

*Pacific
Journal of
Mathematics*

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Volume 194 No. 2

June 2000

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Let D be a bounded balanced domain with C^1 plurisubharmonic defining functions in \mathbf{C}^n . First, we give a necessary and sufficient condition that a locally biholomorphic mapping from D to \mathbf{C}^n is starlike. Next, we give a growth theorem for normalized starlike mappings on D . We also give a quasiconformal extension of some strongly starlike mapping on D .

1. Introduction.

Let f be a univalent mapping in the unit disk Δ with $f(0) = 0$ and $f'(0) = 1$. Then the classical growth theorem is as follows:

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Barnard, FitzGerald and Gong [1] and Chuaqui [2] extended this to normalized starlike mappings on the unit ball \mathbf{B}^n in \mathbf{C}^n . Their proof uses the characterization of the starlikeness due to Suffridge [11]. Chuaqui [2] also obtained a quasiconformal extension of some strongly starlike mapping on \mathbf{B}^n .

In this paper, we will extend the above results to (strongly) starlike mappings on bounded balanced domains with C^1 plurisubharmonic defining functions in \mathbf{C}^n . Since we cannot use the characterization of the starlikeness due to Suffridge [11], we first give a necessary and sufficient condition that a locally biholomorphic mapping on such domains is starlike using the idea of Gong, Wang and Yu [4]. To prove that condition, a Schwarz type lemma on balanced pseudoconvex domains [5], [6] is needed.

2. A Schwarz type lemma.

In this section, we recall a Schwarz type lemma on balanced pseudoconvex domains [5], [6]. The Lempert function \tilde{k}_D for a domain D in \mathbf{C}^n is defined

as follows:

$$\tilde{k}_D(x, y) = \inf\{\rho(\xi, \eta) \mid \xi, \eta \in \Delta, \exists \varphi \in H(\Delta, D) \text{ such that } \varphi(\xi) = x, \varphi(\eta) = y\},$$

where ρ is the Poincaré distance on the unit disk Δ .

Let D be a balanced pseudoconvex domain in \mathbf{C}^n . The Minkowski function h of D is defined as follows:

$$h(z) = \inf \left\{ t > 0 \mid \frac{z}{t} \in D \right\}.$$

Then we have (Proposition 3.1.10. of Jarnicki and Pflug [7]),

$$(2.1) \quad \tilde{k}_D(0, x) = \rho(0, h(x)) \text{ for any } x \text{ in } D.$$

Using (2.1) and the fact that the Lempert functions are contractible with respect to holomorphic mappings, we have the following theorem [5], [6].

Theorem 1. *Let F be a holomorphic mapping from D to D such that $F(0) = 0$. Then*

$$h(F(z)) \leq h(z)$$

holds for all $z \in D$.

3. A necessary and sufficient condition for a locally biholomorphic mapping to be starlike.

Let D be a domain in \mathbf{C}^n which contains 0. A holomorphic mapping from D to \mathbf{C}^n is said to be starlike if f is biholomorphic, $f(0) = 0$ and $f(D)$ is starlike with respect to the origin.

We say that D has C^1 plurisubharmonic defining functions, if for any $\zeta \in \partial D$, there exist a neighborhood U of ζ in \mathbf{C}^n and a C^1 plurisubharmonic function r on U such that $D \cap U = \{z \in U \mid r(z) < 0\}$. Then D is pseudoconvex. From now on, let D be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions. In this section, we give a necessary and sufficient condition for a locally biholomorphic mapping on D to be starlike.

Let

$$u(z_1, z_2, \dots, z_n) = \sum_{i=1}^n |z_i|^{p_i}$$

and let

$$B(p_1, \dots, p_n) = \{z \in \mathbf{C}^n \mid u(z) < 1\},$$

where $2p_n > p_1 \geq p_2 \geq \dots \geq p_n > 1$. Gong, Wang and Yu [4] gave a necessary and sufficient condition that a locally biholomorphic mapping from $B(p_1, \dots, p_n)$ to \mathbf{C}^n is starlike.

Theorem 2. *Suppose that $f : B(p_1, \dots, p_n) \rightarrow \mathbf{C}^n$ is a locally biholomorphic mapping with $f(0) = 0$. Then f is starlike if and only if*

$$(du \cdot f^{-1}) \bullet (d\rho)|_{w=f(z)} \geq 0 \text{ for any } z \in B(p_1, \dots, p_n) \setminus \{0\},$$

where $a \bullet b$ is the inner product in \mathbf{R}^{2n} and $\rho(w)$ is the distance function from the origin in \mathbf{R}^{2n} .

