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HIGHER DIMENSIONAL LINKS IN A SIMPLICIAL COMPLEX EMBEDDED IN A SPHERE

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We show that any embedding of the n-skeleton of a $(2n+3)$-dimensional simplex into the $(2n+1)$-dimensional sphere contains a nonsplittable link of two n-dimensional spheres.

1. Introduction.

Throughout this paper we work in the piecewise linear category. Conway and Gordon showed in [**1**] that any embedding of the complete graph over six vertices into the 3-space contains a pair of nontrivially linked circles. We refer the reader to [**6**], [**2**], [**4**], [**3**] etc. for related works. In this paper we generalize the result of Conway and Gordon to higher dimensions.

Let σ_j^i be the i-skeleton of a j-dimensional simplex $\sigma_j = \langle v_1, v_2, \dots, v_{j+1}\rangle$ where $v_1, v_2, \dots, v_j$ and v_{j+1} are the 0-simplices of σ_j. Let S^k be the k-dimensional unit sphere. Let X and Y be disjoint n-dimensional spheres embedded in S^{2n+1}. Then the linking number $\ell k(X,Y) \in Z$ is defined up to sign, see for example [**7**]. Then the modulo 2 reduction $\ell k_2(X,Y) \in Z/2Z$ of $\ell k(X,Y)$ is well-defined. We note that $\ell k_2(X,Y) \equiv \ell k_2(Y,X) \pmod 2$. Let $\mathcal{L}^n$ be the set of all unordered pairs of disjoint subcomplices of σ_{2n+3}^n each of which is homeomorphic to an n-dimensional sphere. We note that each element (J,K) of $\mathcal{L}^n$ can be written as

$$(J,K) = (\partial\langle v_{a_1}, v_{a_2}, \dots, v_{a_{n+2}}\rangle, \partial\langle v_{b_1}, v_{b_2}, \dots, v_{b_{n+2}}\rangle)$$

where ∂ denotes the boundary and $\{a_1, a_2, \dots, a_{n+2}\} \cup \{b_1, b_2, \dots, b_{n+2}\} = \{1, 2, \dots, 2n+4\}$. Therefore the number of the elements of $\mathcal{L}^n$ is $\binom{2n+4}{n+2}/2$.

Theorem 1.1. *Let n be a non-negative integer. Let $f : \sigma_{2n+3}^n \to S^{2n+1}$ be an embedding. Then*

$$\sum_{(J,K)\in\mathcal{L}^n} \ell k_2(f(J), f(K)) \equiv 1 \pmod 2.$$

We note that σ_5^1 is the complete graph over six vertices and the case $n = 1$ of Theorem 1.1 is what Conway and Gordon actually proved in [**1**]. By Theorem 1.1 we have that there is at least one $(J,K) \in \mathcal{L}^n$ with $\ell k(f(J), f(K)) \equiv 1 \pmod 2$. Thus we have that any embedding of σ_{2n+3}^n into S^{2n+1} contains a nonsplittable link of two n-spheres.

2. Proof of Theorem 1.1.

The idea of the following proof is essentially the same as that of Conway and Gordon in [**1**].

Lemma 2.1. *For any embeddings* $f, g : \sigma^n_{2n+3} \to S^{2n+1}$,

$$\sum_{(J,K)\in\mathcal{L}^n} \ell k_2(f(J), f(K)) \equiv \sum_{(J,K)\in\mathcal{L}^n} \ell k_2(g(J), g(K)) \pmod 2.$$

Proof. Since $n < 2n+1$ we have that both f and g are homotopic to a constant map. Therefore f and g are homotopic. By a standard general position argument we can modify the homotopy between f and g and we may suppose that f and g are connected by a finite sequence of 'crossing changes' of n-simplices of σ^n_{2n+3}. Namely we have a homotopy $H : \sigma^n_{2n+3} \times [0,1] \to S^{2n+1} \times [0,1]$ with $H(x,0) = (f(x),0)$, $H(x,1) = (g(x),1)$ whose multiple points are only finitely many transversal double points of the product of n-simplices and $[0,1]$ and no two of them have the same second entry. Then it is enough to show the case that H has just one double point. If the first entries of the preimage of the double point do not lie in disjoint n-simplices of σ^n_{2n+3} then we have $\ell k_2(f(J), f(K)) \equiv \ell k_2(g(J), g(K)) \pmod 2$ for each $(J,K) \in \mathcal{L}^n$. Thus we may suppose without loss of generality that the first entries of the preimage lie in n-simplices $\langle v_1, v_2, \ldots, v_{n+1}\rangle$ and $\langle v_{n+2}, v_{n+3}, \ldots, v_{2n+2}\rangle$. Let

