\beta + \left(7 - \frac{2}{\alpha}\right) \left(\frac{2}{\alpha} - 6\right)\right) \\ &= \int_{-1}^1 (1 - x^2)(G_2')^2 - 6 \int_{-1}^1 G_2^2 \\ &\geq \frac{1}{2} \int_{-1}^1 (1 - x^2)(G_2')^2 \\ &\geq \frac{1}{2} \left[\int_{-1}^1 (1 - x^2)(G')^2 - \frac{4}{3}\beta^2 - \frac{12}{5}a_2^2 \right] \\ &\geq \frac{1}{2} \left[\left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) \frac{2}{3}\beta - \frac{4}{3}\beta^2 - \frac{12}{5}a_2^2 \right]. \end{aligned}$$

Hence we obtain

$$(2.12) \quad \begin{aligned} & \frac{2}{3}\beta \left[\frac{5}{\alpha} + \left(7 - \frac{2}{\alpha}\right) \left(\frac{2}{\alpha} - 6\right) - \frac{1}{2} \left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) \right] \\ &\geq \frac{10}{3}\beta \left(\frac{1}{\alpha} - \beta\right) - \frac{6}{5}a_2^2 \\ &\geq \frac{10}{3}\beta \left(\frac{1}{\alpha} - \beta\right) - \frac{6}{5} \times \frac{25}{9} \left(\frac{1}{\alpha} - \beta\right)^2 \\ &\geq \frac{10}{3} \left(2\beta - \frac{1}{\alpha}\right) \left(\frac{1}{\alpha} - \beta\right). \end{aligned}$$

Since $(\frac{1}{\alpha} - \beta) \geq 0$, $\alpha \geq 0.5$ and $2\beta - \frac{1}{\alpha} \geq 0$, we conclude that (since $\beta > 0$)

$$(2.13) \quad 0 \leq \frac{5}{\alpha} + \left(7 - \frac{2}{\alpha}\right) \left(\frac{2}{\alpha} - 6\right) - \frac{1}{2} \left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) \leq 1$$

which implies, by a simple computation, that

$$(2.14) \quad \alpha \leq 0.52.$$

Moreover since $\alpha \geq 0.5$ and $\beta \geq 1.5$, we obtain from (2.12) and (2.13) that

$$(2.15) \quad \frac{1}{\alpha} - \beta \leq \frac{\beta}{5(2\beta - \frac{1}{\alpha})} \leq \frac{\beta}{5}.$$

To obtain better estimates, we fix an integer $n \geq 3$. We have by (2.6) and (2.7)

$$\begin{aligned} & \frac{2}{3}\beta \left(4\beta + \left(7 - \frac{2}{\alpha} \right) \left(\frac{2}{\alpha} - 6 \right) \right) \\ &= \sum_{k=3}^{\infty} (\lambda_k - 6)b_k^2 \\ &= \sum_{k=3}^n (\lambda_k - 6)b_k^2 + \sum_{k=n+1}^{\infty} (\lambda_k - 6)b_k^2 \\ &\geq \sum_{k=3}^n (\lambda_k - 6)b_k^2 + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \sum_{k=n+1}^{\infty} \lambda_k b_k^2 \\ &= \sum_{k=3}^n (\lambda_k - 6)b_k^2 \\ &\quad + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{2}{3}\beta \left(\frac{2}{\alpha} - 1 \right) \left(6 - \frac{2}{\alpha} \right) - \frac{4}{3}\beta^2 - \frac{12}{5}a_2^2 - \sum_{k=3}^n \lambda_k b_k^2 \right) \\ &= \sum_{k=3}^n \left(\lambda_k - 6 - \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \lambda_k \right) b_k^2 \\ &\quad + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{2}{3}\beta \left(\frac{2}{\alpha} - 1 \right) \left(6 - \frac{2}{\alpha} \right) - \frac{4}{3}\beta^2 - \frac{12}{5}a_2^2 \right) \\ &= \sum_{k=3}^n 6 \frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}} b_k^2 - \frac{12}{5}a_2^2 \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \\ &\quad + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{2}{3}\beta \left(\frac{2}{\alpha} - 1 \right) \left(6 - \frac{2}{\alpha} \right) - \frac{4}{3}\beta^2 \right). \end{aligned}$$

