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SEIBERG–WITTEN INVARIANTS FOR 3-MANIFOLDS IN THE CASE $b_1 = 0$ OR 1

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In this note we give a detailed exposition of the Seiberg-Witten invariants for closed oriented 3-manifolds paying particular attention to the case of $b_1 = 0$ and $b_1 = 1$. These are extracted from the moduli space of solutions to the Seiberg-Witten equations which depend on choices of a Riemannian metric on the underlying manifold as well as certain perturbation terms in the equations. In favourable circumstances this moduli space is finite and naturally oriented and we may form the algebraic sum of the points. Given any two sets of choices of metric and perturbation which are connected by a 1-parameter family, we analyse in detail the singularities which may develop in the interpolating moduli space. This leads then to an understanding of how the algebraic sum changes. In the case $b_1 = 0$ a topological invariant can be extracted with the addition of a suitable counter-term, which we identify (this idea is attributed to Donaldson). In the case $b_1 = 1$ a topological invariant is defined which depends only on cohomological information related to the perturbation term. We prove a 'wall-crossing' formula which tells us how the invariant changes with different choices of this perturbation. Throughout we pay careful attention to genericity statements and the issue of orientations and signs in all the relations. The equivalence of this invariant in the case of an integral homology sphere with the Casson invariant is treated in Lim, 1999 (see also works of Nicolescu, preprint). The equivalence with Reidemeister Torsion in the case $b_1 > 0$ is a result of Meng & Taubes, 1996. Some related material is in Marcolli, 1996, Froyshov, 1996 and in the survey Donaldson, 1996. Taubes, 1990 contains the originating construction in this article in the context of flat $SU(2)$-connections.

1. The Seiberg-Witten Invariants.

We denote by Y an oriented 3-manifold. Let g be a Riemannian metric on Y and $P \to Y$ a spin-c structure (see for example [**LM**]). Denote by $S \to Y$ the associated positive spinor bundle, i.e., $S = P \times_\varrho \mathbf{C}^2$ where $\varrho : \mathrm{spin}_c(3) \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2)$ is the irreducible representation of the complex Clifford algebra

$\mathbf{C}\mathrm{l}(\mathbf{R}^3)$ with $\varrho(dy) = +1$, where dy is the oriented volume form on Y. Let $\xi : \mathrm{spin}_c(3) = \mathrm{spin}(3) \times_{\{\pm 1\}} U(1) \to U(1)$ be the map which takes the square of the second factor. Then the bundle $L = L_P = P \times_\xi \mathbf{C} \to Y$ is the determinant line bundle of P; it is clearly a $U(1)$-bundle. We remark that a classical fact is $w_2(Y) = 0$ and this gives the existence of a spin-c structure $P \to Y$ with $c_1(L)$ any given class in $2H^2(Y; \mathbf{Z}) \subset H^1(Y; \mathbf{Z})$.

For a pair (A, Φ) consisting of a $U(1)$-connection on L and a section of S the π-perturbed Seiberg-Witten equations (SW_π) read:

$$F_A = \frac{1}{4}\sigma(\Phi, \Phi) + \omega, \quad D_{A+\alpha}\Phi = 0.$$

Here F_A denotes the curvature 2-form of A and D_A the Dirac operator coupled to A. The term $\sigma(\cdot, \cdot)$ is a certain symmetric bilinear form $S \to \Lambda^2(i\mathbf{R})$ (see Section 2 for details). The perturbation π consists of a pair (α, ω) where $\alpha \in \Omega^1(i\mathbf{R})$ and $\omega \in \Omega^2(i\mathbf{R})$, $d\omega = 0$.

$U(1)$ embeds into $\mathrm{spin}_c(3)$ by the map $\iota(z) = [\mathrm{Id}, z]$. Given a smooth map $g : Y \to U(1) \subset \mathbf{C}$, we have an automorphism of P given by the rule $p \mapsto p\iota(g)$. This induces, by pulling back, the action $(A, \Phi) \to g(A, \Phi) = (A + 2g^{-1}dg, g^{-1}\Phi)$. If (A, Φ) is a Seiberg-Witten solution, then $g(A, \Phi)$ is also a solution. Thus the set of Seiberg-Witten solutions is invariant under the above automorphisms of S. The automorphism g called a *gauge transformation* and the group of all gauge transformations is the *gauge group* denoted $\mathcal{G}$.

We will consider the set of L^2_2-SW_π-solutions modulo gauge equivalence (details in Sec. 2). The set of such solutions will be denoted by $Z_\pi(P; g)$ or $Z_{\pi,g}(P)$. When the underlying metric is understood we shall omit it and simply write $Z_\pi(P)$. Fixing a value of $k \geq 1$, denote by $\mathcal{P}_k$ the space of all perturbations π of class C^k.

Theorem 1. *For π from an open dense subset of $\mathcal{P}_k$ the irreducible part $Z^*_\pi(P; g)$ of $Z_\pi(P; g)$ is a finite set of points and these are naturally oriented. Let $\#Z^*_\pi(P; g)$ denote the algebraic sum, assuming π as above. Then:*

- (i) *if $b_1(Y) > 1$, $\#Z^*_\pi(P; g)$ is independent of g and π*
- (ii) *if $b_1(Y) = 1$, $\#Z^*_\pi(P; g)$ depends only on the component of $H^2(Y; \mathbf{R}) \setminus \{c_1(L)_\mathbf{R}\}$ in which $[\frac{i}{2\pi}\omega]$ lies in, ω being the 2-form component of π*
- (iii) *if $b_1(Y) = 0$, $\#Z^*_\pi(P; g)$ is independent of π and g after the addition of a counter-term $\zeta(\pi, g)$ which is a combination of the spectral invariants of Atiyah-Patodi-Singer. $\#Z^*_\pi(P; g) + \zeta(\pi; g)$ takes values in $\mathbf{Z}$ if $H_1(Y; \mathbf{Z}) = \{0\}$ and $\mathbf{Z}\left[\frac{1}{8|H_1(Y;\mathbf{Z})|}\right]$ if $H_1(Y; \mathbf{Z}) \neq \{0\}$.*

The exact expression for the counter-term $\zeta(\pi, g)$ is in Proposition 17. For $b_1(Y) = 1$ the formula for the change in $\#Z_\pi(P; g)$ when we cross the 'wall' in $H^2(Y; \mathbf{R})$ defined by $\{c_1(L)_\mathbf{R}\}$ is given in Corollary 20.

Let $\mathrm{spin_c}(Y)$ denote the equivalence classes of spin-c structures on Y. There is a well-defined map $\mathrm{spin_c}(Y) \to 2H^2(Y;\mathbf{Z})$ which sends a representative P to $c_1(L_P)$.

Corollary 2. *Let Y be connected. The Seiberg-Witten equations define an oriented diffeomorphism invariant τ of Y in the following form:*

(i) *if $b_1(Y) > 0$, $\tau : \mathrm{spin_c}(Y) \to \mathbf{Z}$*

(ii) *if $b_1(Y) = 1$, let $\mathrm{spin_c^*}(Y)$ be the set of pairs $([P],\mathcal{U})$ where $\mathcal{U}$ a component of $H^2(Y;\mathbf{R})\backslash\{c_1(L_P)_{\mathbf{R}}\}$. Then $\tau : \mathrm{spin_c^*}(Y) \to \mathbf{Z}$*

(iii) *if $b_1(Y) = 0$ and $H_1(Y;\mathbf{Z}) = \{0\}$ there is a unique spin-c structure and so $\tau \in \mathbf{Z}$*

(iv) *if $b_1(Y) = 0$ and $H_1(Y;\mathbf{Z}) \neq \{0\}$, then $\tau : \mathrm{spin_c}(Y) \to \mathbf{Z}\left[\frac{1}{8|H_1(Y;\mathbf{Z})|}\right]$.*

In the subsequent sections we work up to a proof of Theorem 1. Section 2 discusses the framework for defining the moduli space and their first properties. Section 3 looks at generic properties. The details of the proof of Theorem 1 are in Section 4.

Addendum. The reviewer has brought to the attention of the author of an alternative exposition of some of the material in this article, in [**Ch**].

2. The Moduli Space.

Throughout this section Y denotes a closed oriented 3-manifold with Riemannian metric g and $P \to Y$ a fixed spin-c structure.

2.1. The Basic Set-up.

As in the usual gauge theory set-up we work with the following spaces. (For more details see, for instance, [**M**].) Let $\mathcal{C}(P)$ denote the space of pairs (A,Φ) consisting of a L^2_2 connection A on L and Φ a L^2_2- section of $S \to Y$. This forms a Hilbert manifold. Let $\mathcal{G}$ denote the space of L^2_3 gauge transformations of S i.e., L^2_3 maps $g : Y \to S^1 \subset \mathbf{C}$. This forms a Hilbert Lie group. $\mathcal{G}$ acts on $\mathcal{C}(P)$ by $g(A,\Phi) = (A + 2g^{-1}dg, g^{-1}\Phi)$. This action is smooth with Hausdoff quotient $\mathcal{B}(P)$.

A pair (A,Φ) is *irreducible* if Φ is not identically 0. Otherwise it is called *reducible.* $\mathcal{G}$ acts freely on $\mathcal{C}^*(P)$, the open set of irreducibles and its quotient is denoted by $\mathcal{B}^*(P)$. The projection map $\mathcal{C}^*(P) \to \mathcal{B}^*(P)$ forms a principle $\mathcal{G}$-bundle. At a reducible $(A,0)$, for which we simply write A, the stabilizer of $\mathcal{G}$ is exactly those gauge transformations g for which $dg = 0$; thus the stabilizer is identified with $U(1) \subset \mathbf{C}$, the constant gauge transformations.

Let $\Omega^p_k(i\mathbf{R})$ denotes the p-forms on Y of class L^2_k, and $\Gamma_k(S)$ the sections of S of class L^2_k. Since $\mathcal{C}(P)$ is an affine space modelled on the vector space $\Omega^1_2(i\mathbf{R}) \times \Gamma_2(S)$ the tangent space at any point is canonically identified with the vector space itself. On the other hand the tangent space to the identity of $\mathcal{G}$ is identified with $\Omega^0_3(i\mathbf{R})$.

The derivative of the map $g \mapsto g(A, \Phi)$ at the identity is given by $\gamma \mapsto (2d\gamma, -\gamma\Phi)$. The tangent bundle of $\mathcal{C}(P)$ carries a natural Riemannian metric which is the L^2-inner product on $\Omega^1_2(i\mathbf{R}) \times \Gamma_2(S)$. This inner product is invariant under complex multiplication in $\Gamma_2(S)$. By taking the L^2-orthogonal to the image of the derivative of the gauge group action we obtain a slice $(A, \Phi) + X_{A,\Phi}$ for the action at (A, Φ). $X_{A,\Phi}$ is defined as

$$\{(a, \phi) \,|\, 2d^*a = i\langle i\Phi, \phi\rangle\} \subset \Omega^1_2(i\mathbf{R}) \times \Gamma_2(S).$$

If $\Phi = 0$ then $X_{A,\Phi}$ reduces to $\ker d^* \times \Gamma_2(S)$. The stabilizer of A preserves X_A and acts by $z(a, \phi) = (a, z^{-1}\phi)$; therefore the stabilizer acts as *the opposite complex structure* on $\Gamma_2(S)$. If $\Phi \neq 0$ then a sufficiently small neighbourhood N of zero in $X_{A,\Phi}$ models an open set for the gauge equivalence class $[(A, \Phi)]$ in $\mathcal{B}(P)$. If $\Phi = 0$ then the same is true except that we should take $N/U(1)$ instead.

The symmetric bilinear form $\sigma(\Phi, \Psi) \in \Lambda^2(i\mathbf{R})$ for us will be defined as the adjoint of Clifford multiplication, that is defined by the condition that for all $\omega \in \Lambda^2(i\mathbf{R})$,

$$\langle \omega \cdot \Psi, \Phi\rangle = \langle \omega, \sigma(\Phi, \Psi)\rangle.$$

The representation $c : \Lambda^2(i\mathbf{R}) \to \mathrm{End}_{\mathbf{C}}(S)$ given by Clifford multiplication is an isomorphism onto its image which is the trace-free Hermitian symmetric endomorphisms of S. If we identify $\Lambda^2(i\mathbf{R})$ with its image under c then

$$\sigma(\Phi, \Psi) = \frac{1}{2}(\Phi \otimes \Psi^* + \Psi \otimes \Phi^* - \langle \Phi, \Psi\rangle_{\mathbf{R}}\mathrm{Id}).$$

In the formula expressions of the form $v \otimes w^*$ mean the endomorphism $v \otimes w^*(u) = v\langle u, w\rangle_{\mathbf{C}}$. We remark that this formula assumes the convention that if τ is unit length in $\Lambda^2(\mathbf{R})$ then $c(i\tau)$ is to be unit length in $\mathrm{End}_{\mathbf{C}}(S)$.