$$(J_1, K_1) = (\partial\langle v_1, v_2, \ldots, v_{n+1}, v_{2n+3}\rangle, \partial\langle v_{n+2}, v_{n+3}, \ldots, v_{2n+2}, v_{2n+4}\rangle)$$

and

$$(J_2, K_2) = (\partial\langle v_1, v_2, \ldots, v_{n+1}, v_{2n+4}\rangle, \partial\langle v_{n+2}, v_{n+3}, \ldots, v_{2n+2}, v_{2n+3}\rangle).$$

Then we have $\ell k_2(f(J_i), f(K_i)) \equiv \ell k_2(g(J_i), g(K_i)) + 1 \pmod 2$ for $i = 1, 2$ and $\ell k_2(f(J), f(K)) \equiv \ell k_2(g(J), g(K)) \pmod 2$ for $(J,K) \in \mathcal{L}^n$, $(J,K) \neq (J_1,K_1), (J_2,K_2)$ as unordered pair. This completes the proof. $\square$

Lemma 2.2. *There is an embedding* $f : \sigma^n_{2n+3} \to S^{2n+1}$ *with*

$$\sum_{(J,K)\in\mathcal{L}^n} \ell k_2(f(J), f(K)) \equiv 1 \pmod 2.$$

Proof. We use the fact that S^{2n+1} is homeomorphic to the join of two n-dimensional spheres, see Chapter 1 of [**5**]. Let P be the join of the two simplicial complices $J_0 = \partial\langle v_1, v_2, \ldots, v_{n+2}\rangle$ and $K_0 = \partial\langle v_{n+3}, v_{n+4}, \ldots, v_{2n+4}\rangle$. Since $\sigma_{2n+3} = \langle v_1, v_2, \ldots, v_{2n+4}\rangle$ is the join of $\langle v_1, v_2, \ldots, v_{n+2}\rangle$ and $\langle v_{n+3}, v_{n+4}, \ldots, v_{2n+4}\rangle$ we have that P is a subcomplex of σ_{2n+3}. Then we have that σ^n_{2n+3} is a subcomplex of P. Since P is homeomorphic to S^{2n+1} we have an embedding, say f, of σ^n_{2n+3} into S^{2n+1}. Let $(J,K) \in \mathcal{L}^n$. Then

$$(J,K) = (\partial\langle v_{a_1}, v_{a_2}, \ldots, v_{a_{n+2}}\rangle, \partial\langle v_{b_1}, v_{b_2}, \ldots, v_{b_{n+2}}\rangle)$$

for some $\{a_1, a_2, \ldots, a_{n+2}\}$ and $\{b_1, b_2, \ldots, b_{n+2}\}$. If $(J, K) \neq (J_0, K_0)$ as unordered pair then we have that the $(n+1)$-simplices $\langle v_{a_1}, v_{a_2}, \ldots, v_{a_{n+2}}\rangle$ and $\langle v_{b_1}, v_{b_2}, \ldots, v_{b_{n+2}}\rangle$ are contained in P. Therefore $f(J)$ and $f(K)$ bound disjoint $(n+1)$-dimensional disks in S^{2n+1} and we have $\ell k_2(f(J), f(K)) \equiv 0 \pmod 2$. It is clear that $\ell k_2(f(J_0), f(K_0)) \equiv 1 \pmod 2$. This completes the proof. $\square$

Theorem 1.1 follows immediately from Lemma 2.1 and Lemma 2.2.

Remark 2.3. If we consider a general position map $f : \sigma^k_{j+k+3} \to S^{j+k+1}$ for $0 \leq j \leq k$ and consider all pair (J, K) of disjoint j-sphere and k-sphere in σ^k_{j+k+3}, then we have a result that is a generalization of Lemma 2.1. The proof is essentially the same. However it turns out that the sum of ℓk_2 is zero whenever $j < k$. In fact, for any finite simplicial complex Q and $j < k$, there is a general position map $f : Q \to S^{j+k+1}$ whose image is contained in the upper hemisphere and whose restriction to the j-skeleton of Q is an embedding into the equator $S^{j+k} \subset S^{j+k+1}$. Then it is easy to see that $\ell k_2(f(J), f(K)) = 0$ for any pair (J, K) of disjoint j-sphere and k-sphere in Q.

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