RIEMANNIAN MANIFOLDS ADMITTING ISOMETRIC IMMERSIONS BY THEIR FIRST EIGENFUNCTIONS

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Given a compact manifold $M$, we prove that every critical Riemannian metric $g$ for the functional “first eigenvalue of the Laplacian” is $\lambda_1$-minimal (i.e., $(M, g)$ can be immersed isometrically in a sphere by its first eigenfunctions) and give a sufficient condition for a $\lambda_1$-minimal metric to be critical. In the second part, we consider the case where $M$ is the 2-dimensional torus and prove that the flat metrics corresponding to square and equilateral lattices of $\mathbb{R}^2$ are the only $\lambda_1$-minimal and the only critical ones.

Introduction.

Many recent works concerning the spectrum of compact Riemannian manifolds have pointed out the importance of a particular class of Riemannian metrics which we called in [5] $\lambda_1$-minimal. Recall that a metric $g$ on a compact $m$-dimensional manifold $M$ is $\lambda_1$-minimal if the eigenspace $E_1(g)$ associated to the first nonzero eigenvalue $\lambda_1(g)$ of the Laplacian of $g$ contains a family $f_1, \ldots, f_k$ of functions satisfying: $\sum_{1 \leq i \leq k} df_i \otimes df_i = g$. It follows from a well known result of Takahashi [8] that this last condition is equivalent to the fact that the map $f = (f_1, \ldots, f_k)$ is a minimal isometric immersion from $(M, g)$ into the Euclidean sphere $S^{k-1}_r$ of radius $r = \sqrt{\frac{m}{\lambda_1(g)}}$.

The best known examples of $\lambda_1$-minimal metrics are the standard metrics of rank one compact symmetric spaces (i.e., spheres and projective spaces). More generally, any Riemannian irreducible homogeneous space is $\lambda_1$-minimal. Also, Yau [9] conjectured that a minimal embedded hypersurface of a Euclidean sphere, carrying the induced metric, must be $\lambda_1$-minimal.

In [2], Berger showed that the $\lambda_1$-minimality of a metric $g$ is strongly related to the extremality of $g$ for a spectral functional involving the $k$-smallest eigenvalues of the Laplacian (where $k$ is the multiplicity of $\lambda_1(g)$). Recently, Nadirashvili [7] considered the functional $\lambda_1 : g \mapsto \lambda_1(g)$ defined on the set of Riemannian metrics of given area on a compact surface $M$ and showed that the extremal metrics of this functional are $\lambda_1$-minimal (here extremality is defined in a generalized sense because of the non-differentiability of $\lambda_1$).
In the first part of this paper we generalize Nadirashvili’s theorem to higher dimensions (Theorem 1.1). We also give a sufficient condition for a $\lambda_1$-minimal metric to be extremal for $\lambda_1$ (Proposition 1.1).

Using results established by us in [4] about $\lambda_1$-minimal metrics we deduce that (Corollary 1.1), if $g$ is an extremal metric of the $\lambda_1$ functional then:

(i) The multiplicity of $\lambda_1(g)$ is at least equal to $m + 1$ and equality holds only for the standard metric of Euclidean spheres.

(ii) The restriction of the $\lambda_1$ functional to the conformal class of $g$ achieves its maximum at $g$. In particular, the $\lambda_1$ functional has no local minima.

(iii) The metric $g$ is, up to dilatation, the unique extremal metric in its conformal class.

(iv) If $g$ is not isometric to the standard metric of a Euclidean sphere then any conformal diffeomorphism of $(M, g)$ is an isometry.

The second part of this paper deals with the classification of $\lambda_1$-minimal metrics and of the extremal metrics of the $\lambda_1$ functional. The only manifold for which this classification was available is the 2-dimensional sphere. Indeed, on $S^2$ the standard metric is (up to dilatation) the only one to be $\lambda_1$-minimal and the only extremal metric for $\lambda_1$ (this follows from the uniqueness of the conformal class on $S^2$ and property (iii) above).