Hence we have

$$(2.16) \quad \begin{aligned} & \frac{2}{3}\beta \left(4\beta + \left(7 - \frac{2}{\alpha} \right) \left(\frac{2}{\alpha} - 6 \right) \right) \\ &\quad - \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{2}{3}\beta \left(\frac{2}{\alpha} - 1 \right) \left(6 - \frac{2}{\alpha} \right) - \frac{4}{3}\beta^2 \right) \\ &\geq \sum_{k=3}^n 6 \frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}} b_k^2 - \frac{12}{5}a_2^2 \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}. \end{aligned}$$

After some simple computations, the left hand of (2.16) equals to

$$12\beta \left(\frac{1}{\alpha} - 2\right) + \frac{4\beta}{\lambda_{n+1}} \left[\left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) - \frac{2}{\alpha} \right] - 4\beta \left(1 - \frac{2}{\lambda_{n+1}}\right) \left(\frac{1}{\alpha} - \beta\right).$$

Thus we have by (2.9), (2.10) and (2.16)

$$\begin{aligned} (2.17) \quad & 12\beta \left(\frac{1}{\alpha} - 2\right) + \frac{4\beta}{\lambda_{n+1}} \left[\left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) - \frac{2}{\alpha} \right] \\ & \geq 4\beta \left(1 - \frac{2}{\lambda_{n+1}}\right) \left(\frac{1}{\alpha} - \beta\right) - \frac{12}{5} a_2^2 \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \\ & \quad + 6 \sum_{k=3}^n \frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}} \frac{2(2k+1)}{9} \left(\frac{1}{\alpha} - \beta\right)^2 \\ & \geq 4\beta \left(1 - \frac{2}{\lambda_{n+1}}\right) \left(\frac{1}{\alpha} - \beta\right) - \frac{20}{3} \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{1}{\alpha} - \beta\right)^2 \\ & \quad - \frac{4}{3} \sum_{k=3}^n \frac{\lambda_{n+1} - \lambda_k}{\lambda_{n+1}} (2k+1) \left(\frac{1}{\alpha} - \beta\right)^2 \\ & \geq \left[4\beta \left(1 - \frac{2}{\lambda_{n+1}}\right) \right. \\ & \quad \left. - \frac{20}{3} \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{1}{\alpha} - \beta\right) - \frac{4}{3} c_n \left(\frac{1}{\alpha} - \beta\right) \right] \left(\frac{1}{\alpha} - \beta\right) \end{aligned}$$

where

$$c_n = \sum_{k=3}^n \frac{\lambda_{n+1} - \lambda_k}{\lambda_{n+1}} (2k+1).$$

Since $1/2 < \alpha \leq 1$ and $\lambda_n > 2$ for $n \geq 1$, we have

$$\begin{aligned} & 12\beta \left(\frac{1}{\alpha} - 2\right) + \frac{4\beta}{\lambda_{n+1}} \left[\left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) - \frac{2}{\alpha} \right] - \frac{8\beta}{\lambda_{n+1}} \\ & = 4\beta \left(\frac{1}{\alpha} - 2\right) \left[3 - \frac{4}{\lambda_{n+1}} \left(\frac{1}{\alpha} - 1\right) \right] \\ & \leq 0. \end{aligned}$$

Thus the left hand side of (2.17) satisfies

$$(2.18) \quad \text{LHS of (2.17)} \leq \frac{8\beta}{\lambda_{n+1}}.$$

We now claim

$$(2.19) \quad \frac{1}{\alpha} - \beta \leq \frac{4}{\lambda_n}, \quad \forall n \geq 4.$$

By (2.18), we just need to show that the right hand side of (2.17) satisfies

$$(2.20) \quad \text{RHS of (2.17)} \geq 2\beta \left(\frac{1}{\alpha} - \beta \right).$$

We prove it by induction.

We first prove $n = 4$. To this end, we iterate the inequality (2.17). Note that the right hand side of (2.17) with $n = 3$ equals

$$(2.21) \quad \begin{aligned} & \left[4\beta \left(1 - \frac{2}{20} \right) - \frac{20}{3} \frac{20-6}{20} \left(\frac{1}{\alpha} - \beta \right) \right. \\ & \quad \left. - \frac{4}{3} \frac{20-12}{20} \times 7 \left(\frac{1}{\alpha} - \beta \right) \right] \left(\frac{1}{\alpha} - \beta \right) \\ & \geq \left[4\beta \frac{9}{10} - \frac{14}{3} \left(\frac{1}{\alpha} - \beta \right) - \frac{56}{15} \left(\frac{1}{\alpha} - \beta \right) \right] \left(\frac{1}{\alpha} - \beta \right) \\ & \geq \left[3.6\beta - \frac{126}{15} \left(\frac{1}{\alpha} - \beta \right) \right] \left(\frac{1}{\alpha} - \beta \right) \\ & \geq \left[3.6\beta - \frac{126}{15} \frac{\beta}{5} \right] \left(\frac{1}{\alpha} - \beta \right) \quad (\text{by (2.15)}) \\ & \geq 1.92\beta \left(\frac{1}{\alpha} - \beta \right). \end{aligned}$$

