HIGHER DIMENSIONAL LINKS IN A SIMPLICIAL COMPLEX EMBEDDED IN A SPHERE

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We show that any embedding of the \( n \)-skeleton of a \((2n + 3)\)-dimensional simplex into the \((2n + 1)\)-dimensional sphere contains a nonsplittable link of two \( n \)-dimensional spheres.

1. Introduction.

Throughout this paper we work in the piecewise linear category. Conway and Gordon showed in [1] that any embedding of the complete graph over six vertices into the 3-space contains a pair of nontrivially linked circles. We refer the reader to [6], [2], [4], [3] etc. for related works. In this paper we generalize the result of Conway and Gordon to higher dimensions.

Let \( \sigma^j \) be the \( i \)-skeleton of a \( j \)-dimensional simplex \( \sigma_j = \langle v_1, v_2, \ldots, v_{j+1} \rangle \) where \( v_1, v_2, \ldots, v_j \) and \( v_{j+1} \) are the 0-simplices of \( \sigma_j \). Let \( S^k \) be the \( k \)-dimensional unit sphere. Let \( X \) and \( Y \) be disjoint \( n \)-dimensional spheres embedded in \( S^{2n+1} \). Then the linking number \( \ell k(X, Y) \in \mathbb{Z} \) is defined up to sign, see for example [7]. Then the modulo 2 reduction \( \ell k_2(X, Y) \equiv \ell k_2(Y, X) \pmod{2} \). Let \( \mathcal{L}^n \) be the set of all unordered pairs of disjoint subcomplexes of \( \sigma^{2n+3} \) each of which is homeomorphic to an \( n \)-dimensional sphere. We note that each element \( (J, K) \) of \( \mathcal{L}^n \) can be written as

\[
(J, K) = (\partial \langle v_{a_1}, v_{a_2}, \ldots, v_{a_{n+2}} \rangle, \partial \langle v_{b_1}, v_{b_2}, \ldots, v_{b_{n+2}} \rangle)
\]

where \( \partial \) denotes the boundary and \( \{a_1, a_2, \ldots, a_{n+2}\} \cup \{b_1, b_2, \ldots, b_{n+2}\} = \{1, 2, \ldots, 2n + 4\} \). Therefore the number of the elements of \( \mathcal{L}^n \) is \( \binom{2n+4}{n+2}/2 \).

**Theorem 1.1.** Let \( n \) be a non-negative integer. Let \( f : \sigma^n_{2n+3} \rightarrow S^{2n+1} \) be an embedding. Then

\[
\sum_{(J, K) \in \mathcal{L}^n} \ell k_2(f(J), f(K)) \equiv 1 \pmod{2}.
\]

We note that \( \sigma^3_1 \) is the complete graph over six vertices and the case \( n = 1 \) of Theorem 1.1 is what Conway and Gordon actually proved in [1]. By Theorem 1.1 we have that there is at least one \( (J, K) \in \mathcal{L}^n \) with \( \ell k(f(J), f(K)) \equiv 1 \pmod{2} \). Thus we have that any embedding of \( \sigma^{2n+3} \) into \( S^{2n+1} \) contains a nonsplittable link of two \( n \)-spheres.

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2. Proof of Theorem 1.1.

The idea of the following proof is essentially the same as that of Conway and Gordon in [1].

**Lemma 2.1.** For any embeddings \( f, g : \sigma_{2n+3}^n \to S^{2n+1} \),

\[
\sum_{(J, K) \in \mathcal{L}^n} \ell k_2(f(J), f(K)) \equiv \sum_{(J, K) \in \mathcal{L}^n} \ell k_2(g(J), g(K)) \pmod{2}.
\]

**Proof.** Since \( n < 2n + 1 \) we have that both \( f \) and \( g \) are homotopic to a constant map. Therefore \( f \) and \( g \) are homotopic. By a standard general position argument we can modify the homotopy between \( f \) and \( g \) and we may suppose that \( f \) and \( g \) are connected by a finite sequence of ‘crossing changes’ of \( n \)-simplices of \( \sigma_{2n+3}^n \). Namely we have a homotopy \( H : \sigma_{2n+3}^n \times [0, 1] \to S^{2n+1} \times [0, 1] \) with \( H(x, 0) = (f(x), 0) \), \( H(x, 1) = (g(x), 1) \) whose multiple points are only finitely many transversal double points of the product of \( n \)-simplices and \([0, 1] \) and no two of them have the same second entry. Then it is enough to show the case that \( H \) has just one double point. If the first entries of the preimage of the double point do not lie in disjoint \( n \)-simplices of \( \sigma_{2n+3}^n \) then we have \( \ell k_2(f(J), f(K)) \equiv \ell k_2(g(J), g(K)) \pmod{2} \) for each \((J, K) \in \mathcal{L}^n\). Thus we may suppose without loss of generality that the first entries of the preimage lie in \( n \)-simplices \( \langle v_1, v_2, \ldots , v_{n+1} \rangle \) and \( \langle v_{n+2}, v_{n+3}, \ldots , v_{2n+3} \rangle \). Let

