STARLIKE MAPPINGS ON BOUNDED BALANCED DOMAINS WITH $C^1$-PLURISUBHARMONIC DEFINING FUNCTIONS

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Let $D$ be a bounded balanced domain with $C^1$ plurisubharmonic defining functions in $\mathbb{C}^n$. First, we give a necessary and sufficient condition that a locally biholomorphic mapping from $D$ to $\mathbb{C}^n$ is starlike. Next, we give a growth theorem for normalized starlike mappings on $D$. We also give a quasiconformal extension of some strongly starlike mapping on $D$.

1. Introduction.

Let $f$ be a univalent mapping in the unit disk $\Delta$ with $f(0) = 0$ and $f'(0) = 1$. Then the classical growth theorem is as follows:

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$  


In this paper, we will extend the above results to (strongly) starlike mappings on bounded balanced domains with $C^1$ plurisubharmonic defining functions in $\mathbb{C}^n$. Since we cannot use the characterization of the starlikeness due to Suffridge [11], we first give a necessary and sufficient condition that a locally biholomorphic mapping on such domains is starlike using the idea of Gong, Wang and Yu [4]. To prove that condition, a Schwarz type lemma on balanced pseudoconvex domains [5], [6] is needed.

2. A Schwarz type lemma.

In this section, we recall a Schwarz type lemma on balanced pseudoconvex domains [5], [6]. The Lempert function $\tilde{k}_D$ for a domain $D$ in $\mathbb{C}^n$ is defined
as follows:
\[
\tilde{k}_D(x, y) = \inf \{ \rho(\xi, \eta) \mid \xi, \eta \in \Delta, \exists \varphi \in H(\Delta, D) \text{ such that } \varphi(\xi) = x, \varphi(\eta) = y \},
\]
where \( \rho \) is the Poincaré distance on the unit disk \( \Delta \).

Let \( D \) be a balanced pseudoconvex domain in \( \mathbb{C}^n \). The Minkowski function \( h \) of \( D \) is defined as follows:
\[
h(z) = \inf \{ t > 0 \mid \frac{z}{t} \in D \}.
\]
Then we have (Proposition 3.1.10. of Jarnicki and Pflug [7]),
\[(2.1) \quad \tilde{k}_D(0, x) = \rho(0, h(x)) \text{ for any } x \in D.\]
Using (2.1) and the fact that the Lempert functions are contractible with respect to holomorphic mappings, we have the following theorem [5], [6].

**Theorem 1.** Let \( F \) be a holomorphic mapping from \( D \) to \( D \) such that \( F(0) = 0 \). Then
\[
h(F(z)) \leq h(z)
\]
holds for all \( z \in D \).

### 3. A necessary and sufficient condition for a locally biholomorphic mapping to be starlike.

Let \( D \) be a domain in \( \mathbb{C}^n \) which contains 0. A holomorphic mapping from \( D \) to \( \mathbb{C}^n \) is said to be starlike if \( f \) is biholomorphic, \( f(0) = 0 \) and \( f(D) \) is starlike with respect to the origin.

We say that \( D \) has \( C^1 \) plurisubharmonic defining functions, if for any \( \zeta \in \partial D \), there exist a neighborhood \( U \) of \( \zeta \) in \( \mathbb{C}^n \) and a \( C^1 \) plurisubharmonic function \( r \) on \( U \) such that \( D \cap U = \{ z \in U \mid r(z) < 0 \} \). Then \( D \) is pseudoconvex. From now on, let \( D \) be a bounded balanced pseudoconvex domain with \( C^1 \) plurisubharmonic defining functions. In this section, we give a necessary and sufficient condition for a locally biholomorphic mapping on \( D \) to be starlike.

Let
\[
u(z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} |z_i|^{p_i}
\]
and let
\[
B(p_1, \ldots, p_n) = \{ z \in \mathbb{C}^n \mid u(z) < 1 \},
\]
where \( 2p_n > p_1 \geq p_2 \geq \ldots \geq p_n > 1 \). Gong, Wang and Yu [4] gave a necessary and sufficient condition that a locally biholomorphic mapping from \( B(p_1, \ldots, p_n) \) to \( \mathbb{C}^n \) is starlike.
Theorem 2. Suppose that \( f : B(p_1, \ldots, p_n) \to \mathbb{C}^n \) is a locally biholomorphic mapping with \( f(0) = 0 \). Then \( f \) is starlike if and only if
\[
(du \cdot f^{-1}) \cdot (dp)|_{w = f(z)} \geq 0 \quad \text{for any } z \in B(p_1, \ldots, p_n) \setminus \{0\},
\]
where \( a \cdot b \) is the inner product in \( \mathbb{R}^{2n} \) and \( \rho(w) \) is the distance function from the origin in \( \mathbb{R}^{2n} \).