Fix a perturbation term $\pi = (\alpha, \omega)$ of class C^k, $k \geq 1$ (we assume this from now on). To set up the moduli space we define the *SW_π-section* $s = s_{\pi,g} : \Omega^0_2(i\mathbf{R}) \times \mathcal{C}(P) \to \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S)$ by

$$s(\eta, A, \Phi) = \left(* \left(F_A - \frac{1}{4}\sigma(\Phi, \Phi) - \omega \right) + 2d\eta, D_{A+\alpha}\Phi - \eta\Phi \right).$$

Since we will want to vary the perturbation term later, we introduce the the Banach space $\mathcal{Q}_k$ of C^k-sections of $\Lambda^1(Y) \otimes i\mathbf{R}$ and the Banach space Ω_k of closed C^k-sections of $\Lambda^2(Y) \otimes i\mathbf{R}$. Then our perturbations π are from the space $\mathcal{P}_k$ which is $\mathcal{Q}_k \times \Omega_k$.

From the definition of the SW_π-section, it would seem that the zeros might capture a much larger set than the SW_π-solutions themselves; but as the following Lemma shows this is only so in a minor way.

Lemma 3. *Let $s(\eta, A, \Phi) = 0$. If $\Phi \neq 0$ then $\eta = 0$ and (A, Φ) is a SW_π-solution. If $\Phi = 0$ then $\eta =$ constant and $(A, 0)$ is a SW_π-solution.*

Proof. We claim that the vector $(*(F_A - \frac{1}{4}\sigma(\Phi,\Phi) - \omega), D_{A+\alpha}\Phi)$ is L^2-orthogonal to $(2d\eta, -\eta\Phi)$. Before we show this we recall some useful identities: (i) if $\eta \in \Omega^0_k(i\mathbf{R})$ then $D_{A+\alpha}(\eta\Phi) = d\eta\cdot\Phi + \eta D_{A+\alpha}\Phi$ (ii) the Clifford action of $a \in \Omega^1_k(i\mathbf{R})$ is equal to the action of $-*a$ (iii) if $\eta \in \Omega^0_k(i\mathbf{R})$ then $\langle \eta\Phi, \Psi\rangle = -\langle \Phi, \eta\Psi\rangle$. To prove the claim we compute:

$$\begin{aligned}
&\langle 2d\eta, *(F_A - \tfrac{1}{4}\sigma(\Phi,\Phi) - \omega)\rangle_{L^2} - \langle \eta\Phi, D_{A+\alpha}\Phi\rangle_{L^2}\\
&= \langle 2\eta, d^* * (F_A - \omega)\rangle_{L^2} - \langle *2d\eta, \tfrac{1}{4}\sigma(\Phi,\Phi)\rangle_{L^2} - \langle \eta\Phi, D_{A+\alpha}\Phi\rangle_{L^2}\\
&= -\tfrac{1}{2}\langle *d\eta\cdot\Phi, \Phi\rangle_{L^2} - \langle \eta\Phi, D_{A+\alpha}\Phi\rangle_{L^2}\\
&= \tfrac{1}{2}\langle d\eta\cdot\Phi, \Phi\rangle_{L^2} - \langle \eta\Phi, D_{A+\alpha}\Phi\rangle_{L^2}\\
&= \tfrac{1}{2}\langle D_{A+\alpha}(\eta\Phi), \Phi\rangle_{L^2} - \tfrac{1}{2}\langle \eta D_{A+\alpha}\Phi, \Phi\rangle_{L^2} - \langle \eta\Phi, D_{A+\alpha}\Phi\rangle_{L^2}\\
&= \tfrac{1}{2}\langle D_{A+\alpha}(\eta\Phi), \Phi\rangle_{L^2} + \tfrac{1}{2}\langle D_{A+\alpha}\Phi, \eta\Phi\rangle_{L^2} - \langle \eta\Phi, D_{A+\alpha}\Phi\rangle_{L^2} = 0.
\end{aligned}$$

The last step follows from $D_{A+\alpha}$ being self-adjoint. Therefore $s(\eta, A, \Phi) = 0$ if and only if (A,Φ) is a SW_π- solution and $(2d\eta, -\eta\Phi) = 0$. $\square$

Thus we identify the space of Seiberg-Witten solutions with

$$s^{-1}(0) \cap \{0\} \times \mathcal{C}(P)$$

and the Seiberg-Witten moduli space $Z_\pi(P)$ is the quotient by $\mathcal{G}$ of this, where $\mathcal{G}$ acts only on the $\mathcal{C}(P)$ factor. The local structure of the moduli space near a solution $(\eta, A, \Phi) \in s^{-1}(0)$ is determined by the elliptic complex associated to the map s:

$$\Omega^0_3(i\mathbf{R}) \xrightarrow{\delta^0_{A,\Phi}} \Omega^0_2(i\mathbf{R}) \oplus \Omega^1_2(i\mathbf{R}) \oplus \Gamma_2(S) \xrightarrow{\delta^1_{\eta,A,\Phi}} \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S)$$

where

$$\begin{aligned}
\delta^0_{A,\Phi}(\gamma) &= (0, 2d\gamma, -\gamma\Phi)\\
\delta^1_{\eta,A,\Phi}(\xi, a, \phi) &= (*(da - \tfrac{1}{2}\sigma(\Phi,\phi)) + 2d\xi, D_{A+\alpha}\phi + \tfrac{1}{2}a\cdot\Phi - \xi\Phi - \eta\phi).
\end{aligned}$$

Since we will be interested only in the case where $\eta = 0$ we will in subsequent notation omit it when that is understood. Thus we let $H^i_{A,\Phi}$, $(i = 0,1,2)$ denote the cohomology of the complex when $\eta = 0$.

Lemma 4. *Let (A,Φ) be a $SW_{\alpha,\omega}$-solution.*

(i) *If $\Phi \neq 0$ then $H^0_{A,\Phi} = 0$, $H^1_{A,\Phi} = H^2_{A,\Phi}$*

(ii) *If $\Phi = 0$ then $H^0_{A,\Phi} = \mathbf{H}^0(i\mathbf{R})$, $H^1_{A,\Phi} = H^0_{A,\Phi} \oplus H^2_{A,\Phi}$, $H^2_{A,\Phi} = \mathbf{H}^1(i\mathbf{R}) \oplus \mathcal{H}_{A+\alpha}$.*

Note. In this article $\mathbf{H}^k(i\mathbf{R})$ always denotes the pure imaginary harmonic forms of degree k and $\mathcal{H}_A$ the kernel of D_A.

Proof. A direct computation shows that the L^2-adjoint of $\delta^1_{A,\Phi}$ is given by $\delta^{1*}_{A,\Phi}(b,\psi) = (2d^*b - i\langle i\Phi,\psi\rangle, \delta^1_{A,\Phi}(a,\psi))$. The adjoint of $\delta^0_{A,\Phi}$ on the other hand, is $\delta^{0*}_{A,\Phi}(\eta,a,\phi) = 2d^*a - i\langle i\Phi,\phi\rangle$. Therefore

$$\begin{aligned}
H^1_{A,\Phi} &= \ker\delta^{0*}_{A,\Phi} \cap \ker\delta^1_{A,\Phi} \\
&= \{(\xi,a,\phi)\,|\,2d^*a - i\langle i\Phi,\phi\rangle = 0, da - \tfrac{1}{2}\sigma(\Phi,\phi) = 0, \\
&\qquad D_{A+\alpha}\phi + \tfrac{1}{2}a\cdot\Phi = 0, d\xi = 0, \xi\Phi = 0\} \\
H^2_{A,\Phi} &= \ker\delta^{1*}_{A,\Phi} \\
&= \{(b,\psi)\,|\,2d^*b - i\langle i\Phi,\psi\rangle = 0, db - \tfrac{1}{2}\sigma(\Phi,\psi) = 0, \\
&\qquad D_{A+\alpha}\psi + \tfrac{1}{2}b\cdot\Phi = 0\}.
\end{aligned}$$

The remainder of the proof follows easily. □

The local structure of the moduli space may now be deduced by the Kuranishi argument. Let $(0,A,\Phi)\in s^{-1}(0)$. If $\Phi\neq 0$, then a neighbourhood of $[(0,A,\Phi)]$ in $Z_\pi(P)$ is modelled on the zeros of a map (the obstruction map) $\Xi : H^1_{A,\Phi}\to H^2_{A,\Phi}$. If $\Phi = 0$ then the same is true except that Ξ is S^1-equivariant and we should take $\Xi^{-1}(0)/S^1$.

If (A,Φ) is a regular solution, i.e., $H^2_{A,\Phi} = \{0\}$, and $\Phi\neq 0$ then $\Xi^{-1}(0)$ is exactly one point. Thus a regular irreducible solution is isolated. Therefore we have:

Proposition 5. *If $Z^*_\pi(P)$ consists solely of gauge equivalence classes of regular solutions then $Z^*_\pi(P)$ is a discrete set, i.e., every point is isolated.*

2.2. Compactness and Regularity.

Proposition 6. *Fix $\pi\in\mathcal{P}_k$, $k\geq 1$.*

(i) *If (A,Φ) is a SW_π-solution then (A,Φ) is gauge equivalent to a SW_π-solution of class L^p_{k+1}, $p\geq 2$.*
(ii) *Let $\{(A_i,\Phi_i)\}_{i=1}^\infty$ be a sequence of SW_π-solutions.*

Then there is a subsequence $\{i'\}\subset\{i\}$ and gauge transformations $\{g_{i'}\}$ such that $\{g_{i'}(A_{i'},\Phi_{i'})\}$ converges in L^2_k to a L^2_k SW_π-solution. In particular, this converges in $\mathcal{C}(P)$, and therefore $\mathcal{B}(P)$.

Proof. The proof is due to [**KM**]. We include it here for completeness. The Bochner formula for the Dirac operator reads (see for instance [**LM**])

$$D^*_{A+\alpha}D_{A+\alpha}\Phi = \nabla^*_{A+\alpha}\nabla_{A+\alpha}\Phi + \frac{1}{4}\kappa\Phi + \frac{1}{2}F_{A+\alpha}\cdot\Phi,$$

κ being scalar curvature. We also have Kato's inequality

$$\frac{1}{2}\Delta|\Phi|^2 \leq \langle\nabla_{A+\alpha}\nabla_{A+\alpha}\Phi\rangle.$$

If (A,Φ) is a $\mathrm{SW}_{\alpha,\omega}$-solution then $D_{A+\alpha}\Phi = 0$ and

$$\begin{aligned}\langle F_{A+\alpha}\cdot\Phi,\Phi\rangle &= \frac{1}{4}|\sigma(\Phi,\Phi)|^2 + \langle(\omega+d\alpha)\cdot\Phi,\Phi\rangle \\ &= \frac{1}{8}|\Phi|^4 + \langle(\omega+d\alpha)\cdot\Phi,\Phi\rangle.\end{aligned}$$

Applying Kato's inequality to the Bochner formula we obtain

$$\frac{1}{2}\Delta|\Phi|^2 \le -\frac{1}{4}\kappa|\Phi|^2 - \frac{1}{8}|\Phi|^4 + |\omega+d\alpha|.|\Phi|^2.$$

At a maximum for Φ, $\Delta|\Phi|^2 \ge 0$. If this is non-zero we obtain

$$|\Phi|^2 \le \max_Y(-2\kappa + 8|\omega+d\alpha|, 0).$$

Since ω and α are in C^1, we obtain a uniform pointwise bound on the spinor component of any $\mathrm{SW}_{\alpha,\omega}$-solution.

Let us prove (ii). Suppose that (A_i,Φ_i) is a given sequence of $\mathrm{SW}_{\alpha,\omega}$-solutions. Choose a fixed reference smooth connection A, and write $A_i = A + a_i$. Then after a gauge transformation we may assume that $da_i = 0$ and the harmonic component of a_i is uniformly bounded, since the component group of maps $Y \to U(1)$ is $H^1(Y;\mathbf{Z})$, and thus $H^1(Y;i\mathbf{R})/H^1(Y;i\mathbf{Z})$ is compact. Let $\hat{a}_i$ be the L^2-component of a_i which is L^2-pependicular to the harmonic forms. Since the harmonic forms are C^∞, $a_i - \hat{a}_i$ must lie in C^k for every k. The $\mathrm{SW}_{\alpha,\omega}$ equations together with the uniform pointwise bound on Φ_i gives (by ellipticity) a uniform L^p_1 bound for the $\hat{a}_i$. Applying this to the equation for Φ_i this gives again by ellipticity a uniform L^p_1 bound on the Φ_i. Circulating inductively we terminate with uniform L^p_{k+1} bounds on both $\hat{a}_i$ and Φ_i, since α and ω are assumed to be in C^k. The uniform bound holds for all $p \ge 2$. Therefore the sequence (a_i,Φ_i) is uniformly bounded in L^p_{k+1}. By Rellich's theorem a subsequence of (a_i,Φ_i) converges in L^p_k. For p sufficiently large, $L^p_k \subset L^2_{k+1}$; thus the sequence converges in $\mathcal{C}(P)$, since the underlying topology in L^2_2.