\[
(J_1, K_1) = (\partial \langle v_1, v_2, \ldots , v_{n+1}, v_{2n+3} \rangle, \partial \langle v_{n+2}, v_{n+3}, \ldots , v_{2n+2}, v_{2n+4} \rangle)
\]

and

\[
(J_2, K_2) = (\partial \langle v_1, v_2, \ldots , v_{n+1}, v_{2n+4} \rangle, \partial \langle v_{n+2}, v_{n+3}, \ldots , v_{2n+2}, v_{2n+3} \rangle).
\]

Then we have \( \ell k_2(f(J_i), f(K_i)) \equiv \ell k_2(g(J_i), g(K_i)) + 1 \pmod{2} \) for \( i = 1, 2 \) and \( \ell k_2(f(J), f(K)) \equiv \ell k_2(g(J), g(K)) \pmod{2} \) for \((J, K) \in \mathcal{L}^n, (J, K) \neq (J_1, K_1), (J_2, K_2)\) as unordered pair. This completes the proof. \( \square \)

**Lemma 2.2.** There is an embedding \( f : \sigma_{2n+3}^n \to S^{2n+1} \) with

\[
\sum_{(J, K) \in \mathcal{L}^n} \ell k_2(f(J), f(K)) \equiv 1 \pmod{2}.
\]

**Proof.** We use the fact that \( S^{2n+1} \) is homeomorphic to the join of two \( n \)-dimensional spheres, see Chapter 1 of [5]. Let \( P \) be the join of the two simplicial complicies \( J_0 = \partial \langle v_1, v_2, \ldots , v_{n+2} \rangle \) and \( K_0 = \partial \langle v_{n+3}, v_{n+4}, \ldots , v_{2n+4} \rangle \). Since \( \sigma_{2n+3}^n = \langle v_1, v_2, \ldots , v_{2n+4} \rangle \) is the join of \( \langle v_1, v_2, \ldots , v_{n+2} \rangle \) and \( \langle v_{n+3}, v_{n+4}, \ldots , v_{2n+4} \rangle \) we have that \( P \) is a subcomplex of \( \sigma_{2n+3}^n \). Then we have that \( \sigma_{2n+3}^n \) is a subcomplex of \( P \). Since \( P \) is homeomorphic to \( S^{2n+1} \) we have an embedding, say \( f \), of \( \sigma_{2n+3}^n \) into \( S^{2n+1} \). Let \((J, K) \in \mathcal{L}^n \). Then

\[
(J, K) = (\partial \langle v_{a_1}, v_{a_2}, \ldots , v_{a_{n+2}} \rangle, \partial \langle v_{b_1}, v_{b_2}, \ldots , v_{b_{n+2}} \rangle)
\]
for some \( \{a_1, a_2, \ldots, a_{n+2}\} \) and \( \{b_1, b_2, \ldots, b_{n+2}\} \). If \((J, K) \neq (J_0, K_0)\) as unordered pair then we have that the \((n+1)\)-simplices \( \langle v_{a_1}, v_{a_2}, \ldots, v_{a_{n+2}} \rangle \) and \( \langle v_{b_1}, v_{b_2}, \ldots, v_{b_{n+2}} \rangle \) are contained in \( P \). Therefore \( f(J) \) and \( f(K) \) bound disjoint \((n+1)\)-dimensional disks in \( S^{2n+1} \) and we have \( \ell k_2(f(J), f(K)) \equiv 0 \pmod{2} \). It is clear that \( \ell k_2(f(J_0), f(K_0)) \equiv 1 \pmod{2} \). This completes the proof. \( \square \)

Theorem 1.1 follows immediately from Lemma 2.1 and Lemma 2.2.

**Remark 2.3.** If we consider a general position map \( f : \sigma_{j+k+3}^k \to S^{j+k+1}_j \) for \( 0 \leq j \leq k \) and consider all pair \((J, K)\) of disjoint \( j\)-sphere and \( k\)-sphere in \( \sigma_{j+k+3}^k \), then we have a result that is a generalization of Lemma 2.1. The proof is essentially the same. However it turns out that the sum of \( \ell k_2 \) is zero whenever \( j < k \). In fact, for any finite simplicial complex \( Q \) and \( j < k \), there is a general position map \( f : Q \to S^{j+k+1}_j \) whose image is contained in the upper hemisphere and whose restriction to the \( j\)-skeleton of \( Q \) is an embedding into the equator \( S^{j+k} \subset S^{j+k+1}_j \). Then it is easy to see that \( \ell k_2(f(J), f(K)) = 0 \) for any pair \((J, K)\) of disjoint \( j\)-sphere and \( k\)-sphere in \( Q \).

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**References**


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