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RICHARD FRIEDERICH ARENS (1919–2000)

Richard Friederich Arens, who made fundamental contributions to many areas of mathematics and mathematical physics, and who was the managing editor of the Pacific Journal of Mathematics for many years, passed away on May 3, 2000. His many friends and colleagues from UCLA and the Pacific Journal mourn his passing away, and remember him as a wonderful human being, full of charm and wit, serene till the end, and above all, a great master of mathematics.

Arens was born in Germany in 1919 and emigrated to the United States in 1925. He attended public schools in Pasadena, California and enrolled at the University of California Los Angeles in 1937. In 1940 Arens won a full scholarship to Harvard University by placing first in the national William Lowell Putnam mathematics competition for college students. After his Ph.D at Harvard under Garrett Birkhoff, Arens went to the Institute for Advanced Study at Princeton as an assistant to Marston Morse. In 1947 he joined the department of mathematics at UCLA. He served with distinction till his retirement in 1989. His work on functional analysis, on Banach algebras and their deep connections with several complex variables, on relativistic particle interactions, on geometric quantization, on Noether currents and other differential geometric aspects of classical field theories, became widely known and established him as a mathematician of the first rank.

He became a member of the editorial board of the Pacific Journal in 1965 and was formally named as the managing editor in 1973, a position he held until 1979. It was during his long stewardship during the years 1965–79 that the Pacific Journal grew out of its local roots and became an internationally recognized mathematics journal of distinction and quality. This transformation of the Pacific Journal was almost entirely due to his broad vision and the unlimited energy with which he looked after the Journal. Even after he left the managing editorship his advice was always available for and eagerly sought after by his successors. His way of running the Journal was relaxed, but there was no compromise with quality. In his dealings with authors, referees, editors and others connected with the operation of the Journal, he was gentle, often humorous, never condescending, and above all, completely human.

The range and depth of what he knew and understood, not only in mathematics but outside of it, were truly astonishing. Yet he wore his distinction lightly. During the memorial service held at UCLA in June a friend remarked to me that after we have said everything his personality still remains elusive. He was truly *sui generis.*

V. S. Varadarajan
Managing Editor

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K_1 OF SEPARATIVE EXCHANGE RINGS AND C*-ALGEBRAS WITH REAL RANK ZERO

P. Ara, K.R. Goodearl, K.C. O'Meara, and R. Raphael

For any (unital) exchange ring R whose finitely generated projective modules satisfy the separative cancellation property ($A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$), it is shown that all invertible square matrices over R can be diagonalized by elementary row and column operations. Consequently, the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective. In combination with a result of Huaxin Lin, it follows that for any separative, unital C*-algebra A with real rank zero, the topological $K_1(A)$ is naturally isomorphic to the unitary group $U(A)$ modulo the connected component of the identity. This verifies, in the separative case, a conjecture of Shuang Zhang.

Introduction.

The extent to which matrices over a ring R can be diagonalized is a measure of the complexity of R, as well as a source of computational information about R and its free modules. Two natural properties offer themselves as "best possible": (1) That an arbitrary matrix can be reduced to a diagonal matrix on left and right multiplication by suitable invertible matrices, or (2) that an arbitrary invertible matrix can be reduced to a diagonal one by suitable elementary row and column operations. The second property has an immediate K-theoretic benefit, in that it implies that the Whitehead group $K_1(R)$ is a natural quotient of the group of units of R. Our main goal here is to prove property (2) for exchange rings (definition below) satisfying a cancellation condition which holds very widely (and conceivably for all exchange rings). This theorem, when applied to C*-algebras with real rank zero (also defined below), verifies a conjecture of Shuang Zhang in an extensive class of C*-algebras.

The class of exchange rings has recently taken on a unifying role for certain direct sum cancellation problems in ring theory and operator algebra. In particular, exchange rings encompass both (von Neumann) regular rings (this is an old and easy observation) on the one hand, and C*-algebras with real rank zero [**3**, Theorem 7.2] on the other. Within this class, a unifying theme for a number of open problems is the property of *separative* *cancellation* for finitely generated projective modules, namely the condition

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$$

(see [**2**, **3**]). For example, if R is a separative exchange ring, then the (K-theoretic) stable rank of R can only be 1, 2, or ∞ [**3**, Theorem 3.3], and every regular square matrix over R is equivalent (via multiplication by invertible matrices) to a diagonal matrix [**2**, Theorem 2.4]. We prove below that invertible matrices over separative exchange rings can be diagonalized via elementary row and column operations. Recently, Perera [**25**] has applied our methods to the problem of lifting units modulo an ideal I in a ring R, assuming that I satisfies non-unital versions of separativity and the exchange property. In this case, a unit u of R/I lifts to a unit of R if and only if the class of u in $K_1(R/I)$ is in the kernel of the connecting homomorphism $K_1(R/I) \to K_0(I)$ [**25**, Theorem 3.1].

We defer discussion of the C*-algebraic aspects of our results to Section 3, except for the following remark. While earlier uses of the exchange property and separativity for C*-algebras can easily be written out in standard C*-theoretic terms – e.g., with direct sums and isomorphisms of finitely generated projective modules replaced by orthogonal sums and Murray-von Neumann equivalences of projections – our present methods do not lend themselves to such a translation. In particular, although our main C*-algebraic application may be stated as a diagonalization result for unitary matrices, all of the steps in our proofs involve manipulations with non-unitary matrices.

Throughout the paper, we consider only unital rings and C*-algebras. We reserve the term *elementary operation* for the row (respectively, column) operation in which a left (respectively, right) multiple of one row (respectively, column) of a matrix is added to a different row (respectively, column). Similarly, we reserve the name *elementary matrix* for a transvection $I + re_{ij}$ where I is an identity matrix, e_{ij} is one of the usual matrix units for some $i \neq j$, and r is an element of the base ring. Thus, as usual, an elementary row (respectively, column) operation on a matrix A corresponds to multiplying A on the left (respectively, right) by an elementary matrix.

Note that while odd permutation matrices usually cannot be expressed as products of elementary matrices, certain signed permutation matrices can be. For example,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In particular, the operation of replacing rows R_i and R_j (respectively, columns C_i and C_j) with the rows R_j and $-R_i$ (respectively, the columns C_j and $-C_i$) can be achieved as a sequence of three elementary operations. Therefore any entry of a matrix can be moved to any other position by a sequence of elementary row and column operations, at the possible expense of moving other entries and multiplying some by -1.

For any ring R, let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by the elementary matrices. If $GL_n(R)$ is generated by $E_n(R)$ together with the subgroup $D_n(R)$ of invertible diagonal matrices, then R is said to be a *GE_n-ring* [**10**, p. 5]. Further, R is a *GE-ring* provided it is a GE_n-ring for all n. If R is a GE_n-ring then $E_n(R)$ is a normal subgroup of $GL_n(R)$, and so $GL_n(R) = D_n(R)E_n(R) = E_n(R)D_n(R)$. Of course, this means that every invertible $n \times n$ matrix over a GE_n-ring can be diagonalized using only elementary row (respectively, column) operations.

It is easy to check that all rings with stable rank 1 are GE-rings. Note that if R is a GE-ring, then the natural homomorphism from $GL_1(R)$, the group of units of R, to $K_1(R)$ is surjective. For comparison, we recall the well-known fact that if R has stable rank d, then the natural map $GL_d(R) \to K_1(R)$ is surjective (e.g., [**12**, Theorem 40.42]).

1. Exchange rings and separativity.

Although our notions and results will be right-left symmetric, all modules considered in this paper will be right modules. A module M over a ring R has the *finite exchange property* [**11**] if for every R-module A and any decompositions

$$A = M' \oplus N = A_1 \oplus \cdots \oplus A_n$$

with $M' \cong M$, there exist submodules $A_i' \subseteq A_i$ such that

$$A = M' \oplus A_1' \oplus \cdots \oplus A_n'.$$

(It follows from the modular law that A_i' must be a direct summand of A_i for all i.) It should be emphasized that the direct sums in this definition are internal direct sums of submodules of A. One advantage of the resulting internal direct sum decompositions (as opposed to isomorphisms with external direct sums) rests on the fact that direct summands with common complements are isomorphic – e.g., $N \cong \bigoplus_{i=1}^n A_i'$ above since each of these summands of A has M' as a complementary summand.

Following Warfield [**29**], we say that R is an *exchange ring* if R_R satisfies the finite exchange property. By [**29**, Corollary 2], this definition is left-right symmetric. If R is an exchange ring, then every finitely generated projective R-module has the finite exchange property (by [**11**, Lemma 3.10], the finite exchange property passes to finite direct sums and to direct summands), and so the endomorphism ring of any such module is an exchange ring. Further, idempotents lift modulo all ideals of an exchange ring [**24**, Theorem 2.1, Corollary 1.3].

The class of exchange rings is quite large. It includes all semiregular rings (i.e., rings which modulo the Jacobson radical are (von Neumann) regular and have idempotent-lifting), all π-regular rings, and more; see [**1**, **28**, **29**]. Further, all unital C*-algebras with real rank zero are exchange rings [**3**, Theorem 7.2].

The following criterion for exchange rings was obtained independently by Nicholson and the second author.

Lemma 1.1 ([**17**, p. 167]; [**24**, Theorem 2.1])**.** *A ring R is an exchange ring if and only if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$.*

In the above lemma, it is equivalent to ask that for any $a_1, a_2 \in R$ with $a_1R + a_2R = R$, there exists an idempotent $e \in a_1R$ such that $1 - e \in a_2R$. We shall also need the analogous property corresponding to sums of more than two right ideals:

Lemma 1.2 ([**24**, Theorem 2.1, Proposition 1.11])**.** *Let R be an exchange ring. If $I_1, \dots, I_n$ are right ideals of R such that $I_1 + \cdots + I_n = R$, then there exist orthogonal idempotents $e_1, \dots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and $e_j \in I_j$ for all j.*

We reiterate that a ring R is *separative* provided the following cancellation property holds for finitely generated projective right (equivalently, left) R-modules A and B:

$$A \oplus A \cong A \oplus B \cong B \oplus B \quad \Longrightarrow \quad A \cong B.$$

See [**3**] for the origin of this terminology and for a number of equivalent conditions. We shall need the following one:

Lemma 1.3 ([**2**, Proposition 1.2]; [**3**, Lemma 2.1])**.** *A ring R is separative if and only if whenever A, B, C are finitely generated projective right R-modules such that $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both A^n and B^n for some n, then $A \cong B$.*

Note, in particular, that if R is separative and A, B, C are finitely generated projective right R-modules, then we can certainly cancel C from $A \oplus C \cong B \oplus C$ whenever A and B are generators in Mod-R.

Separativity seems to hold quite widely within the class of exchange rings; for instance, it holds for all known classes of regular rings (cf. [**3**]). In fact, the existence of non-separative exchange rings is an open problem.

It is clear from either form of the condition that a ring R is separative in case the finitely generated projective R-modules enjoy cancellation with respect to direct sums, which in turn holds in case R has stable rank 1. In fact, for exchange rings, cancellation of finitely generated projective modules is equivalent to stable rank 1 [**31**, Theorem 9]. Separativity, however, is much weaker than stable rank 1. For example, any regular right self-injective ring is separative (e.g., [**15**, Theorem 10.34(b)]), but such rings can have infinite stable rank – e.g., the ring of all linear transformations on an infinite dimensional vector space.

2. K_1 of separative exchange rings.

We use the notation $A \lesssim^{\oplus} B$ to denote that a module A is isomorphic to a direct summand of a module B.

Lemma 2.1. *Let R be an exchange ring and $e_1, \ldots, e_n \in R$ idempotents. Then there exists an idempotent $e \in e_1R + \cdots + e_nR$ such that $e_1R \leq eR$ and $e_iR \lesssim^{\oplus} eR$ for all i. In particular, $ReR = Re_1R + \cdots + Re_nR$.*

Proof. By induction, it suffices to do the case $n = 2$. Now

$$R = e_1R \oplus (1 - e_1)R = e_2R \oplus (1 - e_2)R$$

and e_1R has the finite exchange property, so there exist decompositions $e_2R = A \oplus B$ and $(1 - e_2)R = A' \oplus B'$ such that $R = e_1R \oplus A \oplus A'$. Then we can choose an idempotent $e \in R$ such that $eR = e_1R \oplus A$. Obviously $e_1R \leq eR$, and since

$$e_1R \cong R/(A \oplus A') \cong B \oplus B',$$

we have $e_2R \lesssim^{\oplus} A \oplus e_1R = eR$. □

Corollary 2.2. *Let R be an exchange ring and $a \in R$ such that $RaR = R$. Then there exist idempotents $e \in aR$ and $f \in Ra$ such that $ReR = RfR = R$.*

Proof. Write $R = \sum_{i=1}^{n} x_i aR$ for some x_i. By Lemma 1.2, there exist orthogonal idempotents $g_1, \ldots, g_n \in R$ such that $g_1 + \cdots + g_n = 1$ and $g_i \in x_i aR$ for all i. Set $g_i = x_i a y_i$ with $y_i = y_i g_i$. Then $e_i := a y_i x_i$ is an idempotent in aR and $e_iR \cong g_iR$. By Lemma 2.1, there exists an idempotent $e \in \sum_{i=1}^{n} e_iR$ such that $e_iR \lesssim^{\oplus} eR$ for all i. Then $e \in aR$ and $g_iR \lesssim^{\oplus} eR$ for all i, so all $g_i \in ReR$, and thus $ReR = R$.

The existence of f follows by symmetry. □

Lemma 2.3. *Let R be any ring and $A \in GL_n(R)$. If A has an idempotent entry, then A can be reduced by elementary row and column operations to the form*

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}.$$

Proof. By elementary operations, we can move the idempotent entry, call it e, into the $1,1$ position. If $n = 1$, then e is invertible, so $e = 1$ and we are done. Now assume that $n > 1$, and let

$$\begin{bmatrix} e & b_2 & b_3 & \cdots & b_n \end{bmatrix}$$

be the first row of A. By elementary column operations, we can subtract eb_i from the i-th entry for each $i \geq 2$. Thus, we can assume that $b_2, \dots, b_n \in (1-e)R$. Since A is invertible, $eR + b_2R + \cdots + b_nR = R$, and so it follows that $b_2R + \cdots + b_nR = (1-e)R$. Hence, by elementary column operations we can add $1 - e$ to the first entry. Now we have a 1 in the $1,1$ position, and the rest is routine. □

Since we shall need to perform a number of operations on the top rows of invertible matrices, it is convenient to work with the rows alone. Recall that any row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ of an invertible matrix over a ring R is *right unimodular*, that is, $\sum_{i=1}^n a_iR = R$. Elementary column operations apply to such a row just by viewing it as a $1 \times n$ matrix. Such operations amount to multiplying the row on the right by an elementary matrix. Since our rings need not be commutative, elementary column operations can only introduce right-hand coefficients.

Lemma 2.4. *Let R be an exchange ring and $\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ a right unimodular row over R. Then α can be transformed by elementary column operations to a row $\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$ such that $R = b_1R \oplus \cdots \oplus b_nR$ and each $b_i \in a_iRa_i$.*

Proof. Since $\sum_{i=1}^n a_iR = R$, Lemma 1.2 gives us orthogonal idempotents $e_1, \dots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and $e_i \in a_iR$ for all i, say $e_i = a_ir_i$. By elementary column operations, we can subtract $e_ia_1 = a_ir_ia_1$ from the first entry of α for each $i \geq 2$. This transforms α to $\alpha' = \begin{bmatrix} e_1a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$. Note that $e_1 \in e_1a_1R$. Thus, we can repeat the above process for each entry, and transform α' to the row $\begin{bmatrix} e_1a_1 & e_2a_2 & \cdots & e_na_n \end{bmatrix}$, with entries $e_ia_i \in a_iRa_i$. Moreover, $e_ia_iR = e_iR$, and therefore $R = \bigoplus_{i=1}^n e_ia_iR$. □

Corollary 2.5. *Let R be an exchange ring and $\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ a right unimodular row over R, with $n \geq 2$. Then α can be transformed by elementary column operations to a row $\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$ such that $Rb_1R = R$ and $b_i \in a_iRa_i$ for all $i \geq 2$.*

Proof. By Lemma 2.4, we may assume that $R = \bigoplus_{i=1}^n a_iR$. It follows that all $a_i \in Rb_1$ where $b_1 = a_1 + \cdots + a_n$ (multiply b_1 on the left by the orthogonal idempotents arising from the given decomposition of R_R). Thus $Rb_1R = R$. By elementary column operations, we can add $a_2, \dots, a_n$ to the first entry of α, and thus transform it to $\begin{bmatrix} b_1 & a_2 & \cdots & a_n \end{bmatrix}$. □

Recall that an element x in a ring R is (*von Neumann*) *regular* provided there exists an element $y \in R$ such that $xyx = x$, equivalently, provided xR is a direct summand of R_R. If y can be chosen to be a unit in R, then x is said to be *unit-regular*. A regular element $x \in R$ is unit-regular if and only if $R/xR \cong \text{r.ann}(x)$, where $\text{r.ann}(x)$ denotes the right annihilator of x in R (cf. [**15**, Proof of Theorem 4.1]).

Corollary 2.6. *Let R be an exchange ring and $\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ a right unimodular row over R, with $n \geq 2$. Then α can be transformed by elementary column operations to a row $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ such that c_2 is a regular element, $c_2 \in Ra_2$, and $c_2R = (1-g)R$ for an idempotent g with $RgR = R$.*

Proof. By Corollary 2.5, we may assume that $Ra_1R = R$. By Corollary 2.2, there exists an idempotent $e \in a_1R$ such that $ReR = R$. By elementary column operations, we can subtract ea_2 from the second entry of α, so there is no loss of generality in assuming that $a_2 \in (1-e)R$. (At this stage, our current a_2 is only a <u>left</u> multiple of the original a_2. This is why the conclusions of the lemma state $c_2 \in Ra_2$ rather than $c_2 \in a_2Ra_2$.) Now using Lemma 2.4, we can transform α to a row $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ such that $R = \bigoplus_{i=1}^n c_iR$ and $c_2 \in a_2Ra_2$. Then $c_2R = (1-g)R$ for some idempotent g, and c_2 is regular. Moreover, $(1-g)R = c_2R \subseteq a_2R \subseteq (1-e)R$ and so $Re \subseteq Rg$. Therefore $RgR = R$. □

Lemma 2.7. *Let R be an exchange ring and $A \in GL_n(R)$, with $n \geq 2$. Then A can be transformed by elementary row and column operations to a matrix whose $1,1$ entry d is regular, with $dR = (1-p)R$ and $Rd = R(1-q)$ for some idempotents p, q such that $RpR = RqR = R$.*

Proof. By Corollary 2.6, we can assume that the $1,2$ entry of A is a regular element c such that $cR = (1-g)R$ for some idempotent g with $RgR = R$. With elementary operations, we can move c to the $2,1$ position.

Now apply the transpose of Corollary 2.6 to the first column of A. Thus, A can be transformed by elementary row operations to a matrix whose $2,1$ entry is a regular element d such that $d \in cR$ and $Rd = R(1-q)$ for some idempotent q with $RqR = R$. Since d is regular, $dR = (1-p)R$ for some idempotent p. Then $(1-p)R \subseteq (1-g)R$, whence $Rg \subseteq Rp$ and so $RpR = R$.

Finally, use elementary operations to move d to the $1,1$ position. □

Theorem 2.8. *If R is a separative exchange ring, then R is a GE-ring, and so the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective.*

Proof. We need to show that R is a GE_n-ring for all n. This is trivial for $n = 1$, hence we assume, by induction, that $n \geq 2$ and R is a GE_{n-1}-ring. Let A be an arbitrary invertible $n \times n$ matrix over R.

By Lemma 2.7, we may assume that the $1,1$ entry d of A is regular, with $dR = (1-p)R$ and $Rd = R(1-q)$ for some idempotents p, q such that $RpR = RqR = R$. We claim that d is unit-regular. Note that because $RpR = RqR = R$, the projective modules pR and qR are generators.

Now $R = \text{r.ann}(d) \oplus B = dR \oplus C$ for some B, C, and we have to prove that $\text{r.ann}(d) \cong C$. Since $B \cong dB = dR$, we have $\text{r.ann}(d) \oplus B \cong C \oplus B$. From $Rd = R(1-q)$, we get $\text{r.ann}(d) = qR$ and so $\text{r.ann}(d)$ is a generator. Since $C \cong R/dR \cong pR$, we see that C is a generator too. By Lemma 1.3, $\text{r.ann}(d) \cong C$ as desired.

The unit-regularity of d gives $d = ue$ for some unit u and idempotent e. Set

$$U = \begin{bmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix};$$

then the matrix $U^{-1}A$ has an idempotent entry. By Lemma 2.3, there exist $E, F \in E_n(R)$ such that

$$EU^{-1}AF = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}$$

where $A' \in GL_{n-1}(R)$. By our induction hypothesis, $A' \in E_{n-1}(R)D_{n-1}(R)$. It follows that

$$A \in D_n(R)E_n(R)D_n(R)E_n(R),$$

and therefore we have shown that R is a GE_n-ring. This establishes the induction step and completes the proof. $\square$

Remarks 2.9. (a) Observe that the proof of Theorem 2.8 did not use the full force of separativity, only the cancellation property ($A \oplus C \cong B \oplus C \implies A \cong B$) for finitely generated projective R-modules A, B, C with A and B generators.

(b) Theorem 2.8 includes, in particular, the result of Menal and Moncasi that every factor ring of a right self-injective ring is a GE-ring [**22**, Theorem 2.2]. To make the connection explicit, recall that right self-injective rings are semiregular (e.g., [**13**, Theorem 2.16, Lemma 2.18]) and hence exchange; thus, all their factor rings are exchange rings. Further, any right self-injective ring is separative (e.g., [**14**, Theorem 3]). It follows that factor rings of right self-injective rings are separative [**3**, Theorem 4.2].

(c) As a special case of Theorem 2.8, we obtain that any separative regular ring is a GE-ring, which gives a partial affirmative answer to a question of Moncasi [**23**, Qüestió 5].

(d) In the situation of Theorem 2.8, one naturally asks for a description of the kernel of the epimorphism $GL_1(R) \to K_1(R)$. This has been answered for unit-regular rings and regular right-self-injective rings by Menal and Moncasi [**22**, Theorems 1.6, 2.6], and for exchange rings with primitive factors artinian by Chen and Li [**9**, Theorem 3]. In all the above cases, $K_1(R) \cong GL_1(R)^{\mathrm{ab}}$ provided $\frac{1}{2} \in R$ [**22**, Theorems 1.7, 2.6]; [**9**, Corollary 7]. Further, $K_1(R) \cong GL_1(R)^{\mathrm{ab}}$ when R is either a C*-algebra with unitary 1-stable range or an AW*-algebra [**22**, Theorem 1.3, Corollary 2.11] (here the algebraic K_1 is meant). The unit-regular and AW* results correct and extend earlier work of Handelman [**18**, Theorem 2.4]; [**19**, Theorem 7].

Theorem 2.10. *If R is a separative exchange ring and A is a (von Neumann) regular $n \times n$ matrix over R, then A can be diagonalized using elementary row and column operations.*

Proof. By [**2**, Theorem 2.4], there exist $P, Q \in GL_n(R)$ such that PAQ is diagonal. By Theorem 2.8, $P = U_1V_1$ and $Q = V_2U_2$, where $U_1, U_2 \in D_n(R)$ and $V_1, V_2 \in E_n(R)$. So V_1AV_2 is a diagonal matrix obtained from A by elementary row and column operations. □

Remark 2.11. When applying Theorem 2.10, note the distinction between invertible matrices and general matrices. An invertible matrix over a separative exchange ring can be diagonalized from either side (by Theorem 2.8), whereas the diagonalization of a general regular matrix sometimes requires elementary operations on both the rows and the columns. For example, the 2×2 matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ over a field cannot be diagonalized using only elementary row operations.

Example 2.12. Non-regular matrices over separative exchange rings need not be diagonalizable by elementary operations, even over finite dimensional algebras. For example, choose a field F and let

$$R = F[x_1, x_2, x_3, x_4]/\langle x_1, x_2, x_3, x_4 \rangle^2.$$

Then R has a basis $1, a_1, a_2, a_3, a_4$ such that $a_ia_j = 0$ for all i, j. Since R is clearly semiregular, it is an exchange ring; separativity is an easy exercise. In fact, since R is artinian, it has stable rank 1. Recall that this also implies that R is a GE-ring. Observe that every element of R is a sum of a scalar plus a nilpotent element, and that the product of any two nilpotent elements of R is zero.

Now consider the matrix $A = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]$, whose entries are linearly independent nilpotent elements of R. We claim that any sequence of elementary row or column operations on A can only produce a matrix whose entries are linearly independent nilpotent elements. For instance, consider a product

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} + bc_{21} & c_{12} + bc_{22} \\ c_{21} & c_{22} \end{bmatrix}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are linearly independent nilpotent elements. Then $b = \beta + n$ for some $\beta \in F$ and some nilpotent element n, whence $bc_{21} = \beta c_{21}$ and $bc_{22} = \beta c_{22}$, and so the entries in the matrix product above are linearly independent (they are clearly nilpotent). The same thing happens with other elementary operations, establishing the claim.

Therefore no sequence of elementary operations on A can produce a matrix with a zero entry. In particular, A cannot be diagonalized by elementary operations. Since R is a GE-ring, it follows that A cannot be diagonalized by invertible matrices either, i.e., there do not exist $X, Y \in GL_2(R)$ such that XAY is diagonal. Thus the first of the natural properties discussed in the introduction is not implied by the second.

3. K_1 of separative C*-algebras with real rank zero.

In connection with his work on the structure of multiplier algebras (e.g., [**32**, **33**, **34**]), Shuang Zhang has conjectured [unpublished] that if A is any unital C*-algebra with real rank zero, the topological $K_1(A)$ is isomorphic to the unitary group $U(A)$ modulo the connected component of the identity, $U(A)^\circ$. We confirm this conjecture in case A is separative, which at the same time provides a unified approach to all known cases of the conjecture. The main interest of Zhang's conjecture is in the case when the stable rank of A is greater than 1, since it has long been known that $K_1^{\text{top}}(A) \cong U(A)/U(A)^\circ$ for all unital C*-algebras A with stable rank 1 (e.g., this is equivalent to [**26**, Theorem 2.10]).

We consider only unital, complex C*-algebras in this section, and we refer the reader to [**4**, **30**] for background and notation for C*-algebras. In particular, we use $\sim$ and $\lesssim$ to denote Murray-von Neumann equivalence and subequivalence of projections, and we write $M_\infty(A)$ for the (non-unital) algebra of $\omega \times \omega$ matrices with only finitely many nonzero entries from an algebra A. We write $U(A)$ for the unitary group of a unital C*-algebra A, and $U(A)^\circ$ for the connected component of the identity in $U(A)$.

In the theory of operator algebras, it is customary to write $K_1(A)$ for the topological K_1-group of A (e.g., [**4**, Definition 8.1.1]; [**30**, Definition 7.1.1]), and we shall follow that practice here. Thus, $K_1(A) = GL_\infty(A)/GL_\infty(A)^\circ$. We then use the notation $K_1^{\text{alg}}(A)$ to denote the algebraic K_1-group of A. Since $K_1^{\text{alg}}(A)$ is the abelianization of $GL_\infty(A)$ (e.g., [**27**, Proposition 2.1.4, Definition 2.1.5]) and $K_1(A)$ is abelian (e.g., [**4**, Proposition 8.1.3]; [**30**, Proposition 7.1.2]), there is a natural surjective homomorphism $K_1^{\text{alg}}(A) \to K_1(A)$. Finally, following Brown [**5**, p. 116], we say that A has K_1-*surjectivity* (respectively, K_1-*injectivity*) provided the natural homomorphism $U(A)/U(A)^\circ \to K_1(A)$ is surjective (respectively, injective).

The concept of *real rank zero* for a C*-algebra A has a number of equivalent characterizations (see [**6**]). One is the requirement that each self-adjoint element of A can be approximated arbitrarily closely by real linear combinations of orthogonal projections. (This is usually phrased as saying that the set of self-adjoint elements of A with finite spectrum is dense in the set of all self-adjoint elements.) It was proved in [**3**, Theorem 7.2] that A has real rank zero if and only if it is an exchange ring. Hence, the C*-algebras with real rank zero are exactly the C*-algebras to which our results above can be applied.

Given a C*-algebra A, all idempotents in matrix algebras $M_n(A)$ are equivalent to projections (e.g., [**4**, Proposition 4.6.2]; [**27**, Proposition 6.3.12]). Hence, A is separative if and only if

$$p \oplus p \sim p \oplus q \sim q \oplus q \qquad \Longrightarrow \qquad p \sim q$$

for projections $p, q \in M_\infty(A)$. An equivalent condition (analogous to Lemma 1.3) is that $p \oplus r \sim q \oplus r \implies p \sim q$ whenever $r \lesssim n.p$ and $r \lesssim n.q$ for some n. Separativity in A is equivalent to the requirement that all matrix algebras $M_n(A)$ satisfy the *weak cancellation* introduced by Brown and Pedersen [**5**, p. 116]; [**7**, p. 114]. They have shown that every extremally rich C*-algebra (see [**7**, p. 125]) with real rank zero is separative ([**8**], announced in [**5**, p. 116]). We would like to emphasize the question of whether non-separative exchange rings exist by focusing on the C* case:

Problem. Is every C*-algebra with real rank zero separative?

By combining Theorem 2.8 with a result of Lin, we obtain the following theorem.

Theorem 3.1. *If A is a separative, unital C*-algebra with real rank zero, then the natural map $U(A)/U(A)^\circ \to K_1(A)$ is an isomorphism.*

Proof. Lin proved K_1-injectivity for C*-algebras with real rank zero in [**20**, Lemma 2.2]. Hence, it only remains to show K_1-surjectivity. It is a standard fact that $U(A)$ and $GL_1(A)$ have the same image in $K_1(A)$ (e.g., [**4**, pp. 66, 67] or [**30**, Proof of Proposition 4.2.6]). Now the natural map $GL_1(A) \to K_1(A)$ factors as the composition of natural maps $GL_1(A) \to K_1^{\mathrm{alg}}(A) \to K_1(A)$, the second of which is surjective. Since A has real rank zero, it is an exchange ring, and so the map $GL_1(A) \to K_1^{\mathrm{alg}}(A)$ is surjective by Theorem 2.8. Therefore the image of $U(A)$ in $K_1(A)$ is all of $K_1(A)$, as desired. □

Brown and Pedersen have proved that every separative, extremally rich C*-algebra has K_1-surjectivity ([**8**], announced in [**5**, p. 116]; [**7**, p. 114]). Since there are C*-algebras with real rank zero that are not extremally rich [**5**, Example, p. 117], Theorem 3.1 can be viewed as a partial extension of the Brown-Pedersen result within the class of C*-algebras with real rank zero.

We thank the referee for the following remark.

Remark 3.2. While Theorem 3.1 is neither unexpected nor new in the case of stable rank 1 (cf. the result of Rieffel cited above), it is perhaps surprising that there are many C*-algebras of real rank zero and stable rank 2 to which the theorem applies. To see this, consider C*-algebra extensions

$$0 \to I \to A \to B \to 0$$

in which I and B have real rank zero and A is unital. By theorems of Zhang ([**35**, Lemma 2.4]; cf. [**6**, Theorem 3.14]) and Lin and Rørdam [**21**, Proposition 4], A has real rank zero if and only if projections lift from B to A, if and only if the connecting map $K_0(B) \to K_1(I)$ in topological K-theory vanishes. In this case, by [**3**, Theorem 7.5], A will be separative provided I and B are both separative, and in particular if I and B have stable rank 1. However, by [**21**, Proposition 4], if I and B have stable rank 1, then A will have stable rank 2 provided the connecting map $K_1(B) \to K_0(I)$ does not vanish. It is easy to find specific extensions satisfying the above conditions, such as the examples analyzed in [**21**, End of Section 1] or [**16**].

We conclude with an application of Theorem 3.1 that extends an argument of Brown [**5**, Theorem 1], relating homotopy and unitary equivalence of projections, to a wider context within real rank zero. Projections p and q in a C*-algebra A are *unitarily equivalent* provided there exists a unitary element $u \in A$ such that $upu^* = q$; they are *homotopic* provided there is a continuous path $f : [0,1] \to \{\text{projections in } A\}$ such that $f(0) = p$ and $f(1) = q$. It is a standard fact that homotopic projections are unitarily equivalent (e.g., [**4**, Propositions 4.3.3, 4.6.5]; [**30**, Proposition 5.2.10]).

Theorem 3.3. *Let A be a separative, unital C*-algebra with real rank zero, let $p, q \in A$ be projections, and let $B = \overline{ApA} + \mathbb{C} \cdot 1$. Then p and q are homotopic in A if and only if $q \in B$ and p, q are unitarily equivalent in B.*

Proof. If p and q are homotopic in A, they are connected by a path of projections within A. Each projection along this path is homotopic to p and hence is unitarily equivalent to p. Thus, these projections all lie in ApA. In particular, $q \in B$, and p and q are homotopic in B. Consequently, p and q must be unitarily equivalent in B.

Conversely, assume that $q \in B$ and p, q are unitarily equivalent in B. By [**6**, Corollary 2.8, Theorem 2.5], the closed ideal $I = \overline{ApA}$ has real rank zero (as a non-unital C*-algebra), and so the unital C*-algebras B and pIp have real rank zero. We do not need separativity for B, just K_1-injectivity (by Lin's result). Since I is an ideal of A, any projections in $M_\infty(I)$ which are (Murray-von Neumann) equivalent in $M_\infty(A)$ are also equivalent in $M_\infty(I)$ (any implementing partial isometry necessarily lies in $M_\infty(I)$). Hence, the separativity of A implies that I is separative, and so pIp is separative. Therefore, by Theorem 3.1, pIp has K_1-surjectivity.

With the above information in hand, Brown's proof [**5**, Theorem 1] carries through in the present setting. We sketch the details for the reader's convenience. By hypothesis, $q = upu^*$ for some unitary $u \in U(B)$; let α denote the image of u in $K_1(B)$. Now $K_1(B) = K_1(I^\sim) = K_1(I)$, and because pIp is a full hereditary sub-C*-algebra of I, the natural map $K_1(pIp) \to K_1(I)$ is an isomorphism [**5**, Remark, p. 117]. Thus α is the image of some $\beta \in K_1(pIp)$. Since pIp has K_1-surjectivity, β is the image of some unitary $v_1 \in U(pIp)$. Let $v = v_1 + 1 - p$ and $w = uv^*$. Then w is a unitary in B such that $q = wpw^*$, and the image of w in $K_1(B)$ is zero. Since B has K_1-injectivity, $w \in U(B)^\circ$, from which it follows that p and q are homotopic. □

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K_1 OF SEPARATIVE EXCHANGE RINGS AND C*-ALGEBRAS WITH REAL RANK ZERO

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For any (unital) exchange ring R whose finitely generated projective modules satisfy the separative cancellation property ($A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$), it is shown that all invertible square matrices over R can be diagonalized by elementary row and column operations. Consequently, the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective. In combination with a result of Huaxin Lin, it follows that for any separative, unital C*-algebra A with real rank zero, the topological $K_1(A)$ is naturally isomorphic to the unitary group $U(A)$ modulo the connected component of the identity. This verifies, in the separative case, a conjecture of Shuang Zhang.

Introduction.

The extent to which matrices over a ring R can be diagonalized is a measure of the complexity of R, as well as a source of computational information about R and its free modules. Two natural properties offer themselves as "best possible": (1) That an arbitrary matrix can be reduced to a diagonal matrix on left and right multiplication by suitable invertible matrices, or (2) that an arbitrary invertible matrix can be reduced to a diagonal one by suitable elementary row and column operations. The second property has an immediate K-theoretic benefit, in that it implies that the Whitehead group $K_1(R)$ is a natural quotient of the group of units of R. Our main goal here is to prove property (2) for exchange rings (definition below) satisfying a cancellation condition which holds very widely (and conceivably for all exchange rings). This theorem, when applied to C*-algebras with real rank zero (also defined below), verifies a conjecture of Shuang Zhang in an extensive class of C*-algebras.

The class of exchange rings has recently taken on a unifying role for certain direct sum cancellation problems in ring theory and operator algebra. In particular, exchange rings encompass both (von Neumann) regular rings (this is an old and easy observation) on the one hand, and C*-algebras with real rank zero [**3**, Theorem 7.2] on the other. Within this class, a unifying theme for a number of open problems is the property of *separative* *cancellation* for finitely generated projective modules, namely the condition

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$$

(see [**2**, **3**]). For example, if R is a separative exchange ring, then the (K-theoretic) stable rank of R can only be 1, 2, or ∞ [**3**, Theorem 3.3], and every regular square matrix over R is equivalent (via multiplication by invertible matrices) to a diagonal matrix [**2**, Theorem 2.4]. We prove below that invertible matrices over separative exchange rings can be diagonalized via elementary row and column operations. Recently, Perera [**25**] has applied our methods to the problem of lifting units modulo an ideal I in a ring R, assuming that I satisfies non-unital versions of separativity and the exchange property. In this case, a unit u of R/I lifts to a unit of R if and only if the class of u in $K_1(R/I)$ is in the kernel of the connecting homomorphism $K_1(R/I) \to K_0(I)$ [**25**, Theorem 3.1].

We defer discussion of the C*-algebraic aspects of our results to Section 3, except for the following remark. While earlier uses of the exchange property and separativity for C*-algebras can easily be written out in standard C*-theoretic terms – e.g., with direct sums and isomorphisms of finitely generated projective modules replaced by orthogonal sums and Murray-von Neumann equivalences of projections – our present methods do not lend themselves to such a translation. In particular, although our main C*-algebraic application may be stated as a diagonalization result for unitary matrices, all of the steps in our proofs involve manipulations with non-unitary matrices.

Throughout the paper, we consider only unital rings and C*-algebras. We reserve the term *elementary operation* for the row (respectively, column) operation in which a left (respectively, right) multiple of one row (respectively, column) of a matrix is added to a different row (respectively, column). Similarly, we reserve the name *elementary matrix* for a transvection $I + re_{ij}$ where I is an identity matrix, e_{ij} is one of the usual matrix units for some $i \neq j$, and r is an element of the base ring. Thus, as usual, an elementary row (respectively, column) operation on a matrix A corresponds to multiplying A on the left (respectively, right) by an elementary matrix.

Note that while odd permutation matrices usually cannot be expressed as products of elementary matrices, certain signed permutation matrices can be. For example,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In particular, the operation of replacing rows R_i and R_j (respectively, columns C_i and C_j) with the rows R_j and $-R_i$ (respectively, the columns C_j and $-C_i$) can be achieved as a sequence of three elementary operations. Therefore any entry of a matrix can be moved to any other position by a sequence of elementary row and column operations, at the possible expense of moving other entries and multiplying some by -1.

For any ring R, let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by the elementary matrices. If $GL_n(R)$ is generated by $E_n(R)$ together with the subgroup $D_n(R)$ of invertible diagonal matrices, then R is said to be a *GE_n-ring* [**10**, p. 5]. Further, R is a *GE-ring* provided it is a GE_n-ring for all n. If R is a GE_n-ring then $E_n(R)$ is a normal subgroup of $GL_n(R)$, and so $GL_n(R) = D_n(R)E_n(R) = E_n(R)D_n(R)$. Of course, this means that every invertible $n \times n$ matrix over a GE_n-ring can be diagonalized using only elementary row (respectively, column) operations.

It is easy to check that all rings with stable rank 1 are GE-rings. Note that if R is a GE-ring, then the natural homomorphism from $GL_1(R)$, the group of units of R, to $K_1(R)$ is surjective. For comparison, we recall the well-known fact that if R has stable rank d, then the natural map $GL_d(R) \to K_1(R)$ is surjective (e.g., [**12**, Theorem 40.42]).

1. Exchange rings and separativity.

Although our notions and results will be right-left symmetric, all modules considered in this paper will be right modules. A module M over a ring R has the *finite exchange property* [**11**] if for every R-module A and any decompositions

$$A = M' \oplus N = A_1 \oplus \cdots \oplus A_n$$

with $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus A'_1 \oplus \cdots \oplus A'_n.$$

(It follows from the modular law that A'_i must be a direct summand of A_i for all i.) It should be emphasized that the direct sums in this definition are internal direct sums of submodules of A. One advantage of the resulting internal direct sum decompositions (as opposed to isomorphisms with external direct sums) rests on the fact that direct summands with common complements are isomorphic – e.g., $N \cong \bigoplus_{i=1}^n A'_i$ above since each of these summands of A has M' as a complementary summand.

Following Warfield [**29**], we say that R is an *exchange ring* if R_R satisfies the finite exchange property. By [**29**, Corollary 2], this definition is left-right symmetric. If R is an exchange ring, then every finitely generated projective R-module has the finite exchange property (by [**11**, Lemma 3.10], the finite exchange property passes to finite direct sums and to direct summands), and so the endomorphism ring of any such module is an exchange ring. Further, idempotents lift modulo all ideals of an exchange ring [**24**, Theorem 2.1, Corollary 1.3].

The class of exchange rings is quite large. It includes all semiregular rings (i.e., rings which modulo the Jacobson radical are (von Neumann) regular and have idempotent-lifting), all π-regular rings, and more; see [**1**, **28**, **29**]. Further, all unital C*-algebras with real rank zero are exchange rings [**3**, Theorem 7.2].

The following criterion for exchange rings was obtained independently by Nicholson and the second author.

Lemma 1.1 ([**17**, p. 167]; [**24**, Theorem 2.1]). *A ring R is an exchange ring if and only if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1-e \in (1-a)R$.*

In the above lemma, it is equivalent to ask that for any $a_1, a_2 \in R$ with $a_1R + a_2R = R$, there exists an idempotent $e \in a_1R$ such that $1-e \in a_2R$. We shall also need the analogous property corresponding to sums of more than two right ideals:

Lemma 1.2 ([**24**, Theorem 2.1, Proposition 1.11]). *Let R be an exchange ring. If $I_1, \dots, I_n$ are right ideals of R such that $I_1 + \cdots + I_n = R$, then there exist orthogonal idempotents $e_1, \dots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and $e_j \in I_j$ for all j.*

We reiterate that a ring R is *separative* provided the following cancellation property holds for finitely generated projective right (equivalently, left) R-modules A and B:

$$A \oplus A \cong A \oplus B \cong B \oplus B \quad \Longrightarrow \quad A \cong B.$$

See [**3**] for the origin of this terminology and for a number of equivalent conditions. We shall need the following one:

Lemma 1.3 ([**2**, Proposition 1.2]; [**3**, Lemma 2.1]). *A ring R is separative if and only if whenever A, B, C are finitely generated projective right R-modules such that $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both A^n and B^n for some n, then $A \cong B$.*

Note, in particular, that if R is separative and A, B, C are finitely generated projective right R-modules, then we can certainly cancel C from $A \oplus C \cong B \oplus C$ whenever A and B are generators in Mod-R.

Separativity seems to hold quite widely within the class of exchange rings; for instance, it holds for all known classes of regular rings (cf. [**3**]). In fact, the existence of non-separative exchange rings is an open problem.

It is clear from either form of the condition that a ring R is separative in case the finitely generated projective R-modules enjoy cancellation with respect to direct sums, which in turn holds in case R has stable rank 1. In fact, for exchange rings, cancellation of finitely generated projective modules is equivalent to stable rank 1 [**31**, Theorem 9]. Separativity, however, is much weaker than stable rank 1. For example, any regular right self-injective ring is separative (e.g., [**15**, Theorem 10.34(b)]), but such rings can have infinite stable rank – e.g., the ring of all linear transformations on an infinite dimensional vector space.

2. K_1 of separative exchange rings.

We use the notation $A \lesssim^{\oplus} B$ to denote that a module A is isomorphic to a direct summand of a module B.

Lemma 2.1. *Let R be an exchange ring and $e_1, \dots, e_n \in R$ idempotents. Then there exists an idempotent $e \in e_1R + \cdots + e_nR$ such that $e_1R \le eR$ and $e_iR \lesssim^{\oplus} eR$ for all i. In particular, $ReR = Re_1R + \cdots + Re_nR$.*

Proof. By induction, it suffices to do the case $n = 2$. Now

$$R = e_1R \oplus (1 - e_1)R = e_2R \oplus (1 - e_2)R$$

and e_1R has the finite exchange property, so there exist decompositions $e_2R = A \oplus B$ and $(1 - e_2)R = A' \oplus B'$ such that $R = e_1R \oplus A \oplus A'$. Then we can choose an idempotent $e \in R$ such that $eR = e_1R \oplus A$. Obviously $e_1R \le eR$, and since

$$e_1R \cong R/(A \oplus A') \cong B \oplus B',$$

we have $e_2R \lesssim^{\oplus} A \oplus e_1R = eR$. □

Corollary 2.2. *Let R be an exchange ring and $a \in R$ such that $RaR = R$. Then there exist idempotents $e \in aR$ and $f \in Ra$ such that $ReR = RfR = R$.*

Proof. Write $R = \sum_{i=1}^{n} x_iaR$ for some x_i. By Lemma 1.2, there exist orthogonal idempotents $g_1, \dots, g_n \in R$ such that $g_1 + \cdots + g_n = 1$ and $g_i \in x_iaR$ for all i. Set $g_i = x_iay_i$ with $y_i = y_ig_i$. Then $e_i := ay_ix_i$ is an idempotent in aR and $e_iR \cong g_iR$. By Lemma 2.1, there exists an idempotent $e \in \sum_{i=1}^{n} e_iR$ such that $e_iR \lesssim^{\oplus} eR$ for all i. Then $e \in aR$ and $g_iR \lesssim^{\oplus} eR$ for all i, so all $g_i \in ReR$, and thus $ReR = R$.

The existence of f follows by symmetry. □

Lemma 2.3. *Let R be any ring and $A \in GL_n(R)$. If A has an idempotent entry, then A can be reduced by elementary row and column operations to the form*

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}.$$

Proof. By elementary operations, we can move the idempotent entry, call it e, into the $1,1$ position. If $n = 1$, then e is invertible, so $e = 1$ and we are done. Now assume that $n > 1$, and let

$$\begin{bmatrix} e & b_2 & b_3 & \cdots & b_n \end{bmatrix}$$

be the first row of A. By elementary column operations, we can subtract eb_i from the i-th entry for each $i \geq 2$. Thus, we can assume that $b_2, \dots, b_n \in (1-e)R$. Since A is invertible, $eR + b_2R + \cdots + b_nR = R$, and so it follows that $b_2R + \cdots + b_nR = (1-e)R$. Hence, by elementary column operations we can add $1 - e$ to the first entry. Now we have a 1 in the $1,1$ position, and the rest is routine. □

Since we shall need to perform a number of operations on the top rows of invertible matrices, it is convenient to work with the rows alone. Recall that any row $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ of an invertible matrix over a ring R is *right unimodular*, that is, $\sum_{i=1}^n a_iR = R$. Elementary column operations apply to such a row just by viewing it as a $1 \times n$ matrix. Such operations amount to multiplying the row on the right by an elementary matrix. Since our rings need not be commutative, elementary column operations can only introduce right-hand coefficients.