By using (2.18) and (2.17) again, we obtain

$$(2.22) \quad \frac{1}{\alpha} - \beta \leq \frac{8}{20} \frac{1}{1.92} < 0.25.$$

Similarly, by using (2.22), we have

$$\begin{aligned} \text{RHS of (2.17)} & \geq \left[3.6\beta - \frac{126}{15} \times 0.25 \right] \left(\frac{1}{\alpha} - \beta \right) \quad (\text{by (2.22)}) \\ & \geq 2\beta \left(\frac{1}{\alpha} - \beta \right) \quad (\text{since } \beta \geq 1.5 \text{ by (2.11)}). \end{aligned}$$

Thus (2.20) holds for $n = 4$ and hence (2.19) holds for $n = 4$.

Let us now assume that

$$\frac{1}{\alpha} - \beta \leq \frac{4}{\lambda_k}, \quad k = n \geq 4.$$

We observe that for $n \geq 4$

$$\begin{aligned} c_n & = \sum_{k=3}^n (2k+1) - \frac{1}{\lambda_{n+1}} \sum_{k=3}^n \lambda_k (2k+1) \\ & = \sum_{k=3}^n (2k+1) - \frac{1}{\lambda_{n+1}} \sum_{k=3}^n k(k+1)(2k+1) \end{aligned}$$

$$= \frac{1}{2}\lambda_{n+1} - 9 + \frac{36}{\lambda_{n+1}}.$$

Hence we have by (2.17)

$$\begin{aligned} (2.23) \quad & 12\beta \left(\frac{1}{\alpha} - 2\right) + \frac{4\beta}{\lambda_{n+1}} \left[\left(\frac{2}{\alpha} - 1\right) \left(6 - \frac{2}{\alpha}\right) - \frac{2}{\alpha} \right] \\ & \geq \left[4\beta \left(1 - \frac{2}{\lambda_{n+1}}\right) \right. \\ & \quad \left. - \left(\frac{20}{3} \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} + \frac{4}{3} \left(\frac{1}{2}\lambda_{n+1} - 9 + \frac{36}{\lambda_{n+1}}\right)\right) \left(\frac{1}{\alpha} - \beta\right) \right] \left(\frac{1}{\alpha} - \beta\right). \end{aligned}$$

The right hand of (2.23) satisfies

$$\begin{aligned} & \text{RHS of (2.23)} \\ & \geq \left[4\beta \left(1 - \frac{2}{\lambda_{n+1}}\right) + \frac{64}{3} \frac{1}{\lambda_n} - \frac{32}{\lambda_n \lambda_{n+1}} - \frac{8}{3} \frac{\lambda_{n+1}}{\lambda_n} \right] \left(\frac{1}{\alpha} - \beta\right). \end{aligned}$$

To show (2.20), we only need to show

$$\beta \left(1 - \frac{4}{\lambda_{n+1}}\right) \geq -\frac{32}{3} \frac{1}{\lambda_n} + \frac{16}{\lambda_n \lambda_{n+1}} + \frac{4}{3} \frac{\lambda_{n+1}}{\lambda_n},$$

or

$$\beta \geq \frac{4}{3} \cdot \frac{\lambda_{n+1}^2 - 8\lambda_{n+1} + 12}{\lambda_n(\lambda_{n+1} - 4)}.$$

In view of the inductive assumption, it suffices to show

$$\frac{1}{\alpha} \geq \frac{4}{3} \cdot \frac{\lambda_{n+1}}{\lambda_n} \cdot \frac{\lambda_{n+1} - 5}{\lambda_{n+1} - 4}.$$

Because of (2.14), it is easy to verify that the above inequality holds for $n \geq 4$.

In conclusion, we have obtained (2.19).

Finally we can finish the proof of Theorem 1.1. In fact, if we let $n \rightarrow +\infty$ in (2.19), we obtain

$$\frac{1}{\alpha} - \beta = 0$$

which is a contradiction to (2.8).

This implies that $\beta = 0$ and therefore $G \equiv 0$. Hence $g' \equiv 0$, and $g \equiv \text{Constant}$. □

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