Their proof uses the following properties of \( u \).
(i) \( u(z) = 0 \) if and only if \( z = 0 \),
(ii) \( u \) is \( C^1 \)-smooth on \( B(p_1, \ldots, p_n) \setminus \{0\} \),
(iii) \( u \) is continuous on \( B(p_1, \ldots, p_n) \),
(iv) \( \overline{B}_a = \{z \in B(p_1, \ldots, p_n) \mid u(z) \leq a\} \) for any \( 0 < a < 1 \), where \( \overline{B}_a = \{z \in B(p_1, \ldots, p_n) \mid u(z) < a\} \),
(v) \( \overline{D}_a \) is compact for any \( 0 < a < 1 \),
(vi) \( u(F(z)) \leq u(z) \) for any \( z \in B(p_1, \ldots, p_n) \), where \( F \) is a holomorphic mapping from \( B(p_1, \ldots, p_n) \) into itself with \( F(0) = 0 \) and \( DF(0) = \nu I \),
\( 0 < \nu \leq 1 \), where \( I \) denotes the identity matrix.

We will prove that the Minkowski function \( h \) of \( D \) satisfies the above properties.

Proposition 1. Let \( h \) be the Minkowski function of \( D \), where \( D \) is a bounded balanced pseudoconvex domain in \( \mathbb{C}^n \) with \( C^1 \) plurisubharmonic defining functions. Then:
(i) \( h(z) = 0 \) if and only if \( z = 0 \),
(ii) \( h \) is \( C^1 \)-smooth on \( \mathbb{C}^n \setminus \{0\} \),
(iii) \( h \) is continuous on \( \mathbb{C}^n \),
(iv) \( \overline{D}_a = \{z \in D \mid h(z) \leq a\} \) for any \( 0 < a < 1 \), where \( D_a = \{z \in D \mid h(z) < a\} \),
(v) \( \overline{D}_a \) is compact for any \( 0 < a < 1 \),
(vi) \( h(F(z)) \leq h(z) \) for any \( z \in D \), where \( F \) is a holomorphic mapping from \( D \) into itself with \( F(0) = 0 \).

Proof. (i) Since \( D \) is bounded, \( h(z) = 0 \) if and only if \( z = 0 \).
(ii) There exists a \( R > 0 \) such that the Euclidean closed ball \( \overline{B}(0, R) \) centered at \( 0 \) of radius \( R \) is contained in \( D \). Since \( h(z) = R^{-1}|z|h(Rz/|z|) \) for \( z \neq 0 \), it suffices to prove that \( h \) is \( C^1 \) in a neighborhood of \( z_0 \in \partial D \) and let \( r \) be a \( C^1 \) plurisubharmonic defining function of \( D \) near \( \zeta \). Let \( g(z, s) = r(z/s) \). Since \( g(z, h(z)) = 0 \) in a neighborhood of \( z_0 \), it suffices to show that \( \partial g/\partial s \neq 0 \) at \( (z_0, h(z_0)) \) by the implicit function theorem. We use the idea of a proof of Hopf’s lemma (cf. Krantz [8], p. 61).
Let \( D_0 = \{t \in \mathbb{C} \mid t\zeta \in D\} \). Then \( D_0 = \{t \in \mathbb{C} \mid |t| < 1\} \). Let \( r_0(t) = r(t\zeta) \).
Let \( \mathbf{B}^* \) be the ball in \( \mathbb{C} \) centered at \( c(0 < c < 1) \) of radius \( 1 - c \). Let \( \mathbf{B}_1 \) be a ball in \( \mathbb{C} \) centered at \( 1 \) of sufficiently small radius. Let \( \mathbf{B}' = \mathbf{B}^* \cap \mathbf{B}_1 \).
Let \( \psi(t) = \exp(-\alpha|t - c|^2) - \exp(-\alpha(1 - c)^2) \). Then \( \psi \) is subharmonic on
a neighborhood of $\overline{B'}$ for sufficiently large $\alpha$. Since $r_0 < 0$ on $\overline{B'} \cap \overline{B''}$, there exists an $\varepsilon > 0$ such that $r_0 + \varepsilon \psi < 0$ on $\overline{B'} \cap \overline{B''}$. Since $r_0 + \varepsilon \psi$ is subharmonic, $r_0 + \varepsilon \psi$ attains its maximum on $\overline{B'}$ at 1. Therefore,

\[
\frac{\partial (r_0 + \varepsilon \psi)}{\partial x}(1) \geq 0,
\]

where $x = \text{Re} t$. Since $\partial \psi / \partial x(1) < 0$, we have $\partial r_0 / \partial x(1) > 0$. Then

\[
\frac{\partial g}{\partial s}(z_0, h(z_0)) = -\frac{1}{h(z_0)} \frac{\partial r_0}{\partial x}(1) \neq 0.
\]

(iii) It suffices to show that $h$ is continuous at 0. There exists a $R > 0$ such that the Euclidean closed ball $\overline{B}(0, R)$ centered at 0 of radius $R$ is contained in $D$. Let $M = \sup \{h(z) \mid z \in \partial B(0, R)\}$. Then, for any $\varepsilon > 0$, $h < \varepsilon$ on $B(0, \varepsilon R / M)$.