The proof of (i) follows from the preceding by applying it to the constant sequence. □

2.3. Reducible Solutions.

When $\Phi = 0$, the Seiberg-Witten equations reduce to a single equation for the connection A: $F_A = \omega$. If $\omega = 0$ then the reducible (up to gauge equivalence) is identified with the moduli space of flat $U(1)$-connections on L. Thus a necessary condition is that $c_1(L)_{\mathbf{R}} = 0$. If this is so, then by by a well-known fact in differential geometry, the gauge equivalence classes of flat connections is completely determined by the holonomy representation of $\pi_1(Y)$ and is therefore topologically a product $U(1)\times\cdots\times U(1)$ where the number of factors equals $b_1(Y)$. In particular if $b_1(Y) = 0$ then the reducible is exactly one point. (Note: When $b_1(Y) = 0$, L admits only one flat connection, up to gauge.)

Lemma 7. *The equation $F_A = \omega$ has a solution if and only if the real cohomology classes $[F_A] = [\omega]$ or equivalently $[\frac{i}{2\pi}\omega] = c_1(L)_{\mathbf{R}}$. If the latter holds, then the space of equivalence classses of reducible solutions is topologically $U(1) \times \cdots \times U(1)$ where the number of factors equals $b_1(Y)$, and in the case $b_1(Y) = 0$, a single point.*

Proof. As explained above the condition $[F_A] = [\omega]$ is necessary. For sufficiency, let A_0 be such that $[F_{A_0}] = [\omega]$. Then we only need to solve for a in $F_{A_0+a} = \omega$ which is equivalent to $da = \omega - F_{A_0}$. Since $\omega - F_{A_0}$ is exact such an a can be found. Assuming solutions exist, let $\mathcal{F}$ denote the space of all A's such that $F_A = 0$. Then $\mathcal{F} + a$ describes all the solutions to $F_A = \omega$. Therefore the space of reducible solutions up to gauge are topologically the same as in the case $\omega = 0$. □

2.4. Orientation.

Suppose $Z^*_\pi(P)$ consists only of regular points. It is clear that $Z^*_\pi(P)$ is orientable. We want to produce a procedure for inducing a global orientation. The fundamental elliptic complex can be combined into a single operator $L_{\eta,A,\Phi} : \Omega^0_2(i\mathbf{R}) \oplus \Omega^1_2(i\mathbf{R}) \oplus \Gamma_2(S) \to \Omega^0_1(i\mathbf{R}) \oplus \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S)$, $L_{\eta,A,\Phi} = \delta^1_{\eta,A,\Phi} + \delta^{0*}_{\eta,A,\Phi}$. A direct computation verifies that this operator is formally self-adjoint.

Let $\Lambda = \det \operatorname{Ind}\{L_{\eta,A,\Phi}\}$. Let $g \in \mathcal{G}$ and $(a, \phi) \in \ker L_{\eta,A,\Phi}$. Then $(a, g^{-1}\phi) \in \ker L_{\eta,g(A,\Phi)}$. Therefore the action of $\mathcal{G}$ lifts to an action on Λ. Note that if $\Phi = 0$ then the stabilizer $U(1)$ maps the fibre of Λ at $A = (A, 0)$ back to itself by the identity. Hence Λ decends to a line bundle $\hat{\Lambda}$ over $\Omega^0_2(i\mathbf{R}) \times \mathcal{B}(P)$.

Proposition 8. *The real line bundle $\hat{\Lambda}$ is trivial.*

Proof. We need to show that Λ posseses a $\mathcal{G}$-equivariant trivialization. The substitution of $(1-\varepsilon)\Phi$, $0 \le \varepsilon \le 1$, for Φ and $(1-\varepsilon)\eta$ for η in the definition of $\delta^1_{\eta,A,\Phi}$ and $\delta^{0*}_{\eta,A,\Phi}$ defines a homotopy of $L_{\eta,A,\Phi}$ to an operator $L'_{\eta,A,\Phi}$ given by $L'_{\eta,A,\Phi}(\xi, a, \phi) = (*da + 2d\xi, D_A\phi)$. This homotopy is $\mathcal{G}$-equivariant. We have $\ker L'_{\eta,A,\Phi} = \mathbf{H}^0(i\mathbf{R}) \oplus \mathbf{H}^1(i\mathbf{R}) \oplus \mathcal{H}_A = \operatorname{coker} L'_{\eta,A,\Phi}$. This family has a trivial determinant, and this proves Λ is $\mathcal{G}$-equivariantly trivial. □

Notice that the homotopy given in the proof is the identity over $\{0\} \times \mathcal{C}^{\mathrm{Red}}(P)$. Thus over this set $\det \operatorname{Ind}\{L_{\eta,A,\Phi}\}$ is the determinant of the index of a constant family $\{L_{\mathrm{dRham}}\}$ tensored with the complex family $\{D_A\}$. Since a complex family is canonically oriented, we may ignore it. The kernel and cokernel of L_{dRham} are $\mathbf{H}^0(i\mathbf{R}) \oplus \mathbf{H}^1(i\mathbf{R})$ and by identifying them with each other we obtain a trivialization of $\det \operatorname{Ind}\{L_{\eta,A,\Phi}\}$ over $\{0\} \times \mathcal{C}^{\mathrm{Red}}(P)$.

This orients $\det \operatorname{Ind}\{L_{\eta,A,\Phi}\}$ over all of $\Omega^0_2(i\mathbf{R}) \times \mathcal{C}(P)$. *This is the natural orientation of* $\hat{\Lambda}$.

Consider the trivial real line bundle $\overline{\mathbf{R}}$ over $\Omega^0_2(i\mathbf{R}) \times \mathcal{B}^*(P)$. Then $\hat{\Lambda}$ has the property that over the open set $\mathcal{O}$ of $\Omega^0_2(i\mathbf{R}) \times \mathcal{B}^*(P)$ defined by the condition that $\ker L_{\eta,A,\Phi} = 0$, there is a canonical isomophism $h : \hat{\Lambda}|_{\mathcal{O}} \xrightarrow{\cong} \overline{\mathbf{R}}|_{\mathcal{O}}$.

Proposition 9. *Let $\hat{\Lambda}$ have the natural orientation described above. Assume $Z^*_\pi(P)$ consists only of regular points. Then the following rule defines an orientation $\varepsilon : Z^*_\pi(P) \to \{\pm 1\}$. Let $\mathbf{x} \in Z^*_\pi(P)$. Denote by $o(\mathbf{x}, \overline{\mathbf{R}})$ the canonical orientation of $\overline{\mathbf{R}}|_{\mathbf{x}}$ and $o(\mathbf{x}, \hat{\Lambda})$ the orientation induced by $\hat{\Lambda}$ via the isomorphism h above. Then*

$$\varepsilon(\mathbf{x}) = \begin{cases} 1 & \text{if } o(\mathbf{x}, \overline{\mathbf{R}}) = o(\mathbf{x}, \hat{\Lambda}) \\ -1 & \text{otherwise.} \end{cases}$$

3. Generic Properties.

Let $\{g(t)\}$, $t \in I_\varepsilon = (-\varepsilon, 1+\varepsilon)$ be a 1-parameter family of metrics on Y. In this section we examine the parameters (π, t) in $\mathcal{P} \times I_\varepsilon$ for which the moduli space $Z_\pi(P; g(t))$ consists solely of regular points.

In order to understand how the geometric structures and operators changes with the 1-parameter family of metrics it will be useful to be able to work with a single reference underlying metric, spin-c structure and model for the spinors. Fix an underlying metric which we take to be g, let P_{SO} be the corresponding oriented orthonormal frame bundle and h an automorphism of TY. h induces an automorphism h_* of P_{GL_+}, the component of positively oriented frames of the frame bundle of Y. The image of P_{SO} in P_{GL_+} under h describes the orthonormal frame bundle of another metric. Conversely, the positive orthonormal frame bundle of any other metric can be recovered on this way. Call the second metric g'.

h can be lifted to an isomorphism between P, the spin-c structure for g and P', the spin-c structure for g'. This, in turn, induces a fibrewise isometry $\hat{h}$ between the corresponding spinor bundles S and S'. By changing $\hat{h}$ to $e^u\hat{h}$ where u is a smooth function on Y we can arrange it so that $e^u\hat{h}$ gives an isometry between $\Gamma(S)$ and $\Gamma(S')$ with respect to their L^2-norms.

Given the 1-parameter family $g(t)$ the construction of $e^u\hat{h}$ above can be carried out smoothly in the parameter t, taking for instance $g = g(0)$ to be the reference metric. Therefore using these isomorphisms as identifications we may assume (A, Φ) etc. for every t is defined on a fixed reference bundle. The Dirac operator now depends also on t, and we denote this as $D^{g(t)}_A$. It is always self-adjoint with respect to the reference spinor bundle. Further information regarding the relation between the Dirac operator for different metrics can be found in [**B**], [**BG**] and [**H**].

3.1. Singular Locus of Dirac Operators.

In this section we discuss the singular locus for certain families of Dirac operators, i.e., the parameters for which the Dirac operator is singular. This will be crucial for us later.

Suppose $b_1(Y) = 0$. Since $H^2(Y; i\mathbf{R}) = 0$, we have a bounded right inverse $d^{-1} : \Omega_k \to \Omega^1_2(i\mathbf{R})$ for the operator d. Let θ be a fixed C^∞ flat connection on the determinant L. Then for any given ω, $A = \theta + d^{-1}(\omega)$ solves $F_A = \omega$. Define $\{\mathcal{D}(\alpha,\omega,t)\}$ to be the family of Dirac operators

$$\mathcal{D} : \mathcal{P} \times I_\varepsilon \to \mathrm{Fred}^0(\Gamma(S)), \quad \mathcal{D}(\alpha,\omega,t) = D^{g(t)}_{\theta+\alpha+d^{-1}(\omega)}.$$

Here Fred^0 denotes the Banach space of Fredholm operators of index zero.

In the case $b_1(Y) = 1$, we shall also define a family as follows: This time we keep the metric fixed, so we drop it from the notation. Let A_0 be a fixed C^∞ connection on L and denote by ω_0 its curvature. Fix a choice of non-zero $a_0 \in \mathbf{H}^1(i\mathbf{R})$ such that $\frac{i}{4\pi}a_0$ defines a generator for $H^1(\mathbf{Z})$. (The choice of constants here is so that a_0 is the class of a gauge change $2g^{-1}dg$.) Then the set $\{A_0 + ta_0 \,|\, t \in [0,1)\}$, parametrizes all the reducible $\mathrm{SW}_{\alpha,\omega_0}$ solutions up to gauge equivalence. Let $\mathcal{E}_k$, $k \geq 2$, denote the exact forms in Ω_k. Then on $\mathcal{E}_k$ we can as before define a bounded inverse $d^{-1} : \mathcal{E}_k \to \Omega^1_2(i\mathbf{R})$. Given $\omega \in \mathcal{E}_k$ then $A = A_0 + ta_0 + d^{-1}(\omega)$ solves $F_A = \omega_0$. Thus the set $\{A_0 + ta_0 + d^{-1}(\omega_0) \,|\, t \in [0,1)\}$ parameterizes up to gauge equivalence all the reducible $\mathrm{SW}_{\alpha,\omega+\omega_0}$-solutions, $\omega \in \mathcal{E}_k$. Define the family $\{\overline{\mathcal{D}}(\alpha,\omega,t)\}$ by

$$\overline{\mathcal{D}} : \mathcal{Q} \times \mathcal{E}_k \times I_\varepsilon \to \mathrm{Fred}^0(\Gamma(S)), \quad \overline{\mathcal{D}}(\alpha,\omega,t) = D_{A_0+ta_0+\alpha+d^{-1}(\omega)}.$$

Proposition 10. *Let $\mathcal{N}$ be the subset of $\mathcal{P} \times I_\varepsilon$ consisting of all (α,ω,t) for which $\mathcal{D}(\alpha,\omega,t)$ is singular. Similiarly define the subset $\mathcal{K}$ of $\mathcal{Q} \times \mathcal{E}_k \times I_\varepsilon$ for $\overline{\mathcal{D}}(\alpha,\omega,t)$. Then $\mathcal{N}$ and $\mathcal{K}$ are nowhere dense closed subspaces.*