Lemma 2.4. *Let R be an exchange ring and $\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ a right unimodular row over R. Then α can be transformed by elementary column operations to a row $\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$ such that $R = b_1R \oplus \cdots \oplus b_nR$ and each $b_i \in a_iRa_i$.*

Proof. Since $\sum_{i=1}^n a_iR = R$, Lemma 1.2 gives us orthogonal idempotents $e_1, \dots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and $e_i \in a_iR$ for all i, say $e_i = a_ir_i$. By elementary column operations, we can subtract $e_ia_1 = a_ir_ia_1$ from the first entry of α for each $i \geq 2$. This transforms α to $\alpha' = \begin{bmatrix} e_1a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$. Note that $e_1 \in e_1a_1R$. Thus, we can repeat the above process for each entry, and transform α' to the row $\begin{bmatrix} e_1a_1 & e_2a_2 & \cdots & e_na_n \end{bmatrix}$, with entries $e_ia_i \in a_iRa_i$. Moreover, $e_ia_iR = e_iR$, and therefore $R = \bigoplus_{i=1}^n e_ia_iR$. □

Corollary 2.5. *Let R be an exchange ring and $\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ a right unimodular row over R, with $n \geq 2$. Then α can be transformed by elementary column operations to a row $\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$ such that $Rb_1R = R$ and $b_i \in a_iRa_i$ for all $i \geq 2$.*

Proof. By Lemma 2.4, we may assume that $R = \bigoplus_{i=1}^n a_iR$. It follows that all $a_i \in Rb_1$ where $b_1 = a_1 + \cdots + a_n$ (multiply b_1 on the left by the orthogonal idempotents arising from the given decomposition of R_R). Thus $Rb_1R = R$. By elementary column operations, we can add $a_2, \dots, a_n$ to the first entry of α, and thus transform it to $\begin{bmatrix} b_1 & a_2 & \cdots & a_n \end{bmatrix}$. □

Recall that an element x in a ring R is (*von Neumann*) *regular* provided there exists an element $y \in R$ such that $xyx = x$, equivalently, provided xR is a direct summand of R_R. If y can be chosen to be a unit in R, then x is said to be *unit-regular*. A regular element $x \in R$ is unit-regular if and only if $R/xR \cong \text{r.ann}(x)$, where $\text{r.ann}(x)$ denotes the right annihilator of x in R (cf. [**15**, Proof of Theorem 4.1]).

Corollary 2.6. *Let R be an exchange ring and $\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ a right unimodular row over R, with $n \geq 2$. Then α can be transformed by elementary column operations to a row $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ such that c_2 is a regular element, $c_2 \in Ra_2$, and $c_2R = (1-g)R$ for an idempotent g with $RgR = R$.*

Proof. By Corollary 2.5, we may assume that $Ra_1R = R$. By Corollary 2.2, there exists an idempotent $e \in a_1R$ such that $ReR = R$. By elementary column operations, we can subtract ea_2 from the second entry of α, so there is no loss of generality in assuming that $a_2 \in (1-e)R$. (At this stage, our current a_2 is only a <u>left</u> multiple of the original a_2. This is why the conclusions of the lemma state $c_2 \in Ra_2$ rather than $c_2 \in a_2Ra_2$.) Now using Lemma 2.4, we can transform α to a row $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ such that $R = \bigoplus_{i=1}^n c_iR$ and $c_2 \in a_2Ra_2$. Then $c_2R = (1-g)R$ for some idempotent g, and c_2 is regular. Moreover, $(1-g)R = c_2R \subseteq a_2R \subseteq (1-e)R$ and so $Re \subseteq Rg$. Therefore $RgR = R$. $\square$

Lemma 2.7. *Let R be an exchange ring and $A \in GL_n(R)$, with $n \geq 2$. Then A can be transformed by elementary row and column operations to a matrix whose $1,1$ entry d is regular, with $dR = (1-p)R$ and $Rd = R(1-q)$ for some idempotents p, q such that $RpR = RqR = R$.*

Proof. By Corollary 2.6, we can assume that the $1,2$ entry of A is a regular element c such that $cR = (1-g)R$ for some idempotent g with $RgR = R$. With elementary operations, we can move c to the $2,1$ position.

Now apply the transpose of Corollary 2.6 to the first column of A. Thus, A can be transformed by elementary row operations to a matrix whose $2,1$ entry is a regular element d such that $d \in cR$ and $Rd = R(1-q)$ for some idempotent q with $RqR = R$. Since d is regular, $dR = (1-p)R$ for some idempotent p. Then $(1-p)R \subseteq (1-g)R$, whence $Rg \subseteq Rp$ and so $RpR = R$.

Finally, use elementary operations to move d to the $1,1$ position. $\square$

Theorem 2.8. *If R is a separative exchange ring, then R is a GE-ring, and so the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective.*

Proof. We need to show that R is a GE_n-ring for all n. This is trivial for $n = 1$, hence we assume, by induction, that $n \geq 2$ and R is a GE_{n-1}-ring. Let A be an arbitrary invertible $n \times n$ matrix over R.

By Lemma 2.7, we may assume that the $1,1$ entry d of A is regular, with $dR = (1-p)R$ and $Rd = R(1-q)$ for some idempotents p, q such that $RpR = RqR = R$. We claim that d is unit-regular. Note that because $RpR = RqR = R$, the projective modules pR and qR are generators.

Now $R = \text{r.ann}(d) \oplus B = dR \oplus C$ for some B, C, and we have to prove that $\text{r.ann}(d) \cong C$. Since $B \cong dB = dR$, we have $\text{r.ann}(d) \oplus B \cong C \oplus B$. From $Rd = R(1-q)$, we get $\text{r.ann}(d) = qR$ and so $\text{r.ann}(d)$ is a generator. Since $C \cong R/dR \cong pR$, we see that C is a generator too. By Lemma 1.3, $\text{r.ann}(d) \cong C$ as desired.

The unit-regularity of d gives $d = ue$ for some unit u and idempotent e. Set

$$U = \begin{bmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix};$$

then the matrix $U^{-1}A$ has an idempotent entry. By Lemma 2.3, there exist $E, F \in E_n(R)$ such that

$$EU^{-1}AF = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}$$

where $A' \in GL_{n-1}(R)$. By our induction hypothesis, $A' \in E_{n-1}(R)D_{n-1}(R)$. It follows that

$$A \in D_n(R)E_n(R)D_n(R)E_n(R),$$

and therefore we have shown that R is a GE_n-ring. This establishes the induction step and completes the proof. □

Remarks 2.9. **(a)** Observe that the proof of Theorem 2.8 did not use the full force of separativity, only the cancellation property ($A \oplus C \cong B \oplus C \implies A \cong B$) for finitely generated projective R-modules A, B, C with A and B generators.

(b) Theorem 2.8 includes, in particular, the result of Menal and Moncasi that every factor ring of a right self-injective ring is a GE-ring [**22**, Theorem 2.2]. To make the connection explicit, recall that right self-injective rings are semiregular (e.g., [**13**, Theorem 2.16, Lemma 2.18]) and hence exchange; thus, all their factor rings are exchange rings. Further, any right self-injective ring is separative (e.g., [**14**, Theorem 3]). It follows that factor rings of right self-injective rings are separative [**3**, Theorem 4.2].

(c) As a special case of Theorem 2.8, we obtain that any separative regular ring is a GE-ring, which gives a partial affirmative answer to a question of Moncasi [**23**, Questió 5].

(d) In the situation of Theorem 2.8, one naturally asks for a description of the kernel of the epimorphism $GL_1(R) \to K_1(R)$. This has been answered for unit-regular rings and regular right-self-injective rings by Menal and Moncasi [**22**, Theorems 1.6, 2.6], and for exchange rings with primitive factors artinian by Chen and Li [**9**, Theorem 3]. In all the above cases, $K_1(R) \cong GL_1(R)^{\mathrm{ab}}$ provided $\frac{1}{2} \in R$ [**22**, Theorems 1.7, 2.6]; [**9**, Corollary 7]. Further, $K_1(R) \cong GL_1(R)^{\mathrm{ab}}$ when R is either a C*-algebra with unitary 1-stable range or an AW*-algebra [**22**, Theorem 1.3, Corollary 2.11] (here the algebraic K_1 is meant). The unit-regular and AW* results correct and extend earlier work of Handelman [**18**, Theorem 2.4]; [**19**, Theorem 7].

Theorem 2.10. *If R is a separative exchange ring and A is a (von Neumann) regular $n \times n$ matrix over R, then A can be diagonalized using elementary row and column operations.*

Proof. By [**2**, Theorem 2.4], there exist $P, Q \in GL_n(R)$ such that PAQ is diagonal. By Theorem 2.8, $P = U_1V_1$ and $Q = V_2U_2$, where $U_1, U_2 \in D_n(R)$ and $V_1, V_2 \in E_n(R)$. So V_1AV_2 is a diagonal matrix obtained from A by elementary row and column operations. □

Remark 2.11. When applying Theorem 2.10, note the distinction between invertible matrices and general matrices. An invertible matrix over a separative exchange ring can be diagonalized from either side (by Theorem 2.8), whereas the diagonalization of a general regular matrix sometimes requires elementary operations on both the rows and the columns. For example, the 2×2 matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ over a field cannot be diagonalized using only elementary row operations.

Example 2.12. Non-regular matrices over separative exchange rings need not be diagonalizable by elementary operations, even over finite dimensional algebras. For example, choose a field F and let

$$R = F[x_1, x_2, x_3, x_4]/\langle x_1, x_2, x_3, x_4 \rangle^2.$$

Then R has a basis $1, a_1, a_2, a_3, a_4$ such that $a_ia_j = 0$ for all i, j. Since R is clearly semiregular, it is an exchange ring; separativity is an easy exercise. In fact, since R is artinian, it has stable rank 1. Recall that this also implies that R is a GE-ring. Observe that every element of R is a sum of a scalar plus a nilpotent element, and that the product of any two nilpotent elements of R is zero.

Now consider the matrix $A = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]$, whose entries are linearly independent nilpotent elements of R. We claim that any sequence of elementary row or column operations on A can only produce a matrix whose entries are linearly independent nilpotent elements. For instance, consider a product

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} + bc_{21} & c_{12} + bc_{22} \\ c_{21} & c_{22} \end{bmatrix}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are linearly independent nilpotent elements. Then $b = \beta + n$ for some $\beta \in F$ and some nilpotent element n, whence $bc_{21} = \beta c_{21}$ and $bc_{22} = \beta c_{22}$, and so the entries in the matrix product above are linearly independent (they are clearly nilpotent). The same thing happens with other elementary operations, establishing the claim.

Therefore no sequence of elementary operations on A can produce a matrix with a zero entry. In particular, A cannot be diagonalized by elementary operations. Since R is a GE-ring, it follows that A cannot be diagonalized by invertible matrices either, i.e., there do not exist $X, Y \in GL_2(R)$ such that XAY is diagonal. Thus the first of the natural properties discussed in the introduction is not implied by the second.

3. K_1 of separative C*-algebras with real rank zero.

In connection with his work on the structure of multiplier algebras (e.g., [**32**, **33**, **34**]), Shuang Zhang has conjectured [unpublished] that if A is any unital C*-algebra with real rank zero, the topological $K_1(A)$ is isomorphic to the unitary group $U(A)$ modulo the connected component of the identity, $U(A)^\circ$. We confirm this conjecture in case A is separative, which at the same time provides a unified approach to all known cases of the conjecture. The main interest of Zhang's conjecture is in the case when the stable rank of A is greater than 1, since it has long been known that $K_1^{\text{top}}(A) \cong U(A)/U(A)^\circ$ for all unital C*-algebras A with stable rank 1 (e.g., this is equivalent to [**26**, Theorem 2.10]).

We consider only unital, complex C*-algebras in this section, and we refer the reader to [**4**, **30**] for background and notation for C*-algebras. In particular, we use $\sim$ and $\lesssim$ to denote Murray-von Neumann equivalence and subequivalence of projections, and we write $M_\infty(A)$ for the (non-unital) algebra of $\omega \times \omega$ matrices with only finitely many nonzero entries from an algebra A. We write $U(A)$ for the unitary group of a unital C*-algebra A, and $U(A)^\circ$ for the connected component of the identity in $U(A)$.

In the theory of operator algebras, it is customary to write $K_1(A)$ for the topological K_1-group of A (e.g., [**4**, Definition 8.1.1]; [**30**, Definition 7.1.1]), and we shall follow that practice here. Thus, $K_1(A) = GL_\infty(A)/GL_\infty(A)^\circ$. We then use the notation $K_1^{\text{alg}}(A)$ to denote the algebraic K_1-group of A. Since $K_1^{\text{alg}}(A)$ is the abelianization of $GL_\infty(A)$ (e.g., [**27**, Proposition 2.1.4, Definition 2.1.5]) and $K_1(A)$ is abelian (e.g., [**4**, Proposition 8.1.3]; [**30**, Proposition 7.1.2]), there is a natural surjective homomorphism $K_1^{\text{alg}}(A) \to K_1(A)$. Finally, following Brown [**5**, p. 116], we say that A has K_1-*surjectivity* (respectively, K_1-*injectivity*) provided the natural homomorphism $U(A)/U(A)^\circ \to K_1(A)$ is surjective (respectively, injective).

The concept of *real rank zero* for a C*-algebra A has a number of equivalent characterizations (see [**6**]). One is the requirement that each self-adjoint element of A can be approximated arbitrarily closely by real linear combinations of orthogonal projections. (This is usually phrased as saying that the set of self-adjoint elements of A with finite spectrum is dense in the set of all self-adjoint elements.) It was proved in [**3**, Theorem 7.2] that A has real rank zero if and only if it is an exchange ring. Hence, the C*-algebras with real rank zero are exactly the C*-algebras to which our results above can be applied.

Given a C*-algebra A, all idempotents in matrix algebras $M_n(A)$ are equivalent to projections (e.g., [**4**, Proposition 4.6.2]; [**27**, Proposition 6.3.12]). Hence, A is separative if and only if

$$p \oplus p \sim p \oplus q \sim q \oplus q \qquad \Longrightarrow \qquad p \sim q$$

for projections $p, q \in M_\infty(A)$. An equivalent condition (analogous to Lemma 1.3) is that $p \oplus r \sim q \oplus r \implies p \sim q$ whenever $r \lesssim n.p$ and $r \lesssim n.q$ for some n. Separativity in A is equivalent to the requirement that all matrix algebras $M_n(A)$ satisfy the *weak cancellation* introduced by Brown and Pedersen [**5**, p. 116]; [**7**, p. 114]. They have shown that every extremally rich C*-algebra (see [**7**, p. 125]) with real rank zero is separative ([**8**], announced in [**5**, p. 116]). We would like to emphasize the question of whether non-separative exchange rings exist by focusing on the C* case:

Problem. Is every C*-algebra with real rank zero separative?

By combining Theorem 2.8 with a result of Lin, we obtain the following theorem.

Theorem 3.1. *If A is a separative, unital C*-algebra with real rank zero, then the natural map $U(A)/U(A)^\circ \to K_1(A)$ is an isomorphism.*

Proof. Lin proved K_1-injectivity for C*-algebras with real rank zero in [**20**, Lemma 2.2]. Hence, it only remains to show K_1-surjectivity. It is a standard fact that $U(A)$ and $GL_1(A)$ have the same image in $K_1(A)$ (e.g., [**4**, pp. 66, 67] or [**30**, Proof of Proposition 4.2.6]). Now the natural map $GL_1(A) \to K_1(A)$ factors as the composition of natural maps $GL_1(A) \to K_1^{\mathrm{alg}}(A) \to K_1(A)$, the second of which is surjective. Since A has real rank zero, it is an exchange ring, and so the map $GL_1(A) \to K_1^{\mathrm{alg}}(A)$ is surjective by Theorem 2.8. Therefore the image of $U(A)$ in $K_1(A)$ is all of $K_1(A)$, as desired. □

Brown and Pedersen have proved that every separative, extremally rich C*-algebra has K_1-surjectivity ([**8**], announced in [**5**, p. 116]; [**7**, p. 114]). Since there are C*-algebras with real rank zero that are not extremally rich [**5**, Example, p. 117], Theorem 3.1 can be viewed as a partial extension of the Brown-Pedersen result within the class of C*-algebras with real rank zero.

We thank the referee for the following remark.

Remark 3.2. While Theorem 3.1 is neither unexpected nor new in the case of stable rank 1 (cf. the result of Rieffel cited above), it is perhaps surprising that there are many C*-algebras of real rank zero and stable rank 2 to which the theorem applies. To see this, consider C*-algebra extensions

$$0 \to I \to A \to B \to 0$$

in which I and B have real rank zero and A is unital. By theorems of Zhang ([**35**, Lemma 2.4]; cf. [**6**, Theorem 3.14]) and Lin and Rørdam [**21**, Proposition 4], A has real rank zero if and only if projections lift from B to A, if and only if the connecting map $K_0(B) \to K_1(I)$ in topological K-theory vanishes. In this case, by [**3**, Theorem 7.5], A will be separative provided I and B are both separative, and in particular if I and B have stable rank 1. However, by [**21**, Proposition 4], if I and B have stable rank 1, then A will have stable rank 2 provided the connecting map $K_1(B) \to K_0(I)$ does not vanish. It is easy to find specific extensions satisfying the above conditions, such as the examples analyzed in [**21**, End of Section 1] or [**16**].

We conclude with an application of Theorem 3.1 that extends an argument of Brown [**5**, Theorem 1], relating homotopy and unitary equivalence of projections, to a wider context within real rank zero. Projections p and q in a C*-algebra A are *unitarily equivalent* provided there exists a unitary element $u \in A$ such that $upu^* = q$; they are *homotopic* provided there is a continuous path $f : [0, 1] \to \{\text{projections in } A\}$ such that $f(0) = p$ and $f(1) = q$. It is a standard fact that homotopic projections are unitarily equivalent (e.g., [**4**, Propositions 4.3.3, 4.6.5]; [**30**, Proposition 5.2.10]).

Theorem 3.3. *Let A be a separative, unital C*-algebra with real rank zero, let $p, q \in A$ be projections, and let $B = \overline{ApA} + \mathbb{C} \cdot 1$. Then p and q are homotopic in A if and only if $q \in B$ and p, q are unitarily equivalent in B.*

Proof. If p and q are homotopic in A, they are connected by a path of projections within A. Each projection along this path is homotopic to p and hence is unitarily equivalent to p. Thus, these projections all lie in ApA. In particular, $q \in B$, and p and q are homotopic in B. Consequently, p and q must be unitarily equivalent in B.

Conversely, assume that $q \in B$ and p, q are unitarily equivalent in B. By [**6**, Corollary 2.8, Theorem 2.5], the closed ideal $I = \overline{ApA}$ has real rank zero (as a non-unital C*-algebra), and so the unital C*-algebras B and pIp have real rank zero. We do not need separativity for B, just K_1-injectivity (by Lin's result). Since I is an ideal of A, any projections in $M_\infty(I)$ which are (Murray-von Neumann) equivalent in $M_\infty(A)$ are also equivalent in $M_\infty(I)$ (any implementing partial isometry necessarily lies in $M_\infty(I)$). Hence, the separativity of A implies that I is separative, and so pIp is separative. Therefore, by Theorem 3.1, pIp has K_1-surjectivity.

With the above information in hand, Brown's proof [**5**, Theorem 1] carries through in the present setting. We sketch the details for the reader's convenience. By hypothesis, $q = upu^*$ for some unitary $u \in U(B)$; let α denote the image of u in $K_1(B)$. Now $K_1(B) = K_1(I^\sim) = K_1(I)$, and because pIp is a full hereditary sub-C*-algebra of I, the natural map $K_1(pIp) \to K_1(I)$ is an isomorphism [**5**, Remark, p. 117]. Thus α is the image of some $\beta \in K_1(pIp)$. Since pIp has K_1-surjectivity, β is the image of some unitary $v_1 \in U(pIp)$. Let $v = v_1 + 1 - p$ and $w = uv^*$. Then w is a unitary in B such that $q = wpw^*$, and the image of w in $K_1(B)$ is zero. Since B has K_1-injectivity, $w \in U(B)^\circ$, from which it follows that p and q are homotopic. □

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ON STEKLOFF EIGENVALUE PROBLEM

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Let (M^n, g) be a smooth compact Riemannian manifold with boundary $\partial M \neq \emptyset$. In this article we discuss the first positive eigenvalue of the Stekloff eigenvalue problem

$$\begin{cases} (-\Delta + q)u(x) = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial M, \end{cases}$$

where $q(x)$ is a C^2 function defined on M, $\partial \nu_g$ is the normal derivative with respect to the unit outward normal vector on the boundary ∂M. In particular, when the boundary ∂M satisfies the "interior rolling $R-$ball" condition, we obtain a positive lower bound for the first nonzero eigenvalue in terms of n, the diameter of M, R, the lower bound of the Ricci curvature, the lower bound of the second fundamental form elements, and the tangential derivatives of the second fundamental form elements.

1. Introduction.

Let (M^n, g) be a smooth compact Riemannian manifold with boundary $\partial M \neq \emptyset$. In local coordinates $(x^1, x^2, \dots, x^n)$, the Riemannian metric is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\ dx^i dx^j,$$

and the the Laplace operator is defined by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. We consider the following Stekloff eigenvalue problem:

$$\begin{cases} (-\Delta + q)u(x) = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial M, \end{cases} \tag{1.1}$$

where $q(x)$ is a C^2 function defined on M, $\partial \nu_g$ is the normal derivative with respect to the unit outward normal vector on the boundary ∂M. More specifically, we shall find a lower estimate for the first eigenvalue of the problem (1.1) in terms of the dimension of M, the geometrical data of M and ∂M, and the potential function q.

Problem (1.1) is known as the Stekloff problem as Stekloff first studied it for bounded plane domains with potential function $q \equiv 0$, and he found applications in physics. Also, it is important because the set of eigenvalues for the Stekloff problem is the same as the set of eigenvalues of the well-known Dirichlet-Neumann map. Also, it is well-known that when metrics on manifolds with boundary are conformally deformed, the sign of the Sobolev quotient $Q(M)$ and Sobolev trace quotient $Q(M, \partial M)$ of the manifold M are important conformal invariants and they can be characterized by the sign of the first eigenvalue of the problems (see [**E**])

$$\begin{cases} Lu + \eta_1 u = 0 & \text{in } M \\ Bu = 0 & \text{on } \partial M, \end{cases}$$

and

$$\begin{cases} Lu = 0 & \text{in } M \\ Bu = \lambda_1 u & \text{on } \partial M, \end{cases}$$

respectively, where $L = \Delta_g - \big[(n-2)/4(n-1)\big] R_g$ is the conformal Laplacian, $B = (\partial/\partial\nu_g) + \big[(n-2)/2\big] h_g$ is the boundary operator, h_g denotes mean curvature of the boundary ∂M with respect to ν_g, and R_g denotes the scalar curvature on M. Hence, it is natural to study the first eigenvalue of the associated equation (1.1) without the functions R_g and h_g. From the analysis viewpoint, this problem closely corresponds to the study of the following Neumann eigenvalue.

$$\text{(1.2)} \qquad \begin{cases} (-\Delta + \eta_1) u(x) = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

It is well known that the first nonzero Neumann eigenvalue of the Laplacian on M will provide an optimal upper estimate for the Poincaré constant and it is important from an analysis viewpoint to prove the Poincaré inequality on the manifold M. Therefore, it is interesting to find a positive lower estimate of the first nonzero eigenvalue, and this has been studied extensively by many authors. We will simply refer the reader to [**B**], [**Ch**], [**C**], [**C-L**], [**L1**], [**L2**], [**L-T**], [**L-Y1**], [**L-Y2**], [**W**] for further references. Analogously, it is also interesting whether one may obtain a positive estimate for the lower bound of the first eigenvalue of the problem (1.1).

In a recent paper [**E**], Escobar generalized problem (1.1) with $q \equiv 0$ to a compact manifold (M^n, g) with boundary ∂M. In the two-dimensional case, if M has nonnegative Gaussian curvature and the geodesic curvature k_g of ∂M satisfies $k_g \geq k_0 > 0$, then he proved that the first nonzero eigenvalue λ_1 of the Stekloff problem satisfies $\lambda_1 \geq k_0$. Also, he proved that the equality holds only for the Euclidean ball of radius k_0^{-1}. In higher dimensional cases, if M has non-negative Ricci curvature then he proved that $\lambda_1 > \frac{k_0}{2}$, where $k_0 > 0$ is a lower bound for the second fundamental form elements of the boundary. However, the lower bounds in the paper [**E**] will become zero if one only assumes nonnegative geodesic curvature on the boundary for dimension two case or nonnegative second fundamental form elements on the boundary for higher dimension case.

From the interests of analysis, we shall try to obtain a positive lower bound for λ_1 of the Stekloff problem on a more general class of manifolds. In particular, we shall follow a similar gradient estimate argument as in [**C**] and [**W**] to prove a quantitative generalization of some results in [**E**].

Theorem 1.1. *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. Suppose that ∂M satisfies the "interior rolling $R-$ball" condition. Let K, H and $\bar{H}$ be nonnegative constants such that the Ricci curvature Ric_M of M is bounded below by $-(n-1)K$, the second fundamental form II of ∂M is bounded below by $-H$ and the absolute value of tangential derivatives of II is bounded by $\bar{H}$. By choosing R small, and for $a = 1$ or $a = 3$, we have the following estimate for the first eigenvalue λ_1 of the Stekloff problem (1.1).*

(1.3)
$$\begin{aligned}
&\frac{2(1+C_{14})^{\frac{1}{2}}}{2d^2(n-1)^2(1+H)^2}\exp\left[-1-\left(1+C_{14}\right)^{\frac{1}{2}}\right] \\
&\le \lambda_1^a\Bigg[8(n-1)+(n-1)^2(24+12H)+3\beta\max\Bigg\{12n-8+6H, \\
&\qquad 2(n-1)\left(\sqrt{(n-1)K+(n-2)K_R}+\frac{1}{R}+1\right)\Bigg\} \\
&\qquad +\beta\sqrt{2(n-1)}(96+6C_1+2C_4)+\frac{\beta-1}{\beta}\big(36n+28+12(n-1)H\big)\Bigg] \\
&\qquad +\left[3\beta\sup|q|+\beta\sqrt{2(n-1)\delta}\sup|\nabla q|-2(n-1)\inf q\right],
\end{aligned}$$

where $\delta > 0$ is any constant, C_1, C_4, C_5, C_6, are constants depending on n, K, H, $\bar{H}$, R, $d =$ diameter of M,

$$\begin{aligned}
\frac{\beta}{\beta-1} &= \exp\left[1+(1+C_{14})^{\frac{1}{2}}\right], \\
C_{14} &= d^2(1+H)^2\left[3C_5+C_6+(n-1)K+\frac{1}{2\delta}\right],
\end{aligned}$$

and they can be explicitly computed.

Remark. We shall choose the radius $R < 1$ of the interior rolling ball to satisfy the following inequalities

$$\sqrt{K_R}\tan(R\sqrt{K_R}) \le \frac{H}{2} + \frac{1}{2},$$

$$\frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R}) \le \frac{1}{2},$$

$$\frac{\sqrt{(n-1)K + (n-2)K_R}}{e^{2R\sqrt{(n-1)K+(n-2)K_R}} - 1} > H$$

where K_R denotes the upper bound of the radial curvatures in $\partial M(R) = \{x \in M \cup \partial M | \mathrm{dist}(x, \partial M) \le R\}$.

Corollary 1.2. *Let* (M^n, g) *be as in Theorem* 1.1 *and let* $q \equiv 0$ *in* (1.1). *By choosing* R *small, we have*

$$\lambda_1 \ge C_{15}, \tag{1.4}$$

for some constant C_{15}, *depending on* n, K, H, $\bar{H}$, R, $d =$ *diameter of* M, *and it can be explicitly computed.*

Using the same technique, we may also obtain the following estimate for η_1 in (1.2).

Theorem 1.3. *Let* (M^n, g) *be as in Theorem* 1.1. *By choosing* R *small, we have*

$$\eta_1 \ge C_{16}, \tag{1.5}$$

where C_{16} *is a positive constant depending on* n, K, H, R, $d =$ *diameter of* M *and both can be explicitly computed as in* (3.13).

Corollary 1.4. *Let* (M^n, g) *be as in Theorem* 1.1. *Assume that the Ricci curvature of* M *is nonnegative, the boundary* ∂M *is convex. By choosing* R *small, we have*

$$\eta_1 \ge \frac{C_{17}}{d^2} \tag{1.6}$$

where C_{17} *is a positive constant depending only on* n *and it can be explicitly computed as in* (3.15).

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2. Main Lemma.

We recall the following definition from [**C**].

Definition 2.1. *∂M is said to satisfy the "interior rolling R-ball" condition if for each point $p \in \partial M$ there is a geodesic ball $B_q\left(\frac{R}{2}\right)$, centered at $q \in M$ with radius $\frac{R}{2}$, such that $p = B_q(\frac{R}{2}) \cap \partial M$ and $B_q(\frac{R}{2}) \subset M$.*

We may modify a gradient estimate method as in [**C**] and [**W**] to prove our main lemma for a positive solution of the problem (1.1). In our case, we need to define two functions on M by $\phi(x) = \varphi(r(x))$ and $\psi(x) = \Psi(\frac{r(x)}{R})$, where $r(x)$ denotes the distance from $x \in M$ to ∂M and $\varphi(r)$ and $\Psi(r)$ are nonnegative smooth functions defined on $[0, \infty)$ such that

$$\begin{cases} \varphi(r) \leq \lambda_1 R & \text{if } r \in [0, \frac{R}{2}) \\ \varphi(r) = \lambda_1 R & \text{if } r \in [R, \infty) \end{cases} \tag{2.1}$$

and

$$\begin{cases} \Psi(r) \leq H & \text{if } r \in [0, \frac{1}{2}) \\ \Psi(r) = H & \text{if } r \in [1, \infty) \end{cases} \tag{2.2}$$

with

$$\begin{aligned} &\varphi(0) = 0, \quad 0 \leq \varphi'(r) \leq 2\lambda_1, \quad \varphi'(0) = \lambda_1 \\ &|\varphi''(r)| \leq 2\lambda_1, \quad |\varphi'''(r)| \leq 2\lambda_1, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\Psi(0) = 0, \quad 0 \leq \Psi'(r) \leq 2H, \\ &\Psi'(0) = H, \quad \Psi''(r) \geq -2H. \end{aligned} \tag{2.4}$$

Letting

$$\begin{aligned} &w = (1+\phi)u, \quad f = \log(1+\phi), \\ &p = -q + |\nabla f|^2 - \Delta f, \end{aligned}$$

Equation (1.1) for u is transformed into the following equation for w.

$$\begin{cases} \Delta w - 2\langle \nabla f, \nabla w \rangle + pw = 0 & \text{in } M \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{2.5}$$

Lemma 2.2. *Let (M^n, g) be as in Theorem 1.1. Normalize w such that $1 = \sup w$. For a constant $\beta > 1$, we consider the function*

$$F(x) = (1+\psi(x))^2 \frac{|\nabla w|^2}{(\beta - w)^2}. \tag{2.6}$$

Assume that $F(x_0) = \max_{x\in\bar{M}} F(x)$, *and choose* R *small, we have*

$$F(x_0) \le (1+\psi)^2 \left[(2n-1)\theta + \frac{\sqrt{(2n-1)\gamma}}{\beta-1}\right] \le (1+H)^2 \left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right], \tag{2.7}$$

where $\delta > 0$ *is any constant, and*

$$\begin{aligned}
C_1 &= (n-1)\max\{3H+1, \sqrt{(n-1)K+(n-2)K_R}\}\\
C_2 &= \max\{(3H+1)^2, (n-1)K+(n-2)K_R\} + K_R\\
C_3 &= e^{\frac{RC_1}{n-1}}\left(\bar{H} + Re^{\frac{RC_1}{n-1}}\bar{K}_R\right)\\
C_4 &= \max\{C_2, C_3\}\\
C_5 &= \frac{4H^2}{R^2}\\
C_6 &= \frac{2(n-1)H(3H+1)}{R} - \frac{2H}{2R^2}\\
C_7 &= 4\lambda_1^2\\
C_8 &= \lambda_1(8\lambda_1+6H+4)\\
C_9 &= 2\lambda_1\left(\sqrt{(n-1)K+(n-2)K_R} + \frac{1}{R} + 1\right)\\
C_{10} &= \max\{C_7+(n-1)C_8, (n-1)C_9\}\\
C_{11} &= 64\lambda_1^3 + 32\lambda_1^2 + 2\lambda_1(3C_1+C_4)\\
C_{12} &= \frac{[-\inf q + C_7 + (n-1)C_8]\,\beta}{\beta-w} + \frac{(3n+13)C_7}{2(n-1)}\\
&\quad + C_8 + 3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\\
C_{13} &= \frac{4}{n-1}\left[\sup|q|^2 + C_{10}^2\right] + \delta\left[\sup|\nabla q|^2 + C_{11}^2\right].
\end{aligned}$$

Proof. The proof we give may be divided into the following steps.

(1) In Step (1), we determine the location of the maximum point of x_0 by using a maximum principle.
(2) In Step (2), we apply the maximum principle to obtain an inequality

$$0 \ge aF(x_0)^2 - bF(x_0) - c$$

for $F(x_0)$, where $a > 0$, b, $c \ge 0$ are constants.
(3) In Step (3), we shall find estimates of b and c which lead to an estimate of $F(x_0)$.

Step (1). The point x_0 is either a boundary point or an interior point of M. Suppose that $x_0 \in \partial M$, we let $\{e_i\}$ be a local orthonormal frame field of M^n such that $e_n = \frac{\partial}{\partial \nu}$ on ∂M. If we let h_{ij} denote the second fundamental form elements of ∂M, then

$$\begin{aligned} 0 &\leq \frac{\partial F}{\partial \nu}(x_0) \\ &= \frac{-2\frac{H}{R}|\nabla w|^2 + 2\sum_{i=1}^{n-1} w_i w_{in}}{(\beta - w)^2} \\ &= \frac{-2\frac{H}{R}|\nabla w|^2 - 2\sum_{i,j=1}^{n-1} h_{ij} w_i w_j}{(\beta - w)^2} \\ &\leq \frac{-2\frac{H}{R}|\nabla w|^2 + 2H|\nabla w|^2}{(\beta - w)^2} \\ &< 0, \end{aligned}$$

which is a contradiction, as we may choose R to be smaller than 1. Hence $F(x)$ cannot attain its maximum at the boundary point. Therefore x_0 has to be an interior point of M.

Step (2). Since F attains its maximum value at an point x_0, we have

$$\nabla F(x_0) = 0 \tag{2.8}$$

$$\Delta F(x_0) \leq 0. \tag{2.9}$$

Note that

$$\nabla F \cdot \left(\frac{\beta - w}{1+\psi}\right)^2 + F \cdot \nabla \left(\frac{\beta - w}{1+\psi}\right)^2 = \nabla |\nabla w|^2, \tag{2.10}$$

and

$$\Delta F \cdot \left(\frac{\beta - w}{1+\psi}\right)^2 + 2\nabla F \cdot \nabla \left(\frac{\beta - w}{1+\psi}\right)^2 + F \cdot \Delta \left(\frac{\beta - w}{1+\psi}\right)^2 = \Delta |\nabla w|^2$$

which implies that, at x_0,

$$0 \geq -F(x_0)\Delta \left(\frac{\beta - w}{1+\psi}\right)^2 (x_0) + \Delta |\nabla w|^2 (x_0). \tag{2.11}$$

We may choose an orthonormal frame field $\{e_i\}$ near x_0 such that $w_1(x_0) = |\nabla w|(x_0)$. Note that $|\nabla w|(x_0) \neq 0$, otherwise $F(x) = F(x_0) = 0$ for all $x \in M$ which is a contradiction.

At x_0, Equation (2.8) implies that

$$|\nabla w|^2 \left(-\frac{2w_j}{\beta - w} - \frac{2\psi_j}{1+\psi}\right) = 2w_1 w_{1j}$$

for each $j = 1, \ldots, n$. Therefore, at x_0, we have

$$\begin{cases} w_{11} = -\frac{w_1^2}{\beta - w} - \frac{w_1\psi_1}{1+\psi} \\ w_{1j} = -\frac{w_1\psi_j}{1+\psi} \qquad \text{for } j \neq 1. \end{cases} \tag{2.12}$$

Using the Ricci identity, $w_{ijk} - w_{ikj} = \sum_{l=1}^n w_l R_{lijk}$ and a direct calculation, we get

$$\frac{1}{2}\Delta|\nabla w|^2 = \sum_{i,j=1}^n w_{ij}^2 + \sum_{j=1}^n w_j(\Delta w)_j + \sum_{i,j=1}^n R_{ij} w_i w_j \tag{2.13}$$

$$\frac{1}{2}F \cdot \Delta\left(\frac{\beta - w}{1+\psi}\right)^2 = |\nabla w|^2\left[\frac{-\Delta w}{\beta - w} + \frac{|\nabla w|^2}{(\beta - w)^2} + \frac{2\langle\nabla w \nabla\psi\rangle}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + \frac{3|\nabla\psi|^2}{(1+\psi)^2}\right].$$

Substituting (2.13) into (2.11) and using a direct calculation, we have

$$0 \geq \sum_{i,j=1}^n w_{ij}^2 + \sum_{j=1}^n w_j(\Delta w)_j + \sum_{i,j=1}^n R_{ij}w_iw_j - |\nabla w|^2\left[\frac{-\Delta w}{\beta - w} + \frac{|\nabla w|^2}{(\beta - w)^2} + \frac{2\langle\nabla w\nabla\psi\rangle}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + \frac{3|\nabla\psi|^2}{(1+\psi)^2}\right].$$

Using (2.5), we have

$$\begin{aligned}
&= \sum_{i,j=1}^n w_{ij}^2 + 2w_1\sum_{j=1}^n (f_jw_j)_1 - w_1(pw)_1 + R_{11}w_1w_1 - w_1^2\left[\frac{pw}{\beta - w}\right. \\
&\qquad \left. - \frac{2f_1w_1}{\beta - w} + \frac{w_1^2}{(\beta - w)^2} + \frac{2w_1\psi_1}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + 3\frac{|\nabla\psi|^2}{(1+\psi)^2}\right] \\
&= \sum_{i,j=1}^n w_{ij}^2 + 2w_1^2 f_{11} + 2w_1\sum_{j=1}^n f_j w_{j1} - w_1^2 p - w_1wp_1 + R_{11}w_1w_1 \\
&\qquad - w_1^2\left[\frac{pw}{\beta - w} - \frac{2f_1w_1}{\beta - w} + \frac{w_1^2}{(\beta - w)^2}\right. \\
&\qquad \left. + \frac{2w_1\psi_1}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + 3\frac{|\nabla\psi|^2}{(1+\psi)^2}\right] \\
&= \sum_{i,j=1}^n w_{ij}^2 - \frac{w_1^4}{(\beta - w)^2} - \frac{2w_1^3\psi_1}{(1+\psi)(\beta - w)} \\
&\qquad + 2w_1\sum_{j=1}^n f_j w_{j1} + \frac{2f_1w_1^3}{\beta - w} - w_1wp_1
\end{aligned}$$

$$- w_1^2 \left[\frac{pw}{\beta - w} + p - \frac{\Delta\psi}{1+\psi} + 3\frac{|\nabla\psi|^2}{(1+\psi)^2} - 2f_{11} - R_{11} \right]$$

$$= \sum_{i,j=1}^{n} w_{ij}^2 - \frac{w_1^4}{(\beta - w)^2} - \frac{2w_1^3\psi_1}{(1+\psi)(\beta - w)}$$

$$+ 2w_1 \sum_{j=1}^{n} f_j w_{j1} + \frac{2f_1 w_1^3}{\beta - w} - w_1 w p_1$$

$$- w_1^2 \left[\frac{p\beta}{\beta - w} - \frac{\Delta\psi}{1+\psi} + 3\frac{|\nabla\psi|^2}{(1+\psi)^2} - 2f_{11} - R_{11} \right].$$

Using (2.12), and the inequality $x^2 + y^2 \geq 2xy$, it is easy to see that

$$w_{11}^2 - \frac{w_1^4}{(\beta - w)^2} - \frac{2w_1\psi_1}{(\beta - w)(1+\psi)} = \frac{w_1^2\psi_1^2}{(1+\psi)^2}$$

$$\frac{2f_1 w_1^3}{\beta - w} + 2f_1 w_1 w_{11} = -\frac{2w_1^2 f_1 \psi_1}{1+\psi}$$

$$2\sum_{j=2}^{n} w_{1j}^2 + 2w_1 \sum_{j=2}^{n} f_j w_{1j} \geq -\frac{w_1^2}{2} \sum_{j=2}^{n} f_j^2.$$

Putting these into the above, we have

(2.14)

$$0 \geq \sum_{i,j=2}^{n} w_{ij}^2 - p_1 w w_1 + \frac{w_1^2\psi_1^2}{(1+\psi)^2}$$

$$- w_1^2 \left[\frac{p\beta}{\beta - w} + \frac{1}{2}\sum_{2}^{n} f_j^2 - 2f_{11} - \frac{\Delta\psi}{1+\psi} + \frac{2f_1\psi_1}{1+\psi} + \frac{2|\nabla\psi|^2}{(1+\psi)^2} - R_{11} \right].$$

To prove our claim, we shall find an inequality for $F(x_0)$ of the form

$$0 \geq aF(x_0)^2 - bF(x_0) - c$$

where $a > 0$, b, c are nonnegative constants. To obtain the quadratic term of $F(x_0)$ with positive coefficient, we observe that Cauchy-Schwarz inequality implies that

$$\sum_{i,j=2}^{n} w_{ij}^2 + \frac{w_1^2\psi_1^2}{(1+\psi)^2} \geq \sum_{j=2}^{n} w_{jj}^2 + \frac{w_1^2\psi_1^2}{(1+\psi)^2}$$

$$\geq \frac{1}{n-1}\left(w_{11} - \Delta w\right)^2 + \frac{w_1^2\psi_1^2}{(1+\psi)^2}.$$

Using (2.5), (2.12), and the inequality $(x+y)^2 \geq \frac{1}{2}x^2 - y^2$, the above becomes

$$= \frac{1}{n-1}\left(\frac{w_1^2}{\beta - w} + \frac{w_1\psi_1}{1+\psi} + \Delta w\right)^2 + \frac{w_1^2\psi_1^2}{(1+\psi)^2}$$

$$\geq \frac{1}{2(n-1)}\left(\frac{w_1^2}{\beta-w}+\frac{w_1\psi_1}{1+\psi}\right)^2-\frac{1}{n-1}(\Delta w)^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}.$$

Using the inequality $(x+y)^2 \geq \frac{2(n-1)}{2n-1}x^2-2(n-1)y^2$, the above becomes

$$\begin{aligned}&\geq \frac{1}{2n-1}\left(\frac{w_1^4}{(\beta-w)^2}-\frac{w_1^2\psi_1^2}{(1+\psi)^2}\right)-\frac{1}{n-1}(\Delta w)^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}\\ &= \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{1}{n-1}(\Delta w)^2.\end{aligned}$$

Using (2.5), and the inequality $2(x^2+y^2) \geq (x+y)^2$, we get

$$\sum_{i,j=2}^{n} w_{ij}^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2} \geq \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{2}{n-1}\left(p^2w^2+4f_1^2w_1^2\right). \tag{2.15}$$

Substituting (2.15) into (2.14), we get

(2.16)

$$\begin{aligned}0 &\geq \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{2}{n-1}p^2w^2-p_1ww_1-w_1^2\left[\frac{p\beta}{\beta-w}+\frac{8}{n-1}f_1^2\right.\\ &\quad\left.+\frac{1}{2}\sum_2^n f_j^2-2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}\right]\\ &\geq \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{2}{n-1}p^2w^2-\frac{\delta}{2}p_1^2w^2-\frac{1}{2\delta}w_1^2-w_1^2\left[\frac{p\beta}{\beta-w}\right.\\ &\quad\left.+\frac{8}{n-1}f_1^2+\frac{1}{2}\sum_2^n f_j^2-2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}\right]\\ &= \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\left[\frac{2}{n-1}p^2+\frac{\delta}{2}p_1^2\right]w^2-w_1^2\left[\frac{p\beta}{\beta-w}+\frac{8}{n-1}f_1^2\right.\\ &\quad\left.+\frac{1}{2}\sum_2^n f_j^2-2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}+\frac{1}{2\delta}\right]\end{aligned}$$

where δ is any positive constant. In order to see that we have almost obtained the desired inequality for $F(x_0)$, we shall simplify the notations by setting

$$\begin{aligned}\theta = \frac{p\beta}{\beta-w}+\frac{8}{n-1}f_1^2+\frac{1}{2}\sum_2^n f_j^2\\ -2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}+\frac{1}{2\delta},\end{aligned} \tag{2.17}$$

and

$$\gamma = \frac{2}{n-1}p^2 + \frac{\delta}{2}p_1^2. \tag{2.18}$$

Also, note that if we set $\alpha = \frac{w}{\beta - w}$, then we have

$$\alpha = \frac{w}{\beta - w} \le \frac{1}{\beta - w} \le \frac{1}{\beta - 1}. \tag{2.19}$$

Multiplying (2.16) through by $\dfrac{(1+\psi)^4}{(\beta-w)^2}$ and using (2.17)-(2.19), we obtain

$$0 \ge \frac{1}{2n-1}F^2 - \theta(1+\psi)^2 F - \gamma(1+\psi)^4\alpha^2. \tag{2.20}$$

Step (3). In this step, we shall give estimates on $\theta(1+\psi)^2$ and $\gamma(1+\psi)^4\alpha^2$ in (2.20). The inequality (2.20) implies that we have

$$\begin{aligned} F(x_0) &\le (1+\psi)^2 \left[\frac{(2n-1)\theta}{2} + \sqrt{\frac{(2n-1)^2\theta^2}{4} + (2n-1)\gamma\alpha^2} \right] \\ &\le (1+\psi)^2 \left[(2n-1)\theta + \sqrt{(2n-1)\gamma}\alpha \right]. \end{aligned} \tag{2.21}$$