(iv) Since $h$ is continuous, it suffices to show that $\{z \in D \mid h(z) \leq a\} \subset D_a$. Let $h(z) \leq a$. Since $h(tz) = th(z) < a$ for $0 < t < 1$, $tz \in D_a$ and $tz \rightarrow z$ as $t \rightarrow 1$. This implies that $z \in D_a$.

(v) Since $h$ is continuous on $C^n$, $D_a = \{z \in D \mid h(z) \leq a\} = \{z \in C^n \mid h(z) \leq a\}$. Then $D_a$ is a bounded closed subset of $C^n$.

(vi) See Theorem 1.

Using Proposition 1, we obtain the following theorem as in the proof of Theorem 2 due to Gong, Wang and Yu [4].

**Theorem 3.** Let $h$ be the Minkowski function of $D$, where $D$ is a bounded balanced pseudoconvex domain in $C^n$ with $C^1$ plurisubharmonic defining functions. Suppose that $f : D \rightarrow C^n$ is a locally biholomorphic mapping with $f(0) = 0$. Then $f$ is starlike if and only if

\[
(dh \cdot f^{-1}) \bullet (d\rho)|_{w=f(z)} \geq 0 \text{ for any } z \in D \setminus \{0\},
\]

where $a \bullet b$ is the inner product in $R^{2n}$ and $\rho(w)$ is the distance function from the origin in $R^{2n}$.

**Remark 1.** (i) It is mentioned in Gong, Wang and Yu [4] that FitzGerald pointed out that if the condition $2p_n > p_1$ is dropped, then the Schwarz type lemma does not hold for $u$. So, they cannot obtain Theorem 2 in the case that the condition $2p_n > p_1$ is dropped. However, Theorem 3 holds for all $B(p_1, \ldots, p_n)$ with $p_1, \ldots, p_n > 1$.

(ii) Let $D$ and $f$ be as in Theorem 3. Let $w(z) = (Df(z))^{-1}(f(z))$. Then the condition (3.1) can be written as follows:

\[
\text{Re} \left\langle \frac{\partial h^2}{\partial z}(z), w(z) \right\rangle \geq 0 \text{ for any } z \in D \setminus \{0\},
\]

where $\partial h^2 / \partial z = (\partial h^2 / \partial z_1, \ldots, \partial h^2 / \partial z_n)$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in $C^n$. In particular, Theorem 3 reduces to the Suffridge’s theorem [11] when $D = B(p_1, \ldots, p_n)$ with $p_1 = \cdots = p_n > 1$. 

4. The growth and $1/4$-theorems for normalized starlike mappings.

In this section, we give the growth and $1/4$-theorems for normalized starlike mappings on bounded balanced pseudoconvex domains with $C^1$ plurisubharmonic defining functions using the ideas of Barnard, FitzGerald and Gong [1] and Chuaqui [2]. A holomorphic mapping $f$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

Let $D$ be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ with $C^1$ plurisubharmonic defining functions and let $f$ be a starlike mapping from $D$ to $\mathbb{C}^n$. By the Remark after Theorem 3, we have

$$\text{Re} \left< \frac{\partial h^2}{\partial z}(z), w(z) \right> \geq 0 \text{ for any } z \in D \setminus \{0\},$$

where $\partial h^2/\partial z = (\partial h^2/\partial z_1, \ldots, \partial h^2/\partial z_n)$, $w(z) = (Df(z))^{-1}(f(z))$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in $\mathbb{C}^n$. Let $z \in \partial D$ and let $\zeta \in \Delta \setminus \{0\} = \{|\zeta| < 1\} \setminus \{0\}$. Then

$$0 \leq \text{Re} \left< \frac{\partial h^2}{\partial z}(\zeta z), w(\zeta z) \right> = |\zeta|^2 \text{Re} \left< \frac{\partial h^2}{\partial z}(z), \left( \frac{w(\zeta z)}{\zeta} \right) \right>.$$ 

Let

$$\phi_z(\zeta) = \left< \frac{\partial h^2}{\partial z}(z), \left( \frac{w(\zeta z)}{\zeta} \right) \right>.$$ 