Their proof uses the following properties of u .

- (i) $u(z) = 0$ if and only if $z = 0$,
- (ii) u is C^1 -smooth on $B(p_1, \dots, p_n) \setminus \{0\}$,
- (iii) u is continuous on $B(p_1, \dots, p_n)$,
- (iv) $\overline{B_a} = \{z \in B(p_1, \dots, p_n) \mid u(z) \leq a\}$ for any $0 < a < 1$, where $B_a = \{z \in B(p_1, \dots, p_n) \mid u(z) < a\}$,
- (v) $\overline{B_a}$ is compact for any $0 < a < 1$,
- (vi) $u(F(z)) \leq u(z)$ for any $z \in B(p_1, \dots, p_n)$, where F is a holomorphic mapping from $B(p_1, \dots, p_n)$ into itself with $F(0) = 0$ and $DF(0) = \nu I$, $0 < \nu \leq 1$, where I denotes the identity matrix.

We will prove that the Minkowski function h of D satisfies the above properties.

Proposition 1. *Let h be the Minkowski function of D , where D is a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions. Then:*

- (i) $h(z) = 0$ if and only if $z = 0$,
- (ii) h is C^1 -smooth on $\mathbf{C}^n \setminus \{0\}$,
- (iii) h is continuous on \mathbf{C}^n ,
- (iv) $\overline{D_a} = \{z \in D \mid h(z) \leq a\}$ for any $0 < a < 1$, where $D_a = \{z \in D \mid h(z) < a\}$,
- (v) $\overline{D_a}$ is compact for any $0 < a < 1$,
- (vi) $h(F(z)) \leq h(z)$ for any $z \in D$, where F is a holomorphic mapping from D into itself with $F(0) = 0$.

Proof. (i) Since D is bounded, $h(z) = 0$ if and only if $z = 0$.

(ii) There exists a $R > 0$ such that the Euclidean closed ball $\overline{\mathbf{B}}(0, R)$ centered at 0 of radius R is contained in D . Since $h(z) = R^{-1}|z|h(Rz/|z|)$ for $z \neq 0$, it suffices to prove that h is C^1 in a neighborhood of $z_0 \in D \setminus \{0\}$. Let $\zeta = z_0/h(z_0) \in \partial D$ and let r be a C^1 plurisubharmonic defining function of D near ζ . Let $g(z, s) = r(z/s)$. Since $g(z, h(z)) = 0$ in a neighborhood of z_0 , it suffices to show that $\partial g/\partial s \neq 0$ at $(z_0, h(z_0))$ by the implicit function theorem. We use the idea of a proof of Hopf's lemma (cf. Krantz [8], p. 61). Let $D_0 = \{t \in \mathbf{C} \mid t\zeta \in D\}$. Then $D_0 = \{t \in \mathbf{C} \mid |t| < 1\}$. Let $r_0(t) = r(t\zeta)$. Let \mathbf{B}^* be the ball in \mathbf{C} centered at $c(0 < c < 1)$ of radius $1 - c$. Let \mathbf{B}_1 be a ball in \mathbf{C} centered at 1 of sufficiently small radius. Let $\mathbf{B}' = \mathbf{B}^* \cap \mathbf{B}_1$. Let $\psi(t) = \exp(-\alpha|t - c|^2) - \exp(-\alpha(1 - c)^2)$. Then ψ is subharmonic on

a neighborhood of $\overline{\mathbf{B}'}$ for sufficiently large α . Since $r_0 < 0$ on $\overline{\partial\mathbf{B}' \cap \mathbf{B}^*}$, there exists an $\varepsilon > 0$ such that $r_0 + \varepsilon\psi < 0$ on $\overline{\partial\mathbf{B}' \cap \mathbf{B}^*}$. Since $r_0 + \varepsilon\psi$ is subharmonic, $r_0 + \varepsilon\psi$ attains its maximum on $\overline{\mathbf{B}'}$ at 1. Therefore,

$$\frac{\partial(r_0 + \varepsilon\psi)}{\partial x}(1) \geq 0,$$

where $x = \operatorname{Re} t$. Since $\partial\psi/\partial x(1) < 0$, we have $\partial r_0/\partial x(1) > 0$. Then

$$\frac{\partial g}{\partial s}(z_0, h(z_0)) = -\frac{1}{h(z_0)} \frac{\partial r_0}{\partial x}(1) \neq 0.$$

(iii) It suffices to show that h is continuous at 0. There exists a $R > 0$ such that the Euclidean closed ball $\overline{\mathbf{B}}(0, R)$ centered at 0 of radius R is contained in D . Let $M = \sup\{h(z) \mid z \in \partial\mathbf{B}(0, R)\}$. Then, for any $\varepsilon > 0$, $h < \varepsilon$ on $\mathbf{B}(0, \varepsilon R/M)$.