Proof. We prove only the case for $\mathcal{N}$. The other is done similiarly. Let B be the unit L^2-ball in $\Gamma_2(S)$. Let $V \to \mathcal{P} \times I_\varepsilon \times B$ be the vector bundle whose fibre at (π,ϕ) is the real L^2-orthogonal to ϕ in $\Gamma_1(S)$. By evaluating $\mathcal{D}(\alpha,\omega,t)$ on $\phi \in B$ we obtain a section, call it $\mathcal{D}$, of V. We claim this section is transverse to the zero section. Let $\mathcal{D}(\alpha_0,\omega_0,t_0)\phi_0 = 0$. Let $\psi \in V_{\alpha_0,\omega_0,t_0,\phi_0}$ be L^2-orthogonal to the derivative $d\mathcal{D}$ at $(\alpha_0,\omega_0,t_0,\phi_0)$. By varying ϕ_0 in the tangent direction $\delta\phi$ we find $d\mathcal{D}(\delta\phi) = \mathcal{D}(\alpha_0,\omega_0,t_0)\delta\phi$; thus ψ must also satisfy $\mathcal{D}(\alpha_0,\omega_0,t_0)\psi = 0$ (since $\mathcal{D}(\alpha,\omega,t)$ is self-adjoint). On the other hand, by varying α_0, $d\mathcal{D}(\delta\alpha) = \delta\alpha \cdot_{t_0} \phi_0$ and if ψ is L^2-pependicular to this then $\psi = if\phi_0$ for some real function f. The condition $\mathcal{D}(\alpha_0,\omega_0,t_0)\psi = 0$ then leads to $df \cdot_{t_0} \phi = 0$; but since $\phi_0(x) \neq 0$ on an open set it must be that $df = 0$, and so f is a constant. Finally ψ being in V is necessarily L^2-orthogonal to ϕ_0; thus $f = 0$. Hence $\psi = 0$ and transversality holds and the zeros of $\mathcal{D}$ defines a smooth infinite dimensional submanifold $\mathcal{M}$ of $\mathcal{P} \times I_\varepsilon \times B$. The projection map $p : \mathcal{M} \to \mathcal{P} \times I_\varepsilon$ is proper since the kernel of the Dirac operator is always finite dimensional. Applying the Sard-Smale theorem we can concude that there exists an open dense set $\mathcal{O}$ in $\mathcal{P}\times I_\varepsilon$ with the property that $\mathcal{D}(\alpha,\omega,t)$, $(\alpha,\omega,t)\in\mathcal{O}$ has nullity ≤ 1 over the reals. But since this is a complex linear operator and self-adjoint, $\mathcal{D}(\alpha,\omega,t)$ must be non-singular. The proposition now follows. $\square$

Let $\pi_i=(\alpha_i,\omega_i)$, $i=0,1$, be given perturbations. Denote by $\{\pi(t)\}=\{(\alpha(t),\omega(t))\}$, $t\in I_\varepsilon$ the 1-parameter family of perturbations defined by $\pi(t)=(1-t)\pi_0+t\pi_1$. For a fixed value of π, $\mathcal{D}(\pi(t)+\pi,t)$ defines a 1-parameter family of Dirac operators. We use the notation $\{\mathcal{D}_\pi(t)\}$ for this 1-parameter family. We call this family *transverse* (for the choice of π) if $\mathcal{D}_\pi(0)$ and $\mathcal{D}_\pi(1)$ are non-singular and the family has transverse spectral flow, as t varies over $[0,1]$. Transverse spectral also includes the condition that multiple zero-eigenvalues do not occur as t varies. For the case $b_1(Y)=1$, fix a value of α_0. In a similar way we have a 1-parameter family $\{\overline{\mathcal{D}}_\pi(t)\}$ obtained by considering $\overline{\mathcal{D}}((\alpha_0,\omega_0)+\pi,t)$ for a fixed π. Transversality is defined in the same way as before.

Proposition 11. *Suppose $\{\mathcal{D}_0(t)\}$ has the property that $\mathcal{D}_0(t)$ is non-singular for $t=0,1$. Then there are arbitarily small π such that $\{\mathcal{D}_\pi(t)\}$ is a transverse family. A similiar statement holds for $\{\overline{\mathcal{D}}_\pi(t)\}$.*

Proof. We shall only prove the case of $\{\mathcal{D}_0(t)\}$. The other case is handled by essentially the same argument. Let $x=(\pi_0,t_0)\in\mathcal{P}\times I$, $I=(0,1)$. Consider the map

$$G:\Gamma_2(S)\times\mathcal{P}\times I\to\Gamma_2(S),\quad G(\phi,\pi,t)=\mathcal{D}(\pi,t)\phi.$$

The differential of G at $(0,\pi_0,t_0)$ is given by

$$dG(\delta\phi,\delta\omega,\delta t)=\mathcal{D}(\pi_0,t_0)\delta\phi.$$

Then $\ker dG=\mathcal{H}_{t_0}\oplus\mathcal{P}\oplus\mathbf{R}$ and $\operatorname{coker} dG=\mathcal{H}_{t_0}$. Here $\mathcal{H}_{t_0}$ denotes the kernel of $\mathcal{D}(\pi(t_0)+\pi_0,t_0)$ (acting on $\Gamma_2(S)$). By the implicit function theorem there is a neighbourhood V of $(0,x)\in\mathcal{H}_{t_0}\times\mathcal{P}\times I$ and a unique smooth map $f:V\to\mathcal{H}_{t_0}^\perp$ such that for $(\phi,\pi,t)\in V$,

$$(\mathrm{I}-\Pi)G(\phi+f(\phi,\pi,t),\pi,t)=0,\quad \Pi=L^2\text{-projection onto }\mathcal{H}_{t_0}. \tag{1}$$

Note that the linear extension of f in the ϕ variable continues to satisfy (1) so we may take V to be of the form $\mathcal{H}_{t_0}\times W$, W a neighbourhood of x. Because of (1) the injective/surjective properties of $\mathcal{D}(\pi,t)$, $(\pi,t)\in W$, are completely determined by the finite-dimensional operator finite dimensional operator $\mathcal{H}_{t_0}\to\mathcal{H}_{t_0}$,

$$T(\pi,t)\phi=\Pi G(\phi+f(\phi,\pi,t),\pi,t).$$

We claim $T(\pi,t)$ *is self-adjoint with respect to the* (*real* $\mathbf{C}$*-invariant*) L^2*-inner product on* $\mathcal{H}_{t_0}$. We introduce the notation $\langle\cdot,\cdot\rangle_{\mathcal{H}_{t_0}}$, $\psi,\phi\in\mathcal{H}_{t_0}$, to denote the L^2 inner product on $\mathcal{H}_{t_0}$ and $\langle\cdot,\cdot\rangle_{\Gamma(S)}$ the L^2-inner product on $\Gamma(S)$. We compute:

$$\begin{aligned}
\langle T(\pi,t)\phi,\psi\rangle_{\mathcal{H}_{t_0}} &= \langle G(\psi+f(\psi,\pi,t),\pi,t),\phi\rangle_{\Gamma(S)} \\
&= \langle G(\psi+f(\psi,\pi,t),\pi,t),\phi+f(\phi,\pi,t)\rangle_{\Gamma(S)} \\
&\quad \text{since } G(\psi+f(\psi,\pi,t),\pi,t)\in\mathcal{H}_{t_0},\ f(\phi,\pi,t)\in\mathcal{H}_{t_0}^{\perp} \\
&= \langle \psi+f(\psi,\pi,t),G(\phi+f(\phi,\pi,t),\pi,t)\rangle_{\Gamma(S)} \\
&\quad \text{by self-adjointness of } \mathcal{D}(\pi,t) \\
&= \langle\psi,T(\pi,t)\phi\rangle_{\mathcal{H}_{t_0}}.
\end{aligned}$$

This proves the claim.

Since $T(\pi,t)$ is complex linear and self-adjoint with respect to the real inner product, it is Hermitian with respect to the natural complex extension of the L^2-inner product on $\mathcal{H}_{t_0}$. Let $\mathrm{Herm}(\mathcal{H}_{t_0})$ denote the (real) vector space of Hermitian transformations on $\mathcal{H}_{t_0}$. The determinant function $\det:\mathrm{Herm}(\mathcal{H}_{t_0})\to\mathbf{R}\subset\mathbf{C}$ and $\det^{-1}(0)$ is a closed subvariety of codimension 1 in $\mathrm{Herm}(\mathcal{H}_{t_0})$. Introduce the notation

$$\mathcal{N}^{(k)}=\{l\in\mathrm{Herm}(\mathcal{H}_{t_0})\,|\,\dim\ker(l)\geq k\}.$$

Lemma 12. *Suppose* $\dim_{\mathbf{C}}(\mathcal{H}_{t_0})>0$. *The derivative* $(dT)_{\pi_0,t_0}|_{\mathcal{Q}}$ *of* T *restricted to* $\mathcal{Q}$ *has non-trivial image in* $\mathrm{Herm}(\mathcal{H}_{t_0})$. *If* $\dim_{\mathbf{C}}\mathcal{H}_{t_0}\geq 2$ *then this image is of dimension* ≥ 2.

First let us show that the image of $(dT)_{\pi_0,t_0}$ is non-trivial. The derivative at (π_0,t_0) is computed to be

$$dT_{\pi_0,t_0}(\delta\,\alpha,\delta t)\phi=\Pi(\delta\alpha+d^{-1}\delta\omega)\cdot_{t_0}\phi,\quad \pi=(\alpha,\omega).$$

Suppose that $\langle dT_{\pi_0,t_0}(\delta\alpha)\phi,\phi\rangle_{L^2}=0$ for all $\delta\alpha\in\mathcal{Q}$. Since

$$\langle\delta\alpha\cdot_{t_0}\phi,\phi\rangle_{L^2}=\int_{\widehat{Y}}\langle\delta\alpha,\sigma_{t_0}(\phi,\phi)\rangle.$$

This implies $\sigma_{t_0}(\phi(y),\phi(y))=0$ for all $y\in Y$ and thus $\phi=0$. Thus the image of dT_{π_0,t_0} is non-trivial.

Assume now that $\dim_{\mathbf{C}}(\mathcal{H}_{t_0})\geq 2$. Let $\delta\alpha$ be such that $dT_{\pi_0,t_0}(\delta\alpha)\neq 0$. Let $\phi_1,\ldots,\phi_n$ be an complex orthonormal basis for $\mathrm{Herm}(\mathcal{H}_{t_0})$ such that $dT_{\pi_0,t_0}(\delta\alpha)$ is diagonal with respect to this basis. Thus $\langle\delta\alpha\cdot\phi_i,\phi_j\rangle_{L^2,\mathbf{C}}=0$ for $i\neq j$. Since the ϕ_k are harmonic spinors, unique continuation implies that there is a open set in Y on which $\phi_i\neq\phi_j$, $i\neq j$ on this open set, in particular say at the point $y\in Y$. We can find a $\delta\alpha'$ with support in an arbitarily small neighbourhood of y such that $\int_{\widehat{Y}}\langle\delta\alpha'\cdot\phi_i,\phi_j\rangle_{\mathbf{C}}\neq 0$, $i\neq j$. Thus $dT_{\pi_0,t_0}(\delta\alpha')$ is independent of $dT_{\pi_0,t_0}(\delta\alpha)$. This shows that the image is at least 2-dimensional. This proves the Lemma.

Consider for each $(\alpha,\omega)\in W$,

$$\tau_{\alpha,\omega}(t)=T(\alpha,\omega,t),\quad |t-t_0|<\varepsilon$$

where $\varepsilon>0$ is chosen so that $(\alpha,\omega,t)\in W$. Clearly $\tau_{\alpha,\omega}$ defines a path in $\mathrm{Herm}(\mathcal{H}_{t_0})$.

By an open cover argument the following exists: (1) a finite set $\{t_1,\dots,t_n\}\subset[0,1]$ together with open neighbourhoods V_i of t_i in $\mathbf{R}$ such that $\{V_i\}_i^n$ covers $[0,1]$ (2) an open neighbourhood $W'\subset W$ of $0\in\mathcal{P}$ (3) maps $T^i: W'\times I_i\to\mathrm{Herm}(\mathcal{H}_{t_0})$ as in the preceding which preserves the injectivity/surjectivity properties of $\mathcal{D}(\alpha,\omega,t)$, $(\alpha,\omega,t)\in W'\times I_i$. We denote the corresponding paths by $\tau^i_{\alpha,\omega}(t)$.