Since $w(0) = 0$, $\phi_z$ is a holomorphic function on $\Delta$ and $\text{Re} \phi_z \geq 0$ on $\Delta$ from (4.1). By differentiating $h^2(\zeta z) = \zeta \bar{\zeta} h^2(z)$ with respect to $\zeta$, we have

$$\sum_{i=1}^{n} \frac{\partial h^2}{\partial z_j}(\zeta z) z_j = \bar{\zeta} h^2(z).$$

If $z \in \partial D$ and $\zeta = 1$,

$$\sum_{i=1}^{n} \frac{\partial h^2}{\partial z_j}(z) z_j = 1.$$ 

Since $Dw(0) = I$, this implies that $\phi_z(0) = 1$. If we put

$$\sigma(\zeta) = \frac{\phi_z(\zeta) - 1}{\phi_z(\zeta) + 1},$$

$\sigma$ is a holomorphic function on $\Delta$ such that $\sigma(0) = 0$ and $|\sigma(\zeta)| \leq 1$. The mapping $f$ is said to be strongly starlike if $\phi_z(\Delta)$ is contained in a compact subset of the right half-plane independent of $z \in \partial D$. This condition is equivalent to the condition that $|\sigma(\zeta)| \leq c < 1$ uniformly for $z \in \partial D$. 

Let $f$ be a starlike mapping on $D$ with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Since
\[
\text{Re} \left( \frac{\partial h^2(z)}{\partial z} \right) = h^2(z) \text{Re} \varphi(h(z)) \quad \text{for } z \in D,
\]
where $\tilde{z} = z/h(z)$, we obtain the following lemma by applying the Schwarz lemma to $\sigma$ as in Lemma 2.1 of Pfaltzgraff [9].

**Lemma 1.**
\[
h^2(z) \frac{1 - ch(z)}{1 + ch(z)} \leq \text{Re} \left( \frac{\partial h^2(z)}{\partial z} \right) \leq h^2(z) \frac{1 + ch(z)}{1 - ch(z)} \quad \text{for } z \in D \setminus \{0\}.
\]

Let $v(z, s, t)$ be defined by
\[
v(z, s, t) = f^{-1}(e^{s-t}f(z))
\]
for $0 \leq s \leq t$. Let $z \in D \setminus \{0\}$. Since
\[
\frac{\partial}{\partial t} h(v) = -h(v)^{-1} \text{Re} \left( \frac{\partial h^2}{\partial z}(v) \right),
\]
we have
\[
\frac{\partial}{\partial t} h(v) \leq -h(v) \frac{1 - ch(v)}{1 + ch(v)} < 0
\]
by Lemma 1. Then we have $h(v(z, s, t)) \leq h(v(z, s, s)) = h(z)$. Moreover, we obtain the following inequalities by Lemma 1 as in Lemma 2.2 of Pfaltzgraff [9].

\[
e^t \frac{h(v)}{(1 - ch(v))^2} \leq e^s \frac{h(z)}{(1 - ch(z))^2} \quad \text{on } D
\]

and
\[
e^s \frac{h(z)}{(1 + ch(z))^2} \leq e^t \frac{h(v)}{(1 + ch(v))^2} \quad \text{on } D.
\]

Since $D = \{z \in C^n \mid h(z) < 1\}$ is bounded with respect to the Euclidean distance, a bounded set with respect to $h$ is bounded with respect to the Euclidean distance. By (4.4), we have
\[
h(e^tv) \leq e^s \frac{h(z)}{(1 - ch(z))^2}.
\]

Then \{e^tv\}_{t \geq s} forms a normal family on $D$. If $f$ is normalized, we can show that there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and $e^{t_m}v(z, s, t_m) \to e^s f(z)$ on $D$ as $m \to \infty$ as in Theorem 2.3 of Pfaltzgraff [9]. Substituting $t = t_m$ in (4.4) and (4.5) and letting $m \to \infty$, we have the following theorem.
**Theorem 4.** Let \( D \) be a bounded balanced pseudoconvex domain in \( \mathbb{C}^n \) with \( C^1 \) plurisubharmonic defining functions and let \( f \) be a normalized starlike mapping from \( D \) to \( \mathbb{C}^n \) with \( |\sigma(\zeta)| \leq c \leq 1 \) uniformly for \( z \in \partial D \). Let \( h \) be the Minkowski function of \( D \). Then

\[
\frac{h(z)}{(1 + ch(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1 - ch(z))^2}.
\]

For \( D = B(p_1, \ldots, p_n) \) with \( p_1, \ldots, p_n > 1 \), we can show that the estimates are sharp as in Theorem 2.1 of Barnard, FitzGerald and Gong [1].

**Theorem 5.** Let \( p_1, \ldots, p_n > 1 \). Let \( f \) be a normalized starlike mapping from \( B(p_1, \ldots, p_n) \) to \( \mathbb{C}^n \) with \( |\sigma(\zeta)| \leq c \leq 1 \) uniformly for \( z \in \partial B(p_1, \ldots, p_n) \). Let \( h \) be the Minkowski function of \( B(p_1, \ldots, p_n) \). Then

\[
\frac{h(z)}{(1 + ch(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1 - ch(z))^2}.
\]

Furthermore the estimates are sharp.