The main theorem of Section 2 (Theorem 2.1) states that in genus one (i.e., on the torus $T^2$) there exists, up to dilatation, exactly two $\lambda_1$-minimal metrics: The Clifford metric $g_{cl}$ and the equilateral metric $g_{eq}$ induced from the Euclidean metric respectively on $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{R}^2/\Gamma_{eq}$ with $\Gamma_{eq} = \mathbb{Z}(1, 0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)$. These two metrics are also the only extremal metrics for $\lambda_1$ (Corollary 2.2). Moreover, we prove that for each of them, the standard embedding (in $S^3$ for $g_{cl}$ and $S^5$ for $g_{eq}$) is, up to equivalence, the only full (minimal) isometric immersion by the first eigenfunctions.

Note that a first step towards this classification was achieved by Montiel and Ros [6] who proved that the only minimal torus immersed in $S^3$ by its first eigenfunctions is the Clifford torus. They deduced that if the aforementioned conjecture of Yau is true, then the Clifford torus is the only minimally embedded torus in $S^3$ (Lawson’s conjecture).

1. Extremal metrics for the $\lambda_1$ functional.

Let $M$ be a compact smooth manifold of dimension $m \geq 2$. Denote by $\mathcal{R}_0(M)$ the set of Riemannian metrics of volume 1 on $M$. For any $g \in \mathcal{R}_0(M)$, we denote by $0 < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_k(g) \leq \cdots$ the increasing sequence of eigenvalues of the Laplacian $\Delta_g$ of $g$. The functional:

$$
\lambda_1 : \mathcal{R}_0(M) \rightarrow \mathbb{R} \\
g \mapsto \lambda_1(g)
$$
is continuous but not differentiable in general. However, for any family 
\((g_t)_t\) of metrics, analytic in \(t\), \(\lambda_1(g_t)\) has right and left derivatives w.r.t. \(t\). Indeed, if 
\((g_t)_{t \in \mathbb{R}}\) is such a family and if \(k\) is the multiplicity of \(\lambda_1(g_0)\), 
then there exists \(k\) analytic families \(\Lambda_{1,t}, \ldots, \Lambda_{k,t}\) of real numbers and \(k\) 
analytic families of smooth functions \(u_{1,t}, \ldots, u_{k,t}\) such that: \(\forall i \leq k\) and \(\forall t\), 
\(\Delta_u u_{i,t} = \Lambda_{i,t} u_{i,t}\), \(\Lambda_{i,0} = \lambda_1(g_0)\) and \(\{u_{1,t}, \ldots, u_{k,t}\}\) is \(L_2(g_t)\)-orthonormal 
(see [1] and [2] for details). Moreover, Berger [2] gave the following formula 
for the derivative of \(\Lambda_{i,t}\): 

\[
\frac{d}{dt} \Lambda_{i,t} \bigg|_{t=0} = - \int_M \langle q(u_i), h \rangle \nu_{g_0},
\]

where \(\nu_{g_0}\) is the Riemannian volume element of \(g_0\), \(u_i = u_{i,0}\), \(h = \frac{d}{dt} g_t \big|_{t=0}\), 
\(\langle , \rangle\) is the inner product induced by \(g_0\) on the space \(S^2(M)\) of symmetric 
covariant 2-tensors of \(M\) and where for any \(u \in C^\infty(M)\), 

\[
q(u) = du \otimes du + \frac{1}{4} \Delta_{g_0}(u^2) g_0.
\]

From the continuity of \(\lambda_i(g_t)\) and \(\Lambda_{i,t}\) w.r.t. \(t\), we have for \(t\) small enough 
\(\{\Lambda_{i,t}\}_{1 \leq i \leq k} = \{\lambda_i(g_t)\}_{1 \leq i \leq k}\) and thus \(\lambda_1(g_t) = \min_{1 \leq i \leq k} \{\Lambda_{i,t}\}\). This proves 
the left and right differentiability of \(\lambda_1(g_t)\) and gives: 

\[
\frac{d}{dt} \lambda_1(g_t) \bigg|_{t=0^+} = \min_{1 \leq i \leq k} \frac{d}{dt} \Lambda_{i,t} \bigg|_{t=0} = - \max_{1 \leq i \leq k} \int_M \langle q(u_i), h \rangle \nu_{g_0},
\]

and 

\[
\frac{d}{dt} \lambda_1(g_t) \bigg|_{t=0^-} = \max_{1 \leq i \leq k} \frac{d}{dt} \Lambda_{i,t} \bigg|_{t=0} = - \min_{1 \leq i \leq k} \int_M \langle q(u_i), h \rangle \nu_{g_0}.
\]