Let $N=\max_i\{\dim_{\mathbf{C}}(\mathcal{H}_{t_i})\}$. Suppose $N>1$. Let j be such that $\dim(\mathcal{H}_{t_j})=N$. Note that $\mathcal{N}^{(N)}=\{0\}\subset\mathrm{Herm}(\mathcal{H}_{t_j})$. Thus any non-zero element in $\mathrm{Herm}(\mathcal{H}_{t_0})$ lies in $\mathcal{N}^{(k)}$, $0\le k<N$. Then by the above Lemma we can find a sufficiently small perturbation $(\alpha,0)\in W'$ so that $\tau^i_{\alpha,0}(t)\in\mathcal{N}^{(k')}$, $0\le k'<N$, $t\in I_i$ and for $i\neq j$, $\tau^i_{\alpha,0}(t)\in\mathrm{Herm}(\mathcal{H}_{t_i})$ for all $t\in I_i$. Thus we establish that there is an arbitarily small α so that $\tau^i_{\alpha,0}(t)\in\mathcal{N}^{(k)}$ with $0\le k<N$ for every i. Repeating the above construction over but with W' taken to be an open neighbourhood of $(\alpha,0)$ instead, we inductively prove that we can find an arbitarily small α' so that we have $N=1$. The perturbation argument in this case makes each path $\tau^i_{\alpha',0}$ transverse to $\mathcal{N}^{(1)}=\{0\}\subset\mathrm{Herm}(\mathcal{H}_{t_0})\cong\mathbf{R}$.

Let us show that $\{\mathcal{D}_{\alpha',0}(t)\}$ is a transverse family. Suppose at s, $\tau^i_{\alpha',0}(s)=0$. Let $\phi\in\mathcal{H}_{t_i}$ be unit length. Let $\lambda(t)$ be the 1-parameter family of eigenvalues satisfying $\mathcal{D}_{\alpha',0}(t)\phi=\lambda(t)\phi$ for t close to s. Thus we have

$$\langle\mathcal{D}_{\alpha',0}(t)\phi,\phi\rangle_{L^2}=\lambda(t)\langle\phi,\phi\rangle_{L^2}.$$

Differentiating this equation with respect to t and evaluating at $t=s$ gives

$$\langle dT_{\pi_0,t_0}(0,0,1)\phi,\phi\rangle_{L^2}=\lambda'(s).$$

The left hand term is simply the velocity of $\tau^i_{\alpha',0}$ at $t=s$ and transversality means this is non-zero. Thus $\lambda'(s)\neq 0$ and we have transverse spectral flow at $t=s$. $\square$

3.2. The Parameterized Moduli Space.

As before we assume the 1-parameter families $\{g(t)\}$, $t\in I_\varepsilon$. Define the *parametrized Seiberg-Witten section* to be the map

$$\tilde{s}:\Omega^0_2(i\mathbf{R})\times\mathcal{C}(P)\times\mathcal{P}\times I_\varepsilon\to\Omega^1_1(i\mathbf{R})\oplus\Gamma_1(S),$$
$$\tilde{s}(\eta,A,\Phi,\pi,t)=s_{\pi,g(t)}(\eta,A,\Phi).$$

Then the *parameterized moduli space* is $\mathcal{Z}(P)=\tilde{s}^{-1}(0)/\mathcal{G}$ (with $\mathcal{G}$ acting only on the $\mathcal{C}(P)$ factor). There is the projection map $p:\mathcal{Z}(P)\to\mathcal{P}\times I_\varepsilon$ and clearly $p^{-1}(\pi,t)=Z_\pi(P;g(t))$.

Proposition 13. *The irreducible part $\mathcal{Z}^*(P)$ of the parameterized moduli space is smooth Hilbert space manifold and the projection map $p|_{\mathcal{Z}^*(P)} : \mathcal{Z}^*(P) \to \mathcal{P} \times I_\varepsilon$ is a smooth Fredhom map of index zero.*

If $b_1(Y) = 0$ we determined in Prop. 10 that $p^{-1}(\pi)$ contains no reducibles if we are off the set $\mathcal{N} \subset \mathcal{P} \times I_\varepsilon$. If $b_1(Y) > 0$ we observed in Lemma 7 that there are no reducibles in $p^{-1}(\alpha, \omega)$ if and only if $[\frac{i}{\pi}\omega]$ is not an integral class in cohomology. Thus:

Lemma 14. *Let $q : \mathcal{P}_k \times I_\varepsilon \to H^2(Y; \mathbf{R})$, $q(\alpha, \omega, t) = [\frac{i}{2\pi}\omega]$ and set $\mathcal{W} = q^{-1}(c_1(L_{\mathbf{R}}))$. Then $p^{-1}(\pi, t)$ has reducibles if and only if $(\pi, t) \in \mathcal{W}$. $\mathcal{W}$ is a closed nowhere dense subset of of codimension equal to $b_1(Y)$.*

We remark that in the case $b_1(Y) > 0$ the condition of regularity (i.e., $H^2_{A,\Phi} = \{0\}$) for reducible solutions can never be satisfied. This is because when $\Phi = 0$, $H^2_{A,\Phi}$ reduces to $\mathbf{H}^2(i\mathbf{R}) \oplus \mathcal{H}_A$. So regularity implies *the absence of reducibles* in this case.

Corollary 15. *$p|_{\mathcal{Z}^*(P)} : \mathcal{Z}^*(P) \to \mathcal{P} \times I_\varepsilon$ is proper over $\mathcal{P} \times I_\varepsilon \backslash \mathcal{W}$ where* (i) *$W = \mathcal{N}$ if $b_1(Y) = 0$* (ii) *$W = \mathcal{W}$ if $b_1(Y) > 0$. Therefore for an open dense set $\mathcal{O} \subset \mathcal{P} \times I_\varepsilon$, $p^{-1}(z)$, $z \in \mathcal{O}$, is a finite set of regular points.*

Proof. The properness assertion is the content of Proposition 6 and the fact that a regular reducible point in the case $b_1(Y) = 0$ is necessarily isolated, by the Kuranishi local model. The Sard-Smale theorem then gives the 'open dense set' statement since regularity of an irreducible solution (A, Φ) is equivalent to the derivative $d\tilde{s}$ at $(0, A, \Phi)$ being surjective. (Note: without the properness assertion we can only conclude regularity on a Baire set.) $\square$

Proof of Proposition 13. We have to show that the derivative $d\tilde{s}$ is surjective at every point $(0, A_0, \Phi_0, \pi_0, t_0) \in \tilde{s}^{-1}(0)$ for which $\Phi_0 \neq 0$. Let (b, ψ) lie in the cokernel of ds_{0,A_0,Φ_0}, i.e., $(b, \psi) \in H^1_{A,\Phi}$, thus

$$(2) \qquad \text{(i) } db = \tfrac{1}{2}\sigma(\Phi, \psi), \quad \text{(ii) } D_A\psi + \tfrac{b}{2} \cdot \Phi = 0, \quad \text{(iii) } 2d^*b = i\langle i\Phi, \psi\rangle.$$

Suppose (b, ψ) is L^2-orthogonal to the image of $d\tilde{s}$. The Proposition is proven as soon as we can show $(b, \psi) = 0$. If $\delta\omega \in \Omega_k$, then $d\tilde{s}(\delta\omega) = (*\delta\omega, 0)$. Thus b must be L^2 orthogonal to all the co-closed forms; this implies that b must be closed. Then from (i) we obtain the condition $\sigma_{t_0}(\Phi_0, \psi) = 0$. Working at a point, the kernel of the transformation $v \mapsto \sigma_{t_0}(w, v)$ is of dimension 1 and it is easy to check that $\sigma_{t_0}(w, iw) = 0$. Therefore $\psi = if\Phi_0$ for some real valued function f. Putting this into (ii) of (2) we obtain

$$\begin{aligned} 0 &= D^{g(t_0)}_{A_0}(if\Phi) + \tfrac{b}{2} \cdot_{t_0} \Phi \\ &= (idf + \tfrac{b}{2}) \cdot \Phi \quad (\text{since } D^{g(t_0)}_{A_0}\Phi_0 = 0). \end{aligned}$$

Hence we obtain the pointwise condition $idf + \frac{b}{2} = 0$ on the open dense set $\mathcal{O}$ where $\Phi_0 \neq 0$. By continuity it holds on all of Y. Substituting into (iii) of (2) we get the equation

$$4\Delta f = -|\Phi|^2 f.$$

Taking the product with f and integrating we obtain:

$$\int_Y 4|df|^2 + |\Phi|^2|f|^2 = 0.$$

Thus $f = 0$ on $\mathcal{O}$ and therefore Y and we finally obtain $(b, \psi) = 0$. Finally the index zero assertion follows directly from Lemma 4. $\square$

4. Proof of Theorem 1.

Y is assumed to be a closed oriented 3-manifold with Riemannian metric g and spin-c structure $P \to Y$. According to Corollary 15 applied to the constant family $\{g(t) = g\}$, we may choose a perturbation π from an open dense set in $\mathcal{P}_k$ $(k \geq 3)$ such that $Z_{\pi,g}(P)$ consists of a finite set of regular points, i.e., the cohomology $H^2_{A,\phi}$ is trivial at these points. For this π, if $b_1(Y) = 0$ there is a unique isolated reducible (up to gauge equivalence) and if $b_1(Y) > 0$ there are no reducibles. $Z^*_{\pi,g}(P)$ is then naturally oriented by our conventions (Proposition 9) and we can form the algebraic sum

$$\#Z_{\pi,g}(P).$$

To prove the claimed invariance properties of $\#Z_{\pi,g}(P)$, let g_0, g_1 be two metrics on Y and let π_i be two perturbations which satisfy the above with respect to g_i. We want to relate $\#Z_{\pi_0,g_0}(P)$ and $\#Z_{\pi_1,g_1}(P)$. Consider the 1-parameter family of metrics $\{g(t)\} = \{(1-t)g_0 + tg_1\}$ defined for $t \in I_\varepsilon = (-\varepsilon, 1+\varepsilon)$. Thus as in Section 3 we have a parameterized moduli space $\mathcal{Z}(P)$ and projection map $p : \mathcal{Z}(P) \to \mathcal{P}_k \times I_\varepsilon$.

We consider a smooth path $\sigma : [0,1] \to \mathcal{P}_k \times I_\varepsilon$, $\sigma(0) = (\pi_0, g_0)$, $\sigma(1) = (\pi_1, g_1)$. We introduce the notation $Z_\sigma(P)$ for the σ-parametrized moduli space $\{(x,t) \,|\, x \in p^{-1}(\sigma(t))\}$. If a portion of σ misses the 'singular' sets $\mathcal{N}$ or $\mathcal{W}$ of Sec.3.1 and is transverse p, then that portion of $Z^*_\sigma(P)$ consists purely of regular points and therefore is a smooth arc. This is oriented in the following way. The local deformation theory of $Z_\sigma(P)$ is described by an elliptic complex of the form

$$\Omega^0_3(i\mathbf{R}) \to \Omega^0_2(i\mathbf{R}) \oplus \Omega^1_2(i\mathbf{R}) \oplus \Gamma_2(S) \oplus \mathbf{R} \to \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S).$$

Therefore the orientation is determined by looking at the 'wrapped up' operator

$$L_{\eta,A,\Phi,t} : \Omega^0_2(i\mathbf{R}) \oplus \Omega^1_2(i\mathbf{R}) \oplus \Gamma_2(S) \oplus \mathbf{R} \to \Omega^0_1(i\mathbf{R}) \oplus \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S).$$

The orientation of the regular irreducible points of $Z_\sigma(P)$ is determined by an orientation of the determinant of the index of the family $\{L_{\eta,A,\Phi,t}\}$. We have the short exact sequence of 2-step complexes:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0_2(i\mathbf{R}) \oplus \Omega^1_2(i\mathbf{R}) \oplus \Gamma_2(S) \arrow[r, "L_{\eta,A,\Phi}"] \arrow[d] & \Omega^0_1(i\mathbf{R}) \oplus \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^0_2(i\mathbf{R}) \oplus \Omega^1_2(i\mathbf{R}) \oplus \Gamma_2(S) \oplus \mathbf{R} \arrow[r, "L_{\eta,A,\Phi,t}"] \arrow[d] & \Omega^0_1(i\mathbf{R}) \oplus \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{R} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

This gives rise to a canonical isomorphism

$$(3) \qquad h : \ker L_{\eta,A,\Phi,t} \oplus \operatorname{coker} L_{\eta,A,\Phi} \to \ker L_{\eta,A,\Phi} \oplus \mathbf{R} \oplus \operatorname{coker} L_{\eta,A,\Phi,t}.$$

An orientation for $\det \operatorname{Ind}\{L_{\eta,A,\Phi}\}$ defines an orientation for $\det \operatorname{Ind}\{L_{\eta,A,\Phi,t}\}$ according to this rule: Choose an orientation for $\operatorname{coker} L_{\eta,A,\Phi}$. Then an orientation of $\ker L_{\eta,A,\Phi}$ is determined, since $\det \operatorname{Ind}\{L_{\eta,A,\Phi}\}$ is oriented. Now given an orientation of $\operatorname{coker} L_{\eta,A,\Phi,t}$, then $\ker L_{\eta,A,\Phi,t}$ is oriented so that h is an orientation-preserving isomorphism, where the domain and range spaces are given the product orientation in the order written in (3). With this orientation convention, if Z_σ consists entirely of regular irreducible points and if compact then its boundary is precisely $Z_{\sigma(1)}(P) - Z_{\sigma(0)}(P)$, as oriented spaces.