**Proof.** We will show that the estimates are sharp. Let

\[
f(z) = \left( \frac{z_1}{(1 - cz_1)^2}, \frac{z_2}{(1 - cz_2)^2}, \ldots, \frac{z_n}{(1 - cz_n)^2} \right).
\]

Then \( f \) is a normalized biholomorphic mapping on \( B(p_1, \ldots, p_n) \) and

\[
\phi_z(\zeta) = \sum_{j=1}^n \frac{\partial h^2}{\partial z_j}(z) z_j (1 - c\zeta z_j) \frac{1 - c\zeta z_j}{1 + c\zeta z_j}
\]

for any \( z \in \partial B(p_1, \ldots, p_n) \). Since \( (\partial h^2/\partial z_j)(z) z_j \geq 0 \) and \( \sum_{j=1}^n (\partial h^2/\partial z_j)(z) z_j = 1 \), we have \( |\sigma(\zeta)| \leq c \) for any \( \zeta \in \Delta \). Therefore, \( f \) is a normalized starlike mapping with \( |\sigma(\zeta)| \leq c \) uniformly for \( z \in \partial B(p_1, \ldots, p_n) \). Since

\[
h(f(z_1, 0, \ldots, 0)) = \frac{1}{|1 - cz_1|^2} h((z_1, 0, \ldots, 0))
\]

and

\[
h((z_1, 0, \ldots, 0)) = |z_1|
\]

the estimates are sharp.

**Corollary 1.** Let \( D \) be a bounded balanced pseudoconvex domain in \( \mathbb{C}^n \) with \( C^1 \) plurisubharmonic defining functions and let \( f \) be a normalized starlike mapping from \( D \) to \( \mathbb{C}^n \) with \( |\sigma(\zeta)| \leq c \leq 1 \) uniformly for \( z \in \partial D \). Then the image of \( f \) contains \( 1/(1 + c)^2 D \). If \( D = B(p_1, \ldots, p_n) \) with \( p_1, \ldots, p_n > 1 \), the value \( 1/(1 + c)^2 \) is best possible.
Let $k$ be a positive integer. We say that $f$ has a $k$-fold symmetric image if the image of $f$ is unchanged when multiplied by the scalar complex number $\exp(2\pi i/k)$. If $k$-fold symmetry of $f$ is assumed, then Theorems 4, 5 and Corollary 1 can be strengthened as follows as in Barnard, FitzGerald and Gong [1].

**Corollary 2.** Let $D$ be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ with $C^1$ plurisubharmonic defining functions and let $f$ be a normalized starlike mapping from $D$ to $\mathbb{C}^n$ with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$ and with a $k$-fold symmetric image for some positive integer $k$. Let $h$ be the Minkowski function of $D$. Then

$$
\frac{h(z)}{(1 + ch(z)^k)^{2/k}} \leq h(f(z)) \leq \frac{h(z)}{(1 - ch(z)^k)^{2/k}}.
$$

Therefore, the image of $D$ under $f$ contains $\left(\frac{1}{1 + c}\right)^{2/k}D$. Furthermore, these estimates are sharp when $D = B(p_1, \ldots, p_n)$ with $p_1, \ldots, p_n > 1$.

**Corollary 3.** The only balanced domain which is the image of a bounded balanced pseudoconvex domain $D$ in $\mathbb{C}^n$ with $C^1$ plurisubharmonic defining functions under a normalized biholomorphic mapping is $D$.

5. Quasiconformal extensions.

In this section, we will show that a quasiconformal strongly starlike mapping with $|w|$ uniformly bounded on a bounded balanced pseudoconvex domain $D$ in $\mathbb{C}^n$ with $C^1$ plurisubharmonic defining functions admits a quasiconformal extension to $\mathbb{C}^n$ using the idea of Chuaqui [2].

Let $\Omega, \Omega'$ be domains in $\mathbb{R}^m$. A homeomorphism $f : \Omega \to \Omega'$ is said to be quasiconformal if it is differentiable a.e., ACL(absolutely continuous on lines) and

$$
\|D(f; x)\|^m \leq K|\det D(f; x)| \quad \text{a.e. in } \Omega,
$$

where $D(f; x)$ denotes the (real) Jacobian matrix of $f$, $K$ is a constant and

$$
\|D(f; x)\| = \sup\{|D(f; x)(a)| \mid |a| = 1\}.
$$

**Theorem 6.** Let $D$ be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ with $C^1$ plurisubharmonic defining functions, and let $f$ be a quasiconformal, strongly starlike mapping with $|w|$ uniformly bounded on $D$. Then $f$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