This suggests the following definition:

**Definition 1.1.** A metric \(g \in \mathcal{R}_0(M)\) is said to be extremal for the \(\lambda_1\) 
functional if for any analytic deformation \((g_t)_t \subset \mathcal{R}_0(M)\), with \(g_0 = g\), the 
left and right derivatives of \(\lambda_1(g_t)\) at \(t = 0\) have opposite signs, i.e., 

\[
\frac{d}{dt} \lambda_1(g_t) \bigg|_{t=0^+} \leq 0 \leq \frac{d}{dt} \lambda_1(g_t) \bigg|_{t=0^-}.
\]

This last condition is equivalent to: 

\[
\lambda_1(g_t) \leq \lambda_1(g) + o(t) \quad \text{as} \quad t \to 0.
\]

Hence our definition of extremality is a equivalent formulation of Nadirashvili’s one [7].

The main result of this section is:

**Theorem 1.1.** If a Riemannian metric \(g \in \mathcal{R}_0(M)\) is extremal for \(\lambda_1\) then 
it is \(\lambda_1\)-minimal.
In the 2-dimensional case this result was proved by Nadirashvili [7]. Some of the arguments in our proof are inspired by his. However, the use of the aforementioned result of Berger makes the proof of this theorem simpler and more transparent.

**Lemma 1.1.** If a metric \( g \in \mathcal{R}_0(M) \) is extremal for \( \lambda_1 \) then for any \( h \in S_0^2(M) = \{ h \in S^2(M); \int_M tr_g h \nu_g = 0 \} \) there exists \( u \in E_1(g) \setminus \{0\} \) such that:

\[
\int_M \langle q(u), h \rangle \nu_g = 0.
\]

**Proof.** Suppose that \( g \) is extremal for \( \lambda_1 \) and let \( h \in S_0^2(M) \). We let, for small \( t \), \( g_t = g + th \in \mathcal{R}_0(M) \), with\( V(g + th) \) the Riemannian volume of \( g + th \). Since \( \frac{d}{dt} V(g + th)|_{t=0} = \frac{1}{2} \int_M tr_g h \nu_g = 0 \), we find \( \frac{d}{dt} g_t|_{t=0} = h \). The extremality condition implies that the quadratic form \( u \in E_1(g) \mapsto \int_M \langle q(u), h \rangle \nu_g \) takes on both nonpositive and nonnegative values, and therefore it admits at least one isotropic direction. \( \square \)

**Proof of Theorem 1.1.** Let \( K \) be the convex hull in \( S^2(M) \) of \( \{ q(u), u \in E_1(g) \} \). The set \( K \cup \{ g \} \) is contained in a finite dimensional subspace of \( S^2(M) \). We claim that \( g \in K \). Indeed, if \( g \notin K \) then, since \( K \) is a convex cone, the Hahn-Banach theorem implies the existence of \( s \in S^2(M) \) such that:

\[
\int_M \langle s, g \rangle \nu_g > 0 \quad \text{and for every } l \in K \setminus \{0\}, \int_M \langle l, s \rangle \nu_g < 0.
\]

The 2-tensor \( \bar{s} = s - \frac{(\int_M \langle s, g \rangle \nu_g)}{mV(g)} g \) belongs to \( S_0^2(M) \) and, for any \( u \in E_1(g) \setminus \{0\} \),

\[
\int_M \langle q(u), \bar{s} \rangle \nu_g = \int_M \langle q(u), s \rangle \nu_g - \frac{1}{mV(g)} \left( \int_M \langle s, g \rangle \nu_g \right) \left( \int_M |du|^2 \nu_g \right) < 0.
\]

By Lemma 1.1, this contradicts the extremality of \( g \).