The proof of Theorem 1 in the case $b_1(Y) > 1$ can now be easily established. By Lemma 14 σ may be chosen to be disjoint from the subset of $\mathcal{W}$ for which $Z_\sigma(P)$ has reducibles. Furthermore σ can be assume to be transverse to the projection $p : \mathcal{Z}(P) \to \mathcal{P}_k \times I_\varepsilon$. Thus $Z_\sigma(P)$ defines a smooth compact oriented cobordism between $Z_{\pi_0,g_0}(P)$ and $Z_{\pi_1,g_1}(P)$. This proves the invariance of $\#Z_{\pi,g}(P)$ in this case.

This argument extends to the cases $b_1(Y) = 0, 1$ provided $(\pi_0, 0)$ and $(\pi_1, 1)$ can be connected by a path which missed the 'bad' sets $\mathcal{N}$, $\mathcal{W}$ of Sect. 3.1, Lemmma 14 respectively. However this is not generally true, as we shall describe below.

4.1. The case $\mathbf{b}_1(Y) = 0$.

The argument in the case $b_1(Y) > 1$ may fail here due to the presence of a reducible (unique up to gauge) solution in each $Z_{\sigma(t)}(P)$. The reducible stratum of $Z_\sigma(P)$ is an arc which under p projects diffeomorphically onto I_ε. The path σ may meet the subset $\mathcal{N}$ of Prop. 10 and singularities may occur in Z_σ. Choose σ to be the path defined by the family $\{g(t)\}$ and the family of perturbations $\{\pi(t)\} = \{(t-1)\pi_0 + t\pi_1\}$, $t \in I_\varepsilon$. Fix θ a flat connection on L. Writing $\pi(t) = (\alpha(t), \omega(t))$, in the notation of Sec. 3.1, the reducible solution up to equivalence in $Z_{\sigma(t)}(P)$ is given by $\theta(t) = \theta + d^{-1}(\omega(t))$. Furthermore the associated family of Dirac operators $\{\mathcal{D}_0(t)\}$ determine the cohomology group $H^2_{\theta(t)}$. According to Prop. 11, by an arbitarily small perturbation π we can make this a transverse family $\{\mathcal{D}_\pi(t)\}$. This corresponds to deformation of σ as $\sigma+\pi$ which in turn is induced by perturbations $(\pi_0+\pi,0)$ and $(\pi_1+\pi,0)$ of the end-points of σ. If π is sufficiently small then $\#Z_{\pi_i+\pi}(P)$ coincides with $\#Z_{\pi_i}(P)$; thus without loss we may absorb π and simply assume transversality for $\pi=0$. Then spectral flow for $\{\mathcal{D}(t)\}=\{\mathcal{D}_0(t)\}$ occurs at exactly the values of t where $\sigma(t)$ meets $\mathcal{N}$.

Proposition 16. *Assume $\{\mathcal{D}(t)\}$ is a transverse family and σ is transverse to the projection $p: Z_\sigma(P)\to\mathcal{P}_k\times I_\varepsilon$ away from $\mathcal{N}$. Let $\sigma\cap\mathcal{N}=\{\sigma(t_i)\}_{i=1}^n$. Then for each t_i there is a open neighbourhood N_i of $\theta(t_i)$ such that:*

(i) $\overline{Z_\sigma(P)\setminus\cup N_i}$ *is a smooth compact* 1*-manifold with boundary*
(ii) $Z_\sigma(P)\cap N_i$ *is diffeomorphic to the zeros of the map* $\mathbf{R}\times\mathbf{R}^+\to\mathbf{R}$, $(t,\xi)\mapsto t\xi$
(iii) $Z^*_{\sigma+\pi}(P)\cap N_i=0\times\mathbf{R}^+$ *and the orientation of* $0\times\mathbf{R}^+$ *is* $-\varepsilon_i\frac{\partial}{\partial\xi}$ *where* ε_i *is the sign of the spectral flow of* $\{\mathcal{D}(t)\}$ *at* t_i.

An immediate consequence of this is the formula

$$\#Z_{\sigma(1)}(P)-\#Z_{\sigma(0)}(P)=-\mathrm{SF}\{\mathcal{D}(t)\}$$

where 'SF' on the right denotes the total spectral flow as t varies from 0 to 1. Notice that the left-hand term is actually independent of choice of path, therefore the spectral-flow term only depends on the end-points of the path. (In fact, it is possible to verify this directly as well.)

To define an invariant in this case it is necessary to introduce a counter-term which should be a function of ω and g which has the same change as $\#Z_{\pi,g}(P)$ as we cross from one connected component of $\mathcal{P}_k\times I_\varepsilon\setminus\mathcal{N}$ to another. Such a function can be obtained from the spectral invariants of [**APS**].

Proposition 17. *Assume $b_1(Y)=0$. Let (α,ω,g) be given. Let θ be the unique (up to gauge) flat connection on L and let a be defined by the condition $d^*a=0$, $da=\omega$. Define*

$$\begin{aligned}\zeta(\alpha,\omega,g)&=\frac{1}{8}\eta\left(d*-*d|_{\Omega^{\mathrm{even}}},g\right)\\&\quad+\frac{1}{2}\left(\dim_{\mathbf{C}}\ker D^g_{\theta+a+\alpha}+\eta\left(D^g_{\theta+a+\alpha}\right)\right)+\frac{1}{32\pi^2}\int_Y(a+\alpha)\wedge d(a+\alpha)\end{aligned}$$

where η denotes the Atiyah-Patodi-Singer spectral invariant of the associated operator. Then:

(i) $\zeta(\omega,g)$ *lies in* $\mathbf{Z}\left[\frac{1}{8|H_1(Y;\mathbf{Z})|}\right]$; *if in addition* $H_1(Y,\mathbf{Z})=0$ *it lies in* $\mathbf{Z}$
(ii) *given the path* σ *as Prop.* 16, *we have* $\zeta(\sigma(1))-\zeta(\sigma(0))=SF\{\mathcal{D}(t)\}$.

Thus we see that the combination

$$\# Z_{\pi,g}(P) + \zeta(\pi, g)$$

defines a topological invariant in the case $b_1(Y) = 0$.

Proof of Proposition 17. Every spin-c structure on Y is obtained by tensoring a spin structure on Y with a complex line bundle. By a Theorem of Milnor every spin Y is the oriented spin boundary of an oriented spin 4-manifold X with $b_1(X) = 0$. Every complex line bundle over Y can be extended over X; therefore we may assume a spin-c structure $P' \to X$ which induces the given $P \to Y$. We may also assume X to have a metric which near the boundary which is a product $Y \times [0, \varepsilon)$ of an interval and the metric on Y and with orientation $dy \wedge dt$.

We may extend the connection θ over $L(P') = \det(P')$ as the connection Θ, a as $\hat{a}$ and α as $\hat{\alpha}$ over X. These extensions can be taken to be products over $Y \times [0, \varepsilon)$. The index theorem of [**APS**] applied to the Dirac operator $D^g_{\Theta+\hat{a}+\hat{\alpha}}$ over X associated to P' gives:

$$\begin{aligned}\operatorname{Index} D^g_{\Theta+\hat{a}+\hat{\alpha}} = \int_X \exp\left(\frac{1}{2} c_1(\Theta + \hat{a} + \hat{\alpha})\right) \hat{A} \\ - \frac{1}{2}\left(\dim_{\mathbf{C}} \ker D^g_{\theta+a+\alpha} + \eta\left(D^g_{\theta+a+\alpha}\right)\right).\end{aligned}$$

Here

$$c_1(\Theta + \hat{a} + \hat{\alpha}) = \frac{i}{2\pi} F_{\Theta+\hat{a}+\hat{\alpha}}$$

and $\hat{A}$ is the $\hat{A}$-polynomial in the Pontrjagin classes. On the other hand consider the signature operator on X. This has index

$$\operatorname{sig}(X) = \int_X L - \eta(d* - *d|_{\Omega^{\mathrm{even}}}, g).$$

L is the Hirzebruch L-polynomial in the Pontrjagin classes. Since

$$\begin{aligned}\exp\left(\frac{1}{2} c_1(B)\right) &= 1 + \frac{1}{2} c_1(B) + \frac{1}{8} c_1(B) \wedge c_1(B) + \ldots, \\ \hat{A} &= 1 - \frac{1}{24} p_1 + \ldots, \\ L &= 1 + \frac{1}{3} p_1 + \ldots,\end{aligned}$$

the above index formulas give

$$\begin{aligned}&\frac{1}{8}\eta(d* - *d|_{\Omega^{\mathrm{even}}}, g) + \frac{1}{2}\left(\dim_{\mathbf{C}} \ker D^g_{\theta+a} + \eta(D^g_{\theta+a})\right) \\ &+ \frac{1}{32\pi^2} \int_Y (a + \alpha) \wedge d(a + \alpha)\end{aligned}$$

$$= \frac{1}{8}\int_X c_1(\Theta)\wedge c_1(\Theta) - \frac{1}{8}\mathrm{sig}(X) - \mathrm{Index}\, D^g_{\Theta+\hat{a}+\hat{\alpha}}.$$

If Y is an integral homology sphere then L is trivial and we may choose its extension over X as the trivial bundle; therefore Θ in this case may be assumed trivial. Furthermore the intersection form on X is then unimodular so $\mathrm{sig}(X)$ is divisible by 8. Thus ζ is an integer. When Y is not an integral homology sphere then the term $\int_X c_1(\Theta)\wedge c_1(\Theta)$ depends only on the topological type of the extension of L over X. It can be identified with the $\mathbf{Z}\left[\frac{1}{|H_1(Y;\mathbf{Z})|}\right]$-intersection product of the class $[c_1(\Theta)] \in H^2(X;\mathbf{Z})/\mathrm{torsion}$ with itself. Therefore ζ takes values in $\mathbf{Z}\left[\frac{1}{8|H_1(Y;\mathbf{Z})|}\right]$.

The term $\eta(d*-*d|_{\Omega^{\mathrm{even}}}, g)$ depends continuously on g whereas according to [**APS**] $\frac{1}{2}\eta(D^g_{\theta+a+\alpha})$ jumps by the spectral flow. Thus ζ has the correct behaviour as we cross components of $\mathcal{P}_k \times I_\varepsilon \backslash \mathcal{N}$, as claimed. □

Proof of Proposition 16. The proof of the proposition relies on a detailed understanding of the Kuranishi local model at the singular points $[\theta(t_i)]$. Without loss of generality we assume that there is only one value $t = t_1$ where $\{\mathcal{D}(t)\}$ is singular. The Seiberg-Witten section which gives $Z_\sigma(P)$ is of the form $\hat{s} : \Omega^0_2(i\mathbf{R}) \times \mathcal{C}(P) \times \mathbf{R} \to \Omega^1_1(i\mathbf{R}) \oplus \Gamma_1(S)$,

$$\hat{s}(\eta, A, \Phi, t) = \left(*_t(F_A - \tfrac{1}{4}\sigma_t(\Phi,\Phi) - \omega(t)), D^{g(t)}_{A+\alpha(t)}\Phi - \eta\Phi\right)$$

where the 't' in the notation denotes a dependence on t. (Note: just as in Sec. 3 we work with a fixed $P \to Y$ with respect to a basepoint metric.) The linearization of $\hat{s}$ at $\eta = 0$, $A = \theta(t_1)$, $\Phi = 0$, $t = t_1$ is given by

$$d\hat{s}(\delta\eta, \delta a, \delta\phi, \delta t) = \left(*_{t_1}(d(\delta a) + \omega'(t_1)\delta t), \mathcal{D}(t_1)\delta\phi\right).$$

Let $X_{\theta(t_1),0}$ be the slice of the gauge group action on $\mathcal{C}(P)$ at $(0, \theta(t_1), 0)$ (Sec. 2). Then

$$\begin{aligned} \ker(d\hat{s}) \cap (\Omega^0_2(i\mathbf{R}) \times X_{\theta(t_1),0} \times \mathbf{R}) &= \mathbf{H}^0(i\mathbf{R}) \oplus \mathcal{H}_{\theta(t_1)} \oplus \mathbf{R}_t \\ \mathrm{coker}\,(d\hat{s}) &= \mathcal{H}_{\theta(t_1)}. \end{aligned}$$

Here $\mathbf{R}_t = \mathrm{span}\{(d^{-1}(\omega'(t_1)), 1)\}$ in the $\Omega^1_2(i\mathbf{R}) \oplus \mathbf{R}$ factor. The Kuranishi obstruction map then takes the form

$$\Xi : i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t \supset U \to \mathcal{H}_{\theta(t_1)}.$$

This gives $\hat{s}^{-1}(0)/\mathcal{G}$ near $(0, \theta(t_1), 0, t_1)$ as $\Xi^{-1}(0)/S^1$. A direct verification shows that $\Xi^{-1}(0) \supset (i\mathbf{R} \times 0 \times \mathbf{R}_t) \cap U$. The subset $(i\mathbf{R} \times 0 \times 0) \cap U$ consists of 'virtual' Seiberg-Witten solutions and thus should be ignored to get the Seiberg-Witten moduli space proper (Lemma 3). The subset $(0 \times 0 \times \mathbf{R}_t) \cap U$ are the reducible solutions near $(0, \theta(t_1), 0, t_1)$. Our assumption on σ being transverse to p away from W means that the closure of the irreducible part of $\Xi^{-1}(0)/S^1$, is a compact 1-manifold with boundary except possibly at $(0,0,0)$. Furthermore $\Xi^{-1}(0) \cap U \cap (\mathbf{R} \times 0 \times \mathbf{R}_t - (0,0,0)) = \emptyset$.

