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Professor Dwork passed away on May 9, 1998 after a long illness. The manuscript was completed a few days earlier, and was submitted to the Pacific Journal of Mathematics following his express desire. It is a testimony to his dedication to mathematics even during his last illness - Managing Editor.

Part I.

Our object is to extend earlier work [**D1**] on singular hypersurfaces defined over an algebraic number field to singular hypersurfaces defined over function fields in characteristic zero.

A key role will be played by the results of Bertolin [**B1**] which in turn is based upon the Transfer Theorem of André–Baldassarri–Chiarellotto [**DGS**, Theorem VI 3.2].

Let $h(A,x)$ be the generic form of degree d in $n+1$ variables $x_1, \dots, x_{n+1}$. Thus letting $\mathcal{F}_0 = \left\{ u \in \mathbf{N}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} u_i = d \right\}$,

$$h(A,x) = \sum_{u \in \mathcal{F}_0} A_u x^u$$

where the symbols $\{A_u\}_{u \in \mathcal{F}_0}$ are algebraically independent over $\mathbf{Q}$.

Let $E_i = x_i \frac{\partial}{\partial x_i} (1 \le i \le n+1), h_i = E_i h$. Let $R(A)$ be the resultant of $\{h_1, h_2, \dots, h_n, h\}$.

Let V be an absolutely irreducible subvariety of the discriminant locus, $R(A) = 0$. Let k be the field of definition of V.

Let Ω be a suitable universal domain in characteristic zero, and let $\mathcal{L}^*_\Omega$ be the ring of all formal sums

$$\mathcal{L}^*_\Omega = \left\{ \xi^* = \sum_{u \in \mathcal{F}} C_u \frac{1}{x^u} \,\middle|\, C_u \in \pi^{u_0} \Omega \right\}$$

where $\mathcal{F} = \{u = (u_0, u_1, \dots, u_{n+1}) \mid du_0 = u_1 + \dots + u_{n+1}\}$ and where $\pi^{p-1} = -p$, p a rational prime. (Thus π need not be in Ω.)

For $\lambda \in V$, λ rational over Ω we write

$$D^*_{i,\lambda} = \gamma_- \circ (E_i + \pi x_0 h_i(\lambda, x)) \qquad 1 \le i \le n+1$$

an endomorphism of $\mathcal{L}_\Omega^*$ where γ_- is the projection operator

$$\gamma_- x^v = \begin{cases} 0 & \text{if any } v_i \geq 1 \\ x^v & \text{if all } v_i \leq 0. \end{cases}$$

For each integer ℓ, let $\mathcal{K}_\lambda^{(\ell)}$ be the set of all $\xi^* \in \mathcal{L}_\Omega^*$ such that ξ^* is annihilated by all monomials of degree ℓ in $\{D_{i,\lambda}^*\}_{1\leq i\leq n+1}$.

In the following, ord refers to a rank one valuation of Ω.

For $b \in \mathbf{R}, b > 0$, let $L^*(b) = \left\{\sum_{u\in\mathcal{F}} C_u \frac{1}{x^u} \;\middle|\; \inf_u (\operatorname{ord} C_u + u_0 b) > -\infty\right\}$.

Let Γ be an indeterminate and consider the polynomial $h(\lambda, x) + \Gamma h(A, x)$. Let $R(\lambda, \Gamma, A)$ be the resultant of

$$E_1(h(\lambda, x) + \Gamma h(A, x)), \ldots, E_{n+1}(h(\lambda, x) + \Gamma h(A, x))$$

and write

$$R(\lambda, \Gamma, A) = \Gamma^e(\rho_0(\lambda, A) + \Gamma\rho_1(\lambda, A) + \Gamma^2\rho_2(\lambda, A) + \cdots),$$

where $\rho_0(\lambda, A) \neq 0$. The key result of the research of Bertolin [**B1**, Theorem 3.11] states that:

Theorem 1.

$$\mathcal{K}_\lambda^{(\ell)} \subset L^*(\tau(n, d, e, \ell) \operatorname{ord} \rho_0(\lambda, A) + \varepsilon)$$

for all $\varepsilon > 0$. Here $\tau(n, d, e, \ell)$ depends only on n, d, e and ℓ and is independent of the coefficients of $h(\lambda, X)$.

Remark. Bertolin obtains estimates independent of ℓ. The estimate given here depends upon ℓ but is simpler to state. The slight error in [**B1**, Theorem 3.11] is corrected in [**B2**].

Corollary 1. *If $\lambda \in V$ and $\rho_0(\lambda, A) \neq 0$, then $\mathcal{K}_\lambda^{(\ell)} \subset L^*(\varepsilon)$ for all ℓ and all $\varepsilon > 0$ and for all but a finite set of valuations (depending on λ).*

Corollary 2. *For $\lambda \in V$ with $\rho_0(\lambda, A) \neq 0$, $\dim \mathcal{K}_\lambda^{(\ell)}$ is independent of λ.*

Proof. We choose a valuation v of $k(\lambda)$ such that (extending the valuation of $k(\lambda)$ to $k(\lambda, A)$ via the Gauss norm relative to A)

$$|\rho_0(\lambda, A)|_v = 1$$
$$|\lambda|_v \leq 1.$$

By the Lemma of Appendix B, we may choose a generic point λ' of V over k so close to λ v-adically that $|\lambda - \lambda'|_v < 1$ and hence $|\rho_0(\lambda', A)|_v = 1$. Thus $\mathcal{K}_\lambda^{(\ell)}$ and $\mathcal{K}_{\lambda'}^{(\ell)}$ lie in $L^*(\varepsilon)$ (v-adically) for all $\varepsilon > 0$ and hence $T_{\lambda,\lambda'} = \gamma_- \circ \exp \pi X_0(h(\lambda', x) - h(\lambda, x))$ is an isomorphism between $\mathcal{K}_\lambda^{(\ell)}$ and $\mathcal{K}_{\lambda'}^{(\ell)}$ as vector spaces over Ω. $\square$

Part II: Koszul complex.

In earlier work [**D1**, Theorem 19.2] we discussed the (cohomological) Koszul complex of $D^*_{1,\lambda^{(0)}}, \dots, D^*_{n+1,\lambda^{(0)}}$ operating on $\mathcal{K}^{(\infty)}_{\lambda^{(0)}} = \bigcup_{\ell=1}^{\infty} \mathcal{K}^{(\ell)}_{\lambda^{(0)}}$ where $\lambda^{(0)}$ is algebraic over $\mathbf{Q}$. We denote by $H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda^{(0)}})$ the s-th cohomology group of this complex. We showed:

Theorem 2.

$$\dim H^{(s)}\left(\mathcal{K}^{(\infty)}_{\lambda^{(0)}}\right) < \infty.$$

We also showed [**D1**, Theorem 17.1] that this dimension can be bounded in terms of d and n alone.

Note. Equation 19.4 of [**D1**] is stated without proof. This gap will be filled in Appendix A.

Corollary 3. *For $\lambda' \in V$, $\dim H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda'}) < \infty$ and if $\rho_0(\lambda', A) \neq 0$, then $\dim H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda'})$ is independent of λ'.*

Proof. We choose $\lambda^{(0)}$ algebraic over $\mathbf{Q}$ such that $\lambda^{(0)} \in V$ and $\rho_0(\lambda^{(0)}, A) \neq 0$. We choose a valuation v such that

$$\left|\rho_0\left(\lambda^{(0)}, A\right)\right|_v = 1, \quad \left|\lambda^{(0)}\right|_v \leq 1$$

and then choose a generic point λ of V over k in Ω as in the proof of Corollary 2. Then $T_{\lambda^{(0)},\lambda}$ provides an isomorphism of $\mathcal{K}^{(s)}_{\lambda^{(0)}}$ with $\mathcal{K}^{(s)}_{\lambda}$ which induces an isomorphism of $H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda^{(0)}})$ with $H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda})$ for all s. This shows finiteness for λ generic.

If $\rho_0(\lambda', A) \neq 0$, then by the same argument choosing λ generic close to λ' we conclude that $\dim H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda'}) = \dim H^{(s)}(\mathcal{K}^{(\infty)}_{\lambda})$. If $\rho_0(\lambda', A) = 0$, then λ' lies in a proper subvariety of V and we may use induction on the dimension. □

Notation. For $B = \{1, 2, \dots, n+1\}$ and W a vector space over $k(\lambda)$ let $\mathcal{F}_s(W) = \mathrm{Hom}\left(\bigwedge^s B, W\right)$.

Corollary 4. *For ℓ large enough (depending upon V) and λ a generic point of V over k,*

$$H^s\left(\mathcal{K}^{(\infty)}_{\lambda}\right) \simeq \ker\left(\delta^*_{s+1,\lambda}, \mathcal{F}_s\left(\mathcal{K}^{(\ell)}_{\lambda}\right)\right) \Big/ \left(\mathcal{F}_s\left(\mathcal{K}^{(\ell)}_{\lambda}\right) \cap \delta^*_{s,\lambda}\mathcal{F}_{s-1}\left(\mathcal{K}^{(\infty)}_{\lambda}\right)\right).$$

(*For definition of δ^*_s see* [**D1**].)

Proof. There is a natural injection of the right hand space into the left–hand one induced by the inclusion $\mathcal{K}^{(\ell)}_{\lambda} \hookrightarrow \mathcal{K}^{(\infty)}_{\lambda}$. The left–hand space is of finite dimension and so the mapping is surjective. □

We now give $H^{(s)}(\mathcal{K}_\lambda^{(\infty)})$ the structure of a *differential module* when viewed as a vector space over $k(\lambda)$. Let $\lambda_1, \dots, \lambda_t$ be a transcendence basis over k of $k(\lambda)$. Viewing $\lambda_{t+1}, \lambda_{t+2}$ etc. as dependent variables we define for $1 \leq i \leq t$

$$\sigma_i^* = \gamma_- \circ \left(\frac{\partial}{\partial \lambda_i} - \pi x_0 \frac{\partial h}{\partial \lambda_i} \right).$$

These operators commute with $\{D_{j,\lambda}^*\}_{1 \leq j \leq n+1}$ and hence induce a set of commuting operators on $H^{(s)}(\mathcal{K}_\lambda^{(\infty)})$. If $\lambda^{(1)} \in V$, then horizontal elements are obtained by applying $T_{\lambda^{(1)},\lambda}$ to $H^{(s)}(\mathcal{K}_{\lambda^{(1)}}^{(\infty)})$.

Theorem 3. *Let $\lambda^{(1)}$ be a generic point of V. We consider all extensions to $k(\lambda^{(1)})$ of valuations of k whose restriction to $k(\lambda_1^{(1)}, \dots, \lambda_t^{(1)})$ is given by the Gauss norm of that field relative to $\lambda_1^{(1)}, \dots, \lambda_t^{(1)}$. For almost all such valuations the horizontal elements converge for $|(\lambda_1, \dots, \lambda_t) - (\lambda_1^{(1)}, \dots, \lambda_t^{(1)})| < 1$.*

Corollary 5. *If k is an algebraic number field, then $H^{(s)}(\mathcal{K}_\lambda^{(\infty)})$ is a G-module.*

Appendix A.

Let k be a field of characteristic zero and let $f(x_1, \dots, x_{n+1})$ be a form of degree d in $n+1$ variables. If Ω is an extension of k, let us write $\mathcal{L}_\Omega$ for the ring of all *polynomials* in $x_0, x_1, \dots, x_{n+1}$ of the form

$$\left\{ \sum_{du_0 = u_1 + \cdots + u_{n+1}} C_u \pi^{u_0} x^u \;\middle|\; C_u \in \Omega \right\}.$$

We define $D_i = E_i + \pi x_0 f_i$, $E_i = x_i \frac{\partial}{\partial x_i}$, $f_i = E_i f$. The D_i are commuting endomorphisms of $\mathcal{L}_\Omega$ and likewise by restricting to $\mathcal{L}_k$ we obtain commuting endomorphisms of that ring.

Let $\mathcal{L}_\Omega^*$ (resp: $\mathcal{L}_k^*$) be the adjoint space of $\mathcal{L}_\Omega$ (resp: $\mathcal{L}_k$) and $\mathcal{K}_\Omega^{(\ell)}$ (resp: $\mathcal{K}_k^{(\ell)}$) the set of all $\xi^* \in \mathcal{L}_\Omega^*$ (resp: $\mathcal{L}_k^*$) annihilated by all forms in $\{D_1^*, D_2^*, ..., D_{n+1}^*\}$ of degree ℓ, where $D_i^* = \gamma_- \circ (-E_i + \pi x_0 f_i)$.

Again let $\mathcal{K}_\Omega^{(\infty)}$ (resp: $\mathcal{K}_k^{(\infty)}$) be the union $\bigcup_{\ell=1}^{\infty} \mathcal{K}_\Omega^{(\ell)}$ $\left(\text{resp} : \bigcup_{\ell=1}^{\infty} K_k^{(\ell)}\right)$.

Finally we define $H^{(s)}(\mathcal{K}_\Omega^{(\infty)})$ (resp: $H^{(s)}(\mathcal{K}_k^{(\infty)})$) to be the s-th cohomology group of the (cohomological) Koszul complex of $D_1^*, \dots, D_{n+1}^*$ operating on $\mathcal{K}_\Omega^{(\infty)}$ (resp: $\mathcal{K}_k^{(\infty)}$).

Theorem.

(i) $\mathcal{K}_\Omega^{(\infty)} = \mathcal{K}_k^{(\infty)} \otimes \Omega$

(ii) $H^{(s)}(\mathcal{K}_\Omega^{(\infty)}) = H^{(s)}(\mathcal{K}_k^{(\infty)}) \otimes \Omega$.

This was stated without proof as Equation (19.4) of [**D1**].

Proof. We first show for $\ell < \infty$

(iii) $\mathcal{K}_\Omega^{(\ell)} = \mathcal{K}_k^{(\ell)} \otimes \Omega$.

We know [**D1**, Lemma 7.2] that $\dim_k \mathcal{K}_k^{(\ell)} < \infty, \dim_\Omega \mathcal{K}_\Omega^{(\ell)} < \infty$. We may view each element of $\mathcal{K}_\Omega^{(\ell)}$ as an ∞–tuple $(z_1, z_2, \dots)$ indexed by a countable set I. Indeed $\xi^* \in \mathcal{K}_\Omega^{(\ell)}$ implies $\xi^* = \sum_u C_u/x^u$. The sum being over all u such that $du_0 = u_1 + \cdots + u_{n+1}$. Here $C_u = \pi^{u_0}\overline{C}_u$ with $\overline{C}_u \in \Omega$. Identifying the $\{\overline{C}_u\}$ with the $\{z_i\}$, the condition that $\xi^* \in \mathcal{K}_\Omega^{(\ell)}$ is equivalent to an infinite set of conditions

$$\sum t_{j,i} z_i = 0 \quad \text{for all } j \in J.$$

Here $t_{j,i} \in k$, $t_{j,i} = 0$ for almost all i, for each fixed j. For $\xi^* \in \mathcal{K}_k^{(\ell)}$ we have the same set of conditions. Following a suggestion by Wan, by elementary operations on the rows of the matrix $\{t_{j,i}\}$ the finite dimension of the subspace is given by the number of zero columns in the reduced echelon form. The echelon form is the same for the equation over Ω as over k. It follows that indeed $\dim_k \mathcal{K}_k^{(\ell)} < \infty \Leftrightarrow \dim \mathcal{K}_\Omega^{(\ell)} < \infty$ and both are then equal and $\mathcal{K}_\Omega^{(\ell)} = \mathcal{K}_k^{(\ell)} \otimes \Omega$. The first assertion now follows.

For a vector space W we write $\mathcal{F}_s(W) = \operatorname{Hom}(\bigwedge^s B, W)$ with $B = \{1, 2, \dots, n+1\}$. Then $\xi^* \in \mathcal{F}_s(\mathcal{K}_\Omega^{(\infty)})$ implies $\xi^* = \sum \eta_i \xi_i^*$ a finite sum with $\xi_i^* \in \mathcal{F}_s(\mathcal{K}_k^{(\infty)})$ and $\{\eta_i\}$ a finite set of elements of Ω linearly independent over k.

If $\delta_{s+1}^* \xi^* = 0$ then by linear independence $\delta_{s+1}^* \xi_i^* = 0$ and so $\ker(\delta_{s+1}^*, \mathcal{F}_s(\mathcal{K}_\Omega^{(\infty)})) = \ker(\delta_{s+1}^*, \mathcal{F}_s(\mathcal{K}_k^{(\infty)})) \otimes \Omega$.

Also

$$\delta_s^* \mathcal{F}_{s-1}\left(\mathcal{K}_\Omega^{(\infty)}\right) = \delta_s^* \mathcal{F}_{s-1}\left(\mathcal{K}_k^{(\infty)}\right) \otimes \Omega.$$

The theorem now follows from the following well known proposition. □

Proposition. *Let U be a subspace of a linear k space W. Then*

$$W \otimes \Omega / U \otimes \Omega \simeq (W/U) \otimes \Omega\,.$$

Appendix B. Approximation by generic points.

Lemma. *Let the origin O be on an irreducible affine variety V defined over a field k of characteristic zero. Let Ω be a universal domain complete under a rank one valuation. Then there exists a generic point of V rational over Ω which is as close as you please to the origin.*

We first show the lemma holds if V is a curve in $\mathbb{A}^n$.

Proof. Let $P = (x_1, \ldots, x_n)$ be a generic point of V over k. Then $R = k[x_1, \ldots, x_n]$ has a specialization into k given by $(x_1, \ldots, x_n) \mapsto O$ and hence there exists a place $\mathfrak{p}$ of $k(V)$ with center O. Letting T be a uniformizing parameter of $\mathfrak{p}$, each coordinate x_i as element of $k(V)_{\mathfrak{p}}$, the completion at $\mathfrak{p}$ of $k(V)$, is represented as a power series

$$x_i = a_{i1}T + a_{i2}T^2 + \cdots + \in k'[[T]]$$

where k' is the residue class field at $\mathfrak{p}$ of $k(V)$, a finite extension of k. This series may have zero radius of convergence in the metric of Ω, but if we choose (as we shall) the uniformizing parameter, T, in $k(V)$ then the series represents an algebraic function of T and hence by Eisenstein's Theorem (or more elementarily by Clark's Theorem) the series has a non-trivial radius of convergence.

Since P is a generic point, these series are not all constant. We think of $P(T)$ as function of T for T restricted to a small disk $D(0, r^-)$ in Ω–space. Trivially $P(T) \to 0$ as $T \to 0$. We may suppose x_1 is a non–constant function of T. The theory of Newton polygons shows that the image of $D(0, r^-)$ under x_1 contains elements transcendental over k. This completes the proof for $\dim V = 1$.

We recall [**H**, Chapter I, Proposition 7.1]: □

Proposition. *If V is irreducible of dimension s in $\mathbb{A}^n$ and H is a hypersurface not containing V then each irreducible component of $H \cap V$ has dimension $s - 1$.*

Proof of Lemma. Letting $V_0 = V$ we define inductively $V_1 \supseteq V_2 \supseteq \cdots$ by the condition that V_j be an irreducible component of $V_{j-1} \cap \{x \mid x_j = 0\}$ which contains the origin. Since

$$-1 + \dim V_{j-1} \leq \dim V_j \leq \dim V_{j-1},\ \dim V_n = \{0\}$$

there exists j such that V_j is a curve on V passing through the origin. □

We conclude there exists a curve V' on V passing through the origin. Let $k' \supset k$ be a field of definition of V'. By our previous treatment of curves there exists $P \in V'$, P as close as you please to O such that $k'(P)$ is of transcendence degree unity over k'. Let P_1 be a coordinate of P of transcendence degree unity over k'.

Let $L = k(P_1)$, $\mathfrak{A}$ be the ideal of all $f \in k[x_1, \ldots, x_n]$ which are zero everywhere on V. If $g \in L[x_1, \ldots, x_n]$, $g = 0$ on V then $g \in \mathfrak{A}L[x]$ and hence for each automorphism τ of L/k we have $g^\tau = 0$ on V. In particular $x_1 - P_1$ cannot be zero on V as otherwise $(x_1 - P_1)^\tau$ would also be zero on V and hence $P_1 - P_1^\tau$ would be zero on V for every τ which is impossible as there are nontrivial automorphisms of L/k.

Thus V does not lie in the hyperplane $x_1 = P_1$ and so the intersection has an irreducible component W passing through P of dimension $s-1$. Let k'' be a field of definition of W, $P \in W$. By induction there exists $Q \in W$, $Q - P$ as small as you please with transc deg $k''(Q)/k'' = s - 1$.

Clearly Q is as close as you please to O. It remains to show that $s =$ trans deg $k(Q)/k$.

Since $Q_1 = P_1$, $k(Q) \supset k(P_1)$. Hence

$$\begin{aligned} s \geq \text{ trans deg } k(Q)/k = & \text{ trans deg } k(Q)/k(P_1) + \text{ trans deg } k(P_1)/k \\ & \geq \text{ trans deg } k^{''}(Q)/k^{''} + \text{trans deg } k'(P_1)/k' \\ & \geq (s-1) + 1 = s, \end{aligned}$$

the two inequalities being based on

$$\text{if } k' \supset k \text{ then trans deg } k(P_1)/k \geq \text{ trans deg } k'(P_1)/k'$$

$$\text{if } k'' \supset k(P_1) \text{ then trans deg } k(Q)/k(P_1) \geq \text{trans deg } k^{''}(Q)/k^{''}.$$

This completes the proof of the lemma.

Appendix C: (Generalization of Heaviside's generalized exponential functions).

In this article we examined the Koszul complex of $\{D^*_{1,\lambda}, \dots, D^*_{n+1,\lambda}\}$ operating on $\mathcal{K}^{(\infty)}_\lambda$. In this appendix, we replace $\mathcal{L}^*$ by

$$\mathcal{L}'^* = \left\{ \sum_{n \in \mathcal{F}'} A_u \frac{1}{x^u} \;\middle|\; A_u \in \pi^{u_0}\Omega \right\}$$

and $D^*_{i,\lambda} = \gamma_- \circ (-E_i + \pi x_0 h_i(x_1, x))$ by $D'^*_{i,\lambda} = -E_i + \pi x_0 h_i(\lambda, x)$. Here

$$\mathcal{F}' = \{(u_0, u_1, \dots, u_{n+1}) \big| \in \mathbf{Z}^{n+2} \big| du_0 = u_1 + \cdots + u_{n+1}\}.$$

Thus $\mathcal{L}'^*$ consists of formal Laurent series in $\{x_i, \frac{1}{x_i}\}$ $\quad i = 1, \dots, n+1$. We note that $\mathcal{L}'^*$ is adjoint to $\mathcal{L}'$, the ring of Laurent polynomials with support in

$$du_0 = u_1 + \cdots + u_{n+1}.$$

Let $\mathcal{D}^s$ denote the ideal of all forms of degree s in $D_{1,\lambda}, \dots, D_{n+1,\lambda}$ with coefficients in $k(\lambda)$. We assert that

$$\mathcal{L}' = \mathcal{L} + \mathcal{D}^s \mathcal{L}'.$$

For $s = 1$ this follows by the proof of [**D2**, Lemma 9.7.1]. Assume the formula valid for some given s then $\mathcal{L}' = \mathcal{L} + \mathcal{D}^s(\mathcal{L} + \mathcal{D}\mathcal{L}') \subset \mathcal{L} + \mathcal{D}^{s+1}\mathcal{L}'$, which completes the proof by induction.

This shows that the natural mapping of $\mathcal{L}$ into $\mathcal{L}'$ induces a surjection $\mathcal{L}/\mathcal{D}^s\mathcal{L} \to \mathcal{L}'/\mathcal{D}^s\mathcal{L}'$. We recall that $\mathcal{K}^{(s)}$ denotes the annihilator in $\mathcal{L}^*$ of $\mathcal{D}^s\mathcal{L}$. Let $\mathcal{K}'^{(s)}$ denote the annihilator in $\mathcal{L}'^*$ of $\mathcal{D}^s\mathcal{L}'$. We now know that the dimension of $\mathcal{L}'/\mathcal{D}^s\mathcal{L}'$ is finite and hence the same holds for $\mathcal{K}'^{(s)}$. Thus by duality the mapping of $\mathcal{K}'^{(s)}$ into $\mathcal{K}^{(s)}$ adjoint to the natural mapping is injective. This adjoint mapping is γ_-.

Conclusion. The mapping γ_- maps $\mathcal{K}'^{(s)}$ into $\mathcal{K}^{(s)}$ injectively.

We now restrict our attention to the case where $h(\lambda, x) \in \mathbf{Q}[\lambda, x]$ and consider $\lambda^{(0)}$ algebraic over $\mathbf{Q}$. For $b > 0$, $c \in \mathbf{R}$, w a finite valuation of Ω, let $L'^*(b, c)$ be the set of all formal Laurent series $\xi^* = \sum_{u \in \mathcal{F}'} B_u \frac{1}{x^u}$ such that $B_u \in \Omega$, and $\operatorname{ord}(B_{u-v}) \geq -b(u_0 + v_0) + c$ for all $u, v, \in \mathcal{F}'$. Let $L'^*(b) = \bigcup_{c \in \mathbf{R}} L'^*(b, c)$, a Banach space. For almost all valuations of $\mathbf{Q}(\lambda^{(0)})$ we have a completely continuous mapping of $L'^*(b')$ (giving Ω a valuation extending that of $\mathbf{Q}(\lambda^{(0)})$) defined by putting $F(x) = \exp \pi(x_0 h(\lambda^{(0)}, x) - x_0^q h(\lambda^{(0)}, x^q))$ where q is the order of the residue class field of $\mathbf{Q}(\lambda^{(0)})$ and writing

$$\alpha'^* = F \circ \phi, \qquad \phi : x^u \to x^{qu}$$

$$L'^*(b') \xrightarrow{\phi} L'^*(b'/q) \xrightarrow{F} L'^*(b'/q) \hookrightarrow L'^*(b')$$

where b' is chosen in $[0, \frac{p-1}{p}]$. Here F means multiplication by F and the last map is the inclusion map.

Letting $L^*(b) = \gamma_- L'^*(b)$ we have the completely continuous endomorphism of $L^*(b')$ (for almost all valuations of $\mathbf{Q}(\lambda^{(0)})$) given by

$$\alpha^* = \gamma_- \circ F \circ \phi\ .$$

By the trace formula the two mappings have the same Fredholm determinant. Defining

$$W_z^* = \bigcup_k \ker\Big((I - z\alpha^*)^k, L^*(b')\Big)$$

$$W_z'^* = \bigcup_k \ker\Big((I - z\alpha'^*)^k, L'^*(b')\Big)$$

we conclude equality of dimensions and hence

$$\gamma_- W_z'^* = W_z^*.$$

Now $\mathcal{K}_{\lambda^{(0)}}^{(\ell)}$ is covered by a union of spaces W_z^* and hence by γ_- (finite union of spaces $W_z'^*$) which lies in $\gamma_- \mathcal{K}_{\lambda^{(0)}}'^{(\ell')}$ for suitable ℓ'.

Conclusion. γ_- gives a bijection of $\mathcal{K}_{\lambda^{(0)}}'^{(\infty)}$ onto $\mathcal{K}_{\lambda^{(0)}}^{(\infty)}$ provided $\lambda^{(0)}$ is algebraic over $\mathbf{Q}$.

We propose to remove the restriction that $\lambda^{(0)}$ be algebraic. Again let $\lambda^{(0)} \in V$ be an algebraic point, $\rho_0(\lambda^{(0)}, A) \neq 0$. Excluding a finite set of primes of $\mathbf{Q}(\lambda^{(0)})$ we choose λ generic point of V close to $\lambda^{(0)}$.

If $\xi^* \in \mathcal{K}_{\lambda^{(0)}}'^{(\infty)}$, then $\gamma_- \xi^* \in \mathcal{K}_{\lambda^{(0)}}^{(\ell)}$ for some ℓ and hence $\gamma_- \xi^*$ is a finite sum of elements of spaces W_z^* and hence is the image under γ_- of a finite sum of elements of spaces W'^*_z. But γ_- is injective, and hence ξ^* is a sum of elements of spaces W'^*_z. Thus for almost all valuations of $\mathbf{Q}(\lambda^{(0)})$, $\xi^* \in L'^*(b')$, $b' < \frac{p-1}{p}$. More precisely $\mathcal{K}'^{(\infty)}_{\lambda^{(0)}}$ lies in $L'^*(b')$ for all $b' > 0$ and almost all primes of $\mathbf{Q}$. Hence for almost all primes multiplication by $\exp \pi x_0 \left(h(\lambda, x) - h(\lambda^{(0)}, x)\right)$ provides an isomorphism $T'_{\lambda^{(0)},\lambda}$ of $\mathcal{K}'^{(\infty)}_{\lambda^{(0)}}$ with $\mathcal{K}'^{(\infty)}_{\lambda}$. On the other hand, $T_{\lambda^{(0)},\lambda} = \gamma_- \circ T'_{\lambda^{(0)},\lambda}$ gives an isomorphism between $\mathcal{K}^{(\infty)}_{\lambda^{(0)}}$ and $\mathcal{K}^{(\infty)}_{\lambda}$

$$\begin{array}{ccc} \mathcal{K}'^{(\infty)}_{\lambda^{(0)}} & \xrightarrow{T'_{\lambda^{(0)},\lambda}} & \mathcal{K}'^{(\infty)}_{\lambda} \\ \downarrow \gamma_- & & \downarrow \gamma_- \\ \mathcal{K}^{(\infty)}_{\lambda^{(0)}} & \xrightarrow[T_{\lambda^{(0)},\lambda}]{} & \mathcal{K}^{(\infty)}_{\lambda} \end{array} .$$

The horizontal arrows of this commutative diagram are isomorphisms. The first vertical arrow is also an isomorphism. It follows that the second vertical arrow is also an isomorphism. This completes the proof.

Note. The purpose of the argument involving $\lambda^{(0)}$ is to show that γ_- is injective on $\mathcal{K}'^{(\infty)}_{\lambda}$.

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