**Proof.** We may assume that $f$ is normalized. Let $f_i = u_i + \sqrt{-1}v_i$ and $z_i = x_i + \sqrt{-1}y_i$. We first show that $\|D(u, v; x, y)\|$ is uniformly bounded in $D$. Let $1/2 < h(z) < 1$. By Lemma 1, we have

$$
(5.1) \quad h^2(z) \frac{1 - ch(z)}{1 + ch(z)} \leq \left| \frac{\partial h^2}{\partial z} \right| \cdot |w|.
$$
Using $Df(w) = f$, Theorem 4 and (5.1), we have
\[ h \left( Df \left( \frac{w}{|w|} \right) \right) \leq \left| \frac{\partial h}{\partial z} \right| \frac{1 + ch(z)}{h(z)(1 - ch(z))^3} \leq 2 \left| \frac{\partial h}{\partial z} \right| \frac{1 + c}{(1 - c)^3}. \]
Since $h$ is $C^1$ on $\mathbb{C}^n \setminus \{0\}$, $h(Df(w/|w|))$ is bounded for $1/2 < h(z) < 1$. Since $D = \{h(z) < 1\}$ is bounded, $|Df(w/|w|)|$ is uniformly bounded for $1/2 < h(z) < 1$. By the Cauchy-Riemann equations, this implies that $D(u,v;x,y)^t(\text{Re } w/|w|, \text{Im } w/|w|)$ is uniformly bounded for $1/2 < h(z) < 1$. Since $f$ is quasiconformal, $\|D(u,v;x,y)\|$ is uniformly bounded for $1/2 < h(z) < 1$. Then $\|D(u,v;x,y)\|$ is uniformly bounded in $D$.

Next we will show that $f$ admits a continuous extension to $\overline{D}$, and the extension is univalent in $\overline{D}$. For $a \in \partial D$, let $f(a) = \lim_{j \to \infty} f(t_ja)$, where $t_j < 1$ and $t_j \to 1$. This is well-defined, since $\|D(u,v;x,y)\|$ is uniformly bounded in $D$. Let $g$ be the Riemannian metric induced on $\partial D$ by the Euclidean metric on $\mathbb{R}^{2n}$, and let $d_g$ be the distance function on $\partial D$ with respect to $g$. For any positive $\varepsilon$, let $U_g(a) = \{z \in \partial D \mid d_g(a,z) < \varepsilon/2M\}$, where $M = \sup\{\|D(u,v;x,y)\| \mid (x,y) \in D\}$. Since the topology on $\partial D$ defined by $d_g$ coincides with the topology induced on $\partial D$ by the Euclidean topology on $\mathbb{C}^n$, there exists a $\delta > 0$ such that $U(a) = \{z \in \partial D \mid |z - a| < \delta\} \subset U_g(a)$. Let
\[ V = \{z \in \mathbb{C}^n \mid |z - a| < \delta/2\} \cap \left\{ z \in \overline{D} \mid L \left( \frac{1}{h(z)} - 1 \right) < \min \left( \frac{\delta}{2}, \frac{\varepsilon}{2M} \right) \right\}, \]
where $L = \sup\{|z| \mid z \in \overline{D}\}$. Then $V$ is an open neighborhood of $a$ in $\overline{D}$. Let $z \in V$. Then $z/h(z) \in U(a)$, since
\[ |a - z/h(z)| \leq |a - z| + |z| \left( \frac{1}{h(z)} - 1 \right) < \delta. \]
Then there exists a piecewise $C^1$-curve $\gamma : [0,1] \to \partial D$ such that $\gamma(0) = a$, $\gamma(1) = z/h(z)$ and $L_g(\gamma) < \varepsilon/2M$, where $L_g(\gamma)$ denotes the length of $\gamma$ with respect to $g$. Let $\iota : \partial D \to \mathbb{R}^{2n}$ be the natural inclusion mapping. Then, we have
\[
|f(a) - f(z/h(z))| = \lim_{j \to \infty} \left| \int_0^1 \frac{d}{ds} f(t_j(\iota \circ \gamma)(s)) ds \right|
\leq \lim_{j \to \infty} \int_0^1 \left| \frac{d}{ds} f(t_j(\iota \circ \gamma)(s)) \right| ds
\leq \lim_{j \to \infty} \int_0^1 M \left| t_j \frac{d}{ds} (\iota \circ \gamma)(s) \right| ds
= M \int_0^1 \sqrt{g(\dot{\gamma}(s), \dot{\gamma}(s))} ds
< \frac{\varepsilon}{2}.\]
Then \(|f(z) - f(a)| \leq |f(a) - f(z/h(z))| + |f(z/h(z)) - f(z)| \leq \varepsilon/2 + M|z - z/h(z)| < \varepsilon\) This implies that \(f\) is continuous on \(\overline{D}\). Since
\[
h(v)^{-1} \frac{\partial}{\partial t} h(v) \leq \frac{-(1 - c)}{1 + c}
\]
for \(z \neq 0\) by (4.3), we have
\[
h(v) \leq h(z) \exp \left\{ \frac{-1 - c}{1 + c} (t - s) \right\}
\]
as in Pfaltzgraff [10]. This implies that \(v(z, s, t) \subset D\) for \(0 \leq s < t\).