(iv) Since h is continuous, it suffices to show that $\{z \in D \mid h(z) \leq a\} \subset \overline{D_a}$. Let $h(z) \leq a$. Since $h(tz) = th(z) < a$ for $0 < t < 1$, $tz \in D_a$ and $tz \rightarrow z$ as $t \rightarrow 1$. This implies that $z \in \overline{D_a}$.

(v) Since h is continuous on \mathbf{C}^n , $\overline{D_a} = \{z \in D \mid h(z) \leq a\} = \{z \in \mathbf{C}^n \mid h(z) \leq a\}$. Then $\overline{D_a}$ is a bounded closed subset of \mathbf{C}^n .

(vi) See Theorem 1.

Using Proposition 1, we obtain the following theorem as in the proof of Theorem 2 due to Gong, Wang and Yu [4].

Theorem 3. *Let h be the Minkowski function of D , where D is a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions. Suppose that $f : D \rightarrow \mathbf{C}^n$ is a locally biholomorphic mapping with $f(0) = 0$. Then f is starlike if and only if*

$$(3.1) \quad (dh \cdot f^{-1}) \bullet (d\rho)|_{w=f(z)} \geq 0 \text{ for any } z \in D \setminus \{0\},$$

where $a \bullet b$ is the inner product in \mathbf{R}^{2n} and $\rho(w)$ is the distance function from the origin in \mathbf{R}^{2n} .

Remark 1. (i) It is mentioned in Gong, Wang and Yu [4] that FitzGerald pointed out that if the condition $2p_n > p_1$ is dropped, then the Schwarz type lemma does not hold for u . So, they cannot obtain Theorem 2 in the case that the condition $2p_n > p_1$ is dropped. However, Theorem 3 holds for all $B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$.

(ii) Let D and f be as in Theorem 3. Let $w(z) = (Df(z))^{-1}(f(z))$. Then the condition (3.1) can be written as follows:

$$(3.2) \quad \operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle \geq 0 \text{ for any } z \in D \setminus \{0\},$$

where $\partial h^2/\partial z = (\partial h^2/\partial z_1, \dots, \partial h^2/\partial z_n)$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in \mathbf{C}^n . In particular, Theorem 3 reduces to the Suffridge's theorem [11] when $D = B(p_1, \dots, p_n)$ with $p_1 = \dots = p_n > 1$.

4. The growth and 1/4-theorems for normalized starlike mappings.

In this section, we give the growth and 1/4-theorems for normalized starlike mappings on bounded balanced pseudoconvex domains with C^1 plurisubharmonic defining functions using the ideas of Barnard, FitzGerald and Gong [1] and Chuaqui [2]. A holomorphic mapping f is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

Let D be a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions and let f be a starlike mapping from D to \mathbf{C}^n . By the Remark after Theorem 3, we have

$$\operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle \geq 0 \text{ for any } z \in D \setminus \{0\},$$

where $\partial h^2/\partial z = (\partial h^2/\partial z_1, \dots, \partial h^2/\partial z_n)$, $w(z) = (Df(z))^{-1}(f(z))$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in \mathbf{C}^n . Let $z \in \partial D$ and let $\zeta \in \Delta \setminus \{0\} = \{|\zeta| < 1\} \setminus \{0\}$. Then

$$(4.1) \quad 0 \leq \operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(\zeta z), \overline{w(\zeta z)} \right\rangle = |\zeta|^2 \operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(z), \overline{\left(\frac{w(\zeta z)}{\zeta}\right)} \right\rangle.$$

Let

$$\phi_z(\zeta) = \left\langle \frac{\partial h^2}{\partial z}(z), \overline{\left(\frac{w(\zeta z)}{\zeta}\right)} \right\rangle.$$