In order to prove our claim, we shall estimate each term in (2.17) and (2.18) of θ and γ, respectively. Since $p = -q + |\nabla f|^2 - \Delta f$ and $f = \log(1+\phi(r(x)))$, we shall need to estimate Δr and $|\nabla \Delta r|$ near the boundary ∂M if we want to estimate the term $|\nabla p|$ in γ. Here, we shall first derive some estimates for Δr and $|\nabla \Delta r|$. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame fields of M^n in a neighborhood $\partial M(R) = \{x \in M \cup \partial M | r(x) \le R\}$ of ∂M such that $e_n = -\frac{\partial}{\partial \nu}$, where ν is the unit outward normal vector to ∂M. For any $x \in \partial M \times \{r\}$, where $\partial M \times \{r\} = \{x \in \partial M(R) | r(x) = r\}$, we have $r_n = 1$, and $r_a = 0$ for $a = 1, \dots, n-1$. When R is sufficiently small, we may write each point $x \in \partial M(R)$ as $x = (y, r)$, where $y \in \partial M$ and $\operatorname{dist}(x, y) = r(x)$. A direct calculation shows that

$$r_{na} = 0, \quad r_{nn} = 0, \quad r_{aa} = -h_{aa}$$

for $a = 1, \dots, n-1$, where h_{aa} is a second fundamental form element of $\partial M \times \{r\}$. To estimate $|\nabla \Delta r|$, it suffices to obtain estimates of $|e_n(\Delta r)|$ and $|e_a(\Delta r)|$ for $a = 1, \dots, n-1$. Differentiating r_{aa} in the direction of e_n yields

$$\begin{aligned} e_n(r_{aa}) &= r_{aan} - \sum_{b=1}^{n-1} r_{ab}^2 \\ &= r_{ana} + \sum_{l=1}^{n} r_l R_{laan} - \sum_{b=1}^{n-1} r_{ab}^2 \end{aligned}$$

$$= -\sum_{b=1}^{n-1} r_{ab}^2 - R_{nana},$$

where R_{nana} denotes the curvature tensor of M. Hence, we have

$$e_n(r_{aa}) = -\sum_{b=1}^{n-1} r_{ab}^2 - R_{nana} \tag{2.22}$$

for each $x = (y, r) \in \partial M(R)$. Integrating (2.22) yields

$$r_{aa}(x) = r_{aa}(y) + \int_0^r \left(\sum_{b=1}^{n-1} r_{ab}^2 - R_{nana} \right)(y, t)dt. \tag{2.23}$$

Let K_R and $\bar{K}_R$ denote the upper bounds of the radial curvatures and of the absolute value of covariant derivatives of radial curvatures, respectively, in $\partial M(R)$, i.e., $K_R = \max\,\{R_{nana}(x)|x \in \partial M(R), 1 \le a \le n-1\}$ and $\bar{K}_R = \max\,\{|R_{nana,b}(x)|\,|x \in \partial M(R), 1 \le a, b \le n\}$. Since the boundary ∂M satisfies the "*interior rolling R−ball*" condition, its second fundamental form element II is bounded from above by $\frac{1}{R}$ and is bounded from below by hypothesis. We shall follow an index comparison theorem [**Wa**] to obtain estimates on r_{aa} for $a = 1, \ldots, n-1$. To apply it, we choose R small as in [**C**] such that

$$\sqrt{K_R}\tan(R\sqrt{K_R}) \le \frac{H}{2} + \frac{1}{2},$$
$$\frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R}) \le \frac{1}{2},$$

and

$$\frac{\sqrt{(n-1)K + (n-2)K_R}}{e^{2R\sqrt{(n-1)K+(n-2)K_R}} - 1} > H.$$

Using an index comparison theorem, we have

$$\begin{aligned} r_{aa} &\ge -\frac{H + \sqrt{K_R}\tan(R\sqrt{K_R})}{1 - \frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R})} \\ &\ge -(3H+1), \end{aligned} \tag{2.24}$$

and if we set $\kappa = \sqrt{(n-1)K + (n-2)K_R}$ we have

$$\begin{aligned} r_{aa} &\le \frac{\kappa\left[\left(e^{2\kappa r(x)} - 1\right)\kappa + \left(e^{2\kappa r(x)} + 1\right)\frac{1}{R}\right]}{\left(e^{2\kappa r(x)} + 1\right)\kappa + \left(e^{2\kappa r(x)} - 1\right)\frac{1}{R}} \\ &\le \frac{\kappa\left[\left(e^{2\kappa R} + 1\right)\kappa R + \left(e^{2\kappa R} + 1\right)\right]}{\left(e^{2\kappa r(x)} + 1\right)R} \\ &\le \kappa + \frac{1}{R} = \sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R}, \end{aligned} \tag{2.25}$$

for $a = 1, \dots, n-1$. Hence, we have

$$|\Delta r| \le (n-1)\max\{3H+1, \sqrt{(n-1)K+(n-2)K_R}\} = C_1. \tag{2.26}$$

Combining (2.22), (2.24), and (2.25), we have

$$|e_n(\Delta r)| \le \max\{(3H+1)^2, (n-1)K+(n-2)K_R\} + K_R = C_2. \tag{2.27}$$

Differentiating (2.23), we get

$$r_{aa,b}(y,r) = r_{aa,b}(y,0) + \int_0^r \left(2\sum_{c=1}^{n-1} r_{ac}r_{ac,b} - R_{nana,b}\right)(y,t)dt \tag{2.28}$$

for $b = 1, \dots, n-1$. We may assume that r_{aa} for $a = 1, \dots, n-1$ denotes an eigenvalue for the Hessian of r. Differentiating (2.28) with respect to r and solving the first order differential equation

$$r_{aa,b}(y,r) = 2r_{aa}r_{aa,b}(y,r) - R_{nana,b}(y,r),$$

we have

$$|e_b(\Delta r)|(x) \le e^{\frac{RC_1}{n-1}}\left(\bar{H} + Re^{\frac{RC_1}{n-1}}\bar{K}_R\right) = C_3. \tag{2.29}$$

Combining (2.26) and (2.29), we have

$$|\nabla\Delta r|(x) \le \max\{C_2, C_3\} = C_4. \tag{2.30}$$

We are now ready to give estimates for θ, γ in (2.17) and (2.18). Note that it suffices to find estimates for terms $|\nabla\psi| = \Psi'$, $\Delta\psi$, $|\nabla f|^2 = |\nabla\log(1+\phi)|^2$, f_{jj}, $p = -q + |\nabla f|^2 - \Delta f$, and $|\nabla p|$. In the following, we shall give an estimate for each of these terms. From the definition of ψ and (2.24), it is easy to see that

$$|\nabla\psi|^2 = \frac{1}{R^2}\Psi'^2 \le \frac{4H^2}{R^2} = C_5 \tag{2.31}$$

and

$$\Delta\psi = \frac{1}{R}\Psi'\Delta r + \frac{1}{R^2}\Psi''|\nabla r|^2 \ge -\frac{2(n-1)H(3H+1)}{R} - \frac{2H}{2R^2} = -C_6. \tag{2.32}$$

For the terms $|\nabla f|^2 = |\nabla\log(1+\phi)|^2$ and f_{jj}, we apply (2.24), (2.25) to obtain

$$|\nabla f|^2 = |\nabla\log(1+\phi)|^2 = \frac{\varphi'^2}{(1+\phi)^2} \le 4\lambda_1^2 = C_7, \tag{2.33}$$

$$f_{jj} = \frac{\varphi' r_{jj}}{(1+\phi)^2} + \frac{\varphi'' r_j^2}{(1+\phi)^2} - \frac{2\varphi'^2 r_j^2}{(1+\phi)^3} \tag{2.34}$$
$$\geq -2\lambda_1(3H+1) - 2\lambda_1 - 8\lambda_1^2$$
$$= -\lambda_1(8\lambda_1 + 6H + 4) = -C_8,$$

and

$$f_{jj} \leq 2\lambda_1\left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R}\right) + 2\lambda_1 \tag{2.35}$$
$$= 2\lambda_1\left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1\right) = C_9.$$

For the term $p = -q + |\nabla f|^2 - \Delta f$, we use (2.32)-(2.35) to get

$$p = -q + |\nabla f| - \Delta f \leq -\inf q + C_7 + (n-1)C_8, \tag{2.36}$$
$$p = -q + |\nabla f|^2 - \Delta f \geq -\sup q - (n-1)C_9. \tag{2.37}$$

Combining (2.36), (2.37), we get that

$$|p| \leq \sup|q| + \max\{C_7 + (n-1)C_8, (n-1)C_9\} = \sup|q| + C_{10}. \tag{2.38}$$

For the term $|\nabla p|$, we note that

$$\nabla p = -\nabla q + \nabla\left[\frac{\varphi'^2}{(1+\phi)^2}\right] - \nabla\left[\frac{\varphi'\Delta r}{(1+\phi)^2} + \frac{\varphi''}{(1+\phi)^2} - \frac{2\varphi'^2}{(1+\phi)^3}\right]$$
$$= -\nabla q + \frac{2\varphi'\varphi''\nabla r}{(1+\phi)^2} - \frac{2\varphi'^2\nabla r}{(1+\phi)^3} - \frac{\varphi''\Delta r}{(1+\phi)^2} - \frac{\varphi'\nabla\Delta r}{(1+\phi)^2} + \frac{2\varphi'\Delta r\nabla r}{(1+\phi)^3}$$
$$- \frac{\varphi'''\nabla r}{(1+\phi)^2} + \frac{2\varphi'\varphi''\nabla r}{(1+\phi)^3} + \frac{4\varphi'\varphi''\nabla r}{(1+\phi)^3} - \frac{6\varphi'^3\nabla r}{(1+\phi)^4}.$$

Hence, we use (2.26)and (2.30) to get

$$|\nabla p| \leq 64\lambda_1^3 + 32\lambda_1^2 + 2\lambda_1(3C_1 + C_4) + \sup|\nabla q| = C_{11} + \sup|\nabla q|. \tag{2.39}$$

Note that each constant $C_7, \dots, C_{11}$ contains a factor λ_1. Combining estimates (2.24)-(2.26) and (2.30)-(2.39), we have estimates for θ and γ.

(2.40)

$$\theta = \frac{p\beta}{\beta - w} + \frac{8}{n-1}f_1^2 + \frac{1}{2}\sum_2^n f_j^2$$
$$- 2f_{11} + \frac{2f_1\psi_1}{1+\psi} - \frac{\Delta\psi}{1+\psi} + \frac{2|\nabla\psi|^2}{(1+\psi)^2} - R_{11} + \frac{1}{2\delta}$$
$$\leq \frac{[-\inf q + C_7 + (n-1)C_8]\,\beta}{\beta - 1} + \left(\frac{8}{n-1} + \frac{1}{2}\right)C_7 + C_8 + f_1^2 + \frac{\psi_1^2}{(1+\psi)^2}$$
$$+ \frac{C_6}{1+\psi} + \frac{2C_5}{(1+\psi)^2} + (n-1)K + \frac{1}{2\delta}$$

$$
\begin{aligned}
&\leq \frac{[-\inf q + C_7 + (n-1)C_8]\,\beta}{\beta - 1} + \frac{(3n+13)C_7}{2(n-1)} \\
&\quad + C_8 + 3C_5 + C_6 + (n-1)K + \frac{1}{2\delta} \\
&= C_{12},
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma &= \frac{2}{n-1}p^2 + \frac{\delta}{2}p_1^2 \\
&\leq \frac{4}{n-1}\left[\sup|q|^2 + C_{10}^2\right] + \delta\left[\sup|\nabla q|^2 + C_{11}^2\right] = C_{13}.
\end{aligned}
\tag{2.41}
$$

Finally, we may substitute (2.40), (2.41) into (2.21) to obtain

$$
\begin{aligned}
F(x_0) &\leq (1+\psi)^2\left[(2n-1)\theta + \frac{\sqrt{(2n-1)\gamma}}{\beta-1}\right] \\
&\leq (1+H)^2\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right].
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

3. Proof.

In this section, we shall utilize Lemma 2.2 to give a proof to Theorem 1.1, 1.3 and Corollary 1.2, 1.4.

Proof of Theorem 1.1. Using (2.7), we have

$$
\begin{aligned}
F(x) &\leq (1+\psi)^2\left[(2n-1)\theta + \frac{\sqrt{(2n-1)\gamma}}{\beta-1}\right] \\
&\leq (1+H)^2\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]
\end{aligned}
$$

for any $x \in M \cup \partial M$. This implies that

$$
\frac{|\nabla w|}{\beta - w} \leq (1+H)\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]^{\frac{1}{2}}.
\tag{3.1}
$$

Let x_1, x_2 be two points in M such that $w(x_1) = 0$, $w(x_2) = \sup w = 1$, and let $\gamma \subset M$ be a minimal geodesic joining from x_1 to x_2. Then we have

$$
\begin{aligned}
\log\frac{\beta}{\beta-1} &\leq \int_\gamma \frac{|\nabla w|}{\beta - w} \\
&\leq (1+H)\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]^{\frac{1}{2}} d
\end{aligned}
\tag{3.2}
$$

where d denotes the diameter of M. Recall that constants $C_1, \ldots, C_6$ do not depend on λ_1. We shall group them together. Hence, we have

$$
\begin{aligned}
&\frac{1}{d^2(1+H)^2}\left(\log\frac{\beta}{\beta-1}\right)^2 \\
&\le 2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1} \\
&= \frac{2(n-1)\beta\left[-\inf q + C_7 + (n-1)C_8\right]}{\beta-1} + (3n+13)C_7 \\
&\quad + 2(n-1)C_8 + 2(n-1)\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right] \\
&\quad + \frac{8\left[\sup|q|^2 + C_{10}^2\right] + 2(n-1)\delta\left[\sup|\nabla q|^2 + C_{11}^2\right]}{\beta-1} \\
&\le \left[\frac{2(n-1)\beta}{\beta-1} + (3n+13)\right]C_7 + \left[\frac{2(n-1)^2\beta}{\beta-1} + (2n-1)\right]C_8 \\
&\quad + \frac{3C_{10}}{\beta-1} + \frac{\sqrt{2(n-1)\delta}C_{11}}{\beta-1} + \frac{3\sup|q|}{\beta-1} + \frac{\sqrt{2(n-1)\delta}\sup|\nabla q|}{\beta-1} \\
&\quad - \frac{2(n-1)\beta\inf q}{\beta-1} + 2(n-1)\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right].
\end{aligned} \tag{3.3}
$$

Multiplying (3.3) through by $\frac{\beta-1}{\beta}$, we have

(3.4)

$$
\begin{aligned}
&\frac{\beta-1}{d^2\beta(1+H)^2}\left(\log\frac{\beta}{\beta-1}\right)^2 \\
&\le \left[2(n-1) + \frac{(\beta-1)(3n+13)}{\beta}\right]C_7 + \left[2(n-1)^2 + \frac{2(\beta-1)(n-1)}{\beta}\right]C_8 \\
&\quad + 3\beta C_{10} + \beta\sqrt{2(n-1)\delta}C_{11} + 3\beta\sup|q| + \beta\sqrt{2(n-1)\delta}\sup|\nabla q| \\
&\quad - 2(n-1)\inf q + 2(n-1)\frac{\beta-1}{\beta}\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right] \\
&\le \left[2(n-1)C_7 + 2(n-1)^2C_8 + 3\beta C_{10} + \beta\sqrt{2(n-1)\delta}C_{11}\right] \\
&\quad + \frac{\beta-1}{\beta}\left[(3n+13)C_7 + 2(n-1)C_8\right] \\
&\quad + \left[3\beta\sup|q| + \beta\sqrt{2(n-1)\delta}\sup|\nabla q| - 2(n-1)\inf q\right] \\
&\quad + \frac{2(n-1)(\beta-1)}{\beta}\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right].
\end{aligned}
$$

To finish the proof, we shall estimate constants $C_7, \ldots, C_{11}$ in terms of λ_1. Since we have either $\lambda_1 \ge \lambda_1^3$ or $\lambda_1^3 \ge \lambda_1$, we define a to be the number such that

$$\lambda^a = \max\{\lambda_1, \lambda_1^3\}. \tag{3.5}$$

Using the definitions of the constants $C_7, \dots, C_{11}$ in Lemma 2.2, we have

(3.6)

$$\begin{aligned}
C_7 &= 4\lambda_1^2 \le 4\lambda_1^a \\
C_8 &= \lambda_1 (8\lambda_1 + 6H + 4) \le (12 + 6H)\, \lambda_1^a \\
C_9 &= 2\lambda_1 \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right) \\
&\le 2\lambda_1^a \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right) \\
C_{10} &= \max\{C_7 + (n-1)C_8, (n-1)C_9\} \\
&\le \lambda_1^a \max \left\{ 12n - 8 + 6H, 2(n-1) \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right) \right\} \\
C_{11} &= 64\lambda_1^3 + 32\lambda_1^2 + 2\lambda_1(3C_1 + C_4) \le \lambda_1^a (96 + 6C_1 + 2C_4).
\end{aligned}$$

Substituting these into (3.4), we get

(3.7)

$$\begin{aligned}
&\frac{\beta - 1}{d^2 \beta (1+H)^2} \left(\log \frac{\beta}{\beta - 1} \right)^2 \\
&\quad - \frac{2(n-1)(\beta-1)}{\beta} \left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta} \right] \\
&\le \left[2(n-1)C_7 + 2(n-1)^2 C_8 + 3\beta C_{10} + \beta \sqrt{2(n-1)\delta}\, C_{11} \right] \\
&\quad + \frac{\beta - 1}{\beta} \left[(3n+13)C_7 + 2(n-1)C_8 \right] \\
&\quad + \left[3\beta \sup |q| + \beta \sqrt{2(n-1)\delta} \sup |\nabla q| - 2(n-1) \inf q \right] \\
&\le \lambda_1^a \Bigg[8(n-1) + (n-1)^2 (24 + 12H) + 3\beta \max \Bigg\{ 12n - 8 + 6H, \\
&\qquad 2(n-1) \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right) \Bigg\} \\
&\qquad + \beta \sqrt{2(n-1)} (96 + 6C_1 + 2C_4) + \frac{\beta - 1}{\beta} \big(36n + 28 + 12(n-1)H \big) \Bigg] \\
&\quad + \left[3\beta \sup |q| + \beta \sqrt{2(n-1)\delta} \sup |\nabla q| - 2(n-1) \inf q \right].
\end{aligned}$$

It is clear that the term

$$\frac{\beta-1}{d^2\beta(1+H)^2}\left(\log\frac{\beta}{\beta-1}\right)^2-\frac{2(n-1)(\beta-1)}{\beta}\left[3C_5+C_6+(n-1)K+\frac{1}{2\delta}\right]$$

in (3.7) can be made to be positive by choosing β sufficiently close to 1. It is easy to see that this term attains maximum value with

$$\frac{\beta}{\beta-1}=\exp\left[1+(1+C_{14})^{\frac{1}{2}}\right], \tag{3.8}$$

where

$$C_{14}=2d^2(n-1)^2(1+H)^2\left[3C_5+C_6+(n-1)K+\frac{1}{2\delta}\right]. \tag{3.9}$$

Putting this into (3.7), we have

(3.10)

$$\begin{aligned}
&\frac{2(1+C_{14})^{\frac{1}{2}}}{d^2(1+H)^2}\exp\left[-1-\left(1+C_{14}\right)^{\frac{1}{2}}\right]\\
&\leq\lambda_1^a\Bigg[8(n-1)+(n-1)^2(24+12H)+3\beta\max\Bigg\{12n-8+6H,\\
&\qquad 2(n-1)\left(\sqrt{(n-1)K+(n-2)K_R}+\frac{1}{R}+1\right)\Bigg\}\\
&\qquad+\beta\sqrt{2(n-1)}(96+6C_1+2C_4)+\frac{\beta-1}{\beta}\big(36n+28+12(n-1)H\big)\Bigg]\\
&\quad+\left[3\beta\sup|q|+\beta\sqrt{2(n-1)\delta}\sup|\nabla q|-2(n-1)\inf q\right].
\end{aligned}$$

This completes the proof of Theorem 1.1. □

The proof for Corollary 1.2 is immediate by setting $q=0$ in (3.10).

Proof of Theorem 1.3. In this case $q=-\eta_1\ \lambda_1=0$, then we have $\phi(x)\equiv f(x)\equiv 0$. Hence, the proof of Lemma 2.2 will be simplified by setting constants $C_7,\dots,C_{11}$, and δ to be zero. Then (3.2) will take the form

$$\log\frac{\beta}{\beta-1}\leq(1+H)\left[2(n-1)C_{12}+\frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]^{\frac{1}{2}}d \tag{3.11}$$

with

$$\begin{aligned}
C_{12}&=\frac{\eta_1\beta}{\beta-1}+2C_5+C_6+(n-1)K\\
C_{13}&=\frac{4\eta_1^2}{n-1}
\end{aligned} \tag{3.12}$$

where C_5, C_6 are constants given in Lemma 2.2. Following the argument as in the proof of Theorem 1.1, we obtain

$$\frac{2(1+C_{14})^{\frac{1}{2}}}{d^2(1+H)^2}\exp\left[-1-\left(1+C_{14}\right)^{\frac{1}{2}}\right] \le \eta_1\left[2(n-1)+\sqrt{8}\beta\right] \tag{3.13}$$

where

$$C_{14} = 2d^2(n-1)(1+H)^2[2C_5+C_6+(n-1)K]$$

$$\beta = \frac{C_{14}}{C_{14}-1}. \tag{3.14}$$

□

When the Ricci curvature is nonnegative, the boundary is convex, $q = -\eta_1$, and $\lambda = 0$, it is easy to see that $C_5 = C_6 = 0$ and $K = H = 0$. Therefore, one may apply (3.12) to obtain

$$\frac{e^{\frac{1}{2}}}{4d^2} \le \eta_1\left[2(n-1)+\sqrt{8}\beta\right] \tag{3.15}$$

where

$$\beta = \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}}-1}. \tag{3.16}$$

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BIQUANTIZATION OF LIE BIALGEBRAS

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For any finite-dimensional Lie bialgebra $\mathfrak{g}$, we construct a bialgebra $A_{u,v}(\mathfrak{g})$ over the ring $\mathbf{C}[u][[v]]$, which quantizes simultaneously the universal enveloping bialgebra $U(\mathfrak{g})$, the bialgebra dual to $U(\mathfrak{g}^*)$, and the symmetric bialgebra $S(\mathfrak{g})$. Following Turaev, we call $A_{u,v}(\mathfrak{g})$ a biquantization of $S(\mathfrak{g})$. We show that the bialgebra $A_{u,v}(\mathfrak{g}^*)$ quantizing $U(\mathfrak{g}^*)$, $U(\mathfrak{g})^*$, and $S(\mathfrak{g}^*)$ is essentially dual to the bialgebra obtained from $A_{u,v}(\mathfrak{g})$ by exchanging u and v. Thus, $A_{u,v}(\mathfrak{g})$ contains all information about the quantization of $\mathfrak{g}$. Our construction extends Etingof and Kazhdan's one-variable quantization of $U(\mathfrak{g})$.

Résumé. *Etant donné une bigèbre de Lie $\mathfrak{g}$ de dimension finie, nous construisons une $\mathbf{C}[u][[v]]$-bigèbre $A_{u,v}(\mathfrak{g})$ qui quantifie simultanément la bigèbre enveloppante $U(\mathfrak{g})$, la bigèbre duale de $U(\mathfrak{g}^*)$ et la bigèbre symétrique $S(\mathfrak{g})$. Suivant Turaev, nous appelons $A_{u,v}(\mathfrak{g})$ une biquantification de $S(\mathfrak{g})$. Nous montrons que la bigèbre $A_{u,v}(\mathfrak{g}^*)$ qui quantifie $U(\mathfrak{g}^*)$, $U(\mathfrak{g})^*$ et $S(\mathfrak{g}^*)$ est en dualité avec la bigèbre obtenue à partir de $A_{u,v}(\mathfrak{g})$ en échangeant u et v. La bigèbre $A_{u,v}(\mathfrak{g})$ contient ainsi toutes les informations sur la quantification de $\mathfrak{g}$. Notre construction généralise la quantification en une variable de $U(\mathfrak{g})$ par Etingof et Kazhdan.*

Introduction.

The notion of a Lie bialgebra was introduced by Drinfeld [**Dri82**], [**Dri87**] in the framework of his algebraic formalism for the quantum inverse scattering method. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ provided with a Lie cobracket $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ which is related to the Lie bracket by a certain compatibility condition. The notion of a Lie bialgebra is self-dual: If $\mathfrak{g}$ is a finite-dimensional Lie bialgebra over a field, then the dual $\mathfrak{g}^*$ is also a Lie bialgebra.

Drinfeld raised the question of quantizing Lie bialgebras (see loc. cit. and [**Dri92**]). For any Lie bialgebra $\mathfrak{g}$, its universal enveloping algebra $U(\mathfrak{g})$ is a co-Poisson bialgebra. The quantization problem for $\mathfrak{g}$ consists in finding a (topological) bialgebra structure on the module of formal power series $U(\mathfrak{g})[[h]]$ which induces the given bialgebra structure and Poisson cobracket on $U(\mathfrak{g}) = U(\mathfrak{g})[[h]]/(h)$. This problem is solved in the theory of quantum groups for certain semisimple $\mathfrak{g}$. Recently, P. Etingof and D. Kazhdan [**EK96**] quantized an arbitrary Lie bialgebra $\mathfrak{g}$ over a field $\mathbf{C}$ of characteristic zero. Their construction is based on a delicate analysis of Drinfeld associators.

Besides $U(\mathfrak{g})$, there are other Poisson and co-Poisson bialgebras associated with a Lie bialgebra $\mathfrak{g}$. One can consider, for instance, the (appropriately defined) Poisson bialgebra $U(\mathfrak{g})^*$ dual to $U(\mathfrak{g})$, as well as similar bialgebras $U(\mathfrak{g}^*), U(\mathfrak{g}^*)^*$ associated with $\mathfrak{g}^*$. Note also that the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{n\geq 0} S^n(\mathfrak{g})$ is a bialgebra with Poisson bracket and cobracket extending the Lie bracket and cobracket in $\mathfrak{g}$. The Etingof-Kazhdan theory provides us with quantizations of $U(\mathfrak{g})$ and $U(\mathfrak{g}^*)$ in the category of topological bialgebras. It is essentially clear that, taking the dual bialgebras, we obtain quantizations of $U(\mathfrak{g})^*$ and $U(\mathfrak{g}^*)^*$. The bialgebras $S(\mathfrak{g})$ and $S(\mathfrak{g}^*)$ stay apart and need to be considered separately. At this point, the relationship between all these bialgebras and their quantizations looks a little messy and needs clarification.

The aim of our paper is to sort out and unify these quantizations. We shall show that there is a bialgebra $A(\mathfrak{g})$ quantizing simultaneously $U(\mathfrak{g})$, $U(\mathfrak{g}^*)^*$, and $S(\mathfrak{g})$. Moreover, the bialgebra $A(\mathfrak{g}^*)$ quantizing $U(\mathfrak{g}^*)$, $U(\mathfrak{g})^*$, $S(\mathfrak{g}^*)$ is essentially dual to $A(\mathfrak{g})$. Thus, we can view $A(\mathfrak{g})$ as a "master" bialgebra containing all information about the quantization of $\mathfrak{g}$.

To formalize our results, we appeal to the notion of biquantization introduced in [**Tur89**], [**Tur91**]. It was inspired by a topological study of skein classes of links in the cylinder over a surface. The idea consists in introducing two independent quantization variables, u and v, responsible for the quantization of multiplication and comultiplication, respectively. Let us illustrate this idea with the following construction. Let A be a bialgebra over the ring of formal power series $\mathbf{C}[[u,v]]$. Assume that A is topologically free as a $\mathbf{C}[[u,v]]$-module, commutative modulo u, and cocommutative modulo v. It is clear that A/uA is a commutative bialgebra with Poisson bracket

$$\{p_u(a), p_u(b)\} = p_u\Big(\frac{ab-ba}{u}\Big),$$

where $a,b \in A$ and $p_u : A \to A/uA$ is the projection. The morphism p_u is a quantization of the Poisson bialgebra A/uA. Similarly, the comultiplication Δ in A induces on A/vA the structure of a cocommutative bialgebra with Poisson cobracket

$$\delta(p_v(a)) = (p_v \otimes p_v)\Big(\frac{\Delta(a) - \Delta^{\mathrm{op}}(a)}{v}\Big),$$

where $a \in A$ and $p_v : A \to A/vA$ is the projection. The morphism $p_v : A \to A/vA$ is a quantization of the co-Poisson bialgebra A/vA. By similar formulas, the quotient $A/(u,v) = A/(uA+vA)$ acquires both a Poisson bracket and a Poisson cobracket, and becomes a bi-Poisson bialgebra.

The projections of A/uA and A/vA onto $A/(u,v)$ quantize the comultiplication and the multiplication in $A/(u,v)$, respectively. We sum up these observations in the following commutative diagram of projections

$$\begin{array}{ccc} A & \longrightarrow & A/uA \\ \downarrow & & \downarrow \\ A/vA & \longrightarrow & A/(u,v) \end{array} \tag{0.1}$$

called a biquantization square. This square involves four bialgebras and four bialgebra morphisms quantizing either the multiplication or the comultiplication in their targets. The bialgebra A appears as the summit of the square, quantizing three other bialgebras. We say that A is a biquantization of the bi-Poisson bialgebra $A/(u,v)$. The notion of a biquantization allows us to combine four quantizations of three bialgebras in a single bialgebra. Note that instead of the ring $\mathbf{C}[[u,v]]$ one can use subrings containing u and v. In this paper, as a ground ring for biquantization, we use the ring $\mathbf{C}[u][[v]]$ consisting of the formal power series in v with coefficients in the ring of polynomials $\mathbf{C}[u]$.

Our main result is that, for any finite-dimensional Lie bialgebra $\mathfrak{g}$ over a field $\mathbf{C}$ of characteristic zero, the bi-Poisson bialgebra $S(\mathfrak{g})$ admits a biquantization. More precisely, we construct a topological $\mathbf{C}[u][[v]]$-bialgebra $A_{u,v}(\mathfrak{g})$ biquantizing $S(\mathfrak{g})$. Specifically, $A_{u,v}(\mathfrak{g})$ is free as a topological $\mathbf{C}[u][[v]]$-module, is commutative modulo u and cocommutative modulo v, and $A_{u,v}(\mathfrak{g})/(u,v) = S(\mathfrak{g})$ as bi-Poisson bialgebras. This gives us a biquantization square (0.1) with $A = A_{u,v}(\mathfrak{g})$.

Our second result computes the left-bottom corner A/vA of the biquantization square (0.1), where $A = A_{u,v}(\mathfrak{g})$. Consider the $\mathbf{C}[u]$-algebra $V_u(\mathfrak{g})$ defined in the same way as the universal enveloping algebra $U(\mathfrak{g})$, except that the identity $xy - yx = [x,y]$ is replaced by $xy - yx = u[x,y]$, where $x, y \in \mathfrak{g}$. We view $V_u(\mathfrak{g})$ as a parametrized version of $U(\mathfrak{g})$; note that $V_u(\mathfrak{g})/(u-1) = U(\mathfrak{g})$. Similarly to $U(\mathfrak{g})$, we provide $V_u(\mathfrak{g})$ with the structure of a co-Poisson bialgebra. We prove that $A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) = V_u(\mathfrak{g})$ as co-Poisson bialgebras. According to the remarks above, the projection $A_{u,v}(\mathfrak{g}) \to A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) = V_u(\mathfrak{g})$ is a quantization of $V_u(\mathfrak{g})$. This is a refined version of the Etingof-Kazhdan quantization of $U(\mathfrak{g})$. Indeed, quotienting both $A_{u,v}(\mathfrak{g})$ and $V_u(\mathfrak{g})$ by $u-1$, we obtain the Etingof-Kazhdan quantization of $U(\mathfrak{g})$ (cf. Remark 8.4).

Our third result concerns the right-top corner A/uA of the biquantization square for $A = A_{u,v}(\mathfrak{g})$. Namely, we prove that A/uA is isomorphic to a topological dual of $V_v(\mathfrak{g}^*)$ consisting of $\mathbf{C}[v]$-linear maps $V_v(\mathfrak{g}^*) \to \mathbf{C}[[v]]$ continuous with respect to the v-adic topology in $\mathbf{C}[[v]]$ and a suitable topology in $V_v(\mathfrak{g}^*)$. This dual is a Poisson bialgebra over $\mathbf{C}[[v]]$. It is isomorphic to the Poisson bialgebra $E_v(\mathfrak{g})$ of functions on the Poisson-Lie group associated with $\mathfrak{g}^*\otimes_{\mathbf{C}}\mathbf{C}[[v]]$, cf. [**Tur91**, Sections 11-12]. (As an algebra, $E_v(\mathfrak{g}) = S(\mathfrak{g})[[v]]$.) According to the remarks above, the projection $A_{u,v}(\mathfrak{g}) \to A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) \cong E_v(\mathfrak{g})$ is a quantization of $E_v(\mathfrak{g})$.

To sum up, the $\mathbf{C}[u][[v]]$-bialgebra $A_{u,v}(\mathfrak{g})$ quantizes $S(\mathfrak{g})$, $V_u(\mathfrak{g})$, and the topological dual $E_v(\mathfrak{g})$ of $V_v(\mathfrak{g}^*)$.

We can apply the same constructions to the dual Lie bialgebra $\mathfrak{g}^*$. It is convenient to exchange u and v, i.e., to consider the $\mathbf{C}[v][[u]]$-bialgebra $A_{v,u}(\mathfrak{g}^*)$ obtained from $A_{u,v}(\mathfrak{g}^*)$ via an appropriate tensoring with $\mathbf{C}[v][[u]]$. As above, $A_{v,u}(\mathfrak{g}^*)$ quantizes $S(\mathfrak{g}^*)$, $V_v(\mathfrak{g}^*)$, and the topological dual $E_u(\mathfrak{g}^*)$ of $V_u(\mathfrak{g})$. Observe that the three lower level corners of the biquantization square for $A_{v,u}(\mathfrak{g}^*)$ are dual to the lower level corners of the biquantization square for $A_{u,v}(\mathfrak{g})$. We prove that the bialgebras $A_{u,v}(\mathfrak{g})$ and $A_{v,u}(\mathfrak{g}^*)$ are essentially dual to each other.

Our definition of $A_{u,v}(\mathfrak{g})$ is obtained by an elaboration of Etingof and Kazhdan's quantization of $U(\mathfrak{g})$ and can be regarded as an extension of their work. The definition goes in two steps. First we replace the variable h by the product uv, which allows us to introduce two variables into the game. In particular, the universal R-matrix R_h constructed in [**EK96**] gives rise to a two-variable universal R-matrix $R_{u,v}$. Then we separate the variables u, v in an expression for $R_{u,v}$ by collecting all powers of u (resp. v) in the first (resp. second) tensor factor. The algebra $A_{u,v}(\mathfrak{g})$ is generated by the first tensor factors appearing in such an expression.

The plan of the paper is as follows. In Section 1 we recall the notions of Poisson, co-Poisson, and bi-Poisson bialgebras, as well as the definitions of quantizations and biquantizations. In Section 2 we formulate the main results of the paper (Theorems 2.3, 2.6, 2.9, and 2.11). In Section 3 we recall a construction due to Drinfeld producing certain linear maps out of a bialgebra comultiplication. We use these maps to show that every bialgebra over $\mathbf{C}[[u]]$ has a canonical subalgebra that is commutative modulo u. In Section 4 we collect several useful facts concerning $\mathbf{C}[[u,v]]$-modules. In Section 5 we recall the basic facts concerning Etingof and Kazhdan's quantization $U_h(\mathfrak{g})$ of a Lie bialgebra $\mathfrak{g}$. In Section 6 we define $A_{u,v}(\mathfrak{g})$ and show that it is a topologically free module. The proof that $A_{u,v}(\mathfrak{g})$ is an algebra is also given in Section 6; it uses Lemma 6.10 whose proof is postponed to Section 7. In Section 7 we introduce a completion $\widehat{A}_{u,v}(\mathfrak{g})$ of $A_{u,v}(\mathfrak{g})$ and define a bialgebra structure on $A_{u,v}(\mathfrak{g})$. Section 8 is devoted to the proofs of Theorems 2.3 and 2.6, and the first part of Theorem 2.9. In Section 9 we investigate the two-variable universal R-matrix $R_{u,v}$ and construct a nondegenerate bialgebra pairing between $A_{u,v}(\mathfrak{g})$ and a certain bialgebra A_-^{cop}. In Section 10, using the pairing of Section 9, we relate $S(\mathfrak{g})[[v]]$ to the topological dual of $V_v(\mathfrak{g}^*)$, which allows us to complete the proof of Theorem 2.9. In Section 11 we compare Etingof and Kazhdan's quantization for a Lie bialgebra and the dual Lie bialgebra. In Section 12 we use the results of

Section 11 to show that $A_-^{\mathrm{cop}} \cong A_{v,u}(\mathfrak{g}^*)$ and prove Theorem 2.11. In the appendix we describe explicitly the biquantization of a trivial Lie bialgebra.

We fix once and for all a field $\mathbf{C}$ of characteristic zero.

1. Poisson bialgebras and their quantizations.

We introduce the basic notions used throughout the paper. All objects will be considered over a field $\mathbf{C}$ of characteristic zero. Given a commutative $\mathbf{C}$-algebra κ, we recall that a κ-bialgebra is an associative, unital κ-algebra A equipped with morphisms of algebras $\Delta : A \to A \otimes_\kappa A$, the comultiplication, and $\varepsilon : A \to \kappa$, the counit, such that

$$(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \varepsilon)\Delta = \mathrm{id}_A,$$

where id_A denotes the identity map of A. We shall also consider topological bialgebras. A topological bialgebra A is defined in terms of a two-sided ideal $I \subset A$. The definition is the same as for a κ-bialgebra, except that the comultiplication takes values in the completed tensor product

$$A \mathbin{\widehat{\otimes}_\kappa} A = \varprojlim_n \Big(A/I^n \otimes_\kappa A/I^n \Big).$$

The topological bialgebra A is equipped with the I-adic topology, namely the linear topology for which the powers of I form a fundamental system of neighbourhoods of 0 (see [**Bou61**, Chap. 3]).

1.1. Poisson Bialgebras. A Poisson bracket on a commutative algebra B over the field $\mathbf{C}$ is a Lie bracket $\{\ ,\ \} : B \times B \to B$ satisfying the Leibniz rule, i.e., such that for all $a, b, c \in B$ we have

$$\{ab, c\} = a\{b, c\} + b\{a, c\}. \tag{1.1}$$

A Poisson bracket on B defines a Poisson bracket on $B \otimes B$ by

$$\{a \otimes a', b \otimes b'\} = ab \otimes \{a', b'\} + \{a, b\} \otimes a'b' \tag{1.2}$$

where a, a', b, $b' \in B$.

A *Poisson bialgebra* is a commutative $\mathbf{C}$-bialgebra B equipped with a Poisson bracket such that the comultiplication $\Delta : B \to B \otimes B$ preserves the Poisson bracket:

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\} \tag{1.3}$$

for all $a, b \in B$.

The following well-known construction yields examples of Poisson bialgebras. Let A be a bialgebra over the ring $\mathbf{C}[u]$ of polynomials in a variable u. Assume that A is commutative modulo u in the sense that $ab - ba \in uA$ for all $a, b \in A$. If the multiplication by u is injective on A, then the quotient bialgebra A/uA is a Poisson bialgebra with Poisson bracket defined for all a, $b \in A$ by

$$\{p(a), p(b)\} = p\Big(\frac{ab - ba}{u}\Big), \tag{1.4}$$

where $p : A \to A/uA$ is the projection.

The inverse of this construction is called quantization. More precisely, a *quantization* of a Poisson $\mathbf{C}$-bialgebra B is a $\mathbf{C}[u]$-bialgebra A which is isomorphic as a $\mathbf{C}[u]$-module to the module $B[u]$ of polynomials in u with coefficients in B, is commutative modulo u, and such that A/uA is isomorphic to B as a Poisson bialgebra. The latter condition implies that Equality (1.4) holds for all $a, b \in A$, where $p : A \to A/uA \cong B$ is the projection and $\{\ ,\ \}$ is the Poisson bracket in B.

One can similarly define quantization over the ring $\mathbf{C}[[u]]$ of formal power series. To shorten, we call $\mathbf{C}[[u]]$-bialgebra a topological $\mathbf{C}[[u]]$-algebra A where the topology is the u-adic topology, i.e., is defined by the ideal uA. In this case,

$$A \,\widehat{\otimes}_{\mathbf{C}[[u]]}\, A = \varprojlim_n \Big(A/u^n A \otimes_{\mathbf{C}[[u]]/(u^n)} A/u^n A \Big). \tag{1.5}$$

A *quantization* over $\mathbf{C}[[u]]$ of a Poisson $\mathbf{C}$-bialgebra B is a (topological) $\mathbf{C}[[u]]$-bialgebra A which is isomorphic as a $\mathbf{C}[[u]]$-module to the module $B[[u]]$ of formal power series with coefficients in B, is commutative modulo u, and such that $A/uA = B$ as Poisson bialgebras.

1.2. Co-Poisson Bialgebras. It is straightforward to dualize the definitions of Section 1.1. A Poisson cobracket on a cocommutative $\mathbf{C}$-coalgebra B is a Lie cobracket $\delta : B \to B \otimes B$ satisfying the Leibniz rule, i.e., such that

$$(\mathrm{id} \otimes \Delta)\delta = \big(\delta \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \delta)\big)\Delta, \tag{1.6}$$

where $\Delta : B \to B \otimes B$ is the comultiplication of B and σ is the permutation $a \otimes b \mapsto b \otimes a$ in $B \otimes B$. Recall the notation $\Delta^{\mathrm{op}} = \sigma\Delta$ for the opposite comultiplication.

A *co-Poisson bialgebra* is a cocommutative $\mathbf{C}$-bialgebra B equipped with a Poisson cobracket δ such that

$$\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b) \tag{1.7}$$

for all $a, b \in B$.

We obtain co-Poisson bialgebras by dualizing the constructions of Section 1.1. Here again we have the choice between the ring $\mathbf{C}[v]$ of polynomials and the ring $\mathbf{C}[[v]]$ of formal power series in a variable v. In the context of co-Poisson bialgebras, it will be more relevant to work with formal power series. So let A be a bialgebra over $\mathbf{C}[[v]]$ in the sense of Section 1.1. Assume that A is cocommutative modulo v, i.e., for all $a \in A$ we have

$\Delta(a) - \Delta^{\mathrm{op}}(a) \in vA\widehat{\otimes}_{\mathbf{C}[[v]]}A$, where Δ denotes the comultiplication and Δ^{op} the opposite comultiplication of A. If v acts injectively on $A\,\widehat{\otimes}_{\mathbf{C}[[v]]}\,A$, then the quotient bialgebra A/vA is a co-Poisson bialgebra with cobracket

$$\delta(p(a)) = (p\otimes p)\Big(\frac{\Delta(a) - \Delta^{\mathrm{op}}(a)}{v}\Big) \tag{1.8}$$

for $a \in A$, where $p : A \to A/vA$ is the projection.

A *coquantization* of a co-Poisson $\mathbf{C}$-bialgebra B is a $\mathbf{C}[[v]]$-bialgebra A which is isomorphic to $B[[v]]$ as a $\mathbf{C}[[v]]$-module, is cocommutative modulo v, and such that A/vA is isomorphic to B as a co-Poisson bialgebra. This implies that Formula (1.8) holds for any $a \in A$, where $p : A \to A/vA \cong B$ is the projection and δ is the Poisson cobracket in B.

1.3. Bi-Poisson Bialgebras. Following [**Tur89**, **Tur91**], we combine the definitions given above and define the concepts of bi-Poisson bialgebras and their biquantizations. A *bi-Poisson bialgebra* is a commutative and cocommutative bialgebra B equipped with Poisson bracket $\{\ ,\ \}$ and Poisson cobracket δ turning B into a Poisson and co-Poisson bialgebra, and satisfying the additional condition:

$$\delta(\{a, b\}) = \{\delta(a), \Delta(b)\} + \{\Delta(a), \delta(b)\} \tag{1.9}$$

for all $a, b \in B$.

In order to introduce biquantization, we use two variables u and v and the ring $\mathbf{C}[u][[v]]$ which consists of formal power series in v whose coefficients are polynomials in u. The following definitions can easily be adapted to the rings $\mathbf{C}[u, v]$, $\mathbf{C}[[u, v]]$, and $\mathbf{C}[v][[u]]$.

By a $\mathbf{C}[u][[v]]$-bialgebra A we mean a topological $\mathbf{C}[u][[v]]$-algebra A, where the topology is defined by the ideal vA, so that the comultiplication takes values in

$$A\,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\,A = \varprojlim_{n}\Big(A/v^nA \otimes_{\mathbf{C}[u][[v]]/(v^n)} A/v^nA\Big). \tag{1.10}$$

Let A be a $\mathbf{C}[u][[v]]$-bialgebra that is commutative modulo u and cocommutative modulo v. If u and v act injectively on A, then the quotient bialgebra $A/(uA + vA)$ is a bi-Poisson bialgebra over $\mathbf{C}$ with Poisson bracket given by (1.4) and Poisson cobracket given by (1.8), where $p : A \to A/(uA + vA)$ is the projection. Inverting this construction, we obtain the following notion of biquantization.

Definition 1.4. A biquantization of a bi-Poisson $\mathbf{C}$-bialgebra B is a $\mathbf{C}[u][[v]]$-bialgebra A which is isomorphic to $B[u][[v]]$ as a $\mathbf{C}[u][[v]]$-module, is commutative modulo u and cocommutative modulo v, and such that $A/(uA + vA) = B$ as bi-Poisson bialgebras.

Any biquantization A gives rise to a "biquantization square" as follows. Observe that A/vA is a cocommutative co-Poisson bialgebra over $\mathbf{C}[u]$ and that A/uA is a commutative Poisson bialgebra over $\mathbf{C}[[v]]$. We form the commutative square

$$\begin{array}{ccc} A & \xrightarrow{p_u} & A/uA \\ {\scriptstyle p_v}\downarrow & & \downarrow{\scriptstyle q_v} \\ A/vA & \xrightarrow{q_u} & B \end{array} \tag{1.11}$$

where p_u, p_v, q_u, q_v are the natural projections. The morphisms p_u and q_u are quantizations whereas p_v and q_v are coquantizations. The projection $p : A \to B$ can therefore be factored in two ways as a composition of a quantization and a coquantization: $p = q_v p_u = q_u p_v$.

2. Statement of the main results.

Any Lie bialgebra $\mathfrak{g}$ gives rise to a bi-Poisson bialgebra $S(\mathfrak{g})$. In this section, after recalling the necessary facts on Lie bialgebras, we state our main theorems concerning a biquantization of $S(\mathfrak{g})$.

2.1. Lie Bialgebras (cf. [**Dri82**]). A *Lie cobracket* on a vector space $\mathfrak{g}$ over $\mathbf{C}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\sigma\delta = -\delta \quad \text{and} \quad (\mathrm{id} + \tau + \tau^2)(\delta \otimes \mathrm{id}) = 0 \tag{2.1}$$

where σ (resp. τ) is the automorphism of $\mathfrak{g} \otimes \mathfrak{g}$ (resp. of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$) given by $\sigma(x \otimes y) = y \otimes x$ (resp. $\tau(x \otimes y \otimes z) = y \otimes z \otimes x$). It is clear that the transpose map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \subset (\mathfrak{g} \otimes \mathfrak{g})^* \to \mathfrak{g}^*$ is a Lie bracket in the dual space $\mathfrak{g}^* = \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$.