By construction the derivative of Ξ at $(0,0,0)$ is the zero map. We aim to compute the second derivative: This will give us the quadratic approximation to Ξ which will be sufficient for our purposes. In the following, we identify $\mathbf{R}_t$ with $\mathbf{R}$ via $t \mapsto (d^{-1}(\omega'(t_1)), 1)t$.

Claim 18. The second derivative of Ξ at $(0,0,0)$ is given by

$$D^2\Xi(\delta\eta, \delta\phi, \delta t) = c\delta t\delta\phi - \delta\eta\delta\phi$$

where c is a non-zero real constant and has the same sign as that of the spectral flow of $\{\mathcal{D}(t)\}$ at $t = t_1$.

To prove the claim: the obstruction map Ξ is constructed as a composition of the form $x \mapsto \Pi \circ \hat{s}(x + f(x))$ where $x \in \mathcal{O} \subset i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t$ and $f : \mathcal{O} \to (i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t)^{\perp}$ is given by the implicit function theorem. As such its derivative at 0 is the zero map. It is then seen that $D^2\Xi$ is given by $\Pi \circ D^2\hat{s}$. This is given by the expression

$$D^2\Xi(\delta\eta, \delta\phi, \delta t) = \Pi\left(\mathcal{D}'(t_1)(\delta t\delta\phi)\right) - \delta\eta\delta\phi.$$

The map $\delta\phi \mapsto \Pi(\mathcal{D}'(t_1)\delta\phi)$ defines a Hermitian transformation on $\mathcal{H}_{\theta(t_1)} \cong \mathbf{C}$ with respect to the complex L^2-inner product. Thus it is multiplication by a real constant $c = \langle\mathcal{D}'(t)v, v\rangle$, v being of unit length. Our assumption on σ was that at $t = t_1$ a single eigenvalue $\lambda(t)$, $|t - t_1| < \delta$, for $\mathcal{D}(t)$ changed from negative to positive or vice-versa. In the first case the spectral flow is $+1$ and in the latter -1. We have a 1-parameter family of unit eigenvectors $v(t)$, $|t - t_1| < \delta$, such that

$$\mathcal{D}(t)v(t) = \lambda(t)v(t).$$

Differentiating this equation at $t = t_1$ and taking the inner product with $v(t_1)$ we obtain using self-adjointness of $\mathcal{D}(t)$,

$$\langle\mathcal{D}'(t_1)v(t_1), v(t_1)\rangle = \lambda'(t_1).$$

Thus the sign of the spectral flow is seen to be same as that of $\lambda'(t_1)$. This proves the Claim.

To continue the proof of Proposition 16: by the Claim, $\Xi(\eta, \phi, t)$ is approximated up to second order by

$$(ct - \eta)\phi.$$

The zeros of the quadratic approximation fall into two branches: $0 \times 0 \times \mathbf{R}_t$ and $0 \times \mathcal{H}_{\theta(t_1)} \times 0$. We claim that these two branches gives a complete picture of the zeros of Ξ near $(0,0,0)$. To get a clearer picture, consider restricting the ϕ variable to the real span of a fixed non-zero vector in $\mathcal{H}_{\theta_0}$. Call this map $\tilde{\Xi}$. Then $\Xi^{-1}(0)/S^1 = \tilde{\Xi}^{-1}(0)/\pm 1$. Since $\tilde{\Xi}(\eta, 0, t) = 0$ we may factor out the branch $\{\phi = 0\}$ by setting $\tilde{\Xi}(\eta, \phi, t) = \phi.\Theta(\eta, \phi, t)$. Using

Claim 18 the linearization of Θ is seen to be $d\Theta(\delta\eta, \delta t, \delta\phi) = (c\delta t - \delta\eta)$. We have $\ker d\Theta = \{\delta t = \delta\eta = 0\}$. Invoking the Implicit Function Theorem we see that near $(0,0,0)$, $\Theta^{-1}(0)$ is a smooth arc tangent to $\{\phi = 0\}$ at $(0,0,0)$. This demonstrates the claimed local structure near the singular point. For later we note the following: the implicit function theorem gives a map $G : \mathcal{H}_{\theta(t_1)} \to \mathbf{R}_t = \{0\} \times \mathbf{R}_t \subset i\mathbf{R} \times \mathbf{R}_t$ such that the closure of the irreducible part of $\Xi^{-1}(0)$ is given by $\mathrm{graph}(G) = \{(0, \phi, G(\phi)) \mid \phi \in \mathcal{H}_{\theta(t_1)}\}$.

What remains is to determine the orientation of the above arc of irreducible solutions. We have an orientation of $\det \mathrm{Ind}\{L_{\eta,A,\Phi,t}\}$ determined by that of $\det \mathrm{Ind}\{L_{\eta,A,\Phi}\}$ according to the map h of (3). (See discussion following there.) Since we are working at a point where $\Phi = 0$, the orientation of $\det \mathrm{Ind}\{L_{\eta,A,\Phi}\}$ is determined by the kernel and cokernel of $L_{\theta(t_1),0}$; namely $\mathbf{H}^0(i\mathbf{R}) \oplus \mathcal{H}_{\theta(t_1)}$. The long exact sequence inducing h takes the form

$$0 \to \mathbf{H}^0(i\mathbf{R}) \oplus \mathcal{H}_{\theta(t_1)} \to \ker L_{\theta(t_1),0,t_1} \xrightarrow{\kappa} \mathbf{R} \to \mathbf{H}^0(i\mathbf{R}) \oplus \mathcal{H}_{\theta(t_1)} \xrightarrow{\beta} \mathrm{coker}\, L_{\theta(t_1),0,t_1} \to 0.$$

The map κ in the sequence sends the subspace $\mathbf{R}_t$ isomorphically onto the the target space. This isomorphism sends $(1, d^{-1}(\omega'(t_1)))r \mapsto r$, $r \in \mathbf{R}$. Choose an orientation of $\mathrm{coker}\, L_{\theta(t_1),0} = \mathbf{H}^0(i\mathbf{R}) \oplus \mathcal{H}_{\theta(t_1)}$; then since β in the sequence is an isomorphism, our orientation convention dictates that $\mathrm{coker}\, L_{\theta(t_1),0,t_1}$ has the orientation induced by β, and the orientation on $\ker L_{\theta(t_1),0,t_1}$ is the product orientation $\ker L_{\theta(t_1),0} \oplus \mathbf{R}_t$, where $\mathbf{R}_t$ is oriented via κ.

In order to determined orientations in the local Kuranishi picture correctly we shall need to combine the obstruction map Ξ with a local slice condition coming from the S^1-action, which is the inverse of complex multiplication on the $\mathcal{H}_{\theta(t_1)}$ factor. The set $\mathrm{graph}(G) \subset \Xi^{-1}(0)$ represents the closure of the S^1-orbits of the irreducible solutions near $(0,0,0)$. Let $0 \neq v \in \mathcal{H}_{\theta(t_1)}$. Then the linearization of the S^1-action at $(0, v, G(v))$ is a map $i\mathbf{R} \to i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t$, $\delta\gamma \mapsto (0, -(\delta\gamma)v, 0)$. The adjoint of this map sends $(\delta\eta, \delta\phi, \delta t) \mapsto -i\langle iv, \delta\phi\rangle$. Therefore a further local description for $\Xi^{-1}(0)/S^1$ near $(0, v, G(v))$ is the zeros of the map

$$\chi : i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t \to i\mathbf{R} \times \mathcal{H}_{\theta(t_1)}, \quad \chi(\eta, \phi, t) = (-i\langle iv, \phi\rangle, \Xi(\eta, \phi, t)).$$

In what follows, we may for simplicity assume $G = 0$; the result for general G is obtained by working sufficiently close to $(0,0,0)$ where $\mathrm{graph}(G)$ is approximated to arbitarily high order by $0 \times \mathcal{H}_{\theta(t_1)} \times 0$. With this assumed, the irreducible zeros of χ is the set of positive multiples of $(0, v, 0)$. The normal bundle to $0 \times \mathcal{H}_{\theta(t_1)} \times 0$ at $(0,0,0)$ is $i\mathbf{R} \times 0 \times \mathbf{R}_t$. This is mapped via $d\Theta$ isomorphically onto $\mathcal{H}_{\theta(t_1)}$. If we pull-back the complex orientation by $d\Theta$, then the induced orientation on $i\mathbf{R} \times 0 \times \mathbf{R}_t$ is $c\delta\eta \wedge \delta t$. By continuity this is carried to the point v as the same orientation. The last remaining direction to the normal bundle of $\chi^{-1}(0)$ at v is given by $(0, iv, 0)$. This is mapped by $d\chi$ to $(-i\langle iv, iv\rangle, 0) = (-i, 0)$. Let us take the product orientation on $i\mathbf{R} \times \mathcal{H}_{\theta(t_1)}$ for the range space of χ, in this order (the final answer is independent of this choice); then our orientation convention dictates that the domain space of χ is oriented in the order $i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t$. Then the pull-back of the orientation on $i\mathbf{R} \times \mathcal{H}_{\theta(t_1)}$ to the normal bundle of $\chi^{-1}(0)$ at v is $-\delta\theta \wedge c\delta\eta \wedge \delta t$, where $\delta\theta$ is the $(0, iv, 0)$ direction. Let δr be the direction given by v, and $\varepsilon\delta r$ the induced orientation of $\chi^{-1}(0)$ near v. Then we require that the orientation on $\chi^{-1}(0)$ followed by the orientation in the normal direction equals the orientation on $i\mathbf{R} \times \mathcal{H}_{\theta(t_1)} \times \mathbf{R}_t$, that is

$$\varepsilon\delta r \wedge (-\delta\theta \wedge c\delta\eta \wedge \delta t) = \delta\eta \wedge \delta r \wedge \delta\theta \wedge \delta t.$$

This shows the induced orientation on $\chi^{-1}(0)$ as $-c\delta r$, as claimed.

4.2. The case $\mathbf{b}_1(Y) = 1$.

This case is similiar but with some slight differences to $b_1(Y) = 0$. Any path σ which connects (π_0, g_0) to (π_1, g_1) in $\mathcal{P}_k \times I_\varepsilon$ may cross the codimension 1 subset $\mathcal{W}$. At the points where σ meets $\mathcal{W}$ the corresponding $Z_{\sigma(t)}(P)$ will admit an S^1's worth of reducibles, otherwise $Z_{\sigma(t)}(P)$ contains no reducibles. As following our notation conventions, $\sigma(t)$ in components is $(\pi(t), g(t))$ or $(\alpha(t), \omega(t), g(t))$.

We may by general transversality arguments assume that σ meets $\mathcal{W}$ transversely and orthogonally and transverse to the projection $p : \mathcal{Z}^*(P) \to \mathcal{P}_k \times I_\varepsilon$ away from $\mathcal{W}$. Let $\{t_i\}$ be the finite set of values for which $\sigma(t_i) \in \mathcal{W}$. To simplify matters even more, since $\mathcal{W}$ the preimage of a set in $\mathcal{P}_k$ we may assume near $\mathcal{W}$ that σ lies in the subset $\mathcal{P}_k \times \{t_i\}$. Hence for values near t_i, the metric represented by σ is unchanging.