Since $w(0) = 0$, ϕ_z is a holomorphic function on Δ and $\operatorname{Re} \phi_z \geq 0$ on Δ from (4.1). By differentiating $h^2(\zeta z) = \zeta \bar{\zeta} h^2(z)$ with respect to ζ , we have

$$\sum_{i=1}^n \frac{\partial h^2}{\partial z_j}(\zeta z) z_j = \bar{\zeta} h^2(z).$$

If $z \in \partial D$ and $\zeta = 1$,

$$\sum_{i=1}^n \frac{\partial h^2}{\partial z_j}(z) z_j = 1.$$

Since $Dw(0) = I$, this implies that $\phi_z(0) = 1$. If we put

$$\sigma(\zeta) = \frac{\phi_z(\zeta) - 1}{\phi_z(\zeta) + 1},$$

σ is a holomorphic function on Δ such that $\sigma(0) = 0$ and $|\sigma(\zeta)| \leq 1$. The mapping f is said to be strongly starlike if $\phi_z(\Delta)$ is contained in a compact subset of the right half-plane independent of $z \in \partial D$. This condition is equivalent to the condition that $|\sigma(\zeta)| \leq c < 1$ uniformly for $z \in \partial D$.

Let f be a starlike mapping on D with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Since

$$\operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle = h^2(z) \operatorname{Re} \phi_{\tilde{z}}(h(z)) \quad \text{for } z \in D,$$

where $\tilde{z} = z/h(z)$, we obtain the following lemma by applying the Schwarz lemma to σ as in Lemma 2.1 of Pfaltzgraff [9].

Lemma 1.

$$h^2(z) \frac{1 - ch(z)}{1 + ch(z)} \leq \operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle \leq h^2(z) \frac{1 + ch(z)}{1 - ch(z)} \quad \text{for } z \in D \setminus \{0\}.$$

Let $v(z, s, t)$ be defined by

$$(4.2) \quad v(z, s, t) = f^{-1}(e^{s-t} f(z))$$

for $0 \leq s \leq t$. Let $z \in D \setminus \{0\}$. Since

$$\frac{\partial}{\partial t} h(v) = -h(v)^{-1} \operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(v), \overline{w(v)} \right\rangle,$$

we have

$$(4.3) \quad \frac{\partial}{\partial t} h(v) \leq -h(v) \frac{1 - ch(v)}{1 + ch(v)} < 0$$

by Lemma 1. Then we have $h(v(z, s, t)) \leq h(v(z, s, s)) = h(z)$. Moreover, we obtain the following inequalities by Lemma 1 as in Lemma 2.2 of Pfaltzgraff [9].

$$(4.4) \quad e^t \frac{h(v)}{(1 - ch(v))^2} \leq e^s \frac{h(z)}{(1 - ch(z))^2} \quad \text{on } D$$

and

$$(4.5) \quad e^s \frac{h(z)}{(1 + ch(z))^2} \leq e^t \frac{h(v)}{(1 + ch(v))^2} \quad \text{on } D.$$

Since $D = \{z \in \mathbf{C}^n \mid h(z) < 1\}$ is bounded with respect to the Euclidean distance, a bounded set with respect to h is bounded with respect to the Euclidean distance. By (4.4), we have

$$h(e^t v) \leq e^s \frac{h(z)}{(1 - ch(z))^2}.$$

Then $\{e^t v\}_{t \geq s}$ forms a normal family on D . If f is normalized, we can show that there exists a sequence $\{t_m\}$ such that $t_m \rightarrow \infty$ and $e^{t_m} v(z, s, t_m) \rightarrow e^s f(z)$ on D as $m \rightarrow \infty$ as in Theorem 2.3 of Pfaltzgraff [9]. Substituting $t = t_m$ in (4.4) and (4.5) and letting $m \rightarrow \infty$, we have the following theorem.

Theorem 4. *Let D be a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions and let f be a normalized starlike mapping from D to \mathbf{C}^n with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Let h be the Minkowski function of D . Then*

$$\frac{h(z)}{(1 + ch(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1 - ch(z))^2}.$$

For $D = B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$, we can show that the estimates are sharp as in Theorem 2.1 of Barnard, FitzGerald and Gong [1].

Theorem 5. *Let $p_1, \dots, p_n > 1$. Let f be a normalized starlike mapping from $B(p_1, \dots, p_n)$ to \mathbf{C}^n with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial B(p_1, \dots, p_n)$. Let h be the Minkowski function of $B(p_1, \dots, p_n)$. Then*

$$\frac{h(z)}{(1 + ch(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1 - ch(z))^2}.$$

Furthermore the estimates are sharp.