A *Lie bialgebra* is a vector space over $\mathbf{C}$ equipped with a Lie bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and a Lie cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta([x, y]) = x\delta(y) - y\delta(x) \tag{2.2}$$

for all $x, y \in \mathfrak{g}$. Here $\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by the adjoint action ($x, z, z' \in \mathfrak{g}$):

$$x(z \otimes z') = [x, z] \otimes z' + z \otimes [x, z'].$$

Let $\mathfrak{g}$ be a Lie bialgebra with Lie bracket $[\ ,\ ]$ and Lie cobracket δ. It is easy to check that, if we replace $[\ ,\ ]$ by $-[\ ,\ ]$ without changing the Lie cobracket, then we obtain a new Lie bialgebra, which we denote $\mathfrak{g}^{\mathrm{op}}$. If we leave the Lie bracket in $\mathfrak{g}$ unaltered and replace δ by $-\delta$, then we obtain another Lie bialgebra denoted $\mathfrak{g}^{\mathrm{cop}}$. The opposite $-\mathrm{id}_{\mathfrak{g}}$ of the identity map of $\mathfrak{g}$ is an isomorphism of Lie bialgebras $\mathfrak{g}^{\mathrm{op}} \to \mathfrak{g}^{\mathrm{cop}}$ and $\mathfrak{g} \to (\mathfrak{g}^{\mathrm{op}})^{\mathrm{cop}}$.

When the Lie bialgebra $\mathfrak{g}$ is finite-dimensional, then the dual vector space $\mathfrak{g}^*$ with the transpose bracket and cobracket is also a Lie bialgebra. Clearly, $(\mathfrak{g}^*)^{\mathrm{op}} = (\mathfrak{g}^{\mathrm{cop}})^*$ and $(\mathfrak{g}^*)^{\mathrm{cop}} = (\mathfrak{g}^{\mathrm{op}})^*$.

2.2. A Bi-Poisson Bialgebra Associated to $\mathfrak{g}$ (cf. [**Tur89**, **Tur91**]). For any vector space $\mathfrak{g}$, the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{n\geq 0} S^n(\mathfrak{g})$ has a structure of bialgebra with comultiplication Δ determined by $\Delta(x) = x\otimes 1 + 1\otimes x$ for all $x \in \mathfrak{g} = S^1(\mathfrak{g})$. If $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\ ,\ ]$, then $S(\mathfrak{g})$ is a Poisson bialgebra with Poisson bracket determined by

$$\{x, y\} = [x, y] \tag{2.3}$$

for all $x, y \in \mathfrak{g}$. If $\mathfrak{g}$ is a Lie coalgebra, then $S(\mathfrak{g})$ is a co-Poisson bialgebra with the unique Poisson cobracket such that its restriction to $S^1(\mathfrak{g}) = \mathfrak{g}$ is the Lie cobracket of $\mathfrak{g}$. If, furthermore, $\mathfrak{g}$ is a Lie bialgebra, then $S(\mathfrak{g})$ is a bi-Poisson bialgebra ([**Tur91**, Theorem 16.2.4]).

We now state our first main theorem.

Theorem 2.3. *Given a finite-dimensional Lie bialgebra $\mathfrak{g}$, there exists a biquantization $A_{u,v}(\mathfrak{g})$ for $S(\mathfrak{g})$.*

The construction of $A_{u,v}(\mathfrak{g})$ will be given in Section 6. It is an extension of Etingof and Kazhdan's quantization of $U(\mathfrak{g})$, as constructed in [**EK96**]. As in loc. cit., our definition of $A_{u,v}(\mathfrak{g})$ is based on the choice of a Drinfeld associator. We nevertheless believe that it is unique up to isomorphism. We shall not discuss this point in this paper.

The fundamental feature of our construction is that the bialgebras in the lower left and the upper right corners in the biquantization square (1.11) when $A = A_{u,v}(\mathfrak{g})$ are closely related to the universal enveloping bialgebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and to the dual of $U(\mathfrak{g}^*)$. We shall give precise statements in the remaining part of this section. We begin with a short discussion of $U(\mathfrak{g})$ and its parametrized version $V_u(\mathfrak{g})$.

2.4. The Bialgebra $V_u(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{C}$. Consider the $\mathbf{C}[u]$-algebra $T(\mathfrak{g})[u]$ of polynomials with coefficients in the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$. Let $V_u(\mathfrak{g})$ be the quotient of $T(\mathfrak{g})[u]$ by the two-sided ideal generated by the elements

$$x \otimes y - y \otimes x - u[x, y],$$

where $x, y \in \mathfrak{g}$. The composition of the natural linear maps $\mathfrak{g} = T^1(\mathfrak{g}) \subset T(\mathfrak{g}) \subset T(\mathfrak{g})[u] \to V_u(\mathfrak{g})$ is an embedding whose image generates $V_u(\mathfrak{g})$ as a $\mathbf{C}[u]$-algebra. The algebra $V_u(\mathfrak{g})$ is a bialgebra with comultiplication Δ determined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \tag{2.4}$$

for all $x \in \mathfrak{g}$. Clearly, $V_u(\mathfrak{g})/(u-1)V_u(\mathfrak{g}) = U(\mathfrak{g})$ and $V_u(\mathfrak{g})/uV_u(\mathfrak{g}) = S(\mathfrak{g})$.

In this paper, we will use the fact that $V_u(\mathfrak{g})$ embeds in the polynomial algebra $U(\mathfrak{g})[u]$. The algebra $U(\mathfrak{g})[u]$ is equipped with a $\mathbf{C}[u]$-bialgebra structure whose comultiplication Δ is also given by (2.4). Let $i : V_u(\mathfrak{g}) \to U(\mathfrak{g})[u]$ be the morphism of $\mathbf{C}[u]$-bialgebras defined by $i(x) = ux$ for all

$x \in \mathfrak{g} \subset V_u(\mathfrak{g})$. Using the Poincaré-Birkhoff-Witt theorem (cf. [**Dix74**, Chap. 2]), we see that $V_u(\mathfrak{g})$ is a free $\mathbf{C}[u]$-module and that i is injective. To describe its image, recall the standard filtration $U^0(\mathfrak{g}) = \mathbf{C} \subset U^1(\mathfrak{g}) \subset U^2(\mathfrak{g}) \subset \cdots$ of $U(\mathfrak{g})$: The subspace $U^m(\mathfrak{g})$ is the image of $\bigoplus_{k=0}^m \mathfrak{g}^{\otimes k}$ under the projection $T(\mathfrak{g}) \to U(\mathfrak{g})$. Then

$$i(V_u(\mathfrak{g})) = \left\{ \sum_{m \geq 0} a_m u^m \in U(\mathfrak{g})[u] \;\middle|\; a_m \in U^m(\mathfrak{g}) \text{ for all } m \geq 0 \right\}.$$

We also have $U^m(\mathfrak{g})/U^{m-1}(\mathfrak{g}) = S^m(\mathfrak{g})$ for all $m \geq 0$. From now on, we identify $V_u(\mathfrak{g})$ with $i(V_u(\mathfrak{g}))$ and $S(\mathfrak{g})$ with the graded algebra

$$\bigoplus_{m \geq 0} U^m(\mathfrak{g}) \,/\, U^{m-1}(\mathfrak{g}).$$

Under these identifications, the natural projection $q_u : V_u(\mathfrak{g}) \to S(\mathfrak{g})$ sends any element $\sum_{m \geq 0} a_m u^m \in V_u(\mathfrak{g})$ to $\sum_{m \geq 0} \bar{a}_m \in S(\mathfrak{g})$, where $\bar{a}_m \in S^m(\mathfrak{g})$ is the class of $a_m \in U^m(\mathfrak{g})$ modulo $U^{m-1}(\mathfrak{g})$. These observations lead to the following easy fact.

Lemma 2.5. *The $\mathbf{C}[u]$-bialgebra $V_u(\mathfrak{g})$ is a quantization of the Poisson bialgebra $S(\mathfrak{g})$.*

Suppose now that $\mathfrak{g}$ is a Lie bialgebra with Lie cobracket δ. It was shown in [**Tur91**, Theorem 7.4] that δ induces a co-Poisson bialgebra structure on $V_u(\mathfrak{g})$ with Poisson cobracket δ_u determined for all $x \in \mathfrak{g}$ by

$$\delta_u(ux) = u^2 \delta(x) \in u\mathfrak{g} \otimes u\mathfrak{g} \subset V_u(\mathfrak{g}) \otimes_{\mathbf{C}[u]} V_u(\mathfrak{g}). \tag{2.5}$$

The projection $q_u : V_u(\mathfrak{g}) \to S(\mathfrak{g})$ preserves the co-Poisson structure; in other words, $V_u(\mathfrak{g})$ is a quantization of $S(\mathfrak{g})$ in the category of co-Poisson bialgebras.

Theorem 2.6. *For the bialgebra $A_{u,v}(\mathfrak{g})$ of Theorem 2.3, there is an isomorphism of co-Poisson $\mathbf{C}[u]$-bialgebras*

$$A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) = V_u(\mathfrak{g}).$$

Theorem 2.6 will be proved in Section 8.

2.7. The Bialgebra $E_v(\mathfrak{g})$. Let $\mathfrak{g}$ be a finite-dimensional Lie coalgebra with Lie cobracket δ. By Section 2.2 the cobracket δ induces a co-Poisson bialgebra structure on $S(\mathfrak{g})$.

Turaev ([**Tur89**, Sections 4-5] and [**Tur91**, Sections 11-12]) constructed a (topological) $\mathbf{C}[[v]]$-bialgebra $E_v(\mathfrak{g})$ which may be viewed as the bialgebra of functions on the simply-connected Lie group associated to the dual Lie algebra $\mathfrak{g}^*$. As an algebra, $E_v(\mathfrak{g})$ is the algebra of formal power series with coefficients in $S(\mathfrak{g})$:

$$E_v(\mathfrak{g}) = S(\mathfrak{g})[[v]].$$

To define the comultiplication in $E_v(\mathfrak{g})$, consider the Campbell-Hausdorff series

$$(2.6)\quad \mu(X,Y) = \log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big([X,[X,Y]] + [[X,Y],Y]\big) + \cdots$$

where $X, Y \in \mathfrak{g}^*$. Let us multiply all Lie brackets of length n by v^n. This yields the modified Campbell-Hausdorff series

$$(2.7)\quad \mu_v(X,Y) = \frac{1}{v}\log(e^{vX} e^{vY}) = X + Y + \frac{v}{2}[X,Y] + \frac{v^2}{12}\big([X,[X,Y]] + [[X,Y],Y]\big) + \cdots .$$

The comultiplication Δ' in $E_v(\mathfrak{g})$ is given by $a \mapsto a \circ \mu_v$, which makes sense when we identify elements of $E_v(\mathfrak{g})$ with $\mathbf{C}[[v]]$-valued polynomial functions on $\mathfrak{g}^*$. For $x \in \mathfrak{g} \subset E_v(\mathfrak{g})$ we have

$$(2.8)\quad \Delta'(x) = x \otimes 1 + 1 \otimes x + \frac{v}{2}\delta(x) + \frac{v^2}{12}\sum_i (x_i' x_i'' \otimes x_i''' + x_i''' \otimes x_i' x_i'') + \cdots ,$$

where $(\mathrm{id} \otimes \delta)\delta(x) = \sum_i x_i' \otimes x_i'' \otimes x_i'''$. For details, see loc. cit.

Let $q_v : E_v(\mathfrak{g}) \to S(\mathfrak{g})$ be the algebra morphism sending an element of $E_v(\mathfrak{g})$ to its class modulo $vE_v(\mathfrak{g})$. Formula (2.8) implies that the induced map $E_v(\mathfrak{g})/vE_v(\mathfrak{g}) \to S(\mathfrak{g})$ is an isomorphism of co-Poisson bialgebras. This leads to the following.

Lemma 2.8. *The* $\mathbf{C}[[v]]$*-bialgebra* $E_v(\mathfrak{g})$ *is a coquantization of the co-Poisson bialgebra* $S(\mathfrak{g})$.

If the Lie coalgebra $\mathfrak{g}$ has a Lie bracket [,] turning it into a Lie bialgebra, then $E_v(\mathfrak{g})$ carries a structure of a Poisson bialgebra whose Poisson bracket { , } is uniquely determined by the condition

$$(2.9)\qquad \{x_1, x_2\} \equiv [x_1, x_2] \mod \Big(\bigoplus_{n\geq 2} S^n(\mathfrak{g})\Big)[[v]],$$

for all $x_1, x_2 \in \mathfrak{g}$ (cf. [**Tur91**, Theorem 11.4 and Remark 11.7]).

Theorem 2.9. *For the bialgebra* $A_{u,v}(\mathfrak{g})$ *of Theorem* 2.3, *there is an isomorphism of Poisson* $\mathbf{C}[[v]]$*-bialgebras*

$$A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) = E_v(\mathfrak{g}).$$

Theorem 2.9 will be proved in two steps: In Section 8.2 we prove that $A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) = S(\mathfrak{g})[[v]]$ as an algebra; in Section 10.7 we determine its coalgebra structure.

2.10. Duality. By Theorem 2.3 we have a biquantization square

$$\begin{array}{ccc} A_{u,v}(\mathfrak{g}) & \xrightarrow{p_u} & A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) \\ {\scriptstyle p_v}\downarrow & & \downarrow{\scriptstyle q_v} \\ A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) & \xrightarrow{q_u} & S(\mathfrak{g}). \end{array} \tag{2.10a}$$

Replacing $\mathfrak{g}$ by the Lie bialgebra $\mathfrak{g}' = (\mathfrak{g}^*)^{\mathrm{cop}}$ (see Section 2.1 for the notation) and exchanging u and v, we obtain the biquantization square

$$\begin{array}{ccc} A_{v,u}(\mathfrak{g}') & \xrightarrow{p_v} & A_{v,u}(\mathfrak{g}')/vA_{v,u}(\mathfrak{g}') \\ {\scriptstyle p_u}\downarrow & & \downarrow{\scriptstyle q_u} \\ A_{v,u}(\mathfrak{g}')/uA_{v,u}(\mathfrak{g}') & \xrightarrow{q_v} & S(\mathfrak{g}'). \end{array} \tag{2.10b}$$

We prove that these squares are in duality as follows.

Let K be a commutative $\mathbf{C}$-algebra together with two subalgebras K_1 and K_2. Given a K_1-module A and a K_2-module B, a $\mathbf{C}$-bilinear map $(\ ,\ ) : A \times B \to K$ will be called a *pairing* if

$$(\lambda_1 a, \lambda_2 b) = \lambda_1\lambda_2\,(a, b)$$

for all $\lambda_1 \in K_1 \subset K$, $\lambda_2 \in K_2 \subset K$, $a \in A$, and $b \in B$. We say that the pairing $(\ ,\ )$ is *nondegenerate* if both annihilators

$$\big\{a \in A \mid (a, b) = 0 \ \text{ for all } b \in B\big\} \quad \text{and} \quad \big\{b \in B \mid (a, b) = 0 \ \text{ for all } a \in A\big\}$$

vanish. The pairing $A \times B \to K$ induces a pairing $(\ ,\ ) : (A \otimes_{K_1} A) \times (B \otimes_{K_2} B) \to K$ by

$$(a \otimes a', b \otimes b') = (a, b)\,(a', b')$$

for all $a, a' \in A$ and $b, b' \in B$. Suppose, in addition, that A and B are bialgebras over K_1 and K_2, respectively. The pairing $(\ ,\ ) : A \times B \to K$ is a *bialgebra pairing* if

$$\begin{aligned} (a, bb') &= (\Delta(a), b \otimes b'), \\ (aa', b) &= (a \otimes a', \Delta(b)), \\ (a, 1) &= \varepsilon(a), \\ (1, b) &= \varepsilon(b) \end{aligned} \tag{2.11}$$

for all $a, a' \in A$ and $b, b' \in B$, where Δ denotes the comultiplication and ε the counit.

Theorem 2.11. *Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra and $\mathfrak{g}' = (\mathfrak{g}^*)^{\mathrm{cop}}$. Then there is a nondegenerate bialgebra pairing*

$$A_{u,v}(\mathfrak{g}) \times A_{v,u}(\mathfrak{g}') \to \mathbf{C}[[u, v]],$$

which induces the standard bialgebra pairing

$$S(\mathfrak{g}) \times S(\mathfrak{g}') = A_{u,v}(\mathfrak{g})/(u, v) \times A_{v,u}(\mathfrak{g}')/(u, v) \to \mathbf{C},$$

uniquely determined by $(x, y) = \langle x, y\rangle$ *for all* $x \in \mathfrak{g}$ *and* $y \in \mathfrak{g}' = \mathfrak{g}^*$, *where* $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbf{C}$ *is the evaluation pairing.*

Theorem 2.11 will be proved in Section 12. Note that, quotienting by u (resp. v), we obtain nondegenerate bialgebra pairings

$$E_v(\mathfrak{g}) \times V_v(\mathfrak{g}') \to \mathbf{C}[[v]] \quad \text{and} \quad V_u(\mathfrak{g}) \times E_u(\mathfrak{g}') \to \mathbf{C}[[u]].$$

3. The maps δ^n.

Let A be a $\mathbf{C}[[u]]$-bialgebra in the sense of Section 1.1. In [**Dri87**, Section 7] Drinfeld used a general procedure to construct a $\mathbf{C}[[u]]$-subalgebra A' of A. In Drinfeld's terms, if A is a quantized universal enveloping algebra, then A' is a quantized formal series Hopf algebra. The subalgebra A' is defined using a family of linear maps $(\delta^n : A \to A^{\widehat{\otimes} n})_{n\geq 0}$, whose definition will be recalled below.

In this section, we prove that A' is commutative modulo u. To this end, we establish some properties of the maps δ^n.

3.1. Definition of δ^n. Starting from a bialgebra A over a commutative ring κ with comultiplication Δ and counit ε, we define for each $n \geq 0$ a morphism of algebras $\Delta^n : A \to A^{\otimes n}$ as follows: $\Delta^0 = \varepsilon : A \to \kappa$, $\Delta^1 = \mathrm{id}_A : A \to A$, the map $\Delta^2 : A \to A^{\otimes 2}$ is the comultiplication Δ and, for $n \geq 3$,

$$\Delta^n = (\Delta \otimes \mathrm{id}_A^{\otimes(n-2)})\Delta^{n-1}.$$

Let us embed $A^{\otimes n}$ into $A^{\otimes(n+1)}$ by tensoring on the right by the unit $1 \in A$. We thus get a direct system of algebras

$$A \to A^{\otimes 2} \to A^{\otimes 3} \to \cdots$$

whose limit we denote by $A^{\otimes\infty}$. In this way, each $A^{\otimes n}$ is naturally embedded in $A^{\otimes\infty}$.

Let I be a finite subset of the set of positive integers $\mathbf{N}' = \{1, 2, 3, \dots\}$. If $n = |I|$ is the cardinality of I, we define an algebra morphism $j_I : A^{\otimes n} \to A^{\otimes\infty}$ as follows. If $I = \{i_1, \dots, i_n\}$ with $i_1 < \dots < i_n$, then $j_I(a_1 \otimes \cdots \otimes a_n) = b_1 \otimes b_2 \otimes \cdots$, where $b_i = 1$ if $i \notin I$ and $b_{i_p} = a_p$ for $p = 1, \dots, n$. If $I = \emptyset$, then $j_I : \kappa \to A^{\otimes\infty}$ is the κ-linear map sending the unit of κ to the unit of $A^{\otimes\infty}$.

Suppose we have a κ-linear map $f : A \to A^{\otimes n}$ for some $n \geq 0$. For any set $I \subset \mathbf{N}'$ of cardinality n, we define a linear map $f_I : A \to A^{\otimes\infty}$ by $f_I = j_I \circ f$. If $I = \{1, \dots, n\}$, then f_I is equal to f composed with the standard embedding of $A^{\otimes n}$ in $A^{\otimes\infty}$. This shows that knowing the linear map $f : A \to A^{\otimes n}$ is equivalent to knowing the family of maps $f_I : A \to A^{\otimes\infty}$ indexed by the subsets I of $\mathbf{N}'$ of cardinality n. In particular, from each $\Delta^n : A \to A^{\otimes n}$ we obtain the family of linear maps (Δ_I) indexed by the sets $I \subset \mathbf{N}'$ of cardinality n and defined by $\Delta_I = (\Delta^n)_I : A \to A^{\otimes\infty}$.

After these preliminaries, we define the maps $\delta^n : A \to A^{\otimes n}$ for $n \geq 0$ by the following relation in terms of finite sets $I \subset \mathbf{N}'$:

$$\delta_I = \sum_{J \subset I} (-1)^{|I|-|J|} \Delta_J. \tag{3.1}$$

By the inclusion-exclusion principle, we have the equivalent relation

$$\Delta_I = \sum_{J \subset I} \delta_J. \tag{3.2}$$

It follows immediately from (3.1) that

$$\delta_I(1) = \begin{cases} 1 & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.2. *Let $a, b \in A$ and K be a finite subset of $\mathbf{N}'$. Then*

$$\delta_K(ab) = \sum_{\substack{I,J \subset K \\ I \cup J = K}} \delta_I(a)\delta_J(b). \tag{3.4}$$

Moreover, if $K \neq \emptyset$, then

$$\delta_K(ab - ba) = \sum_{\substack{I,J \subset K \\ I \cup J = K, I \cap J \neq \emptyset}} \big(\delta_I(a)\delta_J(b) - \delta_J(b)\delta_I(a)\big). \tag{3.5}$$

Proof. In order to prove (3.4), we first observe that by (3.2),

$$\sum_{K' \subset K} \delta_{K'}(ab) = \Delta_K(ab) = \Delta_K(a)\Delta_K(b) = \sum_{I,J \subset K} \delta_I(a)\delta_J(b). \tag{3.6}$$

We rewrite (3.6) as follows:

$$\sum_{K' \subset K} \delta_{K'}(ab) = \sum_{K' \subset K} \left(\sum_{\substack{I,J \subset K' \\ I \cup J = K'}} \delta_I(a)\delta_J(b) \right). \tag{3.7}$$

Let us prove (3.4) by induction on the cardinality of K. If $K = \emptyset$, then $\delta_K = j_\emptyset \circ \varepsilon$, which is a morphism of algebras. Suppose now that (3.4) holds for all sets of cardinality $< |K|$, in particular for all proper subsets K' of K. Thus, the right-hand side of (3.7) equals

$$\sum_{\substack{K' \subset K \\ K' \neq K}} \delta_{K'}(ab) + \sum_{\substack{I,J \subset K \\ I \cup J = K}} \delta_I(a)\delta_J(b).$$

We get the desired formula by substracting the summands corresponding to the proper subsets K' of K from both sides of (3.7).

Formula (3.5) follows from (3.4) and the fact that $\delta_I(a)$ and $\delta_J(b)$ commute when $I \cap J = \emptyset$.

3.3. Remark. Note that, if I and $J \subset \mathbf{N}'$ are disjoint finite sets, then

$$(\delta_I \otimes \delta_J) \circ \Delta = \delta_{I \cup J}. \tag{3.8}$$

Eric Müller observed (private communication) that $\delta^n : A \to A^{\otimes n}$ can also be defined as $\delta^n = (\mathrm{id}_A - \varepsilon)^{\otimes n} \circ \Delta^n$.

3.4. Definition of A'. Let A be a bialgebra over $\mathbf{C}[[u]]$ in the sense of Section 1.1. Using the comultiplication $\Delta : A \to A \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u]]} A$, we define $\mathbf{C}[[u]]$-linear maps $\delta^n : A \to A^{\widehat{\otimes} n}$ as in Section 3.1. Observe that Formulas (3.1)-(3.5) hold in this setting as well. Following Drinfeld [**Dri87**, Section 7], we introduce the submodule A' of A by

$$A' = \left\{ a \in A \ \mid \ \delta^n(a) \in u^n A^{\widehat{\otimes} n} \ \text{ for all } n > 0 \right\}. \tag{3.9}$$

It follows from (3.3) and (3.4) that A' is a subalgebra of A.

Proposition 3.5. *If the multiplication by u is injective on $A^{\widehat{\otimes} n}$ for all $n \geq 1$, then the algebra A' is commutative modulo u, i.e., $ab - ba \in uA'$ for all $a, b \in A'$.*

Proof. Let us first observe that there exists $a_1 \in A$ such that $a = ua_1 + \varepsilon(a)1$. This follows from the fact that $\mathrm{id}_A = \Delta^1 = \delta^1 + \delta^0 = \delta^1 + \varepsilon\, 1$ and $\delta^1(a) \in uA$. Similarly, there exists $b_1 \in A$ such that $b = ub_1 + \varepsilon(b)1$. Hence, $ab - ba = uc$, where $c = u(a_1 b_1 - b_1 a_1)$. It suffices to show that $c \in A'$. To this end, it is enough to check that $\delta_K(c)$ is divisible by $u^{|K|}$ for any nonempty finite subset K of $\mathbf{N}'$. Since the multiplication by u is injective on $A^{\widehat{\otimes}|K|}$, it is enough to check that $\delta_K(ab - ba)$ is divisible by $u^{|K|+1}$. We apply Formula (3.5). Let I and J be subsets of K such that $I \cup J = K$ and $I \cap J \neq \emptyset$. Then $|I| + |J| \geq |K| + 1$. Since $\delta_I(a)$ is divisible by $u^{|I|}$ and $\delta_J(b)$ is divisible by $u^{|J|}$, it follows from (3.5) that $\delta_K(ab - ba)$ is divisible by $u^{|I|+|J|}$, hence by $u^{|K|+1}$. □

3.6. Remark. If A is topologically free, i.e., isomorphic to $V[[u]]$ as a $\mathbf{C}[[u]]$-module for some vector space V, then so is A'. A similar, but more complicated statement will be proved in Lemma 7.2.

3.7. Example. Consider a Lie algebra $\mathfrak{g}$ and its universal enveloping bialgebra $U(\mathfrak{g})$. Let $U(\mathfrak{g})[[u]]$ be the $\mathbf{C}[[u]]$-bialgebra consisting of the formal power series over $U(\mathfrak{g})$, with comultiplication Δ given by (2.4). Using the notation of Section 2.4, we introduce a subalgebra $\widehat{V}_u(\mathfrak{g})$ of $U(\mathfrak{g})[[u]]$ by

$$\widehat{V}_u(\mathfrak{g}) = \left\{ \sum_{m \geq 0} a_m\, u^m \in U(\mathfrak{g})[[u]] \ \mid \ a_m \in U^m(\mathfrak{g}) \text{ for all } m \geq 0 \right\}. \tag{3.10}$$

Clearly, $V_u(\mathfrak{g}) \subset \widehat{V}_u(\mathfrak{g})$. Let I_u be the two-sided ideal of $V_u(\mathfrak{g})$ generated by $uV_u(\mathfrak{g})$ and by $u\mathfrak{g} \subset uU^1(\mathfrak{g}) \subset V_u(\mathfrak{g})$; it is the kernel of the morphism of algebras

$$V_u(\mathfrak{g}) \xrightarrow{q_u} S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})/\Big(\bigoplus_{n\geq 1} S^n(\mathfrak{g})\Big) = \mathbf{C},$$

cf. Section 2.4. It is easy to check that $\widehat{V}_u(\mathfrak{g})$ is the I_u-adic completion of $V_u(\mathfrak{g})$.

Proposition 3.8. *If $A = U(\mathfrak{g})[[u]]$, then $A' = \widehat{V}_u(\mathfrak{g})$.*

Proof. Let $a = \sum_{m\geq 0} a_m\, u^m$ be a formal power series with coefficients in $U(\mathfrak{g})$. For $n \geq 1$, the condition $\delta^n(a) \in u^n U(\mathfrak{g})^{\otimes n}[[u]]$ implies that $\delta^n(a_{n-1}) = 0$. We claim that

$$\mathrm{Ker}\big(\delta^n : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}\big) = U^{n-1}(\mathfrak{g}) \tag{3.11}$$

for all $n \geq 1$. It follows from this claim that $a_{n-1} \in U^{n-1}(\mathfrak{g})$, hence, $a \in \widehat{V}_u(\mathfrak{g})$.

Equality (3.11) is probably well known, but we give a proof for the sake of completeness. The standard symmetrization map $\eta : S(\mathfrak{g}) \to U(\mathfrak{g})$ is known to be an isomorphism of coalgebras (cf. [**Dix74**, Chap. 2]). Hence, $\eta^{\otimes n}\delta^n = \delta^n\eta$, where δ^n stands for the corresponding maps both on $S(\mathfrak{g})$ and $U(\mathfrak{g})$. Moreover, $\eta^{-1}(U^{n-1}(\mathfrak{g})) = \oplus_{k=0}^{n-1} S^k(\mathfrak{g})$. Therefore, Equality (3.11) is equivalent to

$$\mathrm{Ker}\big(\delta^n : S(\mathfrak{g}) \to S(\mathfrak{g})^{\otimes n}\big) = \bigoplus_{k=0}^{n-1} S^k(\mathfrak{g}).$$

If $(x_i)_i$ is a totally ordered basis of $\mathfrak{g}$, we get a basis of $S(\mathfrak{g})$ by taking all words $w = x_{i_1} \ldots x_{i_p}$ such that $x_{i_1} \leq \cdots \leq x_{i_p}$. We call subword of a word w any word obtained from w by deleting some letters. With this convention, the comultiplication Δ of $S(\mathfrak{g})$ is given on a basis element w by $\Delta(w) = \sum w_1 \otimes w_2$, where the sum is over all subwords w_1, w_2 of w such that $w = w_1 w_2$. Iterating Δ, we get for all $n \geq 1$

$$\Delta^n(w) = \sum w_1 \otimes \cdots \otimes w_n,$$

where the sum is over all subwords $w_1, \ldots, w_n$ of w such that $w = w_1 \ldots w_n$. This, together with (3.1) or (3.2), implies that

$$\delta^n(w) = \sum w_1 \otimes \cdots \otimes w_n, \tag{3.12}$$

where the sum is now over all *nonempty* subwords $w_1, \ldots, w_n$ of w such that $w = w_1 \ldots w_n$. This shows that, if w is of length $< n$, then the right-hand side of (3.12) is empty and $\delta^n(w) = 0$. Therefore,

$$\bigoplus_{k=0}^{n-1} S^k(\mathfrak{g}) \subset \mathrm{Ker}(\delta^n).$$

To prove the opposite inclusion, it is enough to check that the restriction of δ^n to the subspace $\oplus_{k\geq n} S^k(\mathfrak{g})$ is injective. This is a consequence of the following observation: If w is a basis element of length $\geq n$ and μ is the multiplication in $S(\mathfrak{g})$, then (3.12) implies that $\mu\delta^n(w) = \|w\|\, w$, where $\|w\| > 0$ is the number of summands on the right-hand side of (3.12). □

4. Topologically free $\mathbf{C}[[u,v]]$-modules.

In this section, we establish a few technical results on modules over the ring $\mathbf{C}[[u,v]]$ of formal power series in two commuting variables u and v with coefficients in $\mathbf{C}$. They are modelled on similar results for modules over the ring $\mathbf{C}[[h]]$ of formal power series in h.

4.1. Modules over $\mathbf{C}[[h]]$. We recall a few facts about $\mathbf{C}[[h]]$-modules (see, e. g., [**Kas95**, Sections XVI.2-3]). A $\mathbf{C}[[h]]$-module M is called *topologically free* if it is isomorphic to a module $V[[h]]$ consisting of all formal power series with coefficients in the vector space V. A $\mathbf{C}[[h]]$-module M is topologically free if and only if there is no nonzero element $m \in M$ such that $hm = 0$ and the natural map $M \to \varprojlim_n M/h^nM$ is an isomorphism. We define a topological tensor product $\widehat{\otimes}_{\mathbf{C}[[h]]}$ for $\mathbf{C}[[h]]$-modules M and N by

$$M \,\widehat{\otimes}_{\mathbf{C}[[h]]}\, N = \varprojlim_n \big(M/h^nM \otimes_{\mathbf{C}[[h]]/(h^n)} N/h^nN\big).$$

For all vector spaces V, W, we have $V[[h]] \,\widehat{\otimes}_{\mathbf{C}[[h]]}\, W[[h]] \cong (V \otimes_{\mathbf{C}} W)[[h]]$.

Let us extend these considerations to $\mathbf{C}[[u,v]]$-modules.

4.2. Basic Definitions. Let M be a $\mathbf{C}[[u,v]]$-module. We say that M is *u-torsion-free* (resp. *v-torsion-free*) if there is no nonzero element $m \in M$ such that $um = 0$ (resp. such that $vm = 0$).

We say that M is *admissible* if any element divisible by both u and v in M is divisible by uv in M. In other words, M is admissible if, for any $m \in M$ such that there exists $m_1, m_2 \in M$ with $m = um_1 = vm_2$, there exists $m_0 \in M$ such that $m = uvm_0$.

Observe that, if M is admissible and u-torsion-free, then any element of M divisible by u^n and by v is divisible by $u^n v$, where $n > 0$.

We denote by $\widehat{M}_{(u,v)}$ the (u,v)-adic completion of M: It is the projective limit of the projective system $(M/(u,v)^nM)_{n\geq 1}$, where $(u,v)M = uM + vM$. The projections $M \to M/(u,v)^nM$ induce a natural $\mathbf{C}[[u,v]]$-linear map $i : M \to \widehat{M}_{(u,v)}$. The kernel of i is the intersection of the submodules $((u,v)^nM)_{n\geq 1}$. We say that the module M is *separated* (resp. *complete*) if the map $i : M \to \widehat{M}_{(u,v)}$ is injective (resp. surjective).

Given a vector space V over $\mathbf{C}$, consider the vector space $V[[u,v]]$ consisting of formal power series $\sum_{m,n\geq 0} x_{mn}\, u^m v^n$, where the coefficients x_{mn} $(m, n \geq 0)$ are elements of V. The standard multiplication of formal power series endows $V[[u,v]]$ with a $\mathbf{C}[[u,v]]$-module structure. A $\mathbf{C}[[u,v]]$-module M isomorphic to a module of the form $V[[u,v]]$ will be called *topologically free.*

It is easy to check that a topologically free $\mathbf{C}[[u,v]]$-module is u-torsion-free, v-torsion-free, admissible, separated, and complete. We now prove the converse.

Lemma 4.3. *Any u-torsion-free, v-torsion-free, admissible, separated, complete $\mathbf{C}[[u,v]]$-module M is topologically free.*

Proof. Let V be a vector subspace of M supplementary to the submodule $(u,v)M$. We claim that for all $n \geq 0$ we have the direct sum decomposition of vector spaces

$$(u,v)^n M = (u,v)^{n+1} M \oplus \bigoplus_{\substack{k,\ell\geq 0\\ k+\ell=n}} u^k v^\ell V. \tag{4.1}$$

From (4.1) we derive

$$M = (u,v)^{n+1} M \oplus \bigoplus_{\substack{k,\ell\geq 0\\ k+\ell\leq n}} u^k v^\ell V.$$

Consequently,

$$M/(u,v)^{n+1} M = \bigoplus_{\substack{k,\ell\geq 0\\ k+\ell\leq n}} u^k v^\ell V = V[[u,v]]/(u,v)^{n+1} V[[u,v]].$$

Using the hypotheses, we get the following chain of $\mathbf{C}[[u,v]]$-linear isomorphisms:

$$M \cong \widehat{M}_{(u,v)} \cong \widehat{V[[u,v]]}_{(u,v)} \cong V[[u,v]].$$

It remains to check (4.1). We shall prove it by induction on n. If $n=0$, the identity (4.1) holds by definition of V. If $n>0$, let us first show that

$$(u,v)^n M = (u,v)^{n+1} M + \sum_{\substack{k,\ell\geq 0\\ k+\ell=n}} u^k v^\ell V. \tag{4.2}$$

Indeed, any element of $(u,v)^n M$ is of the form $um' + vm''$, where $m', m'' \in (u,v)^{n-1} M$. By the induction hypothesis, m' and m'' belong to

$$(u,v)^n M + \sum_{\substack{k,\ell\geq 0\\ k+\ell=n-1}} u^k v^\ell V.$$

This implies (4.2).

Suppose now that we have elements $m \in (u,v)^{n+1} M$ and $x_0, x_1, \dots, x_n \in V$ such that

$$m + \sum_{k=0}^{n} u^k v^{n-k} x_{n-k} = 0. \tag{4.3}$$

We have to show that $m = x_0 = x_1 = \cdots = x_n = 0$. The element $m \in (u,v)^{n+1}M$ is of the form $m = u^{n+1}m_0 + vm''$, where $m_0 \in M$ and $m'' \in (u,v)^n M$. The element $u^n x_0 + u^{n+1}m_0 = u^n(x_0 + um_0)$ is divisible by u^n. It follows from (4.3) that it is also divisible by v. Since M is admissible and u-torsion-free, there exists $m_1 \in M$ such that $u^n(x_0 + um_0) = u^n v m_1$. Hence, $x_0 + um_0 - vm_1 = 0$. Now, $x_0 \in V$ and $um_0 - vm_1 \in (u,v)M$ belong to supplementary subspaces. Therefore, $x_0 = um_0 - vm_1 = 0$ and $m = u^{n+1}m_0 + vm'' = vm'$, where $m' = u^n m_1 + m'' \in (u,v)^n M$. Now, (4.3) becomes $v\big(m' + \sum_{k=0}^{n-1} u^k v^{n-1-k} x_{n-k}\big) = 0$. Since M is v-torsion-free, we get $m' + \sum_{k=0}^{n-1} u^k v^{n-1-k} x_{n-k} = 0$. By the induction hypothesis, $m' = x_1 = \cdots = x_n = 0$. □

4.4. Topological Tensor Product. Given $\mathbf{C}[[u,v]]$-modules M and N, we define their topological tensor product over $\mathbf{C}[[u,v]]$ by

$$M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N = \varprojlim_n \Big(M/(u,v)^n M \otimes_{\mathbf{C}[[u,v]]/(u,v)^n} N/(u,v)^n N \Big).$$

For example, $M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} \mathbf{C}[[u,v]] = \widehat{M}_{(u,v)}$.

Lemma 4.5. (a) *If $M \cong V[[u,v]]$ and $N \cong W[[u,v]]$ are topologically free $\mathbf{C}[[u,v]]$-modules, then $M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$ is topologically free:*

$$M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N \cong (V \otimes_{\mathbf{C}} W)[[u,v]].$$

(b) *If $i : M' \to M$ and $j : N' \to N$ are injective $\mathbf{C}[[u,v]]$-maps of topologically free modules, then so is the map $i \otimes j : M' \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N' \to M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$.*

Proof. (a) Proceed as in the proof of [**Kas95**, Proposition XVI.3.2].

(b) Since $i \otimes j = (\mathrm{id} \otimes j)(i \otimes \mathrm{id})$, it is enough to prove Part (b) when $N = N'$ or $M = M'$. We give a proof for $N = N'$.

Let V, V', W be vector spaces such that $M = V[[u,v]]$, $M' = V'[[u,v]]$, and $N = W[[u,v]]$. Take a basis $(f_m)_m$ of W. By Part (a), any element Y of $M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$ can be uniquely written as $Y = \sum_m X_m \otimes f_m$, where $X_m \in M$. Set $j_m(Y) = X_m$. This defines for all m a $\mathbf{C}[[u,v]]$-linear map $j_m : M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N \to M$. Using the same basis of W, we define a linear map $j'_m : M' \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N \to M'$ similarly. Clearly, $j_m \circ (i \otimes \mathrm{id}) = i \circ j'_m$ for all m. Now, take $Y' \in M' \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$ such that $(i \otimes \mathrm{id})(Y') = 0$. By the previous equality, we have $i(j'_m(Y')) = 0$ for all m. The map i being injective, we get $j'_m(Y') = 0$ for all m. Therefore, $Y' = \sum_m j'_m(Y') \otimes f_m = 0$ and $i \otimes \mathrm{id}$ is injective. □

4.6. From One Variable to Two Variables. One of the crucial steps in our constructions will be to transform a module N over $\mathbf{C}[[h]]$ into a module $\widetilde{N}$ over $\mathbf{C}[[u,v]]$. This is done as follows.

Let $\iota : \mathbf{C}[[h]] \to \mathbf{C}[[u,v]]$ be the algebra morphism sending h to the product uv. Observe that ι factors through the subalgebras $\mathbf{C}[u][[v]]$ and $\mathbf{C}[v][[u]]$. The morphism ι sends the ideal (h^n) into the ideal $(u,v)^{2n}$. Given a $\mathbf{C}[[h]]$-module N, we consider the projective system of $\mathbf{C}[[u,v]]$-modules

$$N/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n}$$

where $n = 1, 2, 3, \dots$ and set

$$\widetilde{N} = \varprojlim_n \Big(N/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n} \Big). \tag{4.4}$$

Clearly, for any $x \in N$, there is defined a corresponding element $\widetilde{x} \in \widetilde{N}$.

Lemma 4.7. (a) *If $N = V[[h]]$ for some vector space V over $\mathbf{C}$, then $\widetilde{N} = V[[u,v]]$.*

(b) *If N and N' are topologically free $\mathbf{C}[[h]]$-modules, then*

$$\big(N \widehat{\otimes}_{\mathbf{C}[[h]]} N'\big)^{\sim} \cong \widetilde{N} \widehat{\otimes}_{\mathbf{C}[[u,v]]} \widetilde{N'}.$$

(c) *Let $i : N' \to N$ be an injective map of topologically free $\mathbf{C}[[h]]$-modules. Then the induced $\mathbf{C}[[u,v]]$-map $\widetilde{\imath} : \widetilde{N'} \to \widetilde{N}$ is also injective.*

Proof. (a) We have the following chain of $\mathbf{C}[[u,v]]$-linear isomorphisms

$$\begin{aligned}
\widetilde{N} &= \varprojlim_n V[[h]]/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n} \\
&= \varprojlim_n V \otimes_{\mathbf{C}} \mathbf{C}[[h]]/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n} \\
&= \varprojlim_n V \otimes_{\mathbf{C}} \mathbf{C}[[u,v]]/(u,v)^{2n} \\
&= \varprojlim_n V[[u,v]]/(u,v)^{2n} \\
&= V[[u,v]].
\end{aligned}$$

The first isomorphism follows from the definition of $\widetilde{N}$, the second and the fourth ones from the finite-dimensionality of $\mathbf{C}[[h]]/(h^n)$ and of $\mathbf{C}[[u,v]]/(u,v)^{2n}$.

(b) This is an easy exercise which follows from Part (a) and the properties of the topological tensor products over $\mathbf{C}[[h]]$ and $\mathbf{C}[[u,v]]$ stated in Section 4.1 and in Lemma 4.5 (a).

(c) We assume that $N = V[[h]]$ and $N' = V'[[h]]$ for some vector spaces V and V'. Let $(e_k)_k$ be a basis of V' and $(f_j)_j$ a basis of V. The $\mathbf{C}[[h]]$-linear map $i : N' \to N$ is determined by $i(e_k) = \sum_{\ell \geq 0;\, j} x^j_{k,\ell}\, f_j\, h^\ell$, where $(x^j_{k,\ell})_{j,k,\ell}$ is a family of scalars such that for each couple (k,ℓ) the set of j with $x^j_{k,\ell} \neq 0$

is finite. Any element $X \in N'$ is of the form $X = \sum_{n\geq 0;\, k} \alpha_n^k\, e_k\, h^n$, where $(\alpha_n^k)_{k,n}$ is a family of scalars such that for each $n \geq 0$ the set of k with $\alpha_n^k \neq 0$ is finite. We have

$$i(X) = \sum_{\ell,n\geq 0;\, j,k} x_{k,\ell}^j \alpha_n^k\, f_j\, h^{\ell+n} = \sum_{p\geq 0} \left(\sum_{\substack{\ell,n\geq 0;\, j,k \\ \ell+n=p}} x_{k,\ell}^j \alpha_n^k\, f_j \right) h^p.$$

The coefficient of $f_j h^p$ in $i(X)$ is

$$\sum_{\substack{\ell,n\geq 0;\, k \\ \ell+n=p}} x_{k,\ell}^j \alpha_n^k = \sum_{\substack{\ell,k \\ 0\leq \ell\leq p}} x_{k,\ell}^j \alpha_{p-\ell}^k.$$

This allows us to reformulate the injectivity of i as follows: The equations on a family of scalars $(\alpha_n^k)_{k;\, n\geq 0}$

$$\sum_{\substack{\ell,k \\ 0\leq \ell\leq p}} x_{k,\ell}^j \alpha_{p-\ell}^k = 0 \tag{4.5}$$

holding for all j and $p \geq 0$ imply that $\alpha_n^k = 0$ for all k and $n \geq 0$.

By Part (a) we have $\widetilde{N} = V[[u,v]]$ and $\widetilde{N'} = V'[[u,v]]$. On the basis $(e_k)_k$ the map $\widetilde{\imath}$ is defined by $\widetilde{\imath}(e_k) = \sum_{\ell\geq 0;\, j} x_{k,\ell}^j\, f_j\, u^\ell v^\ell$. Any element $Y \in \widetilde{N'}$ is of the form $Y = \sum_{m,n\geq 0;\, k} \beta_{mn}^k\, e_k\, u^m v^n$, where $(\beta_{mn}^k)_{k,m,n}$ is a family of scalars such that for each $m, n \geq 0$ the set of k with $\beta_{mn}^k \neq 0$ is finite. We have

$$\begin{aligned}\widetilde{\imath}(Y) &= \sum_{\ell,m,n\geq 0;\, j,k} x_{k,\ell}^j \beta_{mn}^k\, f_j\, u^{\ell+m} v^{\ell+n} \\ &= \sum_{p,q\geq 0} \left(\sum_{\substack{\ell,m,n\geq 0;\, j,k \\ \ell+m=p,\ell+n=q}} x_{k,\ell}^j \beta_{mn}^k\, f_j \right) u^p v^q.\end{aligned}$$

Note that the sum in the brackets is finite. Suppose that $\widetilde{\imath}(Y) = 0$. For all $p, q \geq 0$ and all j we have

$$\sum_{\substack{\ell,m,n\geq 0;\, k \\ \ell+m=p,\ell+n=q}} x_{k,\ell}^j \beta_{mn}^k = \sum_{\substack{\ell,k \\ 0\leq \ell\leq \min(p,q)}} x_{k,\ell}^j \beta_{p-\ell,q-\ell}^k = 0.$$

Fixing $q \geq p \geq 0$ and setting $\alpha_n^k = \beta_{n,q-p+n}^k$, we get (4.5) for all j. This implies that $\beta_{n,q-p+n}^k = \alpha_n^k = 0$ for all k, n, p, q. If $p > q \geq 0$, we set $\alpha_n^k = \beta_{p-q+n,n}^k$ and we conclude likewise. Therefore, $Y = 0$. □

We define a $\mathbf{C}[[u,v]]$-*bialgebra* as a topological $\mathbf{C}[[u,v]]$-bialgebra A with respect to the ideal $(u,v) = uA + vA$. As a consequence of Lemma 4.7, we have the following:

Corollary 4.8. *If A is a $\mathbf{C}[[h]]$-bialgebra that is topologically free as a $\mathbf{C}[[h]]$-module, then $\widetilde{A}$ is a $\mathbf{C}[[u,v]]$-bialgebra that is topologically free as a $\mathbf{C}[[u,v]]$-module.*

Proof. The $\mathbf{C}[[u,v]]$-module $\widetilde{A}$ is topologically free by Lemma 4.7 (a). It is a $\mathbf{C}[[u,v]]$-bialgebra as a consequence of Lemma 4.7 (b). ☐

5. On Etingof and Kazhdan's quantization of a Lie bialgebra.

In this section, we recall the results from Etingof and Kazhdan's work [**EK96**] needed in the sequel.