Thus \( g \in K \) and there exists \( w_1, \ldots, w_d \in E_1(g) \) such that:

\[
g = \sum_{1 \leq i \leq d} q(w_i) = \sum_{1 \leq i \leq d} dw_i \otimes dw_i + \frac{1}{4} \left( \sum_{1 \leq i \leq d} \Delta w_i^2 \right) g
\]

\[
= \sum_{1 \leq i \leq d} \left( dw_i \otimes dw_i + \frac{1}{2} \left( \lambda_1(g)w_i^2 - |dw_i|^2 \right) g \right).
\]
The traceless part of the last member of this equation must be zero. Therefore,

\[ \sum_{1 \leq i \leq d} \left( dw_i \otimes dw_i - \frac{|dw_i|^2}{m} g \right) = 0, \]

and then:

\[ \frac{\lambda_1}{2} \sum_{1 \leq i \leq d} w_i^2 = 1 + \left( \frac{m-2}{2m} \right) \sum_{1 \leq i \leq d} |dw_i|^2. \]

The \( \lambda_1 \)-minimality of \( g \) will follow from the fact that \( \sum_{1 \leq i \leq d} |dw_i|^2 \) is constant and equal to \( m \). Indeed, set \( f = \left( \sum_{1 \leq i \leq d} w_i^2 \right) - \frac{m}{\lambda_1(g)} \). From (1) we get:

\[ (m-2) \Delta_g f = 2(m-2) \left( \lambda_1(g) \left( \sum_{1 \leq i \leq d} w_i^2 \right) - \sum_{1 \leq i \leq d} |dw_i|^2 \right) = -4\lambda_1(g)f. \]

This implies that \( f = 0 \) (the Laplacian being a positive operator). Therefore \( \left( \sum_{1 \leq i \leq d} w_i^2 \right) = \frac{m}{\lambda_1(g)} \). Replacing in (1) we obtain \( \sum_{1 \leq i \leq d} |dw_i|^2 = m. \)

In [4] we showed that \( \lambda_1 \)-minimal metrics satisfy certain remarkable conformal properties. Theorem 1.1 tells us that all these properties are still true for extremal metrics:

**Corollary 1.1.** Let \( g \in \mathcal{R}_0(M) \) be an extremal metric for \( \lambda_1 \).

(i) The multiplicity of \( \lambda_1(g) \) satisfies: \( \text{mult}(\lambda_1(g)) \geq m+1 \), where equality holds if and only if \( g \) is isometric to a standard metric of a Euclidean sphere.

(ii) For any \( g' \in C_0(g) = \{ g' \in \mathcal{R}_0(M) \mid g' \text{ conformal to } g \} \) we have \( \lambda_1(g') \leq \lambda_1(g) \), and equality holds if and only if \( g' \) is isometric to \( g \). In particular, the functional \( \lambda_1 \) does not admit a local minimum in \( \mathcal{R}_0(M) \).

(iii) The metric \( g \) is, up to isometry, the only extremal metric of \( \lambda_1 \) in \( C_0(g) \).

(iv) If \( (M,g) \) is not isometric to a Euclidean sphere then any conformal diffeomorphism of \( (M,g) \) is an isometry.

The following is a converse to Theorem 1.1.

**Proposition 1.1.** Let \( g \in \mathcal{R}_0(M) \) and assume there exists an \( L_2(g) \)-orthonormal basis \( \{ \phi_1, \ldots, \phi_k \} \) of \( E_1(g) \) such that the 2-tensor \( \sum_{1 \leq i \leq k} d\phi_i \otimes d\phi_i \) is proportional to \( g \). Then \( g \) is extremal for \( \lambda_1 \).
Proof. Let \((g_t)_t \subset \mathcal{R}_0(M)\) be a family of metrics analytic in \(t\) with \(g_0 = g\) and set \(h = \frac{d}{dt} g_t \big|_{t=0}\). With the same notation as above we have for small \(t\):

\[
\sum_{1 \leq i \leq k} \lambda_i(g_t) = \sum_{1 \leq i \leq k} \Lambda_{i,t}.
\]

Therefore, \(\sum_{1 \leq i \leq k} \lambda_i(g_t)\) is differentiable at \(t = 0\) and

\[
\frac{d}{dt} \sum_{1 \leq i \leq k} \lambda_i(g_t) \bigg|_{t=0} = \frac{d}{dt} \sum_{1 \leq i \leq k} \Lambda_{i,t} \bigg|_{t=0} = \text{trace } Q_h,
\]

where \(Q_h\) is the quadratic form defined on \(E_1(g)\) by:

\[
Q_h(u) = \int_M \langle q(u), h \rangle \nu_g,
\]

and where the trace of \(Q_h\) is taken w.r.t the \(L_2\) inner product induced by \(g\). Now

\[
\text{trace } Q_h = \sum_{1 \leq i \leq k} Q_h(\phi_i)
\]

\[
= \int_M \left( \sum_{1 \leq i \leq k} d\phi_i \otimes d\phi_i, h \right) \nu_g + \frac{1}{4} \sum_{1 \leq i \leq k} \int_M \langle \Delta \phi_i^2, h \rangle \nu_g.
\]

Since \(\sum_{1 \leq i \leq k} d\phi_i \otimes d\phi_i\) is proportional to \(g\) and \(\int_M \langle g, h \rangle \nu_g = 2 \frac{d}{dt} V(g_t) \big|_{t=0} = 0\) we have \(\int_M \left( \sum_{1 \leq i \leq k} d\phi_i \otimes d\phi_i, h \right) \nu_g = 0\). Moreover, by Takahashi’s theorem \(\sum_{1 \leq i \leq k} \phi_i^2\) is constant. Therefore, trace \(Q_h = 0\) and \(\frac{d}{dt} \sum_{1 \leq i \leq k} \lambda_i(g_t) \bigg|_{t=0} = 0\). The extremality of \(g\) then follows from the inequality \(\lambda_1(g_t) \leq \frac{1}{k} \sum_{1 \leq i \leq k} \lambda_i(g_t)\) which is an equality at \(t = 0\). □

Remarks.

1) It is known that compact irreducible homogeneous Riemannian spaces satisfy the hypothesis of Proposition 1.1 (see [8]). Thus, their standard metrics are extremal for \(\lambda_1\).
2) We restricted ourselves to \(\lambda_1\). Nevertheless, the results of this paragraph can be carried over to the case of higher eigenvalues.

2. \(\lambda_1\)-minimal and extremal metrics on the torus.

Let \((M, g)\) be an orientable compact surface of genus one endowed with a Riemannian metric \(g\). It is well known that there exists a lattice \(\Gamma\) of \(\mathbb{R}^2\) such that \((M, g)\) is conformally equivalent to the torus \((\mathbb{R}^2/\Gamma, g_\Gamma)\), where \(g_\Gamma\) is the flat metric induced from the Euclidean metric on \(\mathbb{R}^2\). The Clifford torus \((T^2_{cl} = \mathbb{R}^2/\Gamma_{cl}, g_{cl} = g_{\Gamma_{cl}})\) with \(\Gamma_{cl} = \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)\), and the equilateral torus \((T^2_{eq} = \mathbb{R}^2/\Gamma_{eq}, g_{eq} = g_{\Gamma_{eq}})\) with \(\Gamma_{eq} = \mathbb{Z}(1,0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)\),
each admit a natural homothetic minimal embedding into a sphere. These embeddings, denoted by \( \phi_{cl} \) and \( \phi_{eq} \), are those induced on \( T_{cl}^2 \) and \( T_{eq}^2 \) from \( \tilde{\phi}_{cl} : \mathbb{R}^2 \rightarrow S^3 \), where \( \tilde{\phi}_{cl}(x, y) = \frac{1}{\sqrt{2}} (\exp 2i\pi x, \exp 2i\pi y) \), and \( \tilde{\phi}_{eq} : \mathbb{R}^2 \rightarrow S^5 \), where \( \tilde{\phi}_{eq}(x, y) = \frac{1}{\sqrt{3}} (\exp 4i\pi y/\sqrt{3}, \exp 2i\pi (x - y/\sqrt{3}), \exp 2i\pi (x + y/\sqrt{3})) \).

**Theorem 2.1.** Let \((M, g)\) be a compact orientable surface of genus one and suppose that there exists a full isometric immersion \( \phi = (\phi_1, \ldots, \phi_{n+1}) \) from \((M, g)\) in the \(n\)-dimensional unit sphere \( S^n \) such that \( \forall i \leq n+1, \phi_i \in E_1(g) \). Then either:

(i) \((M, g)\) is isometric to the normalized Clifford torus \((T_{cl}^2, 2\pi^2 g_{cl})\), \( n = 3 \) and \( \phi \) is equivalent to \( \phi_{cl} \), or

(ii) \((M, g)\) is isometric to the normalized equilateral torus \((T_{eq}^2, \frac{8\pi^2}{5} g_{eq})\), \( n = 5 \) and \( \phi \) is equivalent to \( \phi_{eq} \).

Recall that an immersion \( \phi \) into \( S^n \) is full if its image is not contained in a great sphere of \( S^n \). Two immersions \( \phi \) and \( \psi \) into \( S^n \) are called equivalent if there exists an isometry \( R \) of \( S^n \) such that \( \phi = R \circ \psi \). A direct consequence of Theorem 2.1 is:

**Corollary 2.1.** A compact genus one orientable surface \((M, g)\) is \( \lambda_1 \)-minimal if and only if it is homothetic to \((T_{cl}^2, g_{cl})\) or \((T_{eq}^2, g_{eq})\).

As the metrics \( g_{cl} \) and \( g_{eq} \) trivially satisfy the hypothesis of Proposition 1.1 we have the following:

**Corollary 2.2.** Let \( M \) be a compact orientable surface of genus one. A metric \( g \) on \( M \) is extremal for \( \lambda_1 \) if and only if \((M, g)\) is homothetic to \((T_{cl}^2, g_{cl})\) or \((T_{eq}^2, g_{eq})\).

The proof of Theorem 2.1 is based on the following Propositions 2.1 and 2.2 which are valid in a more general setting.

**Proposition 2.1.** Let \((M, g)\) be a \(n\)-dimensional compact Riemannian homogeneous manifold non homothetic to \( S^n \). If a metric \( g = fg_0 \), conformal to \( g_0 \), is \( \lambda_1 \)-minimal, then \( f \) is constant on \( M \).

**Proof.** As \((M, g)\) is \( \lambda_1 \)-minimal non homothetic to \( S^n \) then any conformal diffeomorphism of \((M, g)\) is an isometry (cf. [4]). It follows that any isometry of \((M, g_0)\) is also an isometry of \((M, g)\). Thus the function \( f \) is invariant under the isometry group of \((M, g_0)\). The result follows from the homogeneity of \((M, g_0)\). \( \square \)

**Proposition 2.2.** Let \( \eta_1, \eta_2, \ldots, \eta_N \) be \( N \) continuous functions on a domain \( \Omega \) of \( \mathbb{R}^m \) and assume that the \( N^2 \) functions: \( 2\eta_j \) \((1 \leq j \leq N)\), \( \eta_k + \eta_l \) and \( \eta_k - \eta_l \) \((1 \leq k < l \leq N)\) are non-constant and mutually distinct modulo \( 2\pi \). If \( \phi = (\phi_1, \ldots, \phi_{n+1}) \) is a map from \( \Omega \) to \( S^n \) such that all its components \( \phi_i \) are in the vector space generated by \{\cos \eta_j, \sin \eta_j, 1 \leq j \leq N\},
then there exists an isometry $R$ of $\mathbb{S}^n$ such that

$$R \circ \phi = (\alpha_1 \exp i\eta_1, \alpha_2 \exp i\eta_2, \ldots, \alpha_r \exp i\eta_r, 0, \ldots, 0),$$

where $r \leq (n + 1)/2$, $j_1, \ldots, j_r \in \{1, \ldots, N\}$ and $\alpha_1, \ldots, \alpha_r$ are positive constants satisfying $\sum_{1 \leq j \leq r} \alpha_j^2 = 1$. In particular, $R(\phi(\Omega)) \subset \mathbb{S}^1(\alpha_1) \times \mathbb{S}^1(\alpha_2) \times \cdots \times \mathbb{S}^1(\alpha_r) \times \{0\}$.

The proof of this proposition is quite elementary and can be omitted.

Proof of Theorem 2.1. In view of Proposition 2.1, it suffices to consider the case where the metric $g$ is flat. It is well known that there exists $(a, b) \in \mathbb{R}^2$; $0 \leq a \leq \frac{1}{2}$, $b > 0$ and $a^2 + b^2 \geq 1$, such that $(M, g)$ is homothetic to $\left( T_{a,b} = \mathbb{R}^2/\Gamma(a,b), g_{ab} = g_{\Gamma(a,b)} \right)$ with $\Gamma(a,b) = \mathbb{Z}(1,0) \oplus \mathbb{Z}(a,b)$ (cf. [3]). Now the existence of an isometric immersion from $(M, g)$ into the unit sphere by the first eigenfunctions implies that $\lambda_1(g) = 2$. Since $\lambda_1(g_{ab}) = \frac{4\pi^2}{ab}$, $(M, g)$ is in fact isometric to $\left( T_{a,b}^2, \frac{2\pi^2}{ab}g_{ab} \right)$. Let $E_{a,b}$ be the first eigenspace of $g_{ab}$ and $\phi : \left( T_{a,b}^2, \frac{2\pi^2}{ab}g_{ab} \right) \to \mathbb{S}^n$ a full isometric immersion whose components $\phi_i \in E_{a,b}$.

- If $a^2 + b^2 > 1$ then the dimension of $E_{a,b}$ is 2 and there is no such $\phi$.
- If $a^2 + b^2 = 1$ and $(a, b) \neq (1/2, \sqrt{3}/2)$ then $E_{a,b}$ is generated by $\cos \eta_j$, $\sin \eta_j$, $j \leq 2$, with $\eta_1(x,y) = \frac{2\pi y}{b}$ and $\eta_2(x,y) = 2\pi \left( x - \frac{ay}{b} \right)$. From Proposition 2.2, it follows that $n = 3$ and, up to an isometry of $\mathbb{S}^3$, $\phi$ has the form $\phi = (\alpha_1 \exp(i\eta_1), \alpha_2 \exp(i\eta_2))$ with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1^2 + \alpha_2^2 = 1$. As $\phi$ is isometric we deduce that $a = 0$, $b = 1$ and $\alpha_1 = \alpha_2 = \sqrt{2}/2$. Thus $(M, g)$ is isometric to $(T_{a,b}^2, 2\pi^2g_{cd})$ and $\phi$ is equivalent to $\phi_{eq}$.
- If $(a, b) = (1/2, \sqrt{3}/2)$ then $E_{a,b}$ is generated by $\cos \eta_j$, $\sin \eta_j$, $j \leq 3$, where $\eta_1(x,y) = 4\pi y/\sqrt{3}$, $\eta_2(x,y) = 2\pi \left( x - \frac{ay}{\sqrt{3}} \right)$ and $\eta_3(x,y) = 2\pi \left( x + \frac{ay}{\sqrt{3}} \right)$. As before $n \leq 5$ and, up to isometry, $\phi = (\alpha_1 \exp(i\eta_1), \alpha_2 \exp(i\eta_2), \alpha_3 \exp(i\eta_3))$ where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are nonnegative constants such that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. As $\phi$ is isometric we obtain $\alpha_1 = \alpha_2 = \alpha_3 = \sqrt{3}/3$. Thus $\phi$ is equivalent to $\phi_{eq}$. 

\[\square\]

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