Proof. We will show that the estimates are sharp. Let

$$f(z) = \left(\frac{z_1}{(1 - cz_1)^2}, \frac{z_2}{(1 - cz_2)^2}, \dots, \frac{z_n}{(1 - cz_n)^2} \right).$$

Then f is a normalized biholomorphic mapping on $B(p_1, \dots, p_n)$ and

$$\phi_z(\zeta) = \sum_{j=1}^n \frac{\partial h^2}{\partial z_j}(z) z_j \frac{1 - c\zeta z_j}{1 + c\zeta z_j}$$

for any $z \in \partial B(p_1, \dots, p_n)$. Since $(\partial h^2 / \partial z_j)(z) z_j \geq 0$ and $\sum_{j=1}^n (\partial h^2 / \partial z_j)(z) z_j = 1$, we have $|\sigma(\zeta)| \leq c$ for any $\zeta \in \Delta$. Therefore, f is a normalized starlike mapping with $|\sigma(\zeta)| \leq c$ uniformly for $z \in \partial B(p_1, \dots, p_n)$. Since

$$h(f(z_1, 0, \dots, 0)) = \frac{1}{|1 - cz_1|^2} h((z_1, 0, \dots, 0))$$

and

$$h((z_1, 0, \dots, 0)) = |z_1|,$$

the estimates are sharp.

Corollary 1. *Let D be a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions and let f be a normalized starlike mapping from D to \mathbf{C}^n with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Then the image of f contains $1/(1 + c)^2 D$. If $D = B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$, the value $1/(1 + c)^2$ is best possible.*

Let k be a positive integer. We say that f has a k -fold symmetric image if the image of f is unchanged when multiplied by the scalar complex number $\exp(2\pi i/k)$. If k -fold symmetry of f is assumed, then Theorems 4, 5 and Corollary 1 can be strengthened as follows as in Barnard, FitzGerald and Gong [1].

Corollary 2. *Let D be a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions and let f be a normalized starlike mapping from D to \mathbf{C}^n with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$ and with a k -fold symmetric image for some positive integer k . Let h be the Minkowski function of D . Then*

$$\frac{h(z)}{(1 + ch(z)^k)^{2/k}} \leq h(f(z)) \leq \frac{h(z)}{(1 - ch(z)^k)^{2/k}}.$$

Therefore, the image of D under f contains $(1/(1 + c)^{2/k})D$. Furthermore, these estimates are sharp when $D = B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$.

Corollary 3. *The only balanced domain which is the image of a bounded balanced pseudoconvex domain D in \mathbf{C}^n with C^1 plurisubharmonic defining functions under a normalized biholomorphic mapping is D .*

5. Quasiconformal extensions.

In this section, we will show that a quasiconformal strongly starlike mapping with $|w|$ uniformly bounded on a bounded balanced pseudoconvex domain D in \mathbf{C}^n with C^1 plurisubharmonic defining functions admits a quasiconformal extension to \mathbf{C}^n using the idea of Chuaqui [2].

Let Ω, Ω' be domains in \mathbf{R}^m . A homeomorphism $f : \Omega \rightarrow \Omega'$ is said to be quasiconformal if it is differentiable a.e., ACL(absolutely continuous on lines) and

$$\|D(f; x)\|^m \leq K |\det D(f; x)| \quad \text{a.e. in } \Omega,$$

where $D(f; x)$ denotes the (real) Jacobian matrix of f , K is a constant and

$$\|D(f; x)\| = \sup\{|D(f; x)(a)| \mid |a| = 1\}.$$

Theorem 6. *Let D be a bounded balanced pseudoconvex domain in \mathbf{C}^n with C^1 plurisubharmonic defining functions, and let f be a quasiconformal, strongly starlike mapping with $|w|$ uniformly bounded on D . Then f extends to a quasiconformal homeomorphism of \mathbf{R}^{2n} onto itself.*

Proof. We may assume that f is normalized. Let $f_i = u_i + \sqrt{-1}v_i$ and $z_i = x_i + \sqrt{-1}y_i$. We first show that $\|D(u, v; x, y)\|$ is uniformly bounded in D . Let $1/2 < h(z) < 1$. By Lemma 1, we have

$$(5.1) \quad h^2(z) \frac{1 - ch(z)}{1 + ch(z)} \leq \left| \frac{\partial h^2}{\partial z} \right| \cdot |w|.$$

Using $Df(w) = f$, Theorem 4 and (5.1), we have

$$h \left(Df \left(\frac{w}{|w|} \right) \right) \leq \left| \frac{\partial h^2}{\partial z} \right| \frac{1 + ch(z)}{h(z)(1 - ch(z))^3} \leq 2 \left| \frac{\partial h^2}{\partial z} \right| \frac{1 + c}{(1 - c)^3}.$$

Since h is C^1 on $\mathbf{C}^n \setminus \{0\}$, $h(Df(w/|w|))$ is bounded for $1/2 < h(z) < 1$. Since $D = \{h(z) < 1\}$ is bounded, $|Df(w/|w|)|$ is uniformly bounded for $1/2 < h(z) < 1$. By the Cauchy-Riemann equations, this implies that $D(u, v; x, y)^t(\operatorname{Re} w/|w|, \operatorname{Im} w/|w|)$ is uniformly bounded for $1/2 < h(z) < 1$. Since f is quasiconformal, $\|D(u, v; x, y)\|$ is uniformly bounded for $1/2 < h(z) < 1$. Then $\|D(u, v; x, y)\|$ is uniformly bounded in D .

Next we will show that f admits a continuous extension to \bar{D} , and the extension is univalent in \bar{D} . For $a \in \partial D$, let $f(a) = \lim_{j \rightarrow \infty} f(t_j a)$, where $t_j < 1$ and $t_j \rightarrow 1$. This is well-defined, since $\|D(u, v; x, y)\|$ is uniformly bounded in D . Let g be the Riemannian metric induced on ∂D by the Euclidean metric on \mathbf{R}^{2n} , and let d_g be the distance function on ∂D with respect to g . For any positive ε , let $U_g(a) = \{z \in \partial D \mid d_g(a, z) < \varepsilon/2M\}$, where $M = \sup\{\|D(u, v; x, y)\| \mid (x, y) \in D\}$. Since the topology on ∂D defined by d_g coincides with the topology induced on ∂D by the Euclidean topology on \mathbf{C}^n , there exists a $\delta > 0$ such that $U(a) = \{z \in \partial D \mid |z - a| < \delta\} \subset U_g(a)$. Let

$$V = \{z \in \mathbf{C}^n \mid |z - a| < \delta/2\} \cap \left\{ z \in \bar{D} \mid L \left(\frac{1}{h(z)} - 1 \right) < \min \left(\frac{\delta}{2}, \frac{\varepsilon}{2M} \right) \right\},$$

where $L = \sup\{|z| \mid z \in \bar{D}\}$. Then V is an open neighborhood of a in \bar{D} . Let $z \in V$. Then $z/h(z) \in U(a)$, since

$$|a - z/h(z)| \leq |a - z| + |z| \left(\frac{1}{h(z)} - 1 \right) < \delta.$$

Then there exists a piecewise C^1 -curve $\gamma : [0, 1] \rightarrow \partial D$ such that $\gamma(0) = a$, $\gamma(1) = z/h(z)$ and $L_g(\gamma) < \varepsilon/2M$, where $L_g(\gamma)$ denotes the length of γ with respect to g . Let $\iota : \partial D \rightarrow \mathbf{R}^{2n}$ be the natural inclusion mapping. Then, we have

$$\begin{aligned} |f(a) - f(z/h(z))| &= \lim_{j \rightarrow \infty} \left| \int_0^1 \frac{d}{ds} f(t_j(\iota \circ \gamma)(s)) ds \right| \\ &\leq \lim_{j \rightarrow \infty} \int_0^1 \left| \frac{d}{ds} f(t_j(\iota \circ \gamma)(s)) \right| ds \\ &\leq \lim_{j \rightarrow \infty} \int_0^1 M \left| \left(t_j \frac{d}{ds} (\iota \circ \gamma)(s) \right) \right| ds \\ &= M \int_0^1 \sqrt{g(\dot{\gamma}(s), \dot{\gamma}(s))} ds \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Then $|f(z) - f(a)| \leq |f(a) - f(z/h(z))| + |f(z/h(z)) - f(z)| \leq \varepsilon/2 + M|z - z/h(z)| < \varepsilon$. This implies that f is continuous on \overline{D} . Since

$$h(v)^{-1} \frac{\partial}{\partial t} h(v) \leq \frac{-(1-c)}{1+c}$$

for $z \neq 0$ by (4.3), we have

$$h(v) \leq h(z) \exp \left\{ -\frac{1-c}{1+c}(t-s) \right\}$$

as in Pfaltzgraff [10]. This implies that

$$(5.2) \quad \overline{v(D, s, t)} \subset D \quad \text{for } 0 \leq s < t.$$

Let $f_t(z) = e^t f(z)$ for $t \geq 0$. By (4.2), we have $f_s(z) = f_t(v(z, s, t))$ for $z \in D$. Then by (5.2), $f_s(\overline{D}) \subset f_t(D)$ for $0 \leq s < t$. Therefore

$$v(z, s, t) = f_t^{-1}(f_s(z)) \quad (0 \leq s < t)$$

defines a continuous extension of v to \overline{D} . For $z \in D$, we have

$$(5.3) \quad \begin{aligned} |v(z, s, t) - z| &\leq \int_s^t \left| \frac{\partial}{\partial \tau} v(z, s, \tau) \right| d\tau \\ &= \int_s^t | -w(v(z, s, \tau)) | d\tau \\ &\leq C(t-s) \end{aligned}$$

for some positive constant C , since $|w|$ is uniformly bounded. Since v is continuous on \overline{D} , this estimate holds for $z \in \overline{D}$. Suppose that $f(z_1) = f(z_2)$ for $z_1, z_2 \in \overline{D}$. Then for $t > 0$, we have

$$f_t(v(z_1, 0, t)) = f_t(v(z_2, 0, t)).$$

Since f_t is univalent in D , we obtain $v(z_1, 0, t) = v(z_2, 0, t)$. Letting $t \rightarrow 0$, we have $z_1 = z_2$ by (5.3). Therefore, f is univalent in \overline{D} .

Let

$$F(z) = \begin{cases} f(z) & z \in \overline{D} \\ h(z)f(\frac{z}{h(z)}) & z \notin \overline{D}. \end{cases}$$

We will show that F is the quasiconformal extension of f . It is easy to show that F is continuous and univalent on \mathbf{R}^{2n} . Let $\mathbf{R}^{2n} \cup \{\infty\} = S^{2n}$ be a one point compactification of \mathbf{R}^{2n} . We extend F to S^{2n} by $F(\infty) = \infty$. By Theorem 4, F is a continuous bijective mapping from S^{2n} onto itself. Therefore, F is a homeomorphism from S^{2n} onto itself. Thus F is a homeomorphism from \mathbf{R}^{2n} onto itself. For $0 < r < 1$, let

$$F^r(z) = \begin{cases} f(rz) & z \in \overline{D} \\ h(z)f(r\frac{z}{h(z)}) & z \notin \overline{D}. \end{cases}$$

Then

$$F^r(z/r) = \begin{cases} f(z) & z \in \overline{D_r} \\ r^{-1}h(z)f(\frac{z}{r^{-1}h(z)}) & z \notin \overline{D_r}. \end{cases}$$

Since $r^{-1}h(z)$ is the Minkowski function of D_r , F^r is a homeomorphism from \mathbf{R}^{2n} onto itself. We will show that $F^r \rightarrow F$ uniformly on compact subsets of \mathbf{R}^{2n} , F^r is differentiable a.e., F^r is ACL and

$$\|D(u^r, v^r; x, y)\|^{2n} \leq K|\det D(u^r, v^r; x, y)| \quad \text{a.e. in } \mathbf{R}^{2n},$$

where $F_i^r = u_i^r + \sqrt{-1}v_i^r$ and K is independent of r and x . Then by Corollary 21.3 and Corollary 37.4 of Väisälä [12], F is quasiconformal. Since f is continuous on \overline{D} , $F^r \rightarrow F$ uniformly on compact subsets of \mathbf{R}^{2n} . Since h is C^1 on $\mathbf{R}^{2n} \setminus \{0\}$, F^r is differentiable on $\mathbf{R}^{2n} \setminus \partial D$.