5.1. The Co-Poisson Bialgebra $U(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie bialgebra with Lie cobracket δ. Consider the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ with standard cocommutative comultiplication given by (2.4). By [**Dri87**], the bialgebra $U(\mathfrak{g})$ has a unique co-Poisson bialgebra structure with a Poisson cobracket whose restriction to $\mathfrak{g} \subset U(\mathfrak{g})$ is the Lie cobracket δ. Recall from Section 1.2 that a coquantization A of $U(\mathfrak{g})$ is a $\mathbf{C}[[h]]$-bialgebra A such that $A \cong U(\mathfrak{g})[[h]]$ as a $\mathbf{C}[[h]]$-module and $A/hA = U(\mathfrak{g})$ as co-Poisson bialgebras.

In [**EK96**] Etingof and Kazhdan constructed a coquantization $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ in this sense. To this end, they first constructed a coquantization $U_h(\mathfrak{d})$ of $U(\mathfrak{d})$, where $\mathfrak{d}$ is the double of $\mathfrak{g}$. We recall the definition of $\mathfrak{d}$.

5.2. Double of a Lie Bialgebra. Let $\mathfrak{g} = \mathfrak{g}_+$ be a finite-dimensional Lie bialgebra over $\mathbf{C}$ with Lie bracket $[\ ,\ ]$ and cobracket δ. Let $\mathfrak{g}_- = (\mathfrak{g}_+^{\mathrm{op}})^* = (\mathfrak{g}_+^*)^{\mathrm{cop}}$ be the dual Lie bialgebra modified as in Section 2.1.

Consider the direct sum $\mathfrak{d} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$. Drinfeld [**Dri82**, **Dri87**] showed that there is a unique structure of Lie bialgebra on $\mathfrak{d}$, which he called the *double* of $\mathfrak{g}_+$, such that

(a) the inclusions of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ into $\mathfrak{d}$ are morphisms of Lie bialgebras and

(b) the Lie bracket $[x,y]$ for $x \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$ is given by

$$[x,y] = (y \otimes 1)\,\delta(x) + x \cdot y, \tag{5.1}$$

where $x \cdot y \in \mathfrak{g}_- \subset \mathfrak{d}$ is defined by $(x \cdot y)(x') = -y([x,x'])$ for $x' \in \mathfrak{g}_+$.

The Lie cobracket on $\mathfrak{d}$ (hence on $\mathfrak{g}_\pm$) is given by

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r] = \sum_{i=1}^{d} \Big([X, x_i] \otimes y_i + x_i \otimes [X, y_i] \Big) \tag{5.2}$$

for $X \in \mathfrak{d}$. Here $r = \sum_{i=1}^{d} x_i \otimes y_i$ is the canonical element of $\mathfrak{g}_+ \otimes \mathfrak{g}_- \subset \mathfrak{d} \otimes \mathfrak{d}$, where $(x_i)_{i=1}^{d}$ is a basis of $\mathfrak{g}_+$ and $(y_i)_{i=1}^{d}$ is the dual basis of $\mathfrak{g}_-$.

5.3. The bialgebra $U_h\mathfrak{d}$. By [**EK96**, Section 3] there exists a $\mathbf{C}[[h]]$-bialgebra $U_h(\mathfrak{d})$ with the following features:

(i) As a $\mathbf{C}[[h]]$-algebra, $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$, i.e., the multiplication is the standard formal power series product.

(ii) There exists an invertible element $J_h \in (U\mathfrak{d} \otimes U\mathfrak{d})[[h]]$ with constant term $1\otimes 1$ such that the comultiplication Δ_h of $U_h(\mathfrak{d})$ is given for all $a \in U(\mathfrak{d})$ by

$$\Delta_h(a) = J_h^{-1}\Delta(a)J_h, \tag{5.3}$$

where Δ is the standard comultiplication in $U(\mathfrak{d})$. The first terms of the formal power series J_h are given by

$$J_h \equiv 1\otimes 1 + \frac{h}{2}r \mod h^2 \tag{5.4}$$

where $r \in \mathfrak{d}\otimes\mathfrak{d}$ was defined in Section 5.2. From (5.2–5.4) it follows that for $x \in \mathfrak{d} \subset U_h(\mathfrak{d})$ we have

$$\Delta_h(x) - \Delta_h^{\mathrm{op}}(x) \equiv h\,\delta(x) \mod h^2, \tag{5.5}$$

where Δ_h^{op} is the opposite comultiplication and δ is the Lie cobracket (5.2).

(iii) If we set $t = r + r_{21} = \sum_{i=1}^{d}(x_i \otimes y_i + y_i \otimes x_i)$, then the element

$$R_h = (J_h^{-1})_{21}\exp\Big(\frac{ht}{2}\Big)J_h \in (U\mathfrak{d}\otimes U\mathfrak{d})[[h]] = U_h(\mathfrak{d})\,\widehat{\otimes}_{\mathbf{C}[[h]]}\,U_h(\mathfrak{d}) \tag{5.6}$$

defines a quasitriangular structure on $U_h(\mathfrak{d})$. This means that $\Delta_h^{\mathrm{op}}(a) = R_h\Delta_h(a)R_h^{-1}$ for all $a \in U_h(\mathfrak{d})$ and that

$$(\Delta_h\otimes\mathrm{id})(R_h) = (R_h)_{13}(R_h)_{23} \quad\text{and}\quad (\mathrm{id}\otimes\Delta_h)(R_h) = (R_h)_{13}(R_h)_{12}. \tag{5.7}$$

Formula (5.4) implies

$$R_h = 1\otimes 1 + hR'_h, \tag{5.8}$$

where $R'_h \in U_h(\mathfrak{d})\,\widehat{\otimes}_{\mathbf{C}[[h]]}\,U_h(\mathfrak{d})$ such that $R'_h \equiv r \bmod h$.

From (i) and (ii) it is clear that $U_h(\mathfrak{d})$ is a coquantization of the co-Poisson bialgebra $U(\mathfrak{d})$.

5.4. The bialgebras $U_h(\mathfrak{g}_\pm)$. In [**EK96**, Section 4] Etingof and Kazhdan constructed a $\mathbf{C}[[h]]$-bialgebra $U_h(\mathfrak{g}_\pm)$ (with h-adic topology) with the following properties:

(i) As a $\mathbf{C}[[h]]$-module, $U_h(\mathfrak{g}_\pm)$ is isomorphic to $U(\mathfrak{g}_\pm)[[h]]$.

(ii) $U_h(\mathfrak{g}_\pm)$ is a $\mathbf{C}[[h]]$-subbialgebra of $U_h(\mathfrak{d})$. The map $p_h : U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]] \to U(\mathfrak{d}) = U(\mathfrak{d})[[h]]/hU(\mathfrak{d})[[h]]$ induces a bialgebra isomorphism

$$U_h(\mathfrak{g}_\pm)/hU_h(\mathfrak{g}_\pm) = U(\mathfrak{g}_\pm) \subset U(\mathfrak{d}).$$

(iii) The element $R'_h \in U_h(\mathfrak{d})\widehat{\otimes}_{\mathbf{C}[[h]]}U_h(\mathfrak{d})$ of (5.8) belongs to $U_h(\mathfrak{g}_+)\widehat{\otimes}_{\mathbf{C}[[h]]}U_h(\mathfrak{g}_-)$. So does the universal R-matrix R_h.

(iv) The coalgebra structure on $U_h(\mathfrak{g}_\pm)$ induces an algebra structure on the dual module $U_h^*(\mathfrak{g}_\pm) = \mathrm{Hom}_{\mathbf{C}[[h]]}(U_h(\mathfrak{g}_\pm), \mathbf{C}[[h]])$. By (iii) we can define linear maps $\rho_\pm : U_h^*(\mathfrak{g}_\mp) \to U_h(\mathfrak{g}_\pm)$ by

$$\rho_+(f) = (\mathrm{id}\otimes f)(R_h) \quad \text{and} \quad \rho_-(g) = (g\otimes \mathrm{id})(R_h) \tag{5.9}$$

for all $f \in U_h^*(\mathfrak{g}_-)$ and $g \in U_h^*(\mathfrak{g}_+)$. In [**EK96**, Propositions 4.8 and 4.10] it was shown that ρ_+ is an injective antimorphism of algebras and ρ_- is an injective morphism of algebras.

The construction of $U_h(\mathfrak{d})$ and $U_h(\mathfrak{g}_\pm)$ depends on a Drinfeld associator, see Sections 11.2-11.4. Nevertheless, it was shown in [**EK97**] (and in Section 10 of the revised version of [**EK96**]) that the assignment $(\mathfrak{g}_+, \mathfrak{d}, \mathfrak{g}_-) \mapsto \Big(U_h(\mathfrak{g}_+) \hookrightarrow U_h(\mathfrak{d}) \hookleftarrow U_h(\mathfrak{g}_-)\Big)$ is functorial when the Drinfeld associator is fixed.

5.5. The Linear Forms f_x. Choose a $\mathbf{C}[[h]]$-linear isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ such that $\alpha_-(1) = 1$ and $\alpha_- \equiv \mathrm{id}$ modulo h. Choose also a $\mathbf{C}$-linear projection $\pi_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-) = \mathbf{C}\oplus\mathfrak{g}_-$ that is the identity on $U^1(\mathfrak{g}_-)$. For any $x \in \mathfrak{g}_+$ we define a $\mathbf{C}$-linear form $\langle x, -\rangle : U^1(\mathfrak{g}_-) \to \mathbf{C}$ extending the evaluation map $\langle x, -\rangle : \mathfrak{g}_- \to \mathbf{C}$ and such that $\langle x, 1\rangle = 0$.

Given $x \in \mathfrak{g}_+$ we define a $\mathbf{C}[[h]]$-linear form $f_x : U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ by

$$f_x(b) = \langle x, \pi_-\alpha_-(b)\rangle = \sum_{n\geq 0} \langle x, \pi_-(b_n)\rangle\, h^n, \tag{5.10}$$

where $b \in U_h(\mathfrak{g}_-)$ and the elements $b_n \in U(\mathfrak{g}_-)$ are defined by $\alpha_-(b) = \sum_{n\geq 0} b_n h^n$. It follows from the definition that $f_x(1) = 0$.

Applying the map ρ_+ of (5.9) to $f_x \in U_h^*(\mathfrak{g}_-)$, we get an element $\rho_+(f_x) \in U_h(\mathfrak{g}_+)$. Fix a basis $(x_1, \dots, x_d)$ of $\mathfrak{g}_+$. Given a d-tuple $\underline{j} = (j_1, \dots, j_d)$ of nonnegative integers, we set $|\underline{j}| = j_1 + \cdots + j_d$ and $x_{\underline{j}} = x_1^{j_1} \dots x_d^{j_d} \in U(\mathfrak{g}_+)$. Note that $(x_{\underline{j}})_{\underline{j}}$ is a basis of $U(\mathfrak{g}_+)$.

Lemma 5.6. (a) *For any d-tuple $\underline{j} = (j_1, \dots, j_d)$ of nonnegative integers, there exists an element $t_{\underline{j}} \in U_h(\mathfrak{g}_+)$ such that*

$$\rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d} = h^{|\underline{j}|}\, t_{\underline{j}} \quad \textit{and} \quad p_h(t_{\underline{j}}) = x_{\underline{j}},$$

where $p_h : U_h(\mathfrak{g}_+) \to U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+) = U(\mathfrak{g}_+)$ is the canonical projection.

(b) *For any $a \in U_h(\mathfrak{g}_+)$, there is a unique family of scalars $\lambda_{\underline{j}}^{(n)} \in \mathbf{C}$ indexed by a nonnegative integer n and a finite sequence $\underline{j} = (j_1, \dots, j_d)$ of nonnegative integers such that*

$$a = \sum_{n\geq 0} \left(\sum_{|\underline{j}|\leq c(n)} \lambda_{\underline{j}}^{(n)} t_{\underline{j}} \right) h^n,$$

where $c(n)$ is an integer depending on a and n.

(c) *If* $a \in \operatorname{Im}\rho_+$, *then* $c(n) = n$, *that is,* $\lambda_{\underline{j}}^{(n)} = 0$ *whenever* $n < |\underline{j}|$, *where* $\lambda_{\underline{j}}^{(n)}$ *are the scalars above.*

Proof. (a) For any $x \in \mathfrak{g}_+$, we have $\rho_+(f_x) = ht_x$ for some $t_x \in U_h(\mathfrak{g}_+)$ such that $p_h(t_x) = x$. This follows from (5.8) (cf. [**EK96**, Lemma 4.6]). We set $t_{\underline{j}} = t_{x_1}^{j_1} \dots t_{x_d}^{j_d}$.

(b) The proof of Proposition 4.5 of [**EK96**] implies that any $a \in U_h(\mathfrak{g}_+)$ can be expanded as above. Let us check that such an expression is unique. If

$$\sum_{n \geq 0} \Big(\sum_{\underline{j};\, |\underline{j}| \leq c(n)} \lambda_{\underline{j}}^{(n)} t_{\underline{j}} \Big) h^n = 0, \tag{5.11}$$

then $\sum_{|\underline{j}| \leq c(0)} \lambda_{\underline{j}}^{(0)} x_{\underline{j}} = 0$ by application of the projection p_h. Since the elements $(x_{\underline{j}})_{\underline{j}}$ form a basis of $U(\mathfrak{g}_+)$, we conclude that $\lambda_{\underline{j}}^{(0)} = 0$ for all $\underline{j}$. We may then divide the left-hand side of (5.11) by h and start again. This implies the vanishing of $\lambda_{\underline{j}}^{(1)} = 0$ for all $\underline{j}$, and so on.

(c) Clearly, $U_h^*(\mathfrak{g}_-) = U(\mathfrak{g}_-)^*[[h]]$ where $U(\mathfrak{g}_-)^* = \operatorname{Hom}_{\mathbf{C}}(U(\mathfrak{g}_-), \mathbf{C})$. We provide $U_h^*(\mathfrak{g}_-)$ with the multiplication induced by the comultiplication of $U_h(\mathfrak{g}_-)$. We claim that the family of linear forms $(f_{x_d}^{j_d} \dots f_{x_1}^{j_1})_{\underline{j}} \in U_h^*(\mathfrak{g}_-)$ is linearly independent and that the $\mathbf{C}[[h]]$-module it spans is dense in $U_h^*(\mathfrak{g}_-)$ for the I_h^*-adic topology, where I_h^* is the two-sided ideal of $U_h^*(\mathfrak{g}_-)$ generated by h and f_{x_k} $(k = 1, \dots, d)$. It suffices to prove that the images $\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1} \in U(\mathfrak{g}_-)^*$ of $f_{x_d}^{j_d} \dots f_{x_1}^{j_1}$ under the algebra morphism $U_h^*(\mathfrak{g}_-) \to U_h^*(\mathfrak{g}_-)/hU_h^*(\mathfrak{g}_-) = U(\mathfrak{g}_-)^*$ are linearly independent and that their linear span is dense in $U(\mathfrak{g}_-)^*$ for the I_0^*-adic topology, where I_0^* is the two-sided ideal of $U(\mathfrak{g}_-)^*$ generated by θ_{x_k} $(k = 1, \dots, d)$. Now, by definition of f_{x_i}, we have $\theta_{x_i} = \langle x_i, \pi_-(-)\rangle$. This implies that, for all $i, j = 1, \dots, d$, we have

$$\theta_{x_i}(1) = 0 \quad \text{and} \quad \theta_{x_i}(y_j) = \delta_{ij}, \tag{5.12}$$

where $(y_1, \dots, y_d)$ is the dual basis of the basis $(x_1, \dots, x_d)$. We compute the values of the linear form $\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1}$ on the basis $(y_d^{k_d} \dots y_1^{k_1})_{k_1, \dots, k_d \geq 0}$ of $U(\mathfrak{g}_-)$:

$$(\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1})(y_d^{k_d} \dots y_1^{k_1}) = (\theta_{x_d}^{\otimes j_d} \otimes \dots \otimes \theta_{x_1}^{\otimes j_1})\big(\Delta^{|\underline{j}|}(y_d^{k_d} \dots y_1^{k_1})\big).$$

A simple computation, using (5.12) and the definition of Δ (cf. the proof of Proposition 3.8), shows that

$$(\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1})(y_d^{k_d} \dots y_1^{k_1}) = \begin{cases} 0 & \text{if } k_1 + \dots + k_d < j_1 + \dots + j_d, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d} & \text{if } k_1 + \dots + k_d = j_1 + \dots + j_d. \end{cases} \tag{5.13}$$

The claim about the linear forms $\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1} \in U(\mathfrak{g}_-)^*$ follows immediately from (5.13).

Part (a) of this lemma and the claim established above imply that the $\mathbf{C}[[h]]$-linear span of the set $(\rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d})_{\underline{j}}$ is dense in $\operatorname{Im}\rho_+$ for the h-adic topology. It is enough to prove (c) for an element a in this span. By Part (a), $a = \sum_{n\geq 0,\, \underline{j}} P_{\underline{j}}\, h^{|\underline{j}|}\, t_{\underline{j}}$ with $P_{\underline{j}} \in \mathbf{C}[[h]]$. By Part (b), the element a can be written uniquely as $a = \sum_{n\geq 0,\, \underline{j}} \lambda_{\underline{j}}^{(n)}\, h^n\, t_{\underline{j}}$. Hence, for any $\underline{j}$, the formal power series $\sum_{n\geq 0} \lambda_{\underline{j}}^{(n)} h^n$ is divisible by $h^{|\underline{j}|}$, which implies the vanishing of $\lambda_{\underline{j}}^{(n)}$ for $n < |\underline{j}|$. □

6. The algebra $A_+ = A_{u,v}(\mathfrak{g}_+)$.

We first define a two-variable version $U_{u,v}(\mathfrak{g}_\pm)$ of Etingof and Kazhdan's quantization. Then we construct the algebra $A_+ = A_{u,v}(\mathfrak{g}_+)$ appearing in Theorem 2.3. We use the notation $\mathfrak{g}_\pm$, $\mathfrak{d}$ defined in Section 5.

6.1. The bialgebras $U_{u,v}(\mathfrak{d})$ and $U_{u,v}(\mathfrak{g}_\pm)$. Applying the construction of Section 4.6 to the $\mathbf{C}[[h]]$-bialgebras $U_h(\mathfrak{d})$ and $U_h(\mathfrak{g}_\pm)$, we obtain $\mathbf{C}[[u,v]]$-modules

$$U_{u,v}(\mathfrak{d}) = \widetilde{U_h(\mathfrak{d})} \quad \text{and} \quad U_{u,v}(\mathfrak{g}_\pm) = \widetilde{U_h(\mathfrak{g}_\pm)}. \tag{6.1}$$

As a consequence of Lemma 4.5, Lemma 4.7, Corollary 4.8, and of the results summarized in Sections 5.3 and 5.4, we get the following proposition.

Proposition 6.2. (a) *The $\mathbf{C}[[u,v]]$-modules $U_{u,v}(\mathfrak{d})$ and $U_{u,v}(\mathfrak{g}_\pm)$ are topologically free.*

(b) *$U_{u,v}(\mathfrak{d})$ has a bialgebra structure whose underlying algebra is the algebra $U(\mathfrak{d})[[u,v]]$ of formal power series with coefficients in $U(\mathfrak{d})$.*

(c) *$U_{u,v}(\mathfrak{g}_\pm)$ has a bialgebra structure such that the $\mathbf{C}[[u,v]]$-linear map $U_{u,v}(\mathfrak{g}_\pm) \to U_{u,v}(\mathfrak{d})$ induced by $U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{d})$ is an embedding of bialgebras.*

(d) *There are canonical isomorphisms of bialgebras*

$$U_{u,v}(\mathfrak{d})/(u,v)U_{u,v}(\mathfrak{d}) = U(\mathfrak{d}) \quad \text{and} \quad U_{u,v}(\mathfrak{g}_\pm)/(u,v)U_{u,v}(\mathfrak{g}_\pm) = U(\mathfrak{g}_\pm).$$

By Proposition 6.2 (c) we may view $U_{u,v}(\mathfrak{g}_\pm)$ as a subset (in fact, a subbialgebra) of $U_{u,v}(\mathfrak{d})$. We denote the comultiplication in $U_{u,v}(\mathfrak{d})$ and in $U_{u,v}(\mathfrak{g}_\pm)$ by $\Delta_{u,v}$. To Etingof and Kazhdan's universal R-matrix $R_h \in U_h(\mathfrak{d})\,\widehat{\otimes}_{\mathbf{C}[[h]]}\, U_h(\mathfrak{d})$ corresponds an element $R_{u,v} \in U_{u,v}(\mathfrak{d})\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{d})$. By Section 5.4 (iii) and Lemma 4.5 (b), we have

$$R_{u,v} \in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-).$$

The following is a consequence of (5.7) and (5.8).

Lemma 6.3. (a) *We have*

$$(\Delta_{u,v}\otimes\mathrm{id})(R_{u,v}) = (R_{u,v})_{13}(R_{u,v})_{23} \quad \textit{and}$$
$$(\mathrm{id}\otimes\Delta_{u,v})(R_{u,v}) = (R_{u,v})_{13}(R_{u,v})_{12}.$$

(b) *There is a unique* $R' \in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_-)$ *such that* $R_{u,v} = 1\otimes 1 + uvR'$. *The image of* R' *under the projection*

$$U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_-) \to \big(U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_-)\big)/(u,v)$$
$$= U(\mathfrak{g}_+)\otimes_{\mathbf{C}} U(\mathfrak{g}_-)$$

is the element $r = \sum_{i=1}^{d} x_i\otimes y_i$ *defined in Section* 5.2.

Following 5.4, consider the dual spaces

$$U^*_{u,v}(\mathfrak{g}_\pm) = \mathrm{Hom}_{\mathbf{C}[[u,v]]}(U_{u,v}(\mathfrak{g}_\pm), \mathbf{C}[[u,v]]),$$

and define $\mathbf{C}[[u,v]]$-linear maps $\rho_+ : U^*_{u,v}(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_+)$ and $\rho_- : U^*_{u,v}(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_-)$ by

$$\rho_+(f) = (\mathrm{id}\otimes f)(R_{u,v}) \quad \text{and} \quad \rho_-(g) = (g\otimes\mathrm{id})(R_{u,v}) \tag{6.2}$$

for $f \in U^*_{u,v}(\mathfrak{g}_-)$ and $g \in U^*_{u,v}(\mathfrak{g}_+)$. The dual space $U^*_{u,v}(\mathfrak{g}_\pm)$ carries a $\mathbf{C}[[u,v]]$-algebra structure. The map ρ_+ is an antimorphism of algebras and ρ_- is a morphism of algebras. This follows by a standard argument from Lemma 6.3 (a) (cf. [**EK96**, Proposition 4.8]).

6.4. The Linear Forms $\widetilde{f}_x$. In Section 5.5 we constructed a $\mathbf{C}[[h]]$-linear form $f_x : U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ for all $x \in \mathfrak{g}_+$. The construction depends on the choice of an isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ and a projection $\pi_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-)$. By extension of scalars, we obtain a $\mathbf{C}[[u,v]]$-linear form $\widetilde{f}_x : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$. We have $\widetilde{f}_x(1) = 0$.

Let us apply $\rho_+ : U^*_{u,v}(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_+)$ to $\widetilde{f}_x$. The following is a consequence of Lemma 6.3 (b).

Lemma 6.5. *The element* $\rho_+(\widetilde{f}_x) \in U_{u,v}(\mathfrak{g}_+)$ *is divisible by* uv.

6.6. Definition of A_+. Let $(x_1, \ldots , x_d)$ be the basis of $\mathfrak{g}_+$ fixed in Section 5.5. The set $(u^{|\underline{j}|}\, x_{\underline{j}})$, where $\underline{j} = (j_1, \ldots , j_d)$ runs over all d-tuples of nonnegative integers, is a basis of the free $\mathbf{C}[u]$-module $V_u(\mathfrak{g}_+)$ introduced in Section 2.4. In view of Lemma 6.5, we can define a $\mathbf{C}[u]$-linear map $\psi_+ : V_u(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_+)$ by $\psi_+(1) = 1$ and

$$\psi_+(u^{|\underline{j}|}\, x_{\underline{j}}) = v^{-|\underline{j}|}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \ldots \rho_+(\widetilde{f}_{x_d})^{j_d}, \tag{6.3}$$

where $\underline{j} = (j_1, \ldots , j_d)$ is a d-tuple of nonnegative integers with $|\underline{j}| \geq 1$. This map extends uniquely to a $\mathbf{C}[u][[v]]$-linear map, still denoted ψ_+, from

$V_u(\mathfrak{g}_+)[[v]]$ to $U_{u,v}(\mathfrak{g}_+)$ by

$$\psi_+\left(\sum_{n\geq 0} w_n v^n\right) = \sum_{n\geq 0} \psi_+(w_n)v^n,$$

where $w_0, w_1, w_2, \ldots \in V_u(\mathfrak{g}_+)$. We define the $\mathbf{C}[u][[v]]$-module A_+ by

$$A_+ = \psi_+(V_u(\mathfrak{g}_+)[[v]]) \subset U_{u,v}(\mathfrak{g}_+). \tag{6.4}$$

The remaining part of Section 6 is concerned with the study of A_+. The relevant results are stated in Theorem 6.9.

We choose a $\mathbf{C}[[h]]$-linear isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ such that $\alpha_+(1) = 1$ and $\alpha_+ \equiv \mathrm{id}$ modulo h. Such an isomorphism exists by Section 5.4 (ii). Extending the scalars, we get a $\mathbf{C}[[u,v]]$-linear isomorphism $\widetilde{\alpha}_+ : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u,v]]$ such that $\widetilde{\alpha}_+ \equiv \mathrm{id}$ modulo uv. Let us consider the composed map

$$p_v : U_{u,v}(\mathfrak{g}_+) \xrightarrow{\widetilde{\alpha}_+} U(\mathfrak{g}_+)[[u,v]] \to U(\mathfrak{g}_+)[[u]],$$

where the second map is the projection $v \mapsto 0$. We equip $U(\mathfrak{g}_+)[[u]]$ with the power series multiplication and the comultiplication (2.4).

Lemma 6.7. *The map $p_v : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u]]$ is a morphism of bialgebras.*

Proof. The multiplication and the comultiplication of $U_h(\mathfrak{g}_+)$ transfer, via the $\mathbf{C}[[h]]$-linear isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$, to a multiplication μ_h and a comultiplication Δ_h on $U(\mathfrak{g}_+)[[h]]$. Expanding μ_h and Δ_h into formal power series, we obtain

$$\begin{aligned} \mu_h &= \mu_0 + h\mu_1 + h^2\mu_2 + \cdots \quad \text{and} \\ \Delta_h &= \Delta_0 + h\Delta_1 + h^2\Delta_2 + \cdots, \end{aligned} \tag{6.5}$$

where $\mu_i : U(\mathfrak{g}_+)^{\otimes 2} \to U(\mathfrak{g}_+)$ and $\Delta_i : U(\mathfrak{g}_+) \to U(\mathfrak{g}_+)^{\otimes 2}$ are linear maps for all $i = 0, 1, \ldots$ Since $U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+) = U(\mathfrak{g}_+)$ as bialgebras, we see that μ_0 and Δ_0 are the standard multiplication and comultiplication of $U(\mathfrak{g}_+)$.

The multiplication and the comultiplication of $U_{u,v}(\mathfrak{g}_+)$ give rise, via $\widetilde{\alpha}_+$, to a multiplication $\mu_{u,v}$ and a comultiplication $\Delta_{u,v}$ on $U(\mathfrak{g}_+)[[u,v]]$ of the form

$$\begin{aligned} \mu_{u,v} &= \mu_0 + uv\mu_1 + u^2v^2\mu_2 + \cdots \quad \text{and} \\ \Delta_{u,v} &= \Delta_0 + uv\Delta_1 + u^2v^2\Delta_2 + \cdots, \end{aligned} \tag{6.6}$$

where the maps μ_i and Δ_i are the same as in (6.5). It follows that p_v is a morphism of bialgebras, where $U(\mathfrak{g}_+)[[u]]$ is equipped with μ_0 and Δ_0. □

The following result is an elaboration of Lemma 5.6 (a).

Lemma 6.8. (a) *For any d-tuple* $\underline{j} = (j_1, \dots, j_d)$*, the element* $\psi_+(u^{|\underline{j}|}\, x_{\underline{j}})$ *defined by* (6.3) *belongs to* $u^{|\underline{j}|}\, U_{u,v}(\mathfrak{g}_+)$ *and*

$$p_v\big(\psi_+(u^{|\underline{j}|}\, x_{\underline{j}})\big) = u^{|\underline{j}|}\, x_{\underline{j}} \in U(\mathfrak{g}_+)[[u]].$$

(b) *We have* $p_v(A_+) = V_u(\mathfrak{g}_+)$ *and* $p_v \circ \psi_+ : V_u(\mathfrak{g}_+)[[v]] \to V_u(\mathfrak{g}_+)$ *is the projection sending* v *to* 0.

Proof. (a) By multiplicativity of p_v, it suffices to prove that $v^{-1}\rho_+(\widetilde{f}_x)$ belongs to $u\, U_{u,v}(\mathfrak{g}_+)$ and that $p_v\big(v^{-1}\rho_+(\widetilde{f}_x)\big) = ux$ for any $x \in \mathfrak{g}_+$. The first assertion follows from Lemma 6.5.

Let us compute $p_v\big(v^{-1}\rho_+(\widetilde{f}_x)\big)$. Recall the isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ from Section 5.5 and the isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ defined above. Let $X_i \in U_h(\mathfrak{g}_+)$ be defined by $X_i = \alpha_+^{-1}(x_i)$ and $Y_i \in U_h(\mathfrak{g}_-)$ by $Y_i = \alpha_-^{-1}(y_i)$, where $(x_1, \dots, x_d)$ is the fixed basis of $\mathfrak{g}_+$ and $(y_1, \dots, y_d)$ is the dual basis. By (5.10),

$$f_x(Y_i) = \langle x, \pi_-\alpha_-(Y_i)\rangle = \langle x, \pi_-(y_i)\rangle = \langle x, y_i\rangle. \tag{6.7}$$

It follows from (5.8) that

$$R_h = 1\otimes 1 + h\sum_{i=1}^{d} X_i\otimes Y_i + h^2 Z, \tag{6.8}$$

where $Z \in U_h(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[h]]}\, U_h(\mathfrak{g}_-)$. By extension of scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$, we get

$$R_{u,v} = 1\otimes 1 + uv\sum_{i=1}^{d} \widetilde{X}_i\otimes \widetilde{Y}_i + u^2v^2 \widetilde{Z}, \tag{6.9}$$

where $\widetilde{X}_i \in U_{u,v}(\mathfrak{g}_+)$, $\widetilde{Y}_i \in U_{u,v}(\mathfrak{g}_-)$, and $\widetilde{Z} \in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$. Moreover, using the definition of p_v and Formula (6.7), we have

$$p_v(\widetilde{X}_i) = x_i, \quad \text{and} \quad \widetilde{f}_x(\widetilde{Y}_i) = \langle x, y_i\rangle. \tag{6.10}$$

Applying $\mathrm{id}\otimes\widetilde{f}_x$ to $R_{u,v}$ and using (6.9) and (6.10), we obtain

$$\begin{aligned}\rho_+(\widetilde{f}_x) &= (\mathrm{id}\otimes\widetilde{f}_x)(R_{u,v})\\ &= \widetilde{f}_x(1) + uv\sum_{i=1}^{d} \widetilde{X}_i \widetilde{f}_x(\widetilde{Y}_i) + u^2v^2(\mathrm{id}\otimes\widetilde{f}_x)(\widetilde{Z})\\ &= uv\sum_{i=1}^{d} \langle x, y_i\rangle\, \widetilde{X}_i + u^2v^2(\mathrm{id}\otimes\widetilde{f}_x)(\widetilde{Z}).\end{aligned}$$

Therefore,

$$v^{-1}\rho_+(\widetilde{f}_x) = u\sum_{i=1}^{d} \langle x, y_i\rangle \, \widetilde{X}_i + u^2 v(\mathrm{id}\otimes\widetilde{f}_x)(\widetilde{Z}).$$

This implies, in view of (6.10),

$$p_v\big(v^{-1}\rho_+(\widetilde{f}_x)\big) = u\sum_{i=1}^{d} \langle x, y_i\rangle \, p_v(\widetilde{X}_i) = u\sum_{i=1}^{d} \langle x, y_i\rangle \, x_i = ux.$$

(b) It follows from Part (a) and the definition of A_+. □

Theorem 6.9. (a) *The map* $\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to A_+$ *is an isomorphism of* $\mathbf{C}[u][[v]]$*-modules.*
(b) A_+ *is a subalgebra of* $U_{u,v}(\mathfrak{g}_+)$*.*
(c) *The algebra* A_+ *is independent of the choices made in Section* 5.5*.*

Proof. (a) The map ψ_+ is surjective by definition of A_+. Let us check that it is injective. Let $w = \sum_{n\geq 0} w_n v^n \in V_u(\mathfrak{g}_+)[[v]]$ with w_0, w_1, $w_2, \ldots \in V_u(\mathfrak{g}_+)$. Assume that $w \neq 0$. Take the minimal $N \geq 0$ such that $w_N \neq 0$ and define w' by $w = v^N w'$. By Lemma 6.8, we have $p_v(\psi_+(w')) = w_N \neq 0$, hence $\psi_+(w') \neq 0$. As $A_+ \subset U_{u,v}(\mathfrak{g}_+)$ has no v-torsion, we see that $\psi_+(w) = v^N \psi_+(w') \neq 0$.

(b) Let us check that $\psi_+(u^{|\underline{i}|}x_{\underline{i}})\,\psi_+(u^{|\underline{j}|}x_{\underline{j}}) \in A_+$ for all d-tuples $\underline{i} = (i_1, \ldots, i_d)$ and $\underline{j} = (j_1, \ldots, j_d)$. Since $\rho_+ : U_h^*(\mathfrak{g}_-) \to U_h(\mathfrak{g}_+)$ is an anti-morphism of algebras, the product

$$\rho_+(f_{x_1})^{i_1} \ldots \rho_+(f_{x_d})^{i_d} \rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d}$$

belongs to the image of ρ_+. Therefore, by Lemma 5.6 (b-c), it can be expanded as

$$\rho_+(f_{x_1})^{i_1} \ldots \rho_+(f_{x_d})^{i_d} \rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d} = \sum_{n\geq 0} \left(\sum_{|\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)} t_{\underline{k}} \right) h^n,$$

where $\lambda_{\underline{k}}^{(n)} \in \mathbf{C}$. By Lemma 5.6 (a),

$$\begin{aligned} &\rho_+(f_{x_1})^{i_1} \ldots \rho_+(f_{x_d})^{i_d} \rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d} \\ &\quad = \sum_{n\geq 0;\, \underline{k},\, |\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)} \rho_+(f_{x_1})^{k_1} \ldots \rho_+(f_{x_d})^{k_d} h^{n-|\underline{k}|}. \end{aligned}$$

By extension of scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$, we have $\widetilde{\rho_+(f_{x_i})} = \rho_+(\widetilde{f}_{x_i})$. Therefore,

$$\rho_+(\widetilde{f}_{x_1})^{i_1} \ldots \rho_+(\widetilde{f}_{x_d})^{i_d} \rho_+(\widetilde{f}_{x_1})^{j_1} \ldots \rho_+(\widetilde{f}_{x_d})^{j_d}$$

$$= \sum_{n\geq 0;\, \underline{k},\, |\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)} \rho_+(\widetilde{f}_{x_1})^{k_1} \dots \rho_+(\widetilde{f}_{x_d})^{k_d} u^{n-|\underline{k}|} v^{n-|\underline{k}|}.$$

Using (6.3), we obtain

$$v^{|\underline{i}|+|\underline{j}|}\, \psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}}) = \sum_{n\geq 0;\, \underline{k},\, |\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)} \psi_+(u^{|\underline{k}|}x_{\underline{k}})\, u^{n-|\underline{k}|}\, v^n$$

$$= \sum_{n\geq 0} \left(\sum_{\underline{k};\, |\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)} u^{n-|\underline{k}|}\, \psi_+(u^{|\underline{k}|}x_{\underline{k}}) \right) v^n.$$

Thus, $v^{|\underline{i}|+|\underline{j}|}\, \psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}})$ is a formal power series in v whose coefficients belong to the $\mathbf{C}[u]$-linear span of the elements $\psi_+(u^{|\underline{k}|}x_{\underline{k}})$. Hence, $v^{|\underline{i}|+|\underline{j}|}\, \psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}}) \in A_+$. Applying Lemma 6.10 below $|\underline{i}| + |\underline{j}|$ times, we obtain $\psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}}) \in A_+$.

(c) The definition of A_+ in Section 6.6 was based on the choice of a $\mathbf{C}[[h]]$-linear isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ such that $\alpha_-(1) = 1$ and $\alpha_- \equiv \mathrm{id}$ modulo h, of a $\mathbf{C}$-linear projection $\pi_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-)$ that restricts to the identity on $U^1(\mathfrak{g}_-)$, and of a basis $(x_1, \dots, x_d)$ of $\mathfrak{g}_+$. We have to check that A_+ is independent of these choices as a subset of $U_{u,v}(\mathfrak{g}_+)$.

(i) Suppose that we take another $\mathbf{C}[[h]]$-linear isomorphism $\alpha'_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ such that $\alpha'_-(1) = 1$ and $\alpha'_- \equiv \mathrm{id}$ modulo h. This gives us a new linear form $f'_x : U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ and, by extension of scalars, a new linear form $\widetilde{f}'_x : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$ for all $x \in \mathfrak{g}_+$. Lemma 6.5 also holds for $\widetilde{f}'_x$. By Part (b) it is enough to check that $v^{-1}\, \rho_+(\widetilde{f}'_x)$ belongs to A_+.

Since $\alpha'_- \equiv \alpha_-$ modulo h, we have $f'_x \equiv f_x$ modulo h. By the proof of Lemma 5.6 (c), we see that

$$f'_x = f_x + \sum_{n\geq 1} h^n \left(\sum_{\underline{j}} \lambda_{\underline{j}}^{(n)}\, f_{x_d}^{j_d} \dots f_{x_1}^{j_1} \right), \tag{6.11}$$

where $\lambda_{\underline{j}}^{(n)} \in \mathbf{C}$ are indexed by a nonnegative integer n and a d-tuple $\underline{j} = (j_1, \dots, j_d)$ of nonnegative integers. Applying ρ_+, we get

$$\rho_+(f'_x) = \rho_+(f_x) + \sum_{n\geq 1} h^n \left(\sum_{\underline{j}} \lambda_{\underline{j}}^{(n)}\, \rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d} \right).$$

By extension of scalars, we have

$$\rho_+(\widetilde{f}'_x) = \rho_+(\widetilde{f}_x) + \sum_{n\geq 1} u^n\, v^n \left(\sum_{\underline{j}} \lambda_{\underline{j}}^{(n)}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \right).$$

Using (6.3), we obtain

$$\begin{aligned} v^{-1}\,\rho_+(\widetilde{f}'_x) &= v^{-1}\,\rho_+(\widetilde{f}_x) + \sum_{n\geq 1} u^n\, v^{n-1} \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \right) \\ &= \psi_+(ux) + \sum_{n\geq 1} u^n\, v^{n-1} \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, v^{|\underline{j}|}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \right) \\ &= \psi_+(ux) + \sum_{k\geq 1} v^k \left(\sum_{\underline{j};\, |\underline{j}|\leq k} \lambda^{(n)}_{\underline{j}}\, u^{k-|\underline{j}|+1}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \right). \end{aligned}$$

This shows that $v^{-1}\,\rho_+(\widetilde{f}'_x)$ is a formal power series in v whose coefficients belong to the $\mathbf{C}[u]$-linear span of the elements $\psi_+(u^{|\underline{j}|} x_{\underline{j}})$. Hence, $v^{-1}\,\rho_+(\widetilde{f}'_x) \in A_+$.

(ii) Suppose now that we take another projection $\pi'_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-)$ whose restriction to $U^1(\mathfrak{g}_-)$ is the identity. We denote by f'_x the new linear form $U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ obtained by using π'_-. By extension of scalars, we obtain a new linear form $\widetilde{f}'_x : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$ for $x \in \mathfrak{g}_+$.

Since $\pi'_- - \pi_- = 0$ on $U^1(\mathfrak{g}_-)$, it follows from the proof of Lemma 5.6 (c) that

$$(6.12) \qquad f'_x = f_x + \sum_{|\underline{j}|\geq 2} \lambda^{(0)}_{\underline{j}}\, f^{j_d}_{x_d} \dots f^{j_1}_{x_1} + \sum_{n\geq 1} h^n \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, f^{j_d}_{x_d} \dots f^{j_1}_{x_1} \right),$$

where $\lambda^{(n)}_{\underline{j}} \in \mathbf{C}$ are scalars. Note the difference with (6.11): In (6.12) there are extra terms of degree 0 in h. Nevertheless, the same arguments as in Part (i) allow us to conclude.

(iii) Since $x \mapsto f_x$ is linear, it follows that A_+ is independent of the basis in $\mathfrak{g}_+$. □

Lemma 6.10. *We have $A_+ \cap vU_{u,v}(\mathfrak{g}_+) = vA_+$.*

Lemma 6.10 will be proved in Section 7.7.

7. Bialgebra structure on A_+.

In this section we establish that A_+ has a $\mathbf{C}[u][[v]]$-bialgebra structure. We begin with a $\mathbf{C}[[u,v]]$-subalgebra $\widehat{A}_+$ of $U_{u,v}(\mathfrak{g}_+)$ in which A_+ sits as a dense subalgebra.

7.1. The Algebra $\widehat{A}_+$. Using the comultiplication $\Delta_{u,v}$ of $U_{u,v}(\mathfrak{g}_+)$ and proceeding as in Section 3.1, we obtain $\mathbf{C}[[u,v]]$-linear maps $\delta^n : U_{u,v}(\mathfrak{g}_+) \to$

$U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$ for all $n \geq 1$. Formulas (3.1)–(3.5) hold in this setting. We define a $\mathbf{C}[[u,v]]$-submodule $\widehat{A}_+$ of $U_{u,v}(\mathfrak{g}_+)$ by

$$(7.1) \qquad \widehat{A}_+ = \left\{ a \in U_{u,v}(\mathfrak{g}_+) \ \mid \ \delta^n(a) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n} \ \text{ for all } n \geq 1 \right\}.$$

It follows from (3.3) and (3.4) that $\widehat{A}_+$ is a subalgebra of $U_{u,v}(\mathfrak{g}_+)$.

Lemma 7.2. *$\widehat{A}_+$ is a topologically free $\mathbf{C}[[u,v]]$-module.*

Proof. By Lemma 4.3 it is enough to check that $\widehat{A}_+$ is a u-torsion-free, v-torsion-free, admissible, separated, and complete $\mathbf{C}[[u,v]]$-module.

We use the fact that $\widehat{A}_+$ is a submodule of the topologically free module $U_{u,v}(\mathfrak{g}_+)$. Since the latter is separated, u-torsion-free, and v-torsion-free, so is any of its submodules. We are left with checking admissibility and completeness.

Admissibility: Let $a, a_1, a_2 \in \widehat{A}_+$ be such that $a = ua_1 = va_2$. Since $U_{u,v}(\mathfrak{g}_+)$ is topologically free, hence admissible, there exists $a_0 \in U_{u,v}(\mathfrak{g}_+)$ such that $a = uva_0$. We shall prove that $a_0 \in \widehat{A}_+$, i.e., that $\delta^n(a_0) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. Since $u(va_0 - a_1) = 0$ and $U_{u,v}(\mathfrak{g}_+)$ has no u-torsion, we have $a_1 = va_0$. Therefore, $v\delta^n(a_0) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. In other words, $v\delta^n(a_0)$ is divisible both by v and by u^n in $U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$, which is topologically free. By an observation in Section 4.2, $v\delta^n(a_0) = u^n v Z$ for some $Z \in U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. Since $U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$ has no v-torsion, $\delta^n(a_0) = u^n Z$.

Completeness: Let $(a_n)_{n\geq 0}$ be a sequence of elements of $\widehat{A}_+$ such that for all $n \geq 0$ the image of a_{n+1} in $\widehat{A}_+/(u,v)^{n+1}$ maps onto the image of a_n in $\widehat{A}_+/(u,v)^n$. Since $U_{u,v}(\mathfrak{g}_+)$ is complete, it contains an element a such that $a - a_n \in (u,v)^n U_{u,v}(\mathfrak{g}_+)$ for all $n \geq 0$. We shall show that $a \in \widehat{A}_+$, i.e., that $\delta^p(a)$ is divisible by u^p for all $p \geq 1$. For any $n \geq p$,

$$\delta^p(a) - \delta^p(a_n) \in (u,v)^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p} \quad \text{and} \quad \delta^p(a_n) \in u^p U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p},$$

which implies that $\delta^p(a) \in u^p U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p} + (u,v)^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p}$. Consequently, $\delta^p(a)$ is divisible by u^p in $\varprojlim_n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p}/(u,v)^n = U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p}$. □

Consider the morphism $p_v : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u]]$ of Lemma 6.7. Recall from (3.10) the algebra

$$\widehat{V}_u(\mathfrak{g}_+) = \left\{ \sum_{m \geq 0} a_m \, u^m \ \mid \ a_m \in U^m(\mathfrak{g}_+) \text{ for all } m \geq 0 \right\} \subset U(\mathfrak{g}_+)[[u]].$$

Lemma 7.3. (a) *The morphism p_v sends $\widehat{A}_+$ into $\widehat{V}_u(\mathfrak{g}_+)$.*

(b) *We have* $\mathrm{Ker}\big(p_v : \widehat{A}_+ \to \widehat{V}_u(\mathfrak{g}_+)\big) = \widehat{A}_+ \cap v\, U_{u,v}(\mathfrak{g}_+) = v\, \widehat{A}_+$.