Let \(f_t(z) = e^t f(z)\) for \(t \geq 0\). By (4.2), we have \(f_s(z) = f_t(v(z, s, t))\) for \(z \in D\). Then by (5.2), \(f_s(D) \subset f_t(D)\) for \(0 \leq s < t\). Therefore
\[
v(z, s, t) = f_t^{-1}(f_s(z)) \quad (0 \leq s < t)
\]
defines a continuous extension of \(v\) to \(\overline{D}\). For \(z \in D\), we have
\[
|v(z, s, t) - z| \leq \int_s^t |\frac{\partial}{\partial \tau} v(z, s, \tau)| d\tau
\]
\[
= \int_s^t | - w(v(z, s, \tau))| d\tau
\]
\[
\leq C(t - s)
\]
for some positive constant \(C\), since \(|w|\) is uniformly bounded. Since \(v\) is continuous on \(\overline{D}\), this estimate holds for \(z \in \overline{D}\). Suppose that \(f(z_1) = f(z_2)\) for \(z_1, z_2 \in D\). Then for \(t > 0\), we have
\[
f_t(v(z_1, 0, t)) = f_t(v(z_2, 0, t)).
\]
Since \(f_t\) is univalent in \(D\), we obtain \(v(z_1, 0, t) = v(z_2, 0, t)\). Letting \(t \to 0\), we have \(z_1 = z_2\) by (5.3). Therefore, \(f\) is univalent in \(\overline{D}\).

Let
\[
F(z) = \begin{cases} f(z) & z \in \overline{D} \\ h(z) f(\frac{z}{h(z)}) & z \notin \overline{D}. \end{cases}
\]
We will show that \(F\) is the quasiconformal extension of \(f\). It is easy to show that \(F\) is continuous and univalent on \(\mathbb{R}^{2n}\). Let \(\mathbb{R}^{2n} \cup \{\infty\} = S^{2n}\) be a one point compactification of \(\mathbb{R}^{2n}\). We extend \(F\) to \(S^{2n}\) by \(F(\infty) = \infty\). By Theorem 4, \(F\) is a continuous bijective mapping from \(S^{2n}\) onto itself. Therefore, \(F\) is a homeomorphism from \(S^{2n}\) onto itself. Thus \(F\) is a homeomorphism from \(\mathbb{R}^{2n}\) onto itself. For \(0 < r < 1\), let
\[
F'(z) = \begin{cases} f(rz) & z \in \overline{D} \\ h(z) f(r \frac{z}{h(z)}) & z \notin \overline{D}. \end{cases}
\]
Then
\[
F'(z/r) = \begin{cases} 
  f(z) & z \in \overline{D}_r \\
  r^{-1}h(z)f\left(\frac{z}{r^{-1}h(z)}\right) & z \notin \overline{D}_r.
\end{cases}
\]

Since \(r^{-1}h(z)\) is the Minkowski function of \(D_r\), \(F'(z)\) is a homeomorphism from \(\mathbb{R}^{2n}\) onto itself. We will show that \(F' \to F\) uniformly on compact subsets of \(\mathbb{R}^{2n}\), \(F'\) is differentiable a.e., \(F'\) is ACL and
\[
\|D(u^r, v^r; x, y)\|^{2n} \leq K|\det D(u^r, v^r; x, y)| \text{ a.e. in } \mathbb{R}^{2n},
\]
where \(F'_r = u'_r + \sqrt{-1}v'_r\) and \(K\) is independent of \(r\) and \(x\). Then by Corollary 21.3 and Corollary 37.4 of Väisälä [12], \(F\) is quasiconformal. Since \(f\) is continuous on \(\overline{D}\), \(F' \to F\) uniformly on compact subsets of \(\mathbb{R}^{2n}\). Since \(h\) is \(C^1\) on \(\mathbb{R}^{2n} \setminus \{0\}\), \(F'\) is differentiable on \(\mathbb{R}^{2n} \setminus \partial D\).