Since f is quasiconformal in D , there exists a positive constant K_1 such that

$$(5.4) \quad \|D(u, v; x, y)\|^{2n} \leq K_1|\det D(u, v; x, y)| \text{ in } D.$$

Then we have

$$(5.5) \quad \|D(u^r, v^r; x, y)\|^{2n} \leq K_1|\det D(u^r, v^r; x, y)| \text{ in } D,$$

since

$$(5.6) \quad D(u^r, v^r; x, y) = rD(u, v; rx, ry) \text{ on } D.$$

For $z \notin \overline{D}$, let $\zeta = rh(z)^{-1}z \in D \setminus \{0\}$ and let $\zeta = \xi + \sqrt{-1}\eta$. Then

$$D(u^r, v^r; x, y) = rD(u, v; \xi, \eta)(I + M(\xi, \eta)),$$

where

$$M(\xi, \eta) = r^{-1} \begin{pmatrix} \operatorname{Re}(w(\zeta) - \zeta) \\ \operatorname{Im}(w(\zeta) - \zeta) \end{pmatrix} \operatorname{grad}h(\xi, \eta).$$

Since h is C^1 on $\mathbf{C}^n \setminus \{0\}$ and $\|M(\xi, \eta)\| = r^{-1}|w(\zeta) - \zeta|\operatorname{grad}h(\xi, \eta)|$, $\|M(\xi, \eta)\|$ is uniformly bounded for r near 1. Then

$$(5.7) \quad \begin{aligned} \|D(u^r, v^r; x, y)\| &\leq r\|D(u, v; \xi, \eta)\|\|I + M(\xi, \eta)\| \\ &\leq r\|D(u, v; \xi, \eta)\|(1 + \|M(\xi, \eta)\|) \\ &\leq K_2\|D(u, v; \xi, \eta)\|. \end{aligned}$$

Since $M(\xi, \eta)$ has rank 1,

$$\begin{aligned} \det(I + M(\xi, \eta)) &= 1 + \operatorname{tr} M(\xi, \eta) \\ &= r^{-2}\operatorname{Re} \left\langle \frac{\partial h^2}{\partial z}(\zeta), \overline{w(\zeta)} \right\rangle \\ &\geq r^{-2}h^2(\zeta) \frac{1 - ch(\zeta)}{1 + ch(\zeta)} \\ &\geq \frac{1 - c}{1 + c} \end{aligned}$$

by Lemma 1. Then

$$(5.8) \quad \begin{aligned} |\det D(u^r, v^r; x, y)| &= r^{2n} |\det D(u, v; \xi, \eta)| |\det(I + M(\xi, \eta))| \\ &\geq r^{2n} \frac{1-c}{1+c} |\det D(u, v; \xi, \eta)|. \end{aligned}$$

By (5.4), (5.7) and (5.8), we have

$$(5.9) \quad \begin{aligned} \|D(u^r, v^r; x, y)\|^{2n} &\leq K_2^{2n} \|D(u, v; \xi, \eta)\|^{2n} \\ &\leq K_1 K_2^{2n} |\det D(u, v; \xi, \eta)| \\ &\leq r^{-2n} \frac{1+c}{1-c} K_1 K_2^{2n} |\det D(u^r, v^r; x, y)|. \end{aligned}$$

By (5.5) and (5.9), we have

$$\|D(u^r, v^r; x, y)\|^{2n} \leq K |\det D(u^r, v^r; x, y)| \quad \text{a.e. in } \mathbf{R}^{2n},$$

where K is independent of r and x .

Let $\mathbf{R}_i^{2n-1} = \{x \in \mathbf{R}^{2n} \mid x_i = 0\}$ and let P_i be the orthogonal projection of \mathbf{R}^{2n} onto \mathbf{R}_i^{2n-1} . Let Q be a closed $2n$ -interval. Let $J_y = Q \cap P_i^{-1}(y)$. We will show that F^r is absolutely continuous on J_y for almost every $y \in P_i Q$. Let $A = \{y \in P_i Q \mid J_y \cap \partial D \text{ is uncountable}\}$. By Theorem 30.16 of Väisälä [12], $m_{2n-1}(A) = 0$. For any $y \in P_i Q \setminus A$, $F^r|_{J_y}$ is an injective path, and $J_y \cap \partial D$ is countable. By (5.6) and (5.7), $|\partial_i F^r|$ is bounded on $U \setminus \partial D$ for $1 \leq i \leq 2n$, where U is a neighborhood of $\partial D \cup J_y$, since $\|D(u, v; x, y)\|$ is uniformly bounded in D . Then F^r is absolutely continuous on every closed subinterval of $J_y \setminus (J_y \cap \partial D)$ and

$$\int_{J_y} |\partial_i F^r| dm_1 < \infty.$$

By Theorem 30.12 of Väisälä [12], $F^r|_{J_y}$ is absolutely continuous.

This completes the proof.

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Received April 22, 1997.

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