We can always find connections A_i such that $F_{A_i} = \omega(t_i)$. Using the value $\alpha(t_i)$, we can as in Sec. 3.1 form the 1-parameter family of operators $\{\overline{\mathcal{D}}^i_{\pi_i}(s)\}$. By Prop. 11 we can make this family transverse by an arbitarily small perturbation π_i. This perturbation can be achieved by a perturbation of σ, supported for values of t near t_i, and maintaining the original properties of σ. Thus we can assume $\{\overline{\mathcal{D}}^i_0(s)\}$ is a transverse family and we drop the '0' subscript notation. Finally let $s_{i,j}$ be the values of s for which $\{\overline{\mathcal{D}}^i(s)\}$ has spectral flow. Denote by $A_{i,j}$ the connection $A_i + s_{i,j}a_0$.

A technical issue which will be significant is the orientation of the family $\{\overline{\mathcal{D}}^i(s)\}$. Looking back at the definition in Sec. 3.1 we see that this involved a certain choice of a non-zero element a_0 in $\mathbf{H}^1(i\mathbf{R})$. We shall make a specific choice for each i. The assumption that σ meets $\mathcal{W}$ orthogonally means in particular that the derivative $\omega'(t_i)$ is L^2-orthogonal to the exacts. Thus $d^*(\omega'(t_i)) = 0$ so $*\omega'(t_i)$ is closed. For a chosen i we now make the convention that the a_0 should be a positive multiple of $[-*\omega'(t_i)]$.

Proposition 19. *For each* (i,j) *there is a open neighbourhood* $N_{i,j}$ *of* $[A_{i,j}]$ *such that* (i) $\overline{Z_\sigma(P)\setminus \cup N_{i,j}}$ *is a smooth compact* 1*-manifold with boundary* (ii) $N_{i,j}$ *is diffeomorphic to the zeros of the map* $\mathbf{R}\times\mathbf{R}^+ \to \mathbf{R}$, $(s,\xi)\mapsto s\xi$ *with* $Z^*_\sigma(P)\cap N_{i,j} = 0\times\mathbf{R}^+$ (iii) *the orientation of* $0\times\mathbf{R}^+$ *is* $\varepsilon_{i,j}\frac{\partial}{\partial\xi}$ *where* $\varepsilon_{i,j}$ *is the sign of the spectral flow of* $\{\overline{\mathcal{D}}^i(s)\}$ *at* $s_{i,j}$*, as* s *varies from* 0 *to* 1*.*

Proof. This largely proceeds in the manner of the case $b_1(Y)=0$. We continue to use notation introduced there. Again, without loss, we may assume σ meets $\mathcal{W}$ exactly once, at $t=t_1$. We consider again the map $\hat{s}$ of the case $b_1(Y)=0$. Computed at $\eta=0$, $A=A_{1,j}$, $\Phi=0$, this time we find

$$\begin{aligned}\ker(d\hat{s})\cap(\Omega^0_2(i\mathbf{R})\times X_{A_{1,j}}\times\mathbf{R}) &= i\mathbf{R}\oplus H^1(i\mathbf{R})\oplus\mathcal{H}_{A_{1,j}}\\ \operatorname{coker}(d\hat{s}) &= \mathcal{H}_{A_{1,j}}.\end{aligned}$$

We have the obstruction map $\Xi : i\mathbf{R}\times H^1(i\mathbf{R})\times\mathcal{H}_{A_{1,j}} \to \mathcal{H}_{A_{1,j}}$ whose second derivative at $(0,0,0)$ is

$$D^2\Xi(\delta\eta,\delta a,\delta\phi) = \frac{1}{2}\Pi(\delta a\cdot\delta\phi) - \delta\eta\delta\phi.$$

Then if we let $\delta a = -*\omega'(t_1)\delta t$, the term

$$\frac{1}{2}\Pi(\delta a\cdot\delta\phi) = \frac{1}{2}\delta t\Pi(-*\omega'(t_1)\cdot\delta\phi) = c\delta t\delta\phi$$

where c is a non-zero real constant with the same sign of the spectral flow $\{\overline{\mathcal{D}}^i(s)\}$ at $s_{1,j}$. As in the $b_1=0$ case, the irreducible zeros of Ξ are modelled by the subset $0\times 0\times\mathcal{H}_{A_{1,j}}$ and the reducible zeros by $0\times\mathbf{H}^1(i\mathbf{R})\times 0$.

Let us now deal with the orientations in this case. Looking at the long exact sequence inducing h of (3) we see

$$0\to\mathbf{H}^0(i\mathbf{R})\oplus\mathbf{H}^1(i\mathbf{R})\oplus\mathcal{H}_{A_{1,j}} \xrightarrow{\Xi} \ker L_{A_{1,j},0,t_1} \to \mathbf{R} \xrightarrow{\kappa}$$
$$\mathbf{H}^0(i\mathbf{R})\oplus\mathbf{H}^1(i\mathbf{R})\oplus\mathcal{H}_{A_{1,j}} \xrightarrow{\beta} \operatorname{coker} L_{A_{1,j},0,t_1} \to 0.$$

We note that $\kappa(t) = -\omega'(t_1)t$ maps isomorphically onto the $\mathbf{H}^1(i\mathbf{R})$ factor, and $\operatorname{coker} L_{A_{1,j},0,t_1}$ is $\mathbf{H}^0(i\mathbf{R})\oplus\mathbf{H}^1(i\mathbf{R})$. β is the obvious projection. Let us assume the canonical orientations on $\mathbf{H}^0(i\mathbf{R})$, $\mathcal{H}_{A_{1,j}}$, and the orientation on $\mathbf{H}^1(i\mathbf{R})$ *induced by* κ, which is given by $-*\omega'(t_1)$. Finally choose the product orientation (in the order indicated) on $\mathbf{H}^0(i\mathbf{R})\oplus\mathbf{H}^1(i\mathbf{R})\oplus\mathcal{H}_{A_{1,j}}$. Then $\ker L_{A_{1,j},0,t_1}$ is identically oriented and $\operatorname{coker} L_{A_{1,j},0,t_1}$ is oriented according to the order $\mathbf{H}^0(i\mathbf{R})\oplus\mathcal{H}_{A_{1,j}}$.

Let $v\in\mathcal{H}_{A_{1,j}}$. Then combining the slice condition with Ξ gives the moduli space near $(0,0,v)$ (as before we may assume $G=0$) as the zeros of the map

$$\chi : i\mathbf{R}\times\mathbf{H}^1(i\mathbf{R})\times\mathcal{H}_{A_{1,j}} \to i\mathbf{R}\times\mathcal{H}_{A_{1,j}},$$
$$\chi(\eta,a,\phi) = (-i\langle iv,\phi\rangle, \Xi(\eta,a,\phi)).$$

As before the zeros of χ are the positive multiples of $(0,0,v)$. The pull-back of the orientation on the target space onto the normal bundle of $\chi^{-1}(0)$ is given by $-\delta\theta\wedge c\delta\eta\wedge\delta a$ where $\delta\theta$ is the angular coordinate on $\mathcal{H}_{A_{1,j}}$. Letting r be the direction determined by v and $\varepsilon\delta r$ the induced orientation, then we require

$$\varepsilon\delta r\wedge(-c\delta\theta\wedge\delta\eta\wedge\delta a)=\delta\eta\wedge\delta a\wedge\delta r\wedge\delta\theta$$

which gives the induced orientation on $\chi^{-1}(0)$ as $-c\delta r$. □

As mentioned before, $Z_{\sigma(t)}(P)$ admits reducible solutions exactly $\sigma(t)\in\mathcal{W}$. This corresponds to when $[\frac{i}{2\pi}\omega(t)]$ coincides with the class $c_1(L)_{\mathbf{R}}$. Let $\mathcal{U}$ denote a connected component of $H^1(Y;\mathbf{R})-\{c_1(L)_{\mathbf{R}}\}$. Then if our path σ has the property that $[\frac{i}{2\pi}\omega(t)]\in\mathcal{U}$ for all t, then $\#Z_{\sigma(0)}(P)=\#Z_{\sigma(0)}$. Therefore $\#Z_{\alpha,\omega,g}(P)$ is an integer-valued function depending only on the choice of $\mathcal{U}$. Denote this function as $\tau(\mathcal{U})$.

We think of $\{c_1(L)_{\mathbf{R}}\}$ as a 'wall' in $H^2(Y;\mathbf{R})$. Then as we cross this wall τ changes. This change can be determined from the previous proposition to give a 'wall-crossing' formula.

Corollary 20. *Let $a\in H^2(Y;\mathbf{Z})/\mathrm{torsion}$ be an indivisible class and let $c_1(L)_{\mathbf{R}}=2na$. Let $\mathcal{U}_\pm$ be the component of $H^2(Y;\mathbf{R})-\{c_1(L)_{\mathbf{R}}\}$ containing $(2n\pm1/2)a$. Then*

$$\tau(\mathcal{U}_+)-\tau(\mathcal{U}_-)=n.$$

Proof. Take (π_0,g_0) and (π_1,g_1) which define the values $\tau(\mathcal{U}_+)$ and $\tau(\mathcal{U}_-)$ respectively. Choose our connecting path σ with properties as used for as for Prop. 19. Without loss, We may suppose that σ crosses W exactly once, say at $t=t_1$. We now follow the notation and ideas in the proof of Prop. 19. According to Prop. 19 We need then to compute the total spectral flow of the family $\{\overline{\mathcal{D}}^1(s)\}$ as s varies from 0 to 1. The orientation of this family is determined by $-*\omega'(t_1)$. We shall choose a to be consistent with this orientation, but the statement of the corollary is actually independent of this choice. Take a positive multiple ω of $\omega'(t)$ such that with $[\frac{i}{2\pi}\omega]=2a$. Thus $A_1-*\omega s$, $0\le s<1$ parameterizes all the reducibles in $Z_{\sigma(t_1)}(P)$. We may deform the family $\{\overline{\mathcal{D}}^1(s)\}$ preserving self-adjointness to the family $\{D^{g(t_1)}_{A_1-*\omega s}\}$, $s\in[0,1]$. Thus it suffices to compute the spectral flow for this family. Notice that there is a gauge transformation g such that $g(A_1)=A_1-*\omega$, or equivalently $g^{-1}dg=-*\omega$. A theorem of [**APS**] says that the spectral flow of the Dirac operators $\{D^{g(t_1)}_{A_1-*\omega s}\}$ is equivalent to computing the index of a Dirac operator $D^{(4)}_A$ on $Y\times S^1$ with a spin-c structure obtained by taking the product $P\times[0,1]$ over $Y\times[0,1]$ and identifying via $g:P\times\{1\}\to P\times\{0\}$. A is a connection which is in temporal gauge and coincides with $A-*\omega s$ on $L\times\{s\}$. (Remark: We follow the orientation conventions of [**APS**] closely.

In particular $Y \times S^1$ has the product orientation $dy \wedge ds$ where dy is the orientation form on Y and s the real coordinate on S^1 thinking of it as $\mathbf{R}/\mathbf{Z}$.) Denote the resulting determinant on $Y \times S^1$ by L'. The index of $D_A^{(4)}$ is given by

$$\frac{1}{8}\langle c_1^2(L'), [Y \times S^1]\rangle + \frac{1}{8}\mathrm{sig}(Y \times S^1).$$

To compute the first term, we notice $F_A = d(A_1 - *\omega s) = F_{A_1} - *\omega ds$. Then

$$\begin{aligned}
\langle c_1^2(L'), [Y \times S^1]\rangle &= \int_{Y\times[0,1]} \frac{i}{2\pi}F_A \wedge \frac{i}{2\pi}F_A \\
&= -\frac{1}{4\pi^2}\int_{Y\times[0,1]} (F_{A_1} - *\omega ds) \wedge (F_{A_1} - *\omega ds) \\
&= -\frac{1}{4\pi^2}(-2)\int_{Y\times[0,1]} F_{A_1} \wedge *\omega ds \\
&= -\frac{1}{4\pi^2}(-2)\int_{Y\times[0,1]} n\omega \wedge *\omega ds \\
&= -\frac{1}{4\pi^2}(-2n)\int_Y \omega \wedge *\omega \int_0^1 ds \\
&= -\frac{1}{4\pi^2}(-2n)4\pi i a \cdot \mathrm{PD}(4\pi i a) \\
&= 8n.
\end{aligned}$$

Here 'PD' denotes Poincare Duality. Since $\mathrm{sig}(Y \times S^1) = 0$, the index of $D_A^{(4)}$ is n and the corollary follows. $\square$

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University of California
Santa Barbara, CA 93106
E-mail address: ylim@gauss.math.ucsb.edu