Since \(f\) is quasiconformal in \(D\), there exists a positive constant \(K_1\) such that
\[
(5.4) \quad \|D(u, v; x, y)\|^{2n} \leq K_1|\det D(u, v; x, y)| \text{ in } D.
\]
Then we have
\[
(5.5) \quad \|D(u^r, v^r; x, y)\|^{2n} \leq K_1|\det D(u^r, v^r; x, y)| \text{ in } D,
\]
and
\[
(5.6) \quad D(u^r, v^r; x, y) = rD(u, v; rx, ry) \text{ on } D.
\]
For \(z \notin \overline{D}\), let \(\zeta = rh(z)^{-1}z \in D \setminus \{0\}\) and let \(\zeta = \xi + \sqrt{-1}\eta\). Then
\[
D(u^r, v^r; x, y) = rD(u, v; \xi, \eta)(I + M(\xi, \eta)),
\]
where
\[
M(\xi, \eta) = r^{-1} \begin{pmatrix} \text{Re}(w(\zeta) - \zeta) \\ \text{Im}(w(\zeta) - \zeta) \end{pmatrix} \text{grad}h(\xi, \eta).
\]
Since \(h\) is \(C^1\) on \(\mathbb{C}^n \setminus \{0\}\) and \(\|M(\xi, \eta)\| = r^{-1}|w(\zeta) - \zeta||\text{grad}h(\xi, \eta)|\), \(\|M(\xi, \eta)\|\) is uniformly bounded for \(r\) near 1. Then
\[
(5.7) \quad \|D(u^r, v^r; x, y)\| \leq r\|D(u, v; \xi, \eta)\||I + M(\xi, \eta)|| \leq r\|D(u, v; \xi, \eta)\||(1 + \|M(\xi, \eta)\|) \leq K_2\|D(u, v; \xi, \eta)\|.
\]
Since \(M(\xi, \eta)\) has rank 1,
\[
\det(I + M(\xi, \eta)) = 1 + \text{tr } M(\xi, \eta)
\]
\[
= r^{-2}\text{Re}\left(\frac{\partial^2}{\partial z^2}(\zeta, \overline{w(\zeta)})\right)
\]
\[
\geq r^{-2}h^2(\zeta) \frac{1 - c_h(\zeta)}{1 + c_h(\zeta)}
\]
\[
\geq \frac{1 - c}{1 + c}
\]
by Lemma 1. Then

\[ | \det D(u^r, v^r; x, y) | = r^{2n} | \det D(u, v; \xi, \eta) | | \det (I + M(\xi, \eta)) | \]

\[ \geq r^{2n} \frac{1 - c}{1 + c} | \det D(u, v; \xi, \eta) |. \]

By (5.4), (5.7) and (5.8), we have

\[ \| D(u^r, v^r; x, y) \|^2 \leq K_2^{2n} \| D(u, v; \xi, \eta) \|^2 \]

\[ \leq K_1 K_2^{2n} | \det D(u, v; \xi, \eta) | \]

\[ \leq r^{-2n} \frac{1 + c}{1 - c} K_1 K_2^{2n} | \det D(u^r, v^r; x, y) |. \]

By (5.5) and (5.9), we have

\[ \| D(u^r, v^r; x, y) \|^2 \leq K | \det D(u^r, v^r; x, y) | \quad \text{a.e. in } \mathbb{R}^{2n}, \]

where \( K \) is independent of \( r \) and \( x \).

Let \( \mathbb{R}^{2n-1}_i = \{ x \in \mathbb{R}^{2n} \mid x_i = 0 \} \) and let \( P_i \) be the orthogonal projection of \( \mathbb{R}^{2n} \) onto \( \mathbb{R}^{2n-1}_i \). Let \( Q = \bigcap P_i^{-1}(y) \). We will show that \( F^r \) is absolutely continuous on \( J_y \) for almost every \( y \in P_i Q \).

Let \( A = \{ y \in P_i Q \mid J_y \cap \partial D \text{ is uncountable} \} \). By Theorem 30.16 of Väisälä [12], \( m_{2n-1}(A) = 0 \). For any \( y \in P_i Q \setminus A \), \( F^r|_{J_y} \) is an injective path, and \( J_y \cap \partial D \) is countable. By (5.6) and (5.7), \( | \partial_i F^r | \) is bounded on \( U \setminus \partial D \) for \( 1 \leq i \leq 2n \), where \( U \) is a neighborhood of \( \partial D \cup J_y \), since \( \| D(u, v; x, y) \| \) is uniformly bounded in \( D \). Then \( F^r \) is absolutely continuous on every closed subinterval of \( J_y \setminus (J_y \cap \partial D) \) and

\[ \int_{J_y} | \partial_i F^r | dm_1 < \infty. \]

By Theorem 30.12 of Väisälä [12], \( F^r|_{J_y} \) is absolutely continuous.

This completes the proof.

References


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Kyushu Kyoritsu University
1-8, Jiyugaoka, Yahatanishi-ku
Kitakyushu 807-8585
Japan

E-mail address: hamada@kyukyo-u.ac.jp