Proof. (a) By (3.1) and (6.6) the map δ^n for $U_{u,v}(\mathfrak{g}_+)$ is of the form

$$\delta^n = \delta_0^n + uv\delta_1^n,$$

where δ_0^n is obtained by (3.1) from the standard comultiplication Δ of $U(\mathfrak{g}_+)[[u]]$. Hence, $p_v^{\otimes n}\delta^n = \delta_0^n p_v$. Therefore, Part (a) follows from the definitions and Proposition 3.8.

(b) Let $a \in \widehat{A}_+$ and $b \in U_{u,v}(\mathfrak{g}_+)$ be such that $a = vb$. We have to check that $b \in \widehat{A}_+$. For any $n \geq 1$, the element $\delta^n(a) = v\delta^n(b)$ is divisible both by v and by u^n in $U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. Since the latter is topologically free, there exists $Z \in U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$ such that $v\delta^n(b) = u^n vZ$. Hence, $\delta^n(b) = u^n Z$, which shows that $b \in \widehat{A}_+$. □

Lemma 7.4. *We have $A_+ \subset \widehat{A}_+$.*

Proof. Let us first prove that $\psi_+(ux) = v^{-1}\rho_+(\widetilde{f}_x)$ belongs to $\widehat{A}_+$ for all $x \in \mathfrak{g}_+$. Given $n \geq 1$, we have to check that $\delta^n(v^{-1}\rho_+(\widetilde{f}_x))$ is divisible by u^n. Formula $(\Delta_{u,v}\otimes\mathrm{id})(R) = R_{13}R_{23}$ for $R = R_{u,v}$ implies

$$(\Delta_{u,v}^n\otimes\mathrm{id})(R) = R_{1,n+1}R_{2,n+1}\cdots R_{n-1,n+1}R_{n,n+1}.$$

Therefore,

$$(\delta^n\otimes\mathrm{id})(R) = (R_{1,n+1}-1)(R_{2,n+1}-1)\cdots(R_{n-1,n+1}-1)(R_{n,n+1}-1).$$

Since $R = 1\otimes 1 + uvR'$, we have

$$(\delta^n\otimes\mathrm{id})(R) = u^n v^n\, R'_{1,n+1}R'_{2,n+1}\cdots R'_{n-1,n+1}R'_{n,n+1}.$$

It follows that

$$\begin{aligned}
&\delta^n(\rho_+(\widetilde{f}_x))\\
&= \delta^n\big((\mathrm{id}\otimes\widetilde{f}_x)(R)\big)\\
&= (\delta^n\otimes\widetilde{f}_x)(R)\\
&= (\mathrm{id}\otimes\widetilde{f}_x)\big((\delta^n\otimes\mathrm{id})(R)\big)\\
&= u^n v^n(\mathrm{id}\otimes\widetilde{f}_x)(R'_{1,n+1}R'_{2,n+1}\cdots R'_{n-1,n+1}R'_{n,n+1}) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}.
\end{aligned}$$

Hence, for $n \geq 1$,

$$\begin{aligned}
&\delta^n(v^{-1}\rho_+(\widetilde{f}_x))\\
&= u^n v^{n-1}(\mathrm{id}\otimes\widetilde{f}_x)(R'_{1,n+1}R'_{2,n+1}\cdots R'_{n-1,n+1}R'_{n,n+1}) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}.
\end{aligned}$$

Since $\widehat{A}_+$ is a subalgebra of $U_{u,v}(\mathfrak{g}_+)$, $\psi_+(u^{|\underline{j}|}\, x_{\underline{j}}) \in \widehat{A}_+$ for any d-tuple $\underline{j}$. Since $\widehat{A}_+$ is topologically free (hence complete) by Lemma 7.2, the map

$$\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to U_{u,v}(\mathfrak{g}_+)$$

takes its values in $\widehat{A}_+$. We conclude with Formula (6.4). □

Lemma 7.5. *The* $\mathbf{C}[u][[v]]$*-linear map* $\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to U_{u,v}(\mathfrak{g}_+)$ *extends to a* $\mathbf{C}[[u,v]]$*-linear map* $\widehat{\psi}_+ : \widehat{V}_u(\mathfrak{g}_+)[[v]] \to U_{u,v}(\mathfrak{g}_+)$*. The map* $\widehat{\psi}_+$ *is injective, its image is* $\widehat{A}_+$*:*

$$\widehat{\psi}_+(\widehat{V}_u(\mathfrak{g}_+)[[v]]) = \widehat{A}_+,$$

and $p_v \circ \widehat{\psi}_+ : \widehat{V}_u(\mathfrak{g}_+)[[v]] \to \widehat{V}_u(\mathfrak{g}_+)$ *is the projection sending* v *to* 0*.*

Proof. Any element of $\widehat{V}_u(\mathfrak{g}_+)$ is of the form $w = \sum_{m\geq 0} a_m\, u^m$, where

$$a_m = \sum_{\underline{j};\, |\underline{j}|\leq m} \nu_{\underline{j}}^{(m)} x_{\underline{j}}$$

and $\nu_{\underline{j}}^{(m)} \in \mathbf{C}$. By Lemma 6.8 (a), the element $\psi_+(a_m\, u^m)$ belongs to $u^m U_{u,v}(\mathfrak{g}_+)$. Since $U_{u,v}(\mathfrak{g}_+)$ is topologically free over $\mathbf{C}[[u,v]]$, the series $\sum_{m\geq 0} \psi_+(a_m\, u^m)$ converges in $U_{u,v}(\mathfrak{g}_+)$, so that we can define

$$\widehat{\psi}_+(w) = \sum_{m\geq 0} \psi_+(a_m\, u^m).$$

By Lemma 7.4 and (7.1), for each $m \geq 0$, $\delta^n(\psi_+(a_m\, u^m))$ is divisible by u^n for all $n \geq 1$. It follows that $\delta^n(\widehat{\psi}_+(w))$ is also divisible by u^n for all $n \geq 1$. Therefore, $\widehat{\psi}_+(w) \in \widehat{A}_+$. Now any element of $\widehat{V}_u(\mathfrak{g}_+)[[v]]$ is of the form $\sum_{n\geq 0} w_n v^n$, where $w_n \in \widehat{V}_u(\mathfrak{g}_+)$ for all $n \geq 0$. Clearly, $\sum_{n\geq 0} \widehat{\psi}_+(w_n) v^n$ converges in $\widehat{A}_+$. We set $\widehat{\psi}_+\left(\sum_{n\geq 0} w_n v^n\right) = \sum_{n\geq 0} \widehat{\psi}_+(w_n) v^n$.

Lemma 6.8 (b) implies that $p_v \circ \widehat{\psi}_+$ is the identity on $\widehat{V}_u(\mathfrak{g}_+)$. Proceeding as in the proof of Theorem 6.9 (a), we see that $\widehat{\psi}_+$ is injective on $\widehat{V}_u(\mathfrak{g}_+)[[v]]$.

It remains to prove that the image of $\widehat{\psi}_+$ is $\widehat{A}_+$. For $a \in \widehat{A}_+$, set $w_0 = p_v(a) \in \widehat{V}_u(\mathfrak{g}_+)$, cf. Lemma 7.3 (a). Viewing w_0 as a constant formal power series in $\widehat{V}_u(\mathfrak{g}_+)[[v]]$, we consider the element $a - \widehat{\psi}_+(w_0) \in \widehat{A}_+$; it clearly sits in the kernel of p_v, which is $v\widehat{A}_+$ by Lemma 7.3 (b). Therefore, there exists $a_1 \in \widehat{A}_+$ such that $a - \widehat{\psi}_+(w_0) = va_1$. Similarly, there exist $w_1 \in \widehat{V}_u(\mathfrak{g}_+)$ and $a_2 \in \widehat{A}_+$ such that $a_1 - \widehat{\psi}_+(w_1) = va_2$. Repeating this construction and using the separatedness of $\widehat{A}_+$, we obtain an element $w = \sum_{n\geq 0} w_n v^n \in \widehat{V}_u(\mathfrak{g}_+)[[v]]$ such that $a = \widehat{\psi}_+(w)$. □

Corollary 7.6. *We have*

$$A_+ \cap v\widehat{A}_+ = vA_+ \quad \text{and} \quad A_+ \cap u\widehat{A}_+ = uA_+.$$

Proof. By Theorem 6.9 (a) and Lemma 7.5, it is enough to check that

$$V_u(\mathfrak{g}_+)[[v]] \cap v\widehat{V}_u(\mathfrak{g}_+)[[v]] = vV_u(\mathfrak{g}_+)[[v]]$$

and

$$V_u(\mathfrak{g}_+)[[v]] \cap u\widehat{V}_u(\mathfrak{g}_+)[[v]] = uV_u(\mathfrak{g}_+)[[v]].$$

The former is clear; the latter is a consequence of $V_u(\mathfrak{g}_+) \cap u\widehat{V}_u(\mathfrak{g}_+) = uV_u(\mathfrak{g}_+)$, which is easy to check. □

7.7. Proof of Lemma 6.10. It is a consequence of Lemmas 7.3 (b) and 7.4, and the first inclusion of Corollary 7.6. □

We can now show that A_+ has a bialgebra structure. (For the definition of $\widehat{\otimes}_{\mathbf{C}[u][[v]]}$ and $\widehat{\otimes}_{\mathbf{C}[[u,v]]}$, see Sections 1.3 and 4.4.)

Proposition 7.8. (a) *We have the inclusions*

$$A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+ \subset \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+ \subset U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_+).$$

(b) *If $\Delta_{u,v}$ denotes the comultiplication of $U_{u,v}(\mathfrak{g}_+)$, then*

$$\Delta_{u,v}\big(A_+\big) \subset A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+$$

and

$$\Delta_{u,v}\big(\widehat{A}_+\big) \subset \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+.$$

Proof. (a) The inclusion $\widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+ \subset U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_+)$ follows from Proposition 6.2 (a), Lemma 7.2, and Lemma 4.5 (b).

Let us consider the first inclusion. By Theorem 6.9 (a) and Lemma 7.5, it is enough to prove that the natural map

$$V_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, V_u(\mathfrak{g}_+)[[v]] \to \widehat{V}_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{V}_u(\mathfrak{g}_+)[[v]] \tag{7.2}$$

induced by the inclusion $V_u(\mathfrak{g}_+)[[v]] \subset \widehat{V}_u(\mathfrak{g}_+)[[v]]$ is injective. By definition of $\widehat{\otimes}_{\mathbf{C}[u][[v]]}$, we see that

$$\begin{aligned} &V_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, V_u(\mathfrak{g}_+)[[v]] \\ &= \big(V_u(\mathfrak{g}_+) \otimes_{\mathbf{C}[u]} V_u(\mathfrak{g}_+)\big)[[v]] = V_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\widehat{V}_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{V}_u(\mathfrak{g}_+)[[v]] \\ &= \varprojlim_n \Big(\widehat{V}_u(\mathfrak{g}_+)[[v]]/(u,v)^n \otimes_{\mathbf{C}[[u,v]]/(u,v)^n} \widehat{V}_u(\mathfrak{g}_+)[[v]]/(u,v)^n\Big) \\ &= \varprojlim_n \Big(V_u(\mathfrak{g}_+)[v]/(u,v)^n \otimes_{\mathbf{C}[u,v]/(u,v)^n} V_u(\mathfrak{g}_+)[v]/(u,v)^n\Big) \\ &= \varprojlim_n \Big(V_u(\mathfrak{g}_+) \otimes_{\mathbf{C}[u]} V_u(\mathfrak{g}_+)\Big)[v]/(u,v)^n \\ &= \varprojlim_n V_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[v]/(u,v)^n \\ &= \varprojlim_n \widehat{V}_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]]/(u,v)^n \\ &= \widehat{V}_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]]. \end{aligned}$$

The last equality holds because $\widehat{V}_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]]$ is a topologically free $\mathbf{C}[[u,v]]$-module. The injectivity of (7.2) follows.

(b) In order to prove that the image of A_+ under $\Delta_{u,v}$ lies in the subalgebra $A_+ \mathbin{\widehat{\otimes}}_{\mathbf{C}[u][[v]]} A_+$, it is enough to show that $\Delta_{u,v}\big(\psi_+(ux)\big)$ belongs to this subalgebra for all $x \in \mathfrak{g}_+$.

Let us consider the linear form $f_x \in U_h^*(\mathfrak{g}_-)$ of Section 5.5. Since $\rho_+ : U_h^*(\mathfrak{g}_-) \to U_h(\mathfrak{g}_+)$ is a morphism of coalgebras (see [**EK96**, Proposition 4.8]), we have $\Delta_h(\rho_+(f_x)) \in \operatorname{Im}\rho_+ \mathbin{\widehat{\otimes}}_{\mathbf{C}[[h]]} \operatorname{Im}\rho_+$.

It follows from Lemma 5.6 that for any element $a \in U_h(\mathfrak{g}_+) \mathbin{\widehat{\otimes}}_{\mathbf{C}[[h]]} U_h(\mathfrak{g}_+)$, there exists a unique family $\nu^{(n)}_{\underline{j},\underline{k}} \in \mathbf{C}$ indexed by a nonnegative integer n and two d-tuples $\underline{j}$ and $\underline{k}$ such that

$$a = \sum_{n\geq 0} \left(\sum_{|\underline{j}|+|\underline{k}|\leq c(n)} \nu^{(n)}_{\underline{j},\underline{k}}\, t_{\underline{j}} \otimes t_{\underline{k}} \right) h^n,$$

where $c(n)$ is an integer depending on a and n. If, in addition, $a \in \operatorname{Im}\rho_+ \mathbin{\widehat{\otimes}}_{\mathbf{C}[[h]]} \operatorname{Im}\rho_+$, then $c(n) = n$, i.e., $\nu^{(n)}_{\underline{j},\underline{k}} = 0$ whenever $n < |\underline{j}| + |\underline{k}|$. Applying this to $a = \Delta_h(\rho_+(f_x))$, we obtain a family $\nu^{(n)}_{\underline{j},\underline{k}} \in \mathbf{C}$ as above such that

$$\begin{aligned}
&\Delta_h(\rho_+(f_x)) \\
&= \sum_{n\geq 0} \left(\sum_{|\underline{j}|+|\underline{k}|\leq n} \nu^{(n)}_{\underline{j},\underline{k}}\, t_{\underline{j}} \otimes t_{\underline{k}} \right) h^n \\
&= \sum_{\substack{n\geq 0, \underline{j}, \underline{k} \\ |\underline{j}|+|\underline{k}|\leq n}} \nu^{(n)}_{\underline{j},\underline{k}}\, \rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d} \otimes \rho_+(f_{x_1})^{k_1} \dots \rho_+(f_{x_d})^{k_d}\, h^{n-|\underline{j}|-|\underline{k}|},
\end{aligned}$$

where $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$. Extending the scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$ and using (6.3), we obtain

$$\begin{aligned}
&\Delta_{u,v}\big(\rho_+(\widetilde{f}_x)\big) \\
&= \sum_{\substack{n\geq 0;\, \underline{j}, \underline{k} \\ |\underline{j}|+|\underline{k}|\leq n}} \nu^{(n)}_{\underline{j},\underline{k}}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \otimes \rho_+(\widetilde{f}_{x_1})^{k_1} \dots \rho_+(\widetilde{f}_{x_d})^{k_d} (uv)^{n-|\underline{j}|-|\underline{k}|} \\
&= \sum_{n\geq 0;\, \underline{j},\underline{k},\, |\underline{j}|+|\underline{k}|\leq n} \nu^{(n)}_{\underline{j},\underline{k}}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \otimes \psi_+(u^{|\underline{k}|} x_{\underline{k}})\, u^{n-|\underline{j}|-|\underline{k}|}\, v^n \\
&= \sum_{n\geq 0} \left(\sum_{\underline{j},\underline{k};\, |\underline{j}|+|\underline{k}|\leq n} \nu^{(n)}_{\underline{j},\underline{k}}\, u^{n-|\underline{j}|-|\underline{k}|}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \otimes \psi_+(u^{|\underline{k}|} x_{\underline{k}}) \right) v^n.
\end{aligned}$$

Therefore, $v\,\Delta_{u,v}\big(\psi_+(ux)\big) = \Delta_{u,v}\big(\rho_+(\widetilde{f_x})\big)$ is a formal power series in v whose coefficients belong to the $\mathbf{C}[u]$-linear span of the elements $\psi_+(u^{|\underline{j}|}x_{\underline{j}}) \otimes \psi_+(u^{|\underline{k}|}x_{\underline{k}})$. Hence, $v\,\Delta_{u,v}\big(\psi_+(ux)\big)$ belongs to $A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+$.

The element $\Delta_{u,v}\big(\psi_+(ux)\big) \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_+)$ can be expanded as

$$\Delta_{u,v}\big(\psi_+(ux)\big) = \sum_i a_i \otimes z_i,$$

where $(a_i)_i$ is a basis of the topologically free $\mathbf{C}[[u,v]]$-module $U_{u,v}(\mathfrak{g}_+)$ and $z_i \in U_{u,v}(\mathfrak{g}_+)$. Since

$$\sum_i a_i \otimes v z_i = v\,\Delta_{u,v}\big(\psi_+(ux)\big) \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+,$$

we have $vz_i \in \widehat{A}_+$ for all i. By Lemma 7.3 (b) it follows that $z_i \in \widehat{A}_+$ for all i. Now taking a basis $(b_j)_j$ of the topologically free $\mathbf{C}[[u,v]]$-module $\widehat{A}_+$, we can write

$$\Delta_{u,v}\big(\psi_+(ux)\big) = \sum_j z'_j \otimes b_j,$$

where $z'_j \in U_{u,v}(\mathfrak{g}_+)$. Since $\sum_j v z'_j \otimes b_j = v\,\Delta_{u,v}\big(\psi_+(ux)\big) \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+$, we have $vz'_j \in \widehat{A}_+$, hence $z'_j \in \widehat{A}_+$ for all j. Therefore,

$$\Delta_{u,v}\big(\psi_+(ux)\big) \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+.$$

The desired inclusion $\Delta_{u,v}\big(\psi_+(ux)\big) \in A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+$ follows from

$$A_+^{\widehat{\otimes}2} \cap v\Big(\widehat{A}_+^{\widehat{\otimes}2}\Big) = v\Big(A_+^{\widehat{\otimes}2}\Big). \tag{7.3}$$

In view of Theorem 6.9 (a) and Lemma 7.5, Equality (7.3) is equivalent to

$$V_u(\mathfrak{g}_+)[[v]]^{\widehat{\otimes}2} \cap v\Big(\widehat{V}_u(\mathfrak{g}_+)[[v]]^{\widehat{\otimes}2}\Big) = v\Big(V_u(\mathfrak{g}_+)[[v]]^{\widehat{\otimes}2}\Big),$$

which is proved by using the identifications of the proof of Part (a). We have thus established that $\Delta_{u,v}\big(A_+\big) \subset A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+$.

We now check that $\Delta_{u,v}\big(\widehat{A}_+\big) \subset \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+$. By Lemma 7.5 any element of $\widehat{A}_+$ is of the form $\widehat{\psi}_+(a)$, where $a \in \widehat{V}_u(\mathfrak{g}_+)[[v]]$. For any $N > 0$, there exists $b \in V_u(\mathfrak{g}_+)[[v]]$ such that $a - b = \sum_{n\geq 0} a_n v^n$ with $a_n \in \bigoplus_{p\geq N} U^p(\mathfrak{g}_+)u^p$. Now, $\widehat{\psi}_+(b) = \psi_+(b) \in A_+$, and $\widehat{\psi}_+(a-b) \in u^N U_{u,v}(\mathfrak{g}_+)$ by Lemma 6.8 (a). Therefore,

$$\Delta_{u,v}(\widehat{\psi}_+(a)) \equiv \Delta_{u,v}(\psi_+(b)) \mod u^N. \tag{7.4}$$

It follows from the considerations above that

$$\Delta_{u,v}(\psi_+(b)) \in A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+ \subset \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+.$$

The latter $\mathbf{C}[[u,v]]$-module being topologically free, Formula (7.4) for all $N>0$ implies

$$\Delta_{u,v}(\widehat{\psi_+}(a)) \in \widehat{A}_+ \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} \widehat{A}_+ .$$

□

Corollary 7.9. *The algebras A_+ and $\widehat{A}_+$ are subbialgebras of $U_{u,v}(\mathfrak{g}_+)$.*

7.10. Remark. The bialgebra A_+ has the following alternative definition. Define the $\mathbf{C}[u][[v]]$-bialgebra

$$U'_{u,v}(\mathfrak{g}_+) = \varprojlim_n U_h(\mathfrak{g}_+) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[u][[v]]/(v^n),$$

where $\mathbf{C}[u][[v]]$ is a $\mathbf{C}[[h]]$-module by the morphism ι of Section 4.6. One can check that $U'_{u,v}(\mathfrak{g}_+)$ embeds as a subbialgebra into the bialgebra $U_{u,v}(\mathfrak{g}_+)$ of Section 6.1, that the map $\widetilde{\alpha}_+$ of Section 6.6 sends the $\mathbf{C}[u][[v]]$-module $U'_{u,v}(\mathfrak{g}_+)$ isomorphically onto $U(\mathfrak{g}_+)[u][[v]]$, and that the bialgebra morphism p_v of Lemma 6.7 maps $U'_{u,v}(\mathfrak{g}_+)$ onto the bialgebra $U(\mathfrak{g}_+)[u]$ of polynomials with coefficients in $U(\mathfrak{g}_+)$.

Adapting the proofs of Sections 6–7, one can prove that A_+ is in $U'_{u,v}(\mathfrak{g}_+)$ and that

$$A_+ = \left\{ a \in U'_{u,v}(\mathfrak{g}_+) \ \mid\ \delta^n(a) \in u^n U'_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n} \text{ for all } n \geq 1 \right\}.$$

8. Proofs of Theorems 2.3, 2.6, and 2.9 (I).

Let $A_{u,v}(\mathfrak{g}_+) = A_+$ be the bialgebra constructed in Sections 6–7. We first prove Theorem 2.6 and then determine A_+/uA_+ as an algebra (Part I of Theorem 2.9). The proof of Theorem 2.3 follows.

8.1. Proof of Theorem 2.6. It follows from Lemma 6.7, Lemma 6.8 (b), Theorem 6.9, and Corollary 7.9 applied to $\mathfrak{g}_+ = \mathfrak{g}$ that the morphism of bialgebras $p_v : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u]]$ restricts to a surjective morphism of bialgebras $p_v : A_+ \to V_u(\mathfrak{g}_+)$ whose kernel is vA_+. Therefore, the induced map $A_+/vA_+ \to V_u(\mathfrak{g}_+)$ is an isomorphism of bialgebras. It remains to check that this isomorphism preserves the cobracket.

The bialgebra structure on A_+ induces on $V_u(\mathfrak{g}_+)$ a Poisson cobracket δ' given by (1.8), where $p = p_v$. We have to check that δ' coincides with the Poisson cobracket δ_u of $V_u(\mathfrak{g}_+)$ defined by (2.5). Since the algebra $V_u(\mathfrak{g}_+)$ is generated by the elements ux with $x \in \mathfrak{g}_+$, it suffices to show that $\delta'(ux) = \delta_u(ux)$ for all $x \in \mathfrak{g}_+$.

We identify the module $U_{u,v}(\mathfrak{g}_+)$ with $U(\mathfrak{g}_+)[[u,v]]$ via the isomorphism $\widetilde{\alpha}_+$ of Section 6.6. Let $a \in \widetilde{\alpha}_+^{-1}(ux) \subset U_{u,v}(\mathfrak{g}_+)$. We have $p_v(a) = ux$. Viewing $U_{u,v}(\mathfrak{g}_+)$ as a subbialgebra of $U_{u,v}(\mathfrak{d})$, we see by (5.3)–(5.4) that the comultiplication $\Delta_{u,v}$ of $U_{u,v}(\mathfrak{g}_+)$ satisfies

$$\Delta_{u,v}(a) \equiv \Delta(a) + uv \left[\Delta(a), \frac{r}{2}\right] \mod u^2v^2\, U_{u,v}(\mathfrak{d})^{\widehat{\otimes} 2},$$

where Δ is the standard comultiplication (2.4) on $U_{u,v}(\mathfrak{d}) = U(\mathfrak{d})[[u,v]]$. Therefore,

$$\frac{\Delta_{u,v}(a) - \Delta_{u,v}^{\mathrm{op}}(a)}{v} \equiv u \left[\Delta(a), \frac{r - r_{21}}{2}\right] \quad \operatorname{mod} u^2 v\, U_{u,v}(\mathfrak{d})^{\widehat{\otimes} 2}.$$

It follows that

$$\begin{aligned}
\delta'(ux) &= (p_v \otimes p_v)\left(\frac{\Delta_{u,v}(a) - \Delta_{u,v}^{\mathrm{op}}(a)}{v}\right) \\
&= u \left[\Delta(ux), \frac{r - r_{21}}{2}\right] \\
&= u^2 \left[x\otimes 1 + 1\otimes x, \frac{r - r_{21}}{2}\right] \\
&= u^2 \left([x\otimes 1 + 1\otimes x, r] - \frac{1}{2}\,[x\otimes 1 + 1\otimes x, r + r_{21}]\right) \\
&= u^2\,[x\otimes 1 + 1\otimes x, r] = u^2 \delta(x) = \delta_u(ux).
\end{aligned}$$

The vanishing of $[x\otimes 1 + 1\otimes x, r + r_{21}]$ is due to the invariance of the 2-tensor $r + r_{21}$. The identity $\delta(x) = [x\otimes 1 + 1\otimes x, r]$ follows from (5.2). □

8.2. Proof of Theorem 2.9. Part I. We prove here that $A_+/uA_+ = S(\mathfrak{g}_+)[[v]]$ as a $\mathbf{C}[[v]]$-algebra. We first observe that the algebra A_+/uA_+ is commutative. Indeed, $A_+/uA_+ \subset \widehat{A}_+/u\widehat{A}_+$ by the second equality of Corollary 7.6. By Proposition 3.5, the quotient algebra $\widehat{A}_+/u\widehat{A}_+$ is commutative; hence, so is A_+/uA_+.

Consider the $\mathbf{C}[u][[v]]$-linear isomorphism $\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to A_+$ of Theorem 6.9 (a). It induces a $\mathbf{C}[[v]]$-linear isomorphism

$$\Psi_+ : S(\mathfrak{g}_+)[[v]] = V_u(\mathfrak{g}_+)[[v]]/uV_u(\mathfrak{g}_+)[[v]] \to A_+/uA_+.$$

By definition,

$$\Psi_+(x_1^{j_1} \dots x_d^{j_d}) = v^{-|\underline{j}|}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \quad \text{modulo } u\,A_+ \tag{8.1}$$

for all d-tuples $\underline{j} = (j_1, \dots, j_d)$. (Recall that $(x_1, \dots, x_d)$ is a fixed basis of $\mathfrak{g}_+$.) Since A_+/uA_+ is commutative, Ψ_+ is an algebra morphism. □

8.3. Proof of Theorem 2.3. By Theorem 6.9 (a), the $\mathbf{C}[u][[v]]$-module A_+ is isomorphic to $V_u(\mathfrak{g}_+)[[v]]$, hence to $S(\mathfrak{g}_+)[u][[v]]$ (see Section 2.4 and Lemma 2.5). As a consequence of Theorem 2.6 and Section 8.2, the bialgebra A_+ is commutative modulo u and cocommutative modulo v. It follows from Theorem 2.6 and Lemma 2.5 that $A_+/(u,v) = S(\mathfrak{g})$ as bi-Poisson bialgebras. □

8.4. Remark. Since A_+ is a $\mathbf{C}[u][[v]]$-module, we may set $u = 1$. We claim that the quotient bialgebra $A_+/(u-1)$ is isomorphic to Etingof and Kazhdan's bialgebra $U_v(\mathfrak{g}_+)$ of Section 5.4 (with h replaced by v). Indeed, the bialgebra inclusion $A_+ \subset U'_{u,v}(\mathfrak{g}_+)$ of Remark 7.10 induces a bialgebra morphism $\xi : A_+/(u-1) \to U'_{u,v}(\mathfrak{g}_+)/(u-1) = U_v(\mathfrak{g}_+)$. It remains to show that ξ is an isomorphism. The isomorphism ψ_+ of Theorem 6.9 (a) induces a $\mathbf{C}[[v]]$-linear isomorphism $\overline{\psi}_+ : U(\mathfrak{g}_+)[[v]] = V_u(\mathfrak{g}_+)[[v]]/(u-1) \to A_+/(u-1)$. It now suffices to check that the composite map $\xi \circ \overline{\psi}_+$ is an isomorphism. By Sections 5.5, 6.4, and 6.6 the map $\xi \circ \overline{\psi}_+$ sends $x_{\underline{j}} = x_1^{j_1} \dots x_d^{j_d} \in U(\mathfrak{g}_+)[[v]]$ to $v^{-|\underline{j}|}\, \rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d}$ for all d-tuples $(j_1, \dots , j_d)$. In view of Lemma 5.6 (a) it follows that $\xi \circ \overline{\psi}_+$ is an isomorphism modulo v; hence, it is an isomorphism of topologically free $\mathbf{C}[[v]]$-modules.

9. A nondegenerate bialgebra pairing.

In this section, we construct a pairing between A_+ and a $\mathbf{C}[v][[u]]$-bialgebra A_-, using the element $R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$ introduced in Section 6. We start by defining A_-, then we prove an important property of $R_{u,v}$. We resume the notation of Sections 5-8.

9.1. The Bialgebras A_- and $\widehat{A}_-$. They are defined by analogy with A_+ and $\widehat{A}_+$. Let us begin with the definition of A_-. Consider the $\mathbf{C}[[h]]$-linear isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ of Section 6.6. We have $\alpha_+(1) = 1$ and $\alpha_+ \equiv \mathrm{id}$ modulo h. Choose a $\mathbf{C}$-linear projection $\pi_+ : U(\mathfrak{g}_+) \to U^1(\mathfrak{g}_+) = \mathbf{C} \oplus \mathfrak{g}_+$ that is the identity on $U^1(\mathfrak{g}_+)$. For any $y \in \mathfrak{g}_-$ we define a $\mathbf{C}$-linear form $\langle -, y\rangle : U^1(\mathfrak{g}_+) \to \mathbf{C}$ extending the evaluation map $\langle -, y\rangle : \mathfrak{g}_+ \to \mathbf{C}$ and such that $\langle 1, y\rangle = 0$. We obtain a $\mathbf{C}[[h]]$-linear form $g_y : U_h(\mathfrak{g}_+) \to \mathbf{C}[[h]]$ by

$$g_y(a) = \langle \pi_+\alpha_+(a), y\rangle = \sum_{n\geq 0} \langle \pi_+(a_n), y\rangle\, h^n, \tag{9.1}$$

where $a \in U_h(\mathfrak{g}_+)$ and the elements $a_n \in U(\mathfrak{g}_+)$ are defined by $\alpha_+(a) = \sum_{n\geq 0} a_n h^n$. We have $g_y(1) = 0$.

By extension of scalars, we obtain a $\mathbf{C}[[u, v]]$-linear form $\widetilde{g}_y : U_{u,v}(\mathfrak{g}_+) \to \mathbf{C}[[u, v]]$ such that $\widetilde{g}_y(1) = 0$. We apply the map $\rho_- : U^*_{u,v}(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_-)$ of (6.2) to $\widetilde{g}_y$. By Lemma 6.5 adapted to this situation, $\rho_-(\widetilde{g}_y) \in U_{u,v}(\mathfrak{g}_-)$ is divisible by uv.

Let $V_v(\mathfrak{g}_-)$ be the $\mathbf{C}[v]$-bialgebra introduced in Section 2.4, where we have now replaced u by v. Let $(y_1, \dots , y_d)$ be the basis of $\mathfrak{g}_-$ dual to the fixed basis $(x_1, \dots , x_d)$ of $\mathfrak{g}_+$. The family $(v^{|\underline{k}|}\, y_{\underline{k}})$, where $\underline{k}$ runs over all d-tuples of nonnegative integers, is a $\mathbf{C}[v]$-basis of $V_v(\mathfrak{g}_-)$. We define a $\mathbf{C}[v]$-linear map $\psi_- : V_v(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_-)$ by $\psi_-(1) = 1$ and

$$\psi_-(v^{|\underline{k}|}\, y_{\underline{k}}) = u^{-|\underline{k}|}\, \rho_-(\widetilde{g}_{y_1})^{k_1} \dots \rho_-(\widetilde{g}_{y_d})^{k_d}, \tag{9.2}$$

where $\underline{k} = (k_1, \ldots, k_d)$ is a d-tuple with $|\underline{k}| \geq 1$. This map extends uniquely to a $\mathbf{C}[v][[u]]$-linear map, still denoted ψ_-, from $V_v(\mathfrak{g}_-)[[u]]$ to $U_{u,v}(\mathfrak{g}_-)$ by

$$\psi_-\Big(\sum_{n\geq 0} w_n u^n\Big) = \sum_{n\geq 0} \psi_-(w_n) u^n,$$

where $w_0, w_1, w_2, \ldots \in V_v(\mathfrak{g}_-)$. We then define the $\mathbf{C}[v][[u]]$-module A_- by

$$A_- = \psi_-(V_v(\mathfrak{g}_-)[[u]]) \subset U_{u,v}(\mathfrak{g}_-). \tag{9.3}$$

Recall the isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \cong U(\mathfrak{g}_-)[[h]]$ of Section 5.5. It induces a $\mathbf{C}[[u,v]]$-linear isomorphism $\widetilde{\alpha}_- : U_{u,v}(\mathfrak{g}_-) \cong U(\mathfrak{g}_-)[[u,v]]$ such that $\widetilde{\alpha}_- \equiv \mathrm{id}$ modulo uv. Consider the composed map

$$p_u : U_{u,v}(\mathfrak{g}_-) \xrightarrow{\widetilde{\alpha}_-} U(\mathfrak{g}_-)[[u,v]] \to U(\mathfrak{g}_-)[[v]],$$

where the second map is the projection $u \mapsto 0$. The map p_u is a morphism of bialgebras when we equip $U(\mathfrak{g}_-)[[v]]$ with the power series multiplication and the comultiplication (2.4). Moreover, p_u sends A_- onto $V_v(\mathfrak{g}_-)$ and $p_u \circ \psi_- : V_v(\mathfrak{g}_-)[[u]] \to V_v(\mathfrak{g}_-)$ is the projection sending u to 0. This is proved as in Section 6.

By analogy with Section 7.1, we define a $\mathbf{C}[[u,v]]$-subalgebra $\widehat{A}_-$ of $U_{u,v}(\mathfrak{g}_-)$ by

$$\widehat{A}_- = \Big\{a \in U_{u,v}(\mathfrak{g}_-) \;\mid\; \delta^n(a) \in v^n U_{u,v}(\mathfrak{g}_-)^{\widehat{\otimes} n} \text{ for all } n \geq 1\Big\}. \tag{9.4}$$

It is clear that the results of Sections 6–8 apply to A_- and $\widehat{A}_-$, namely we have the following properties.

(i) The map $\psi_- : V_v(\mathfrak{g}_-)[[u]] \to A_-$ is an isomorphism of $\mathbf{C}[v][[u]]$-modules. It extends to an isomorphism of $\mathbf{C}[[u,v]]$-modules $\widehat{\psi}_- : \widehat{V}_v(\mathfrak{g}_-)[[u]] \to \widehat{A}_-$.

(ii) $A_- \subset \widehat{A}_-$ are subalgebras of $U_{u,v}(\mathfrak{g}_-)$.

(iii) A_- is independent of the choices of the isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$, of the projection $\pi_+ : U(\mathfrak{g}_+) \to U^1(\mathfrak{g}_+)$, and of the basis of $\mathfrak{g}_-$.

(iv) A_- and $\widehat{A}_-$ are topological bialgebras for the u-adic topology and the (u,v)-adic topology, respectively.

(v) A_- and $\widehat{A}_-$ are commutative modulo v and cocommutative modulo u. There are isomorphisms of co-Poisson bialgebras

$$A_-/uA_- = V_v(\mathfrak{g}_-), \tag{9.5}$$

isomorphisms of bi-Poisson bialgebras

$$A_-/(u,v)A_- = S(\mathfrak{g}_-), \tag{9.6}$$

and isomorphisms of algebras

$$A_-/vA_- = S(\mathfrak{g}_-)[[u]]. \tag{9.7}$$

Recall the two-variable universal R-matrix

$$R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$$

of Section 6. We now give a stronger version of Lemma 6.3 (b).

Lemma 9.2. *The element $R_{u,v} - 1\otimes 1$ belongs to the submodules*

$$v\,\widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-) \quad \text{and} \quad U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, u\,\widehat{A}_-$$

of $U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$.

Proof. Recall the element $R' \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$ of Lemma 6.3 (b). It is enough to show that

$$uR' \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-) \quad \text{and} \quad vR' \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_-.$$

We shall prove the first inclusion. The second one has a similar proof.

Let $(b_j)_j$ be a basis over $\mathbf{C}[[u,v]]$ of the (topologically free) $\mathbf{C}[[u,v]]$-module $U_{u,v}(\mathfrak{g}_-)$. We can expand R' as $R' = \sum_j z_j \otimes b_j$, where z_j are elements of $U_{u,v}(\mathfrak{g}_+)$. The proof of Lemma 7.4 shows that $(\delta^n \otimes \mathrm{id})(uvR')$ is divisible by u^n for any $n \geq 1$. Hence,

$$(\delta^n \otimes \mathrm{id})(R') = \sum_j \delta^n(z_j) \otimes b_j$$

is divisible by u^{n-1}. The elements b_j being linearly independent, it follows that $\delta^n(z_j)$ is divisible by u^{n-1} for all $n \geq 1$ and all j. Therefore, $uz_j \in \widehat{A}_+$ for all j and $uR' \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$. □

Corollary 9.3. *The element $R_{u,v}$ belongs to the submodules*

$$\widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-) \quad \text{and} \quad U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_-.$$

We consider the dual $\mathbf{C}[[u,v]]$-modules $\widehat{A}_+^* = \mathrm{Hom}_{\mathbf{C}[[u,v]]}(\widehat{A}_+, \mathbf{C}[[u,v]])$ and $\widehat{A}_-^* = \mathrm{Hom}_{\mathbf{C}[[u,v]]}(\widehat{A}_-, \mathbf{C}[[u,v]])$. In view of Corollary 9.3, Formulas (6.2) now define $\mathbf{C}[[u,v]]$-linear maps $\widehat{A}_-^* \to U_{u,v}(\mathfrak{g}_+)$ and $\widehat{A}_+^* \to U_{u,v}(\mathfrak{g}_-)$, which we still denote by ρ_+ and ρ_-, respectively. The comultiplications of $\widehat{A}_+$ and of $\widehat{A}_-$ induce algebra structures on $\widehat{A}_+^*$ and $\widehat{A}_-^*$. As in Section 6, the map ρ_+ is an antimorphism of algebras and ρ_- is a morphism of algebras.

Lemma 9.4. *We have*

$$A_+ \subset \rho_+(\widehat{A}_-^*) \subset \widehat{A}_+ \quad \text{and} \quad A_- \subset \rho_-(\widehat{A}_+^*) \subset \widehat{A}_-.$$

Proof. Let us prove the first two inclusions. The other two inclusions have similar proofs.

(a) We use the notation of Sections 6.4 and 6.6. We first show that, for any $x \in \mathfrak{g}_+$, the element $v^{-1}\,\rho_+(\widetilde{f}_x) \in A_+$ sits in $\rho_+\big(\widehat{A}_-^*\big)$. Indeed, if $b \in \widehat{A}_-$, then $\delta^1(b) = b - \varepsilon(b)1$ is divisible by v in $U_{u,v}(\mathfrak{g}_-)$. Hence,

$\widetilde{f}_x(b) = \widetilde{f}_x(b) - \varepsilon(b)\widetilde{f}_x(1) \in \mathbf{C}[[u,v]]$ is divisible by v. We then define $\widehat{f}_x \in \widehat{A}_-^*$ by

$$\widehat{f}_x(b) = v^{-1}\, \widetilde{f}_x(b) \in \mathbf{C}[[u,v]] \tag{9.8}$$

for any $b \in \widehat{A}_-$. It follows that the restriction of $\widetilde{f}_x$ to $\widehat{A}_-$ equals $v\,\widehat{f}_x$. Therefore, $v^{-1}\,\rho_+(\widetilde{f}_x) = \rho_+(\widehat{f}_x) \in \rho_+(\widehat{A}_-^*)$.

By Section 6.6, any element $a \in A_+$ is of the form

$$a = \sum_{n \geq 0} v^n \left(\sum_{\underline{j}} P_{\underline{j}}(u)\, v^{-|\underline{j}|}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \right),$$

where the sums inside the brackets are finite and $P_{\underline{j}}(u) \in \mathbf{C}[u]$. The formal power series

$$\sum_{n \geq 0} v^n \left(\sum_{\underline{j}} P_{\underline{j}}(u)\, \widehat{f}_{x_d}^{j_d} \dots \widehat{f}_{x_1}^{j_1} \right)$$

converges to an element f in the topologically free $\mathbf{C}[[u,v]]$-module $\widehat{A}_-^*$. Since $\rho_+ : \widehat{A}_-^* \to U_{u,v}(\mathfrak{g}_+)$ is an antimorphism of algebras, we have $\rho_+(f) = a$. This implies that $A_+ \subset \rho_+(\widehat{A}_-^*)$.

(b) Let us prove that $\rho_+(\widehat{A}_-^*) \subset \widehat{A}_+$. Given $f \in \widehat{A}_-^*$, we have to check that $\delta^n(\rho_+(f))$ is divisible by u^n for all $n \geq 1$. By Lemma 9.2, $vR' \in U_{u,v}(\mathfrak{g}_+) \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} \widehat{A}_-$, hence

$$v^n R'_{1,n+1} R'_{2,n+1} \cdots R'_{n-1,n+1} R'_{n,n+1} \in U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n} \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} \widehat{A}_-.$$

This allows us to apply $\mathrm{id} \otimes f$ to $v^n R'_{1,n+1} R'_{2,n+1} \cdots R'_{n-1,n+1} R'_{n,n+1}$. A computation similar to the one in the proof of Lemma 7.4 yields

$$\begin{aligned} &\delta^n(\rho_+(f)) \\ &\quad = u^n (\mathrm{id} \otimes f)(v^n R'_{1,n+1} R'_{2,n+1} \cdots R'_{n-1,n+1} R'_{n,n+1}) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}. \end{aligned}$$

□

Lemma 9.5. *For $a \in A_+$ and $b \in A_-$, the formulas*

$$(a,b)_{u,v} = \big(\rho_+^{-1}(a)\big)(b) = \big(\rho_-^{-1}(b)\big)(a), \tag{9.9}$$

yield a well-defined bialgebra pairing $A_+ \times A_-^{\mathrm{cop}} \to \mathbf{C}[[u,v]]$.

Here A_-^{cop} denotes the bialgebra A_- with the opposite comultiplication. The pairing $(\ ,\ )_{u,v}$ is in the sense of Section 2.10 with $K_1 = \mathbf{C}[u][[v]]$, $K_2 = \mathbf{C}[v][[u]]$, and $K = \mathbf{C}[[u,v]]$.

Proof. Let us prove that the expression $\big(\rho_-^{-1}(b)\big)(a)$ is well defined. It suffices to check that, if $g \in \widehat{A}_+^*$ satisfies $\rho_-(g) = 0$, then $g(a) = 0$. Suppose first that $a = \psi_+(u^{|\underline{j}|} x_{\underline{j}})$ for some d-tuple $\underline{j}$. By (6.3), $v^{|\underline{j}|}\, a = \rho_+(f)$, where

$f = \widetilde{f}_{x_d}^{j_d} \dots \widetilde{f}_{x_1}^{j_1} \in U_{u,v}^*(\mathfrak{g}_-)$. Applying $g \otimes f$ to $R_{u,v} \in \widehat{A}_+ \widehat{\otimes}_{\mathbf{C}[[u,v]]} U_{u,v}(\mathfrak{g}_-)$, we obtain

$$v^{|\underline{j}|}\, g(a) = g(\rho_+(f)) = (g \otimes f)(R_{u,v}) = f(\rho_-(g)) = 0.$$

Since $\mathbf{C}[[u,v]]$ is v-torsion-free, we obtain $g(a) = 0$. By $\mathbf{C}[[u,v]]$-linearity, $g(a) = 0$ for all $a \in \widehat{A}_+$.

A similar argument proves that $\big(\rho_+^{-1}(a)\big)(b)$ is well defined. Let us show that

$$\big(\rho_+^{-1}(a)\big)(b) = \big(\rho_-^{-1}(b)\big)(a). \tag{9.10}$$

By linearity, it suffices to consider the case $a = \psi_+(u^{|\underline{j}|} x_{\underline{j}})$ as above. We have $v^{|\underline{j}|}\, a = \rho_+(f)$ with $f \in U_{u,v}^*(\mathfrak{g}_-)$. Let $g \in \rho_-^{-1}(b) \subset \widehat{A}_+^*$. Then

$$\begin{aligned} v^{|\underline{j}|}\, \big(\rho_+^{-1}(a)\big)(b) &= f(b) = f(\rho_-(g)) = (g \otimes f)(R_{u,v}) \\ &= g(\rho_+(f)) = v^{|\underline{j}|}\, g(a) = v^{|\underline{j}|}\, \big(\rho_-^{-1}(b)\big)(a). \end{aligned}$$

Hence, (9.10) holds.

That $(\ ,\ )_{u,v}$ is a bialgebra pairing follows directly from the fact that ρ_+ is an antimorphism of algebras and ρ_- is a morphism of algebras. □

9.6. Remark. Proceeding as in the proof of Lemma 9.5, we can show that the maps $\rho_+ : \widehat{A}_-^* \to \widehat{A}_+$ and $\rho_- : \widehat{A}_+^* \to \widehat{A}_-$ are injective.

9.7. Induced Bialgebra Pairings. Passing to the quotient modulo u, the pairing $(\ ,\ )_{u,v}$ induces a bialgebra pairing

$$(\ ,\ )_v : A_+/uA_+ \times A_-/uA_- \to \mathbf{C}[[v]]. \tag{9.11}$$

(The bialgebra A_-/uA_- is cocommutative by (9.5), so that $(A_-/uA_-)^{\mathrm{cop}} = A_-/uA_-$.) Recall the isomorphism of algebras $\Psi_+ : S(\mathfrak{g}_+)[[v]] \to A_+/uA_+$ defined by (8.1). On the other hand, the composition of $\psi_- : V_v(\mathfrak{g}_-) \to A_-$ defined by (9.2) and the projection $A_- \to A_-/uA_-$ is an isomorphism of $\mathbf{C}[v]$-bialgebras $\Psi'_- : V_v(\mathfrak{g}_-) \to A_-/uA_-$, which is defined on the $\mathbf{C}[v]$-basis $(v^{|\underline{k}|}\, y_{\underline{k}})_{\underline{k}}$ of $V_v(\mathfrak{g}_-)$ by

$$\begin{aligned} \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}) &= \psi_-(v^{|\underline{k}|}\, y_{\underline{k}}) \mod uA_- \\ &= u^{-|\underline{k}|}\, \rho_-(\widetilde{g}_{y_1})^{k_1} \cdots \rho_-(\widetilde{g}_{y_d})^{k_d} \mod uA_-, \end{aligned} \tag{9.12}$$

where $\underline{k} = (k_1, \dots, k_d)$ and the maps $\widetilde{g}_{y_i}$ were introduced in Section 9.1.

Lemma 9.8. *If $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$ are d-tuples of non-negative integers, then*

$$(\Psi_+(x_{\underline{j}}), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v = \begin{cases} 0 & \text{if } |\underline{j}| > |\underline{k}|, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d}\, j_1! \dots j_d! & \text{if } |\underline{j}| = |\underline{k}|, \\ \in v^{|\underline{k}|-|\underline{j}|}\, \mathbf{C}[[v]] & \text{if } |\underline{j}| < |\underline{k}|. \end{cases}$$

Proof. We first claim that for any $x \in \mathfrak{g}_+$ and any d-tuple $\underline{k} = (k_1, \dots, k_d)$,

$$(\Psi_+(x), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v = \begin{cases} 0 & \text{if } |\underline{k}| = 0, \\ v^{|\underline{k}|-1}\, \langle x, \pi_-(y_{\underline{k}})\rangle & \text{if } |\underline{k}| \geq 1. \end{cases} \tag{9.13}$$

Indeed, consider the diagram

$$\begin{array}{ccccc} U_{u,v}(\mathfrak{g}_-) & \xrightarrow{\widetilde{\alpha}_-} & U(\mathfrak{g}_-)[[u,v]] & \xrightarrow{\langle x, \pi_-(-)\rangle} & \mathbf{C}[[u,v]] \\ \downarrow{\scriptstyle p_u} & & \downarrow & & \downarrow \\ U(\mathfrak{g}_-)[[v]] & \xrightarrow{\mathrm{id}} & U(\mathfrak{g}_-)[[v]] & \xrightarrow{\langle x, \pi_-(-)\rangle} & \mathbf{C}[[v]] \end{array}$$

where the unmarked vertical maps are the projections sending u to 0. The left-hand and the right-hand squares commute by definition of p_u and by linearity, respectively. It follows that, for any $b \in U_{u,v}(\mathfrak{g}_-)$,

$$\widetilde{f}_x(b) \mod u\mathbf{C}[[u,v]] = \langle x, \pi_-(p_u(b))\rangle. \tag{9.14}$$

Since $\Psi_+(x) = v^{-1}\, \rho_+(\widetilde{f}_x) \mod uA_+$ and $\Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}) = \psi_-(v^{|\underline{k}|}\, y_{\underline{k}}) \mod uA_-$, we have

$$\begin{aligned} (\Psi_+(x), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v &= v^{-1}\, \widetilde{f}_x(\psi_-(v^{|\underline{k}|}\, y_{\underline{k}})) \mod u\mathbf{C}[[u,v]] \\ &= v^{-1}\, \langle x, \pi_-(p_u(\psi_-(v^{|\underline{k}|}\, y_{\underline{k}})))\rangle \\ &= v^{-1}\, \langle x, v^{|\underline{k}|}\, \pi_-(y_{\underline{k}})\rangle = v^{|\underline{k}|-1}\, \langle x, \pi_-(y_{\underline{k}})\rangle \end{aligned}$$

for all $\underline{k}$. If $|\underline{k}| = 0$, then $v^{|\underline{k}|}\, y_{\underline{k}} = 1$, on which $\langle x, -\rangle$ vanishes. This proves (9.13).

Formula (9.13) implies that Lemma 9.8 holds for any $\underline{j}$ and $\underline{k}$ such that $|\underline{j}| = 1$. For the general case, observe that

$$\begin{aligned} &(\Psi_+(x_{\underline{j}}), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v \\ &= (\Psi_+(x_1)^{j_1} \dots \Psi_+(x_d)^{j_d}, \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v \\ &= (\Psi_+(x_1)^{\otimes j_1} \otimes \cdots \otimes \Psi_+(x_d)^{\otimes j_d}, \Delta_{u,v}^{|\underline{j}|}(\Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}})))_v \\ &= (\Psi_+(x_1)^{\otimes j_1} \otimes \cdots \otimes \Psi_+(x_d)^{\otimes j_d}, (\Psi'_-)^{\otimes |\underline{j}|}(\Delta^{|\underline{j}|}(v^{|\underline{k}|}\, y_{\underline{k}})))_v \end{aligned} \tag{9.15}$$

in view of Lemma 9.5, and the fact that Ψ_+ preserves the multiplication and Ψ'_- preserves the comultiplication. Here Δ is given by (2.4). Then the formulas of Lemma 9.8 for a general $\underline{j}$ follow from (2.4), (9.15), and the formulas for $\underline{j}$ such that $|\underline{j}| = 1$. □

Passing to the quotients modulo v and modulo (u, v), the pairing $(\ ,\ )_{u,v}$ induces bialgebra pairings

$$(\ ,\ )_u : A_+/vA_+ \times (A_-/vA_-)^{\mathrm{cop}} \to \mathbf{C}[[u]] \tag{9.16}$$

and

$$A_+/(u,v) \times A_-/(u,v) \to \mathbf{C}. \tag{9.17}$$

The latter can also be obtained from the pairing $(\ ,\ )_v$ of (9.11) by setting $v = 0$.

The isomorphism $\Psi_+ : S(\mathfrak{g}_+)[[v]] \to A_+/uA_+$ defined by (8.1) induces a canonical isomorphism of bialgebras $S(\mathfrak{g}_+) = A_+/(u,v)$. The isomorphism $\Psi'_- : V_v(\mathfrak{g}_-) \to A_-/uA_-$ defined above induces a canonical isomorphism of bialgebras $S(\mathfrak{g}_-) = A_-/(u,v)$. We denote by $(\ ,\ )_0$ the bialgebra pairing $S(\mathfrak{g}_+) \times S(\mathfrak{g}_-) \to \mathbf{C}$ obtained from (9.17) under these identifications. Lemma 9.8 implies that

$$(x_{\underline{j}}, y_{\underline{k}})_0 = \begin{cases} 0 & \text{if } |\underline{j}| \neq |\underline{k}|, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d}\, j_1! \dots j_d! & \text{if } |\underline{j}| = |\underline{k}| \end{cases} \tag{9.18}$$

for all d-tuples $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$.

Corollary 9.9. *The pairings*

$$(\ ,\ )_{u,v} : A_+ \times A_-^{\mathrm{cop}} \to \mathbf{C}[[v]], \quad (\ ,\ )_v : A_+/uA_+ \times A_-/uA_- \to \mathbf{C}[[v]],$$

$$(\ ,\ )_u : A_+/vA_+ \times (A_-/vA_-)^{\mathrm{cop}} \to \mathbf{C}[[u]], \quad \textit{and} \quad (\ ,\ )_0 : S(\mathfrak{g}_+) \times S(\mathfrak{g}_-) \to \mathbf{C}$$

are nondegenerate.

Proof. It follows from (9.18) that $(\ ,\ )_0$ is nondegenerate. (Actually, $(\ ,\ )_0$ is the standard pairing between $S(\mathfrak{g}_+)$ and $S(\mathfrak{g}_-)$.)

We check that $(\ ,\ )_v$ is nondegenerate. Let $a \in A_+/uA_+$ such that $(a,-)_v = 0$. If $\bar{a}$ denotes the image of a under the projection $A_+/uA_+ \to S(\mathfrak{g}_+)$, then $(\bar{a},-)_0 = 0$. It follows from the nondegeneracy of $(\ ,\ )_0$ that $\bar{a} = 0$, which implies that $a \in vA_+/uA_+$. Let $a_1 \in A_+/uA_+$ be such that $a = va_1$. We now have $(a_1,-)_v = 0$. A similar argument shows that a_1 is divisible by v, hence a is divisible by v^2 in A_+/uA_+. Proceeding in the same way, we see that a is divisible by any power of v, which is possible only if $a = 0$. A similar argument shows that $(-,b)_v = 0$ implies $b = 0$.

The nondegeneracy of $(\ ,\ )_{u,v}$ and $(\ ,\ )_u$ is proved in a similar fashion. □

10. Completion of the proof of Theorem 2.9.

Before proceeding to prove Theorem 2.9, we establish a few facts about a topological dual of the $\mathbf{C}[v]$-bialgebra

$$V_v(\mathfrak{g}_-) = \left\{ \sum_{n \geq 0} b_n v^n \in U(\mathfrak{g}_-)[v] \ \middle|\ b_n \in U^n(\mathfrak{g}_-) \text{ for all } n \geq 0 \right\}.$$

10.1. A Topological Dual. Inside the dual

$$V_v^*(\mathfrak{g}_-) = \mathrm{Hom}_{\mathbf{C}[v]}(V_v(\mathfrak{g}_-), \mathbf{C}[[v]])$$

of $V_v(\mathfrak{g}_-)$ there is a $\mathbf{C}[[v]]$-submodule $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ consisting of all $f \in V_v^*(\mathfrak{g}_-)$ satisfying the following condition: For every $m \geq 0$ there exists $N \geq 0$ such that

$$f(U^p(\mathfrak{g}_-)\, v^p) \subset v^m \mathbf{C}[[v]] \tag{10.1}$$

for all $p \geq N$. In other words, $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ consists of all $\mathbf{C}[v]$-linear forms that are continuous when we equip $\mathbf{C}[[v]]$ with the v-adic topology and $V_v(\mathfrak{g}_-)$ with the I-adic topology, where I is the two-sided ideal

$$I = \bigoplus_{p \geq 1} U^p(\mathfrak{g}_-)\, v^p \subset V_v(\mathfrak{g}_-).$$

Lemma 10.2. *The* $\mathbf{C}[[v]]$*-module* $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ *is topologically free and*

$$V_v^{\mathrm{o}}(\mathfrak{g}_-) \bigcap v\, V_v^*(\mathfrak{g}_-) = v\, V_v^{\mathrm{o}}(\mathfrak{g}_-).$$

Proof. For the first statement, it is enough to check that, if $(f_n)_{n\geq 0}$ is a family of elements of $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ such that $f_n \equiv f_{n+1} \bmod v^n$ for all $n \geq 0$, then there exists a unique $f \in V_v^{\mathrm{o}}(\mathfrak{g}_-)$ such that $f \equiv f_n \bmod v^n$ for all $n \geq 0$.

Indeed, since the linear forms f_n are with values in $\mathbf{C}[[v]]$, there exists a unique $f \in V_v^*(\mathfrak{g}_-)$ such that $f \equiv f_n \bmod v^n$ for all $n \geq 0$. Let us show that f belongs to $V_v^{\mathrm{o}}(\mathfrak{g}_-)$. Fix $m \geq 0$. By definition of $V_v^{\mathrm{o}}(\mathfrak{g}_-)$, there exists $N \geq 0$ such that $f_m(U^p(\mathfrak{g}_-)\, v^p) \subset v^m \mathbf{C}[[v]]$ for all $p \geq N$. Since $f \equiv f_m \bmod v^m$, we have $f(a) \equiv f_m(a) \bmod v^m$ for all $a \in V_v(\mathfrak{g}_-)$, hence

$$f(U^p(\mathfrak{g}_-)\, v^p) \equiv f_m(U^p(\mathfrak{g}_-)\, v^p) \equiv 0 \mod v^m$$

for all p. Therefore, $f(U^p(\mathfrak{g}_-)\, v^p) \subset v^m \mathbf{C}[[v]]$ for all $p \geq N$.

The second statement is an easy exercise left to the reader. □

We now relate $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ to $S(\mathfrak{g}_+)[[v]]$. As before, we fix a basis $(x_1, \dots, x_d)$ of $\mathfrak{g}_+$ and the dual basis $(y_1, \dots, y_d)$ of $\mathfrak{g}_-$. The family of elements $x_{\underline{j}} = x_1^{j_1} \dots x_d^{j_d}$ indexed by all d-tuples $\underline{j} = (j_1, \dots, j_d)$ of nonnegative integers is a $\mathbf{C}$-basis of $S(\mathfrak{g}_+)$; the family of elements $(v^{|\underline{k}|}\, y_{\underline{k}})$ indexed by all d-tuples $\underline{k}$ of nonnegative integers is a $\mathbf{C}[v]$-basis of $V_v(\mathfrak{g}_-)$.

Suppose there exists a pairing $(\ ,\ ) : S(\mathfrak{g}_+)[[v]] \times V_v(\mathfrak{g}_-) \to \mathbf{C}[[v]]$ (in the sense of Section 2.10 with $K = K_1 = \mathbf{C}[[v]] \supset K_2 = \mathbf{C}[v]$) such that for all $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$ we have

$$(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) = \begin{cases} 0 & \text{if } |\underline{j}| > |\underline{k}|, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d}\, j_1! \dots j_d! & \text{if } |\underline{j}| = |\underline{k}|, \\ \in v^{|\underline{k}|-|\underline{j}|}\, \mathbf{C}[[v]] & \text{if } |\underline{j}| < |\underline{k}|. \end{cases} \tag{10.2}$$

The pairing $(\ ,\ )$ induces a $\mathbf{C}[[v]]$-linear map $\varphi : S(\mathfrak{g}_+)[[v]] \to V_v^*(\mathfrak{g}_-)$ defined for $a \in S(\mathfrak{g}_+)[[v]]$ by $\varphi(a) = (a, -)$.

Proposition 10.3. *Under Condition* (10.2) *the map* φ *sends* $S(\mathfrak{g}_+)[[v]]$ *isomorphically onto* $V_v^{\rm o}(\mathfrak{g}_-)$.

Proof. The same argument as in the proof of Corollary 9.9 shows that the pairing (,) is nondegenerate. This implies the injectivity of φ.

Let us prove that φ sends $S(\mathfrak{g}_+)[[v]]$ into $V_v^{\rm o}(\mathfrak{g}_-)$. Any element of $S(\mathfrak{g}_+)[[v]]$ is of the form

$$a = \sum_{n\geq 0;\, \underline{j}} \mu_{\underline{j},n}\, x_{\underline{j}}\, v^n,$$

where $(\mu_{\underline{j},n})_{n\geq 0;\,\underline{j}}$ is a family of scalars indexed by a nonnegative integer n and a d-tuple $\underline{j}$ of nonnegative integers, such that for all n there exists an integer N_n with $\mu_{\underline{j},n} = 0$ whenever $|\underline{j}| \geq N_n$.

In order to check that $\varphi(a)$ lies in $V_v^{\rm o}(\mathfrak{g}_-)$, we have to prove that, given $m \geq 0$, there exists N such that for all $p \geq N$ we have

$$\varphi(a)(U^p(\mathfrak{g}_+)\, v^p) \subset v^m \mathbf{C}[[v]].$$

Let N'_m be any integer such that $N'_m \geq N_n$ for all $n = 0, \dots, m-1$. It is clear that $\mu_{\underline{j},n} = 0$ when $|\underline{j}| \geq N'_m$ and $0 \leq n \leq m-1$. For any $p \geq 1$, the family $(v^p\, y_{\underline{k}})$ with $|\underline{k}| \leq p$ is a basis of $U^p(\mathfrak{g}_-)\, v^p$. Let us compute $\varphi(a)(v^p\, y_{\underline{k}})$ when $|\underline{k}| \leq p$. Using (10.2), we get

$$\begin{aligned}\varphi(a)(v^p\, y_{\underline{k}}) &= (a, v^p\, y_{\underline{k}})\\ &= \sum_{n\geq 0;\,\underline{j}} \mu_{\underline{j},n}\, (x_{\underline{j}}, v^p\, y_{\underline{k}})\, v^n\\ &= \sum_{n\geq 0;\,\underline{j}} \mu_{\underline{j},n}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})\, v^{n+p-|\underline{k}|}\\ &= \sum_{\substack{\underline{j}\\ |\underline{j}|\leq|\underline{k}|}} P_{\underline{j}}(v),\end{aligned}$$

where $P_{\underline{j}}(v) = \Big(\sum_{n\geq 0} \mu_{\underline{j},n}\, v^n\Big)(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})\, v^{p-|\underline{k}|}$. If $|\underline{j}| \geq N'_m$, then

$$\sum_{n\geq 0} \mu_{\underline{j},n}\, v^n = \sum_{n\geq m} \mu_{\underline{j},n}\, v^n$$

is divisible by v^m. Hence $P_{\underline{j}}(v)$ is divisible by v^m. If $|\underline{j}| < N'_m$ and $|\underline{j}| \leq |\underline{k}|$, then by (10.2) $(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})$ is divisible by $v^{|\underline{k}|-|\underline{j}|}$. Therefore, $P_{\underline{j}}(v)$ is divisible by $v^{p-|\underline{j}|}$, hence by $v^{p-N'_m+1}$. If $|\underline{j}| < N'_m$ and $|\underline{j}| > |\underline{k}|$, then $p - |\underline{k}| \geq p - N'_m + 1$. Therefore, $P_{\underline{j}}(v)$ is divisible by $v^{p-N'_m+1}$. Summing up, we see that $\varphi(a)(U^p(\mathfrak{g}_+)\, v^p) \subset v^m\mathbf{C}[[v]]$ for all $p \geq m + N'_m - 1$. Hence, $\varphi(a) \in V_v^{\rm o}(\mathfrak{g}_-)$.

It remains to show that $V_v^{\rm o}(\mathfrak{g}_-) \subset \varphi(S(\mathfrak{g}_+)[[v]])$. Since $(v^{|\underline{j}|}\, y_{\underline{j}})_{\underline{j}}$ is a $\mathbf{C}[v]$-basis of $V_v(\mathfrak{g}_-)$, a $\mathbf{C}[v]$-linear form $f \in V_v^*(\mathfrak{g}_-)$ is uniquely determined by the family $(\nu_{\underline{j}}(v))_{\underline{j}}$ of formal power series defined by

$$\nu_{\underline{j}}(v) = f(v^{|\underline{j}|}\, y_{\underline{j}}) \in \mathbf{C}[[v]].$$

Suppose that $f \in V_v^{\rm o}(\mathfrak{g}_-)$. Then for every m there exists N such that for all $\underline{j}$ with $|\underline{j}| \geq N$ the formal power series $\nu_{\underline{j}}(v)$ is divisible by v^m. Consider the formal sum

$$a_0 = \sum_{\underline{j}} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, x_{\underline{j}},$$

where $\underline{j}! = j_1! \dots j_d!$ if $\underline{j} = (j_1, \dots , j_d)$. By the divisibility property of $\nu_{\underline{j}}(v)$ obtained above, a_0 is a well-defined element of $S(\mathfrak{g}_+)[[v]]$. Let us compute $\varphi(a_0) \in V_v^{\rm o}(\mathfrak{g}_-)$.

Given a d-tuple $\underline{k} = (k_1, \dots , k_d)$, we have

$$\begin{aligned}\varphi(a_0)(v^{|\underline{k}|}\, y_{\underline{k}}) &= (a_0, v^{|\underline{k}|}\, y_{\underline{k}}) \\ &= \sum_{\underline{j}} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) \\ &= \sum_{\underline{j};\, |\underline{j}|=|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) + \sum_{\underline{j};\, |\underline{j}|<|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}).\end{aligned}$$

From (10.2) we derive

$$\sum_{\underline{j};\, |\underline{j}|=|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) = \sum_{\underline{j};\, |\underline{j}|=|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, \delta_{\underline{j},\underline{k}}\, \underline{k}! = \nu_{\underline{k}}(v),$$

where $\delta_{\underline{j},\underline{k}} = \delta_{j_1,k_1} \dots \delta_{j_d,k_d}$. On the other hand, by (10.2), $(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})$ is divisible by v if $|\underline{j}| < |\underline{k}|$. It follows that, for all $\underline{k}$,

$$\varphi(a_0)(y_{\underline{k}} v^{|\underline{k}|}) = \nu_{\underline{k}}(v) + v\, \mathbf{C}[[v]] = f(y_{\underline{k}} v^{|\underline{k}|}) + v\, \mathbf{C}[[v]].$$

Therefore, $f = \varphi(a_0) + v f_1$, where f_1 is a linear form on $V_v(\mathfrak{g}_-)$ such that $v f_1$ belongs to the subspace $V_v^{\rm o}(\mathfrak{g}_-)$. By Lemma 10.2, this implies that $f_1 \in V_v^{\rm o}(\mathfrak{g}_-)$. Starting all over again, we get an element $f_2 \in V_v^{\rm o}(\mathfrak{g}_-)$ and an element $a_1 \in S(\mathfrak{g}_+)[[v]]$ such that $f_1 = \varphi(a_1) + v f_2$. Hence, $f = \varphi(a_0 + v a_1) + v^2 f_2$. Proceeding in this way, we see that for all $n \geq 0$

$$V_v^{\rm o}(\mathfrak{g}_-) = \varphi(S(\mathfrak{g}_+)[[v]]) + v^n V_v^{\rm o}(\mathfrak{g}_-).$$

Together with the topological freeness of $V_v^{\rm o}(\mathfrak{g}_-)$ proved in Lemma 10.2, this implies that $V_v^{\rm o}(\mathfrak{g}_-)$ sits inside the image of φ. □

Recall the nondegenerate bialgebra pairing (9.11)

$$(\, , \,)_v : A_+/uA_+ \times A_-/uA_- \to \mathbf{C}[[v]]$$

and the bialgebra isomorphism $\Psi'_v : V_v(\mathfrak{g}_-) \to A_-/uA_-$ of Section 9.7. They give rise to a $\mathbf{C}[[v]]$-linear morphism of algebras $\varphi : A_+/uA_+ \to V_v^*(\mathfrak{g}_-)$ defined for $a \in A_+/uA_+$ and $b \in V_v(\mathfrak{g}_-)$ by

$$\varphi(a)(b) = (a, \Psi'_-(b))_v. \tag{10.3}$$

Corollary 10.4. *$V_v^{\mathrm{o}}(\mathfrak{g}_-)$ is a subalgebra of $V_v^*(\mathfrak{g}_-)$ and $\varphi : A_+/uA_+ \to V_v^*(\mathfrak{g}_-)$ is an injective morphism of algebras whose image is $V_v^{\mathrm{o}}(\mathfrak{g}_-)$.*

Proof. By Lemma 9.8 the pairing

$$(-,-) = (\Psi_+(-), \Psi'_-(-))_v : S(\mathfrak{g}_+)[[v]] \times V_v(\mathfrak{g}_-) \to \mathbf{C}[[v]]$$

satisfies Condition (10.2). By Proposition 10.3 the map $\varphi \circ \Psi_+$ is injective with image $V_v^{\mathrm{o}}(\mathfrak{g}_-)$. Since $\varphi \circ \Psi_+$ is an algebra morphism, its image $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ is necessarily a subalgebra of $V_v^*(\mathfrak{g}_-)$. One concludes by recalling that $\Psi_+ : S(\mathfrak{g}_+)[[v]] \to A_+/uA_+$ is an algebra isomorphism. □

Consider the Poisson $\mathbf{C}[[v]]$-bialgebra $E_v(\mathfrak{g}_+)$ of Section 2.7. As an algebra, $E_v(\mathfrak{g}_+) = S(\mathfrak{g}_+)[[v]]$. By (2.8) its comultiplication Δ' fulfills the following condition: For all $x \in \mathfrak{g}_+ \subset E_v(\mathfrak{g}_+)$,

$$\Delta'(x) = x \otimes 1 + 1 \otimes x + \sum_{k \geq 1} X_k v^k, \tag{10.4}$$

where $X_k \in \bigoplus_{p+q=k+1} S^p(\mathfrak{g}_+) \otimes S^q(\mathfrak{g}_+)$ for all $k \geq 1$. The Poisson bracket $\{ \, , \, \}$ of $E_v(\mathfrak{g}_+)$ is uniquely determined by Condition (2.9).

In [**Tur91**, Section 12] a bialgebra pairing $(\, , \,)'_v : E_v(\mathfrak{g}_+) \times V_v(\mathfrak{g}_-) \to \mathbf{C}[[v]]$ was constructed such that

$$(x, vy)'_v = \langle x, y \rangle \in \mathbf{C} \tag{10.5}$$

for all $x \in \mathfrak{g}_+ \subset S(\mathfrak{g}_+)[[v]] = E_v(\mathfrak{g}_+)$ and $vy \in v\mathfrak{g}_- \subset V_v(\mathfrak{g}_-)$, where $\langle \, , \, \rangle : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbf{C}$ is the evaluation pairing. The pairing $(\, , \,)'_v$ has the following properties.

Lemma 10.5. *Let $X_1, \dots, X_m \in \mathfrak{g}_+$ and $Y_1, \dots, Y_n \in \mathfrak{g}_-$. If $m > n$, then*

$$(X_1 \cdots X_m, v^n Y_1 \cdots Y_n)'_v = 0. \tag{10.6}$$

If $m = n$, then

$$(X_1 \cdots X_m, v^n Y_1 \cdots Y_n)'_v = \sum_\sigma \langle X_{\sigma(1)}, Y_1 \rangle \cdots \langle X_{\sigma(m)}, Y_m \rangle, \tag{10.7}$$

where σ runs over all permutations of $\{1, \dots, n\}$.

If $m < n$, then

$$(X_1 \cdots X_m, v^n Y_1 \cdots Y_n)'_v \subset v^{n-m} \mathbf{C}[[v]]. \tag{10.8}$$

Proof. (i) We prove (10.6) and (10.7) by induction on n, using (2.11) and (10.5). The case $m = n = 1$ follows from (10.5). If $m > n = 1$, then by (2.4) and (2.11)

$$\begin{aligned}
&(X_1 \cdots X_m, v\,Y_1)'_v \\
&= (X_1 \otimes X_2 \cdots X_m, \Delta(v\,Y_1))'_v = (X_1 \otimes X_2 \cdots X_m, vY_1 \otimes 1 + 1 \otimes vY_1)'_v \\
&= (X_1, vY_1)'_v\,(X_2 \cdots X_m, 1)'_v + (X_1, 1)'_v\,(X_2 \cdots X_m, vY_1)'_v = 0.
\end{aligned}$$

Suppose we have proved (10.6) and (10.7) for $1, \ldots, n-1$. By (2.4),

$$\begin{aligned}
&\Delta(Y_1 \cdots Y_n) \\
&= 1 \otimes Y_1 \cdots Y_n + \sum_{p=1}^{n-1} \sum_{\sigma} Y_{\sigma(1)} \cdots Y_{\sigma(p)} \otimes Y_{\sigma(p+1)} \cdots Y_{\sigma(n)} + Y_1 \cdots Y_n \otimes 1,
\end{aligned}$$

where σ runs over all $(p, n-p)$-shuffles, i.e., all permutations of $\{1, \ldots, n\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$. Therefore,

$$\begin{aligned}
&(X_1 \cdots X_m, v^n\,Y_1 \cdots Y_n)'_v \\
&= (X_1 \otimes X_2 \cdots X_m, \Delta(v^n\,Y_1 \cdots Y_n))'_v \\
&= (X_1, 1)'_v\,(X_2 \cdots X_m, Y_1 \cdots Y_n)'_v \\
&\quad + \sum_{p=1}^{n-1} \sum_{\sigma} (X_1, v^p\,Y_{\sigma(1)} \cdots Y_{\sigma(p)})'_v\,(X_2 \cdots X_m, v^{n-p}\,Y_{\sigma(p+1)} \cdots Y_{\sigma(n)})'_v \\
&\quad + (X_1, v^n\,Y_1 \cdots Y_n)'_v\,(X_2 \cdots X_m, 1)'_v,
\end{aligned}$$

where σ runs over the same set of permutations as above. The first and last terms vanish by (2.11). If $m > n$, the middle sum is zero by the induction hypothesis on (10.6). If $m = n$, by (10.6), the only nonzero term is for $p = 1$, so that

$$\begin{aligned}
&(X_1 \cdots X_m, v^n\,Y_1 \cdots Y_n)'_v \\
&= \sum_{\sigma} (X_1, v\,Y_{\sigma(1)})'_v\,(X_2 \cdots X_m, v^{n-1}\,Y_{\sigma(2)} \cdots Y_{\sigma(n)})'_v,
\end{aligned}$$

where σ runs over all permutations of $\{1, \ldots, n\}$ such that $\sigma(2) < \cdots < \sigma(n)$. Therefore,

$$\begin{aligned}
&(X_1 \cdots X_m, v^n\,Y_1 \cdots Y_n)'_v \\
&= \sum_{i=1}^{n} (X_1, vY_i)'_v\,(X_2 \cdots X_m, v^{n-1}\,Y_1 \cdots \widehat{Y_i} \cdots Y_n)'_v,
\end{aligned}$$

where the hat on Y_i means that it is omitted from the product. We conclude with (10.5) and the induction hypothesis on (10.7).

We also prove (10.8) by induction on n. If $n = 1$, then necessarily $m = 0$ and the claim follows from (2.11). For the inductive step, observe that (10.4) implies that, for $X_1, \dots , X_m \in \mathfrak{g}_+$,

$$\Delta'(X_1 \dots X_m) = \sum_{k\geq 0} X'_k \otimes X''_k \, v^k,$$

where $X'_k \otimes X''_k \in \bigoplus_{p+q=k+m} S^p(\mathfrak{g}_+)\otimes S^q(\mathfrak{g}_+)$. By (2.11), we obtain

$$\begin{aligned}(X_1 \cdots X_m, v^n\, Y_1 \cdots Y_n)'_v &= (\Delta'(X_1 \cdots X_m), vY_1 \otimes v^{n-1} Y_2 \cdots Y_n)'_v \\ &= \sum_{k\geq 0} v^k (X'_k, vY_1)'_v \, (X''_k, v^{n-1}\, Y_2 \cdots Y_n)'_v.\end{aligned}$$

By (10.6) the only case where $(X'_k, vY_1)'_v$ may be nonzero is when $X'_k \in S^1(\mathfrak{g}_+)$, therefore when $X''_k \in S^{k+m-1}(\mathfrak{g}_+)$. If $k + m - 1 \leq n - 1$, we use (10.7) and the induction hypothesis on (10.8). Thus, $(X''_k, v^{n-1}\, Y_2 \cdots Y_n)'_v$ is divisible by v^{n-m-k}. If $k + m - 1 > n - 1$, then $(X''_k, v^{n-1}\, Y_2 \cdots Y_n)'_v = 0$ by (10.6). Therefore, $(X''_k, v^{n-1}\, Y_2 \cdots Y_n)'_v$ is divisible by v^{n-m-k} in all cases. Hence, $(X_1 \cdots X_m, v^n\, Y_1 \cdots Y_n)'_v$ is divisible by v^{n-m}. □

From the bialgebra pairing $(\ ,\ )'_v$ we get a morphism of algebras $\varphi' : E_v(\mathfrak{g}_+) \to V_v^*(\mathfrak{g}_-)$ defined by $\varphi'(a) = (a, -)'_v$ for $a \in E_v(\mathfrak{g}_+)$.

Corollary 10.6. *The bialgebra pairing* $(\ ,\ )'_v$ *is nondegenerate and the morphism of algebras* φ' *induces an isomorphism*

$$\varphi' : E_v(\mathfrak{g}_+) \to V_v^{\mathrm{o}}(\mathfrak{g}_-) \subset V_v^*(\mathfrak{g}_-).$$

Proof. By Proposition 10.3 it is enough to check that the pairing $(\ ,\ )'_v$ satisfies Condition (10.2). An easy computation shows that (10.2) is equivalent to (10.6–10.8). □

10.7. Proof of Theorem 2.9. Part II. By Corollaries 10.4 and 10.6 we have two algebra isomorphisms $\varphi : A_+/uA_+ \to V_v^{\mathrm{o}}(\mathfrak{g}_-)$ and $\varphi' : E_v(\mathfrak{g}_+) \to V_v^{\mathrm{o}}(\mathfrak{g}_-)$. Composing φ with the inverse of φ', we obtain an algebra isomorphism

$$\chi = \varphi'^{-1}\varphi : A_+/uA_+ \to E_v(\mathfrak{g}_+).$$

Let us check that χ is a morphism of coalgebras. By definition of φ, φ' and χ,

$$(a, \Psi'_-(b))_v = \varphi(a)(b) = \varphi'\big(\chi(a)\big)(b) = (\chi(a), b)'_v \tag{10.9}$$

for all $a \in A_+/uA_+$ and $b \in V_v(\mathfrak{g}_-)$. (For the definition of Ψ'_-, see Section 9.7.) Using (2.11) and (10.9), we obtain

$$\begin{aligned}(\Delta'(\chi(a)), b_1 \otimes b_2)'_v &= (\chi(a), b_1 b_2)'_v \\ &= (a, \Psi'_-(b_1 b_2))_v \\ &= (a, \Psi'_-(b_1)\Psi'_-(b_2))_v\end{aligned}$$

$$= (\Delta(a), \Psi'_-(b_1)\otimes\Psi'_-(b_2))_v$$
$$= \big((\chi\otimes\chi)(\Delta(a)), b_1\otimes b_2\big)'_v$$

for all $a \in A_+/uA_+$ and b_1, $b_2 \in V_v(\mathfrak{g}_-)$. Here Δ' is the comultiplication in $E_v(\mathfrak{g}_+)$ and Δ is the comultiplication in A_+/uA_+ induced by $\Delta_{u,v}$. Since the pairing $(\ ,\ )'_v$ is nondegenerate, $\Delta'\chi = (\chi\otimes\chi)\Delta$.

The bialgebra A_+/uA_+ is a (commutative) Poisson bialgebra with Poisson bracket $\{\ ,\ \}_v$ defined for $a_1, a_2 \in A_+$ by

$$\{p(a_1), p(a_2)\}_v = p\left(\frac{a_1a_2 - a_2a_1}{u}\right), \tag{10.10}$$

where $p : A_+ \to A_+/uA_+$ is the projection. The bialgebra isomorphism $\chi : A_+/uA_+ \to E_v(\mathfrak{g}_+)$ transfers this Poisson bracket to a Poisson bracket $\{\ ,\ \}'$ on $E_v(\mathfrak{g}_+)$. In order to show that χ is a morphism of Poisson bialgebras, we have to prove that $\{\ ,\ \}' = \{\ ,\ \}$. It suffices to check that $\{\ ,\ \}'$ satisfies Condition (2.9).

The pairing of Lemma 9.5 pairs the bialgebras A_+ and A_-^{cop}. Consequently,

$$(a_1a_2 - a_2a_1, b)_{u,v} = \big(a_1\otimes a_2, \Delta^{\mathrm{op}}_{u,v}(b) - \Delta_{u,v}(b)\big)_{u,v}$$

for all $a_1, a_2 \in A_+$ and $b \in A_-$. The bialgebra A_- being cocommutative modulo u (see Section 9.1), it follows that $\Delta^{\mathrm{op}}_{u,v}(b) - \Delta_{u,v}(b)$ is divisible by u; hence,

$$\left(\frac{a_1a_2 - a_2a_1}{u}, b\right)_{u,v} = \left(a_1\otimes a_2, \frac{\Delta^{\mathrm{op}}_{u,v}(b) - \Delta_{u,v}(b)}{u}\right)_{u,v}. \tag{10.11}$$

By Section 8.1 applied to A_- and by (9.5), the isomorphism $\psi_- : V_v(\mathfrak{g}_-)[[u]] \to A_-$ of Section 9.1 induces the isomorphism $\Psi'_- : V_v(\mathfrak{g}_-) \to A_-/uA_-$ of co-Poisson bialgebras. Therefore,

$$(\Psi'_-\otimes\Psi'_-)(\delta_v(vy)) = \frac{\Delta_{u,v}(b) - \Delta^{\mathrm{op}}_{u,v}(b)}{u} \mod u\, A_- \,\widehat{\otimes}_{\mathbf{C}[v][[u]]}\, A_- \tag{10.12}$$

for $vy \in v\mathfrak{g}_- \subset V_v(\mathfrak{g}_-)$ and $b \in A_-$ mapped onto $\Psi'_-(vy)$ under the projection $A_- \to A_-/uA_-$. Here, $\delta_v : V_v(\mathfrak{g}_-) \to V_v(\mathfrak{g}_-)\otimes_{\mathbf{C}[v]} V_v(\mathfrak{g}_-)$ is the Poisson cobracket defined by (2.5), where we have replaced u by v, and the Lie cobracket δ of $\mathfrak{g}$ by the Lie cobracket δ_- of $\mathfrak{g}_-$. By definition of $\mathfrak{g}_- = (\mathfrak{g}_+^{\mathrm{op}})^*$,

$$\langle x_1\otimes x_2, \delta_-(y)\rangle = -\langle [x_1\otimes x_2], y\rangle \tag{10.13}$$

for all $x_1, x_2 \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$.

Combining (10.10)-(10.12), we obtain

$$(\{p(a_1), p(a_2)\}_v, \Psi'_-(vy))_v = -\big(p(a_1)\otimes p(a_2), (\Psi'_-\otimes\Psi'_-)(\delta_v(vy))\big)_v \tag{10.14}$$

for all $a_1, a_2 \in A_+$ and $y \in \mathfrak{g}_-$. It follows from (2.5), (10.9), (10.13), and (10.14) that

$$(\{x_1, x_2\}', vy)'_v = \big(\chi^{-1}(\{x_1, x_2\}'), \Psi'_-(vy)\big)_v \tag{10.15}$$

$$
\begin{aligned}
&= \big(\{\chi^{-1}(x_1), \chi^{-1}(x_2)\}_v, \Psi'_-(vy)\big)_v \\
&= -\big(\chi^{-1}(x_1)\otimes\chi^{-1}(x_2), (\Psi'_-\otimes\Psi'_-)(\delta_v(vy))\big)_v \\
&= -\big(x_1\otimes x_2, \delta_v(vy)\big)'_v \\
&= -\big(x_1\otimes x_2, v^2\,\delta_-(y)\big)'_v \\
&= \big([x_1, x_2], vy\big)'_v
\end{aligned}
$$

for all x_1, $x_2 \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$.

On the other hand, the Poisson bracket $\{\ ,\ \}'$ induces the Poisson bracket (2.3) on $E_v(\mathfrak{g}_+)/vE_v(\mathfrak{g}_+) = S(\mathfrak{g}_+)$. Consequently, for all $x_1, x_2 \in \mathfrak{g}_+$,

$$
\{x_1, x_2\}' = [x_1, x_2] + \sum_{m\geq 1} X_m\, v^m, \tag{10.16}
$$

where $X_m \in S(\mathfrak{g}_+)$. Let $X_m^{(p)}$ be the component of X_m in $S^p(\mathfrak{g}_+)$. In order to check Condition (2.9) for $\{\ ,\ \}'$, it is enough to show that $X_m^{(p)} = 0$ for all $p = 0, 1$ and $m \geq 1$.

For the case $p = 0$, we use the counits ε of the bialgebras involved. Since ε vanishes on commutators in A_+, we have $\varepsilon(\{a_1, a_2\}_v) = 0$ in the quotient bialgebra A_+/uA_+. The map χ being also a morphism of bialgebras, $\varepsilon(\{x_1, x_2\}') = 0$ for all x_1, $x_2 \in \mathfrak{g}_+$. The map ε vanishing on $S^p(\mathfrak{g}_+)$ for $p \geq 1$ and being the identity on $S^0(\mathfrak{g}_+)$, Formula (10.16) implies

$$
0 = \varepsilon(\{x_1, x_2\}') = \varepsilon([x_1, x_2]) + \sum_{m\geq 1} \varepsilon(X_m)\, v^m = \sum_{m\geq 1} X_m^{(0)}\, v^m.
$$

Hence, $X_m^{(0)} = 0$ for all $m \geq 1$.

For $p = 1$, we use Lemma 10.5, (10.2), (10.15) and (10.16) in the following computation holding for all x_1, $x_2 \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$:

$$
\begin{aligned}
0 &= (\{x_1, x_2\}' - [x_1, x_2], vy)'_v \\
&= \sum_{m\geq 1} (X_m^{(1)}, vy)'_v\, v^m + \sum_{m\geq 1;\, p\geq 2} (X_m^{(p)}, vy)'_v\, v^m \\
&= \sum_{m\geq 1} \langle X_m^{(1)}, y\rangle\, v^m.
\end{aligned}
$$

Hence, $\langle X_m^{(1)}, y\rangle = 0$ for all $y \in \mathfrak{g}_-$ and all $m \geq 1$. Therefore, $X_m^{(1)} = 0$ for all $m \geq 1$. □

10.8. Remark. Our definition of the Poisson bracket $\{\ ,\ \}'$ gives a construction of a Poisson bracket on $E_v(\mathfrak{g}_+)$ that is independent of [**Tur91**, Theorem 11.4]. We have also proved that the topological dual $V_v^{\circ}(\mathfrak{g}_-)$ has a natural structure of a Poisson $\mathbf{C}[[v]]$-bialgebra.

10.9. Remark. There are similar versions of Theorems 2.3, 2.6, and 2.9 for the bialgebra $\widehat{A}_+$ of Section 7.1. To state them, we need the bi-Poisson bialgebra $\widehat{S}(\mathfrak{g}_+)$. As an algebra, it is the completion of $S(\mathfrak{g})$ with respect to its augmentation ideal $I_0 = \bigoplus_{m\geq 1} S^m(\mathfrak{g}_+)$:

$$\widehat{S}(\mathfrak{g}_+) = \prod_{n\geq 0} S^n(\mathfrak{g}_+).$$

The bi-Poisson bialgebra structure on $S(\mathfrak{g}_+)$ defined in Section 2.2 extends to a topological bi-Poisson bialgebra structure on $\widehat{S}(\mathfrak{g}_+)$, where the comultiplication and the Poisson cobracket take values in the completed tensor product

$$\widehat{S}(\mathfrak{g})\,\widehat{\otimes}_{\mathbf{C}}\,\widehat{S}(\mathfrak{g}) = \varprojlim_n \Big(S(\mathfrak{g})/I_0^n \otimes_{\mathbf{C}} S(\mathfrak{g})/I_0^n\Big).$$

The natural projection $q_u : V_u(\mathfrak{g}_+) \to S(\mathfrak{g}_+)$ of Section 2.4 extends to a bialgebra morphism $\widehat{V}_u(\mathfrak{g}_+) \to \widehat{S}(\mathfrak{g}_+)$ that induces a canonical isomorphism of bi-Poisson bialgebras

$$\widehat{V}_u(\mathfrak{g}_+)/u\widehat{V}_u(\mathfrak{g}_+) = \widehat{S}(\mathfrak{g}_+).$$

Similarly, the Poisson $\mathbf{C}[[v]]$-bialgebra structure on $E_v(\mathfrak{g}_+) = S(\mathfrak{g}_+)[[v]]$ extends uniquely to a topological Poisson $\mathbf{C}[[v]]$-bialgebra structure on $\widehat{E}_v(\mathfrak{g}_+) = \widehat{S}(\mathfrak{g}_+)[[v]]$. The projection $\widehat{E}_v(\mathfrak{g}_+) \to \widehat{S}(\mathfrak{g}_+)$ sending v to 0 induces a canonical isomorphism of bi-Poisson bialgebras

$$\widehat{E}_v(\mathfrak{g}_+)/v\widehat{E}_v(\mathfrak{g}_+) \to \widehat{S}(\mathfrak{g}_+).$$

Proceeding for $\widehat{A}_+$ as we did for A_+ in Sections 8–10, we can prove that there is an isomorphism of co-Poisson bialgebras $\widehat{A}_+/v\widehat{A}_+ = \widehat{V}_u(\mathfrak{g}_+)$, an isomorphism of Poisson bialgebras $\widehat{A}_+/u\widehat{A}_+ \cong \widehat{E}_v(\mathfrak{g}_+)$, and an isomorphism of bi-Poisson bialgebras $\widehat{A}_+/(u,v) = \widehat{S}(\mathfrak{g}_+)$.

11. Exchanging $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

Consider the Lie bialgebra $\mathfrak{g}'_+ = \mathfrak{g}_-$ and its double $\mathfrak{d}'$. By definition of the double, $\mathfrak{d}'$ contains $\mathfrak{g}'_- = ({\mathfrak{g}'_+}^*)^{\mathrm{cop}}$ as a Lie subbialgebra. Following Sections 5.3-5.4 for $\mathfrak{g}'_+$, we obtain three $\mathbf{C}[[h]]$-bialgebras $U_h(\mathfrak{g}'_+) \hookrightarrow U_h(\mathfrak{d}') \hookleftarrow U_h(\mathfrak{g}'_-)$. The aim of this section is to prove the following addition to [**EK96**], [**EK97**]. Here, for a bialgebra A, we denote by A^{cop} the bialgebra A obtained by replacing the comultiplication by the opposite comultiplication.

Theorem 11.1. *There is an isomorphism of* $\mathbf{C}[[h]]$*-bialgebras*

$$U_h(\mathfrak{d}') \cong U_h(\mathfrak{d})^{\mathrm{cop}}$$

sending $U_h(\mathfrak{g}'_+)$ *onto* $U_h(\mathfrak{g}_-)^{\mathrm{cop}}$ *and* $U_h(\mathfrak{g}'_-)$ *onto* $U_h(\mathfrak{g}_+)^{\mathrm{cop}}$.

Theorem 11.1 does not follow directly from the functoriality of Etingof and Kazhdan's quantization because in general there is no isomorphism between the triples $(\mathfrak{g}_+, \mathfrak{d}, \mathfrak{g}_-)$ and $(\mathfrak{g}'_+, \mathfrak{d}', \mathfrak{g}'_-)$ (nevertheless, see the proof of Theorem 1.18 in [**EK98**]). We have chosen to give a proof of this theorem using the original definitions of the bialgebras $U_h(\mathfrak{d})$, $U_h(\mathfrak{g}_+)$, $U_h(\mathfrak{g}_-)$ as given in [**EK96**]. These definitions will be recalled in Sections 11.2-11.4 below.

11.2. A Braided Monoidal Category. Consider the double Lie bialgebra $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of $\mathfrak{g}_+$ and let $\mathcal{S}$ be the category of $U(\mathfrak{d})$-modules. This is a symmetric monoidal category: The tensor product of two $U(\mathfrak{d})$-modules is given by $M \otimes N = M \otimes_{\mathbf{C}} N$ on which $U(\mathfrak{d})$ acts through its comultiplication, and the symmetry $\sigma_{M,N} : M\otimes N \to N \otimes M$ by the standard transposition $m\otimes n \mapsto n \otimes m$. The category $\mathcal{S}$ has an infinitesimal braiding $t_{M,N} : M\otimes N \to M\otimes N$ in the sense of Cartier [**Car93**] (see also [**Kas95**, Definition XX.4.1]). The morphism $t_{M,N}$ is given by the action of the two-tensor $t = r + r_{21} = \sum_{i=1}^{d} (x_i \otimes y_i + y_i \otimes x_i)$ of Section 5.3.

We now fix a Drinfeld associator Φ, as defined, e.g., in [**Dri89**], [**Dri90**], [**Kas95**, Section XIX.8], [**KT98**, Section 4.6]. This is a series $\Phi(A, B)$ in two non-commuting variables A and B with coefficients in $\mathbf{C}$ and constant term 1, subject to a certain set of equations (for details see the references above). Such a Φ exists by [**Dri90**] and can be assumed to be the exponential of a Lie series in A and B.

From $\mathcal{S}$ and Φ we construct a braided monoidal category $\mathcal{C}$ as follows: The objects of $\mathcal{C}$ are the same as the objects of $\mathcal{S}$. A morphism from M to N in $\mathcal{C}$ is a formal power series $\sum_{n\geq 0} f_n h^n$, where $f_n \in \mathrm{Hom}_{\mathcal{S}}(M,N) = \mathrm{Hom}_{U(\mathfrak{d})}(M,N)$ for all n. The composition in $\mathcal{C}$ is defined using the composition in $\mathcal{S}$ and the standard multiplication of formal power series. The identity morphism of an object M in $\mathcal{C}$ is the constant formal power series $\sum_{n\geq 0} f_n h^n$, where $f_0 = \mathrm{id}_M$ and $f_n = 0$ when $n > 0$. The category $\mathcal{C}$ has a tensor product: On objects it is the same as on the objects of $\mathcal{S}$; on morphisms it is obtained by extending $\mathbf{C}[[h]]$-linearly the tensor product of morphisms of $\mathcal{S}$. The unit object is the same as in $\mathcal{S}$, namely the trivial module $\mathbf{C}$ on which $U(\mathfrak{g})$ acts by the counit.

For any triple (L, M, N) of objects in $\mathcal{C}$ we define an associativity isomorphism $a_{L,M,N}$ and a braiding $c_{M,N}$ by

$$a_{L,M,N} = \Phi(h\, t_{L,M}\otimes \mathrm{id}_N, h\, \mathrm{id}_L \otimes t_{M,N}) : (L\otimes M)\otimes N \xrightarrow{\cong} L\otimes(M\otimes N) \tag{11.1}$$

and

$$c_{M,N} = \sigma_{M,N} \exp\Big(\frac{h}{2}\, t_{M,N}\Big) : M\otimes N \xrightarrow{\cong} N \otimes M, \tag{11.2}$$

where $\sigma_{M,N}$ is the transposition. For details, see [**Kas95**, XX.6].

The construction of $\mathcal{C}$, Formulas (11.1-11.2), and $\Phi(0,0) = 1$ imply that the braided monoidal category obtained as the quotient of $\mathcal{C}$ by the subclass of morphisms whose constant term as a formal power series in h is 0 is nothing else than the category $\mathcal{S}$ we started from. In this sense, $\mathcal{C}$ is a quantization of $\mathcal{S}$.

11.3. Definition of J_h. Following [**EK96**, Section 2.3], we first define $U(\mathfrak{d})$-modules $M_\pm = U(\mathfrak{d}) \otimes_{U(\mathfrak{g}_\pm)} \mathbf{C}$, where $\mathbf{C}$ is the trivial $U(\mathfrak{g}_\pm)$-module. The Verma module $M_\pm$ is a free $U(\mathfrak{g}_\mp)$-module on a generator $1_\pm$ such that $a \cdot 1_\pm = \varepsilon(a)1_\pm$ for all $a \in U(\mathfrak{g}_\pm)$, where ε is the counit of $U(\mathfrak{g}_\pm)$. There is an isomorphism $\varphi : U(\mathfrak{d}) \to M_+ \otimes M_-$ of $U(\mathfrak{d})$-modules such that

$$\varphi(1) = 1_+ \otimes 1_-. \tag{11.3}$$

There are also $U(\mathfrak{d})$-linear maps $i_\pm : M_\pm \to M_\pm \otimes M_\pm$ defined by $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$.

In the braided monoidal category $\mathcal{C}$ of Section 11.2 consider the isomorphism

$$\chi = \beta^{-1} \circ (\mathrm{id}_+ \otimes c_{M_+,M_-} \otimes \mathrm{id}_-) \circ \alpha : (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \\ \to (M_+ \otimes M_-) \otimes (M_+ \otimes M_-),$$

where $\mathrm{id}_\pm$ is the identity morphism of $M_\pm$, $c_{M_+,M_-} : M_+ \otimes M_- \to M_- \otimes M_+$ is the braiding, α is the composition of the associativity isomorphisms

$$\begin{array}{ccc} (M_+\otimes M_+)\otimes(M_-\otimes M_-) & \xrightarrow{a^{-1}_{M_+\otimes M_+, M_-, M_-}} & \big((M_+\otimes M_+)\otimes M_-\big)\otimes M_- \\ & & \Big\downarrow{\scriptstyle a_{M_+,M_+,M_-}\otimes \mathrm{id}_-} \\ & & \big(M_+\otimes(M_+\otimes M_-)\big)\otimes M_- \end{array}$$

and β is the composition of the isomorphisms

$$\begin{array}{ccc} (M_+\otimes M_-)\otimes(M_+\otimes M_-) & \xrightarrow{a^{-1}_{M_+\otimes M_-, M_+, M_-}} & \big((M_+\otimes M_-)\otimes M_+\big)\otimes M_- \\ & & \Big\downarrow{\scriptstyle a_{M_+,M_-,M_+}\otimes \mathrm{id}_-} \\ & & \big(M_+\otimes(M_-\otimes M_+)\big)\otimes M_-. \end{array}$$

Then, by [**EK96**, Formula (3.1)], the element $J_h \in (U(\mathfrak{d}) \otimes U(\mathfrak{d}))[[h]]$ determining the comultiplication of $U_h(\mathfrak{d})$ in (5.3) is defined by

$$(\varphi \otimes \varphi)(J_h) = \chi(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-) = \chi(i_+ \otimes i_-)(\varphi(1)). \tag{11.4}$$

11.4. Definition of $U_h(\mathfrak{g}_\pm)$. For any $f \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_-)$ consider the endomorphism $\mu_+(f) \in \mathrm{End}_{\mathcal{C}}(M_+ \otimes M_-)$ defined as the following composition of morphisms in the monoidal category $\mathcal{C}$ of Section 11.2:
(11.5)

$$M_+ \otimes M_- \xrightarrow{i_+ \otimes \mathrm{id}_-} (M_+ \otimes M_+) \otimes M_- \xrightarrow{a} M_+ \otimes (M_+ \otimes M_-) \xrightarrow{\mathrm{id}_+ \otimes f} M_+ \otimes M_-,$$

where $a = a_{M_+,M_+,M_-}$ is the associativity isomorphism defined by (11.1). Conjugating by the isomorphism φ of (11.3), we obtain the endomorphism $\varphi^{-1}\mu_+(f)\varphi \in \mathrm{End}_{\mathcal{C}}(U(\mathfrak{d}))$. Applying this endomorphism to the unit element in $U(\mathfrak{d})[[h]]$, we get the formal power series

$$f^+ = \big(\varphi^{-1}\mu_+(f)\varphi\big)(1) \in U(\mathfrak{d})[[h]].$$

By [**EK96**, Section 4.1], $U_h(\mathfrak{g}_+)$ is the image of the map $f \mapsto f^+$ from $\mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_-)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$.

There is a similar definition for $U_h(\mathfrak{g}_-)$. For any $g \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_+)$ define $\mu_-(g) \in \mathrm{End}_{\mathcal{C}}(M_+ \otimes M_-)$ as the following composition of morphisms in $\mathcal{C}$:

(11.6)

$$M_+ \otimes M_- \xrightarrow{\mathrm{id}_+ \otimes i_-} M_+ \otimes (M_- \otimes M_-) \xrightarrow{a^{-1}} (M_+ \otimes M_-) \otimes M_- \xrightarrow{g \otimes \mathrm{id}_-} M_+ \otimes M_-.$$

Applying the endomorphism $\varphi^{-1}\mu_+(f)\varphi \in \mathrm{End}_{\mathcal{C}}(U(\mathfrak{d}))$ to $1 \in U(\mathfrak{d})[[h]]$, we obtain

$$g^- = \big(\varphi^{-1}\mu_-(g)\varphi\big)(1) \in U(\mathfrak{d})[[h]].$$

By [**EK96**, Section 4.1], $U_h(\mathfrak{g}_-)$ is the image of the map $g \mapsto g^-$ from $\mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_+)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$.

11.5. Proof of Theorem 11.1. By Section 2.1,

$$\mathfrak{g}'_- = ({\mathfrak{g}'_+}^*)^{\mathrm{cop}} = (\mathfrak{g}_-^*)^{\mathrm{cop}} = (\mathfrak{g}_+^{\mathrm{op}})^{\mathrm{cop}}$$

is isomorphic to $\mathfrak{g}_+$ via the map $-\mathrm{id}_{\mathfrak{g}_+}$. Let $\mathfrak{d}' = \mathfrak{g}'_+ \oplus \mathfrak{g}'_-$ be the double Lie bialgebra of $\mathfrak{g}'_+$. We have $\mathfrak{d}' = \mathfrak{g}_- \oplus \mathfrak{g}_+ = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces. The following lemma is easily checked.

Lemma 11.6. *The endomorphism σ of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ that is the identity on $\mathfrak{g}_-$ and the opposite of the identity on $\mathfrak{g}_+$ is an isomorphism of Lie bialgebras $\sigma : \mathfrak{d} \to \mathfrak{d}'$ which fits in the following commutative diagram of Lie bialgebras, where the horizontal morphisms are the natural injections:*

$$\begin{array}{ccccc} \mathfrak{g}_- & \hookrightarrow & \mathfrak{d} & \hookleftarrow & \mathfrak{g}_+ \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \sigma}\downarrow & & {\scriptstyle -\mathrm{id}}\downarrow \\ \mathfrak{g}'_+ = \mathfrak{g}_- & \hookrightarrow & \mathfrak{d}' & \hookleftarrow & \mathfrak{g}'_- = (\mathfrak{g}_+^{\mathrm{op}})^{\mathrm{cop}}. \end{array}$$

The morphism σ sends the 2-tensor $r = \sum_{i=1}^d x_i \otimes y_i \in \mathfrak{d} \otimes \mathfrak{d}$ to

$$\sigma(r) = \sum_{i=1}^{d} (-x_i) \otimes y_i = -r \in \mathfrak{d}' \otimes \mathfrak{d}'.$$

Consequently, for the symmetric 2-tensor $t = r + r_{21}$, we have $\sigma(t) = -t$.

The Lie bialgebra isomorphism $\sigma : \mathfrak{d} \to \mathfrak{d}'$ induces a bialgebra isomorphism $\sigma : U(\mathfrak{d}) \to U(\mathfrak{d}')$, hence an algebra isomorphism between their quantizations (cf. Section 5.3):

$$\sigma : U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]] \to U(\mathfrak{d}')[[h]] = U_h(\mathfrak{d}').$$

For the definition of the comultiplication Δ'_h of $U_h(\mathfrak{d}')$ we follow Section 11.2 and construct a braided monoidal category $\mathcal{C}'$, using now the double Lie bialgebra $\mathfrak{d}' = \sigma(\mathfrak{d})$, the same Drinfeld associator Φ as above, and the two-tensor $t' = \sigma(t)$. The morphism σ induces a canonical isomorphism $\mathcal{C} = \mathcal{C}'$ of braided monoidal categories.

We also need Verma modules for $\mathfrak{d}'$. Following Section 11.4, they are defined by $M'_\pm = U(\mathfrak{d}') \otimes_{U(\mathfrak{g}'_\pm)} \mathbf{C}$. As a $U(\mathfrak{g}'_\mp)$-module, $M'_\pm$ is free on a generator $1'_\pm$. There is an isomorphism $\varphi' : U(\mathfrak{d}') \to M'_+ \otimes M'_-$ defined by $\varphi'(1) = 1'_+ \otimes 1'_-$. The homomorphism $\sigma : \mathfrak{d} \to \mathfrak{d}'$ induces canonical algebra isomorphisms $U(\mathfrak{g}_\pm) = U(\mathfrak{g}'_\mp)$, hence canonical isomorphisms

$$M_\pm = U(\mathfrak{d}) \otimes_{U(\mathfrak{g}_\pm)} \mathbf{C} = U(\mathfrak{d}') \otimes_{U(\mathfrak{g}'_\mp)} \mathbf{C} = M'_\mp.$$

Using these isomorphisms, we henceforth identify $\mathfrak{d}'$ with $\mathfrak{d}$, M'_+ with M_-, M'_- with M_+, $\varphi' : U(\mathfrak{d}') \to M'_+ \otimes M'_-$ with the isomorphism of $U(\mathfrak{d})$-modules $\varphi' : U(\mathfrak{d}) \to M_- \otimes M_+$ determined by

$$\varphi'(1) = 1_- \otimes 1_+. \tag{11.7}$$

By (5.3) the comultiplication Δ'_h of the bialgebra $U_h(\mathfrak{d}') = U_h(\mathfrak{d})$ is given for $a \in U(\mathfrak{d})[[h]]$ by

$$\Delta'_h(a) = (J'_h)^{-1} \Delta(a) J'_h,$$

where Δ is the standard comultiplication and J'_h is the element in $(U(\mathfrak{d}') \otimes U(\mathfrak{d}'))[[h]] = (U(\mathfrak{d}) \otimes U(\mathfrak{d}))[[h]]$ defined, according to (11.4) and using the above identifications, by

$$(\varphi' \otimes \varphi')(J'_h) = \chi'(1_- \otimes 1_- \otimes 1_+ \otimes 1_+) = \chi'(i_- \otimes i_+)(\varphi'(1)) \tag{11.8}$$

where χ' is obtained from the morphism χ of Section 11.3 by exchanging M_+ and M_-.

Consider the $U(\mathfrak{d})$-linear automorphism ν of $U(\mathfrak{d})$ defined by

$$\nu = (\varphi')^{-1} c_{M_+, M_-} \varphi, \tag{11.9}$$

where $c_{M_+, M_-} : M_+ \otimes M_- \to M_- \otimes M_+$ is the braiding. The morphism ν is the right multiplication by the invertible element $\omega = \nu(1) \in U(\mathfrak{d})[[h]]$:

$$\nu(a) = a\omega \tag{11.10}$$

for all $a \in U(\mathfrak{d})[[h]]$.

Lemma 11.7. *We have* $\omega \equiv 1$ *mod* h *and*

$$J'_h = \Delta(\omega)^{-1} \exp(ht/2) (J_h)_{21} (\omega \otimes \omega).$$

Proof. By (11.2), (11.3), (11.7), and (11.9) we have

$$\begin{aligned} \omega &= (\varphi')^{-1} \big(\exp(ht/2)(1_+ \otimes 1_-)\big)_{21} \\ &\equiv (\varphi')^{-1} (1_- \otimes 1_+) \equiv 1 \mod h. \end{aligned}$$

Let us compute J'_h. Below we shall prove that

(11.11)
$$c_{M_-\otimes M_+,M_-\otimes M_+}(c_{M_+,M_-}\otimes c_{M_+,M_-})\chi(i_+\otimes i_-) = \chi'(i_-\otimes i_+)c_{M_+,M_-},$$

where $c_{M_-\otimes M_+,M_-\otimes M_+} : (M_-\otimes M_+)\otimes(M_-\otimes M_+) \to (M_-\otimes M_+)\otimes(M_-\otimes M_+)$ is the braiding. Then, (11.9), (11.11) and the naturality of the braiding imply

(11.12)
$$\begin{aligned}((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\nu &= ((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)c_{M_+,M_-}\varphi\\ &= ((\varphi')^{-1}\otimes(\varphi')^{-1})c_{M_-\otimes M_+,M_-\otimes M_+}(c_{M_+,M_-}\otimes c_{M_+,M_-})\chi(i_+\otimes i_-)\varphi\\ &= c_{U(\mathfrak{d}),U(\mathfrak{d})}((\varphi')^{-1}\otimes(\varphi')^{-1})(c_{M_+,M_-}\otimes c_{M_+,M_-})\chi(i_+\otimes i_-)\varphi\\ &= c_{U(\mathfrak{d}),U(\mathfrak{d})}(\nu\otimes\nu)(\varphi^{-1}\otimes\varphi^{-1})\chi(i_+\otimes i_-)\varphi.\end{aligned}$$

Let us apply both sides of (11.12) to the unit in $U(\mathfrak{d})[[h]]$. By (11.8) and (11.10), we obtain for the left-hand side

$$\begin{aligned}&\Big(((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\nu\Big)(1)\\ &= \big(((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\big)(\omega)\\ &= \Delta(\omega)\big(((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\big)(1)\\ &= \Delta(\omega)J'_h.\end{aligned}$$

For the right-hand side, using (11.2), (11.4), (11.10), and the symmetry of t, we obtain

$$\begin{aligned}\big(c_{U(\mathfrak{d}),U(\mathfrak{d})}(\nu\otimes\nu)(\varphi^{-1}\otimes\varphi^{-1})\chi(i_+\otimes i_-)\varphi\big)(1) &= c_{U(\mathfrak{d}),U(\mathfrak{d})}\big((\nu\otimes\nu)(J_h)\big)\\ &= \big(\exp(ht/2)J_h(\omega\otimes\omega)\big)_{21}\\ &= \exp(ht/2)(J_h)_{21}(\omega\otimes\omega).\end{aligned}$$

Putting both computations together, we obtain the desired formula for J'_h.

Let us prove (11.11). By a well-known result of Mac Lane's, any braided monoidal category is equivalent to a strict braided monoidal category. It is therefore licit to omit the associativity isomorphisms in the computations. To simplify notation, we replace in the braidings the subscripts $M_\pm$ by $\pm$ and we omit the tensor product signs. With these conventions, $\chi = \mathrm{id}_+\otimes c_{+,-}\otimes\mathrm{id}_-$ and $\chi' = \mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+$. In $\mathcal{C}$ we have the following sequence of equalities implying (11.11) and justified below:

(11.13)
$$\begin{aligned}&c_{-+,-+}(c_{+,-}\otimes c_{+,-})\chi(i_+\otimes i_-)\\ &= c_{-+,-+}(c_{+,-}\otimes c_{+,-})(\mathrm{id}_+\otimes c_{+,-}\otimes\mathrm{id}_-)(i_+\otimes i_-)\\ &= (\mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+)c_{++,--}(c_{+,+}\otimes c_{-,-})(i_+\otimes i_-)\\ &= (\mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+)c_{++,--}(i_+\otimes i_-)\\ &= (\mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+)(i_-\otimes i_+)c_{+,-}\end{aligned}$$

$$= \chi'(i_-\otimes i_+)c_{+,-}$$

Here, the first and the last equalities hold by definition of χ and χ'. The second one is a consequence of the equality

$$c_{-+,-+}(c_{+,-}\otimes c_{+,-})(\mathrm{id}_+\otimes c_{+,-}\otimes \mathrm{id}_-) = (\mathrm{id}_-\otimes c_{-,+}\otimes \mathrm{id}_+)c_{++,--}(c_{+,+}\otimes c_{-,-}), \tag{11.14}$$

which holds in any braided monoidal category. This equality follows from the identity

$$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 = \sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3, \tag{11.15}$$

which holds in Artin's braid group on four strands B_4, where $\sigma_1, \sigma_2, \sigma_3$ are the standard generators of B_4.

The third equality in (11.13) is a consequence of

$$c_{\pm,\pm} = \mathrm{id}_\pm\otimes \mathrm{id}_\pm. \tag{11.16}$$

Since both sides of (11.16) are $U(\mathfrak{d})$-linear, it suffices to check this equality on the generator $1_\pm\otimes 1_\pm$ of $M_\pm\otimes M_\pm$. Now, by (11.2) and the vanishing of $t(1_\pm\otimes 1_\pm)$, we have

$$c_{\pm,\pm}(1_\pm\otimes 1_\pm) = \big(\exp(ht/2)(1_\pm\otimes 1_\pm)\big)_{21} = (1_\pm\otimes 1_\pm)_{21} = 1_\pm\otimes 1_\pm.$$

This proves (11.16). The fourth equality in (11.13) holds by naturality of the braiding. □

Corollary 11.8. *Let $\sigma_\omega : U_h(\mathfrak{d}) \to U_h(\mathfrak{d}')$ be the algebra isomorphism defined by $\sigma_\omega(a) = \sigma(\omega^{-1}a\omega)$ for all $a \in U_h(\mathfrak{d})$. Then σ_ω is a bialgebra isomorphism $U_h(\mathfrak{d})^{\mathrm{cop}} \cong U_h(\mathfrak{d}')$.*

Proof. We have to check that

$$\Delta'_h\sigma_\omega = (\sigma_\omega\otimes\sigma_\omega)\Delta_h^{\mathrm{op}}. \tag{11.17}$$

It follows from Lemma 11.7 that, for all $a \in U(\mathfrak{d})[[h]]$,

$$\begin{aligned}
&(\omega^{-1}\otimes\omega^{-1})\Delta_h^{\mathrm{op}}(a)(\omega\otimes\omega)\\
&= (\omega^{-1}\otimes\omega^{-1})(J_h^{-1})_{21}\Delta(a)(J_h)_{21}(\omega\otimes\omega)\\
&= (J'_h)^{-1}\Delta(\omega)^{-1}\exp(ht/2)\Delta(a)\exp(-ht/2)\Delta(\omega)J'_h.
\end{aligned}$$

The 2-tensor t being invariant, $\Delta(a)t = t\Delta(a)$, hence $\Delta(a)\exp(ht/2) = \exp(ht/2)\Delta(a)$. Therefore,

$$\begin{aligned}
(\omega^{-1}\otimes\omega^{-1})\Delta_h^{\mathrm{op}}(a)(\omega\otimes\omega) &= (J'_h)^{-1}\Delta(\omega)^{-1}\Delta(a)\Delta(\omega)J'_h\\
&= (J'_h)^{-1}\Delta(\omega^{-1}a\omega)J'_h\\
&= \Delta'_h(\omega^{-1}a\omega).
\end{aligned}$$

This implies (11.17). □

We now complete the proof of Theorem 11.1 by establishing that the bialgebra isomorphism $\sigma_\omega : U_h(\mathfrak{d})^{\mathrm{cop}} \to U_h(\mathfrak{d}')$ sends $U_h(\mathfrak{g}_\mp)$ onto $U_h(\mathfrak{g}'_\pm)$. We give the proof only for $\mathfrak{g}'_+$. The proof for $\mathfrak{g}'_-$ is similar.

For $f' \in \mathrm{Hom}_{\mathcal{C}'}(M'_+ \otimes M'_-, M'_-)$ consider the endomorphism $\mu'_+(f') \in \mathrm{End}_{\mathcal{C}}(M'_+ \otimes M'_-)$ defined as the following composition of morphisms in $\mathcal{C}'$:

(11.18)
$$M'_+ \otimes M'_- \xrightarrow{i'_+ \otimes \mathrm{id}'_-} (M'_+ \otimes M'_+) \otimes M'_- \xrightarrow{a'} M'_+ \otimes (M'_+ \otimes M'_-) \xrightarrow{\mathrm{id}'_+ \otimes f'} M'_+ \otimes M'_-.$$

Here $\mathrm{id}'_\pm$ is the identity morphism of $M'_\pm$, $i'_+ : M'_+ \to M'_+ \otimes M'_+$ is the analogue of $i_+ : M_+ \to M_+ \otimes M_+$, and a' is the corresponding associativity isomorphism. Conjugating by the isomorphism $\varphi' : U(\mathfrak{d}') \to M'_+ \otimes M'_-$, we obtain the endomorphism $(\varphi')^{-1} \mu'_+(f') \varphi' \in \mathrm{End}_{\mathcal{C}'}(U(\mathfrak{d}'))$, hence the formal power series

$$(f')^+ = \big((\varphi')^{-1} \mu'_+(f') \varphi'\big)(1) \in U(\mathfrak{d}')[[h]].$$

By definition, $U_h(\mathfrak{g}'_+)$ is the image of the map $f' \mapsto (f')^+$ from $\mathrm{Hom}_{\mathcal{C}}(M'_+ \otimes M'_-, M'_-)$ to $U_h(\mathfrak{d}') = U(\mathfrak{d}')[[h]]$. Under the above identifications, the morphism (11.18) in $\mathcal{C}'$ becomes for $f \in \mathrm{Hom}_{\mathcal{C}}(M_- \otimes M_+, M_+)$ the composition of morphisms in $\mathcal{C}$

$$\text{(11.19)} \quad \mu(f) : M_- \otimes M_+ \xrightarrow{i_- \otimes \mathrm{id}_+} (M_- \otimes M_-) \otimes M_+ \xrightarrow{a} M_- \otimes (M_- \otimes M_+) \xrightarrow{\mathrm{id}_- \otimes f} M_- \otimes M_+.$$

Therefore, the submodule $\sigma^{-1}(U_h(\mathfrak{g}'_+))$ of $U_h(\mathfrak{d})$ is the image of the map

$$f \mapsto f_- = \big((\varphi')^{-1} \mu(f) \varphi'\big)(1)$$

from $\mathrm{Hom}_{\mathcal{C}}(M_- \otimes M_+, M_+)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$, where $\varphi' : U(\mathfrak{d}) \to M_- \otimes M_+$ is defined by (11.7).

Let us compare the map $f \mapsto f_-$ with the map $g \mapsto g^-$ of Section 11.4. We shall prove below that

$$\text{(11.20)} \qquad c_{M_+,M_-} \mu_-(g) = \mu(g c^{-1}_{M_+,M_-}) c_{M_+,M_-}$$

for all $g \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_+)$. It follows from (11.9), (11.10), (11.20), and from the definitions of g^- and of f_- that

$$\begin{aligned}
(g c^{-1}_{M_+,M_-})_- &= \big((\varphi')^{-1} \mu(g c^{-1}_{M_+,M_-}) \varphi'\big)(1) \\
&= \big(\nu \varphi^{-1} c^{-1}_{M_+,M_-} \mu(g c^{-1}_{M_+,M_-}) c_{M_+,M_-} \varphi \nu^{-1}\big)(1) \\
&= \big(\nu \varphi^{-1} \mu_-(g) \varphi \nu^{-1}\big)(1) \\
&= \big(\nu \varphi^{-1} \mu_-(g) \varphi\big)(\omega^{-1}) \\
&= \nu\big(\omega^{-1} (\varphi^{-1} \mu_-(g) \varphi)(1)\big) \\
&= \nu(\omega^{-1} g^-) = \omega^{-1} g^- \omega = \sigma_\omega(g^-).
\end{aligned}$$

Consequently, $\sigma_\omega(U_h(\mathfrak{g}_-)) = U_h(\mathfrak{g}'_+)$.

It remains to prove (11.20). We use the simplified notation introduced in the proof of (11.11). By functoriality of the braiding in $\mathcal{C}$, we have

(11.21)
$$(i_-\otimes \mathrm{id}_+)c_{+,-} = c_{+,--}(\mathrm{id}_+\otimes i_-) \quad \text{and} \quad c_{+,-}(g\otimes \mathrm{id}_-) = (\mathrm{id}_-\otimes g)c_{+-,-}$$

for $g \in \mathrm{Hom}_{\mathcal{C}}(M_+\otimes M_-, M_+)$. Therefore, by definition of μ,

$$\begin{aligned} c_{+,-}^{-1}\mu(gc_{+,-}^{-1})c_{+,-} &= c_{+,-}^{-1}(\mathrm{id}_-\otimes(gc_{+,-}^{-1}))a(i_-\otimes \mathrm{id}_+)c_{+,-} \\ &= c_{+,-}^{-1}(\mathrm{id}_-\otimes g)(\mathrm{id}_-\otimes c_{+,-}^{-1})a(i_-\otimes \mathrm{id}_+)c_{+,-} \\ &= (g\otimes \mathrm{id}_-)c_{+-,-}^{-1}(\mathrm{id}_-\otimes c_{+,-}^{-1})ac_{+,--}(\mathrm{id}_+\otimes i_-). \end{aligned}$$

Since $\mu_-(g) = (g\otimes \mathrm{id}_-)a^{-1}(\mathrm{id}_+\otimes i_-)$, it suffices to observe that by the general properties of braided categories and (11.16),

$$ac_{+,--} = ac_{+,--}(\mathrm{id}_+\otimes c_{-,-}) = (\mathrm{id}_-\otimes c_{+,-})c_{+-,-}\, a^{-1}. \tag{11.22}$$

This completes the proof of (11.20) and Theorem 11.1. □

We end this section by computing the universal R-matrix R'_h of $U_h(\mathfrak{d}')$ in terms of the universal R-matrix R_h of $U_h(\mathfrak{d})$ and the invertible element $\omega \in U_h(\mathfrak{d})$.

Lemma 11.9. *We have* $R'_h = (\sigma_\omega\otimes\sigma_\omega)(R_h)_{21}$.

Proof. By (5.6) and Lemma 11.7 we have

$$\begin{aligned} R'_h &= (J'_h)_{21}^{-1} \exp(ht/2)\, J'_h \\ &= (\omega^{-1}\otimes\omega^{-1})J_h^{-1} \exp(-ht/2)\Delta(\omega) \\ &\qquad \cdot \exp(ht/2)\,\Delta(\omega)^{-1} \exp(ht/2)(J_h)_{21}(\omega\otimes\omega). \end{aligned}$$

As observed in the proof of Corollary 11.8, $\Delta(a)$ commutes with $\exp(ht/2)$ for any $a \in U_h(\mathfrak{d})$. Hence,

$$R'_h = (\omega^{-1}\otimes\omega^{-1})J_h^{-1} \exp(ht/2)(J_h)_{21}(\omega\otimes\omega) = (\omega^{-1}\otimes\omega^{-1})(R_h)_{21}(\omega\otimes\omega).$$

□

12. Proof of Theorem 2.11.

The aim of this section is to identify the bialgebra A_- of Section 9. As an application, we prove Theorem 2.11.

Let us apply the constructions of Sections 6–7 to the Lie bialgebra $\mathfrak{g}'_+ = \mathfrak{g}_-$ of Sections 5.2 and 11. We obtain a $\mathbf{C}[[u,v]]$-bialgebra $U_{u,v}(\mathfrak{g}'_+)$ containing a $\mathbf{C}[u][[v]]$-bialgebra $A_{u,v}(\mathfrak{g}'_+)$.

12.1. Exchanging u and v. Any $\mathbf{C}[[u,v]]$-module M gives rise to a $\mathbf{C}[[u,v]]$-module $\tau(M)$ defined as follows. As a vector space $\tau(M) = M$, but the action of u, v is different: The new action of u is defined as the multiplication by v and the new action of v is defined as the multiplication by u. Clearly, $\tau(\tau(M)) = M$. Similarly, exchanging the actions of u and v, we transform any $\mathbf{C}[u][[v]]$-module M into a $\mathbf{C}[v][[u]]$-module $\tau(M)$.

For the Lie bialgebra $\mathfrak{g}'_+ = \mathfrak{g}_-$, we obtain a $\mathbf{C}[v][[u]]$-bialgebra $A_{v,u}(\mathfrak{g}'_+)$ and a $\mathbf{C}[[u,v]]$-bialgebra $U_{v,u}(\mathfrak{g}'_+)$ by

$$A_{v,u}(\mathfrak{g}'_+) = \tau\big(A_{u,v}(\mathfrak{g}'_+)\big) \quad \text{and} \quad U_{v,u}(\mathfrak{g}'_+) = \tau\big(U_{u,v}(\mathfrak{g}'_+)\big). \tag{12.1}$$

It is clear that $A_{v,u}(\mathfrak{g}'_+) \subset U_{v,u}(\mathfrak{g}'_+)$.

Theorem 12.2. *There is an isomorphism of* $\mathbf{C}[[u,v]]$*-bialgebras*

$$\sigma_{\widetilde{\omega}} : U_{u,v}(\mathfrak{g}_-)^{\mathrm{cop}} \to U_{v,u}(\mathfrak{g}'_+)$$

sending A_-^{cop} *onto* $A_{v,u}(\mathfrak{g}'_+)$.

Proof. After extending the scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$ and exchanging u and v, the $\mathbf{C}[[h]]$-bialgebra isomorphism $\sigma_\omega : U_h(\mathfrak{d})^{\mathrm{cop}} \cong U_h(\mathfrak{d}')$ of Theorem 11.1 gives rise to a $\mathbf{C}[[u,v]]$-bialgebra isomorphism

$$\sigma_{\widetilde{\omega}} : U_{u,v}(\mathfrak{d})^{\mathrm{cop}} \to U_{v,u}(\mathfrak{d}') \tag{12.2}$$

sending $U_{u,v}(\mathfrak{g}_-)^{\mathrm{cop}}$ onto $U_{v,u}(\mathfrak{g}'_+)$ and $U_{u,v}(\mathfrak{g}_+)^{\mathrm{cop}}$ onto $U_{v,u}(\mathfrak{g}'_-)$. The isomorphism $\sigma_{\widetilde{\omega}}$ is given by $a \mapsto \widetilde{\sigma}(\widetilde{\omega}^{-1} a \widetilde{\omega})$, where $\widetilde{\sigma} : U_{u,v}(\mathfrak{g}_-) \cong U_{v,u}(\mathfrak{g}'_+)$ is the algebra isomorphism induced by extension of scalars from the algebra isomorphism $\sigma : U_h(\mathfrak{d}) \cong U_h(\mathfrak{d}')$ of Section 11.5, and where $\widetilde{\omega}$ is the invertible element of $U_{u,v}(\mathfrak{d}) = U(\mathfrak{d})[[u,v]]$ coming from the element $\omega \in U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$, cf. Section 4.6. As a consequence of Lemma 11.7, we have

$$\widetilde{\omega} \equiv 1 \mod uv. \tag{12.3}$$

The bialgebra $U_{u,v}(\mathfrak{d}')$ contains a universal R-matrix

$$R'_{u,v} \in U_{u,v}(\mathfrak{d}') \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} U_{u,v}(\mathfrak{d}')$$

defined in the same way as the element $R_{u,v} \in U_{u,v}(\mathfrak{d}) \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} U_{u,v}(\mathfrak{d})$ in Section 6. As an immediate corollary of Lemma 11.9,

$$R'_{u,v} = (\sigma_{\widetilde{\omega}} \otimes \sigma_{\widetilde{\omega}})(R_{u,v})_{21}. \tag{12.4}$$

We have to show that $\sigma_{\widetilde{\omega}}$ maps A_- onto $A_{v,u}(\mathfrak{g}'_+)$. We first describe $A_{u,v}(\mathfrak{g}'_+)$ following Sections 5.5 and 6.6. To begin with, we need a $\mathbf{C}[[h]]$-linear isomorphism $\alpha'_- : U_h(\mathfrak{g}'_-) \to U(\mathfrak{g}'_-)[[h]]$ such that $\alpha'_-(1) = 1$ and $\alpha'_- \equiv \mathrm{id}$ modulo h, and a $\mathbf{C}$-linear projection $\pi'_- : U(\mathfrak{g}'_-) \to U^1(\mathfrak{g}'_-) = \mathbf{C} \oplus \mathfrak{g}'_-$

that is the identity on $U^1(\mathfrak{g}'_-)$. We choose them in such a way that the following squares commute:

$$\begin{array}{ccccccc} U_h(\mathfrak{g}_+) & \xrightarrow{\alpha_+} & U(\mathfrak{g}_+)[[h]] & \qquad & U(\mathfrak{g}_+) & \xrightarrow{\pi_+} & U^1(\mathfrak{g}_+) \\ \sigma_\omega\big\downarrow & & \sigma\big\downarrow & & \sigma\big\downarrow & & \sigma\big\downarrow \\ U_h(\mathfrak{g}'_-) & \xrightarrow{\alpha'_-} & U(\mathfrak{g}'_-)[[h]], & & U(\mathfrak{g}'_-) & \xrightarrow{\pi'_-} & U^1(\mathfrak{g}'_-) \end{array} \tag{12.5}$$

where $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ has been chosen in Section 6.6 and $\pi_+ : U(\mathfrak{g}_+) \to U^1(\mathfrak{g}_+)$ in Section 9.1.

For any $y \in \mathfrak{g}_-$, let $\sigma(y)$ be the corresponding element in $\mathfrak{g}'_+$ and $\langle \sigma(y), - \rangle'$: $U^1(\mathfrak{g}'_-) \to \mathbf{C}$ be the $\mathbf{C}$-linear form extending the standard evaluation map $\langle \sigma(y), - \rangle' : \mathfrak{g}'_- \to \mathbf{C}$ and such that $\langle \sigma(y), 1 \rangle' = 0$. Following Section 5.5, given $y \in \mathfrak{g}_-$, we define a $\mathbf{C}[[h]]$-linear form $f'_{\sigma(y)} : U_h(\mathfrak{g}'_-) \to \mathbf{C}[[h]]$ for $a \in U_h(\mathfrak{g}'_-)$ by

$$f'_{\sigma(y)}(a) = \langle \sigma(y), \pi'_- \alpha'_-(a) \rangle'. \tag{12.6}$$

By extension of scalars, we obtain a $\mathbf{C}[[u,v]]$-linear form $\widetilde{f}'_{\sigma(y)} : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$. By Lemma 6.5, the element

$$\rho'_+(\widetilde{f}'_{\sigma(y)}) = (\mathrm{id} \otimes \widetilde{f}'_{\sigma(y)})(R'_{u,v}) \in U_{u,v}(\mathfrak{g}'_+) \tag{12.7}$$

is divisible by uv. Let $(y_1, \ldots, y_d)$ be the basis of $\mathfrak{g}_-$ dual to the basis $(x_1, \ldots, x_d)$ of $\mathfrak{g}_+$. In view of Section 6.6, $A_{u,v}(\mathfrak{g}'_+)$ is the $\mathbf{C}[u][[v]]$-submodule of $U_{u,v}(\mathfrak{g}'_+)$ generated by the elements

$$v^{-|\underline{k}|}\, \rho'_+(\widetilde{f}'_{\sigma(y_1)})^{k_1} \cdots \rho'_+(\widetilde{f}'_{\sigma(y_d)})^{k_d},$$

where $\underline{k}$ runs over all finite sequences of nonnegative integers.

Therefore, $A_{v,u}(\mathfrak{g}'_+) = \tau(A_{u,v}(\mathfrak{g}'_+))$ is the $\mathbf{C}[v][[u]]$-submodule of $U_{v,u}(\mathfrak{g}'_+)$ generated by the elements

$$u^{-|\underline{k}|}\, \rho'_+(\widetilde{f}'_{\sigma(y_1)})^{k_1} \cdots \rho'_+(\widetilde{f}'_{\sigma(y_d)})^{k_d}, \tag{12.8}$$

where $\underline{k}$ runs over all finite sequences of nonnegative integers.

In view of the definition of A_- (see Section 9.1), in order to prove that $\sigma_{\widetilde{\omega}}(A_-) = A_{v,u}(\mathfrak{g}'_+)$, it suffices to check that for all $y \in \mathfrak{g}_-$

$$\sigma_{\widetilde{\omega}}(\rho_-(\widetilde{g}_y)) = -\rho'_+(\widetilde{f}'_{\sigma(y)}), \tag{12.9}$$

where ρ_- is defined by (6.2) and $\widetilde{g}_y : U_{u,v}(\mathfrak{g}_+) \to \mathbf{C}[[u,v]]$ is the $\mathbf{C}[[u,v]]$-linear form extended from the linear form $g_y : U_h(\mathfrak{g}_+) \to \mathbf{C}[[h]]$ defined by (9.1).

Let us prove (12.9). First observe that, since $\sigma = -\mathrm{id}$ on $\mathfrak{g}_+$ and $\sigma = \mathrm{id}$ on $\mathfrak{g}_-$, we have

$$\langle \sigma(y), \sigma(x) \rangle' = -\langle x, y \rangle \tag{12.10}$$

for all $x \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$. It follows from (9.1), (12.5), (12.6), and (12.10) that

$$\begin{aligned} f'_{\sigma(y)}(\sigma_\omega(a)) &= \langle \sigma(y), \pi'_- \alpha'_-(\sigma_\omega(a))\rangle' \\ &= \langle \sigma(y), \sigma\pi_+\alpha_+(a)\rangle' \\ &= -\langle \pi_+\alpha_+(a), y\rangle = -g_y(a) \end{aligned}$$

for all $y \in \mathfrak{g}_-$ and $a \in U_h(\mathfrak{g}_+)$. By extension of scalars, we obtain

$$\widetilde{f}'_{\sigma(y)}(\sigma_{\widetilde{\omega}}(a)) = -\widetilde{g}_y(a) \tag{12.11}$$

for all $y \in \mathfrak{g}_-$ and $a \in U_{u,v}(\mathfrak{g}_+)$.

As a consequence of (6.2), (12.4), (12.7), and (12.11),

$$\begin{aligned} \rho'_+(\widetilde{f}'_{\sigma(y)}) &= (\mathrm{id}\otimes \widetilde{f}'_{\sigma(y)})(R'_{u,v}) \\ &= (\mathrm{id}\otimes \widetilde{f}'_{\sigma(y)})\Big((\sigma_{\widetilde{\omega}}\otimes\sigma_{\widetilde{\omega}})(R_{u,v})_{21}\Big) \\ &= (\widetilde{f}'_{\sigma(y)}\otimes \mathrm{id})(\sigma_{\widetilde{\omega}}\otimes\sigma_{\widetilde{\omega}})(R_{u,v}) \\ &= \sigma_{\widetilde{\omega}}\Big(\big(\widetilde{f}'_{\sigma(y)}\sigma_{\widetilde{\omega}}\otimes \mathrm{id}\big)(R_{u,v})\Big) \\ &= -\sigma_{\widetilde{\omega}}\Big((\widetilde{g}_y\otimes \mathrm{id})(R_{u,v})\Big) \\ &= -\sigma_{\widetilde{\omega}}(\rho_-(\widetilde{g}_y)). \end{aligned}$$

This proves (12.9) and completes the proof of Theorem 12.2. □

12.3. Proof of Theorem 2.11. Under the bialgebra isomorphism $A_-^{\mathrm{cop}} \cong A_{v,u}(\mathfrak{g}'_+)$ of Theorem 12.2, the nondegenerate bialgebra pairing $(\ ,\ )_{u,v}$ of Lemma 9.5 and Corollary 9.9 gives rise to a nondegenerate bialgebra pairing $A_{u,v}(\mathfrak{g}_+) \times A_{v,u}(\mathfrak{g}'_+) \to \mathbf{C}[[u,v]]$. The second assertion in Theorem 2.11 follows from (9.18) and (12.3). □

Appendix A. Biquantization of the trivial bialgebra.

Let $\mathfrak{g}_+$ be a d-dimensional Lie bialgebra with basis $(x_1, \ldots, x_d)$ and with dual basis $(y_1, \ldots, y_d)$. Assume throughout the appendix that $\mathfrak{g}_+$ is the trivial Lie bialgebra, i.e., with zero Lie bracket and cobracket:

$$[x_i, x_j] = 0 \quad \text{and} \quad \delta(x_i) = 0 \tag{A.1}$$

for all i and $j = 1, \ldots, d$. We now give a complete description of the biquantization $A_{u,v}(\mathfrak{g}_+)$ and of the pairing (9.9) under the hypothesis (A.1).

The dual Lie bialgebra $\mathfrak{g}_- = (\mathfrak{g}_+^*)^{\mathrm{cop}}$ is also trivial, whereas the double Lie bialgebra $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is not: It follows from (5.1) and (5.2) that the Lie bracket of $\mathfrak{d}$ is equal to zero, but not its Lie cobracket, which is given by $\delta(u) = [u\otimes 1 + 1\otimes u, r]$, where $r = \sum_{i=1}^d x_i\otimes y_i$.

We first determine the bialgebras $U_h(\mathfrak{d})$ and $U_h(\mathfrak{g}_\pm)$ of Section 5. Since $\mathfrak{d}$ is a trivial Lie algebra, we have

$$U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]] = S(\mathfrak{d})[[h]]. \tag{A.2}$$

This is not only an isomorphism of algebras, but also of bialgebras. Indeed, since $U_h(\mathfrak{d})$ is commutative, it follows from (5.3) that its comultiplication is the standard one: $\Delta_h = \Delta$.

In order to determine the subbialgebras $U_h(\mathfrak{g}_\pm)$ of $U_h(\mathfrak{d})$, we need Sections 11.2–11.4, whose notation we use freely. Consider the braided monoidal category $\mathcal{C}$ of Section 11.2. We claim that the associativity isomorphims are trivial:

$$a_{L,M,N} = \mathrm{id}_{L\otimes M\otimes N} \tag{A.3}$$

for any triple (L, M, N) of objects in $\mathcal{C}$. Indeed, since the Lie algebra $\mathfrak{d}$ is abelian, the morphisms $t_{L,M} \otimes \mathrm{id}_N$ and $\mathrm{id}_L \otimes t_{M,N}$ coming up in (11.1) commute with one another. Now, the Drinfeld associator $\Phi(A, B)$, being the exponential of a Lie series in the variables A and B, is equal to 1 if A and B commute. This proves (A.3).

On the Verma modules $M_\pm$, the braiding c_{M_+,M_-} is given by

$$c_{M_+,M_-}(1_+\otimes 1_-) = \exp(ht/2)(1_-\otimes 1_+)$$

in view of (11.2) and the symmetry of t. Since $\mathfrak{d}$ is abelian, we have

$$\exp(ht/2) = \prod_{i=1}^{d} \exp(h(x_i\otimes y_i)/2) \exp(hr_{21}/2).$$

Now, $r_{21}(1_-\otimes 1_+) = \sum_{i=1}^{d} y_i 1_-\otimes x_i 1_+ = 0$. Therefore

$$c_{M_+,M_-}(1_+\otimes 1_-) = \prod_{i=1}^{d} \exp(h(x_i\otimes y_i)/2)(1_-\otimes 1_+). \tag{A.4}$$

Let us give a formula for the isomorphism $\varphi : U(\mathfrak{d}) \to M_+\otimes M_-$ of (11.3). Since $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Lie algebras, any element of $U(\mathfrak{d}) = S(\mathfrak{d})$ is a linear combination of elements of the form ab, where $a \in S(\mathfrak{g}_+) \subset S(\mathfrak{d})$ and $b \in S(\mathfrak{g}_-) \subset S(\mathfrak{d})$. We have

$$\varphi(ab) = b1_+\otimes a1_-. \tag{A.5}$$

Indeed, using Sweedler's notation, the definition of $M_\pm$ as modules, and the commutativity of $U(\mathfrak{d}) = S(\mathfrak{d})$, we have

$$\begin{aligned}\varphi(ab) &= \Delta(ab)(1_+ \otimes 1_-)\\ &= \sum_{(a)(b)} a_{(1)}b_{(1)}1_+\otimes a_{(2)}b_{(2)}1_-\\ &= \sum_{(a)(b)} b_{(1)}a_{(1)}1_+\otimes a_{(2)}b_{(2)}1_-\end{aligned}$$

$$
\begin{aligned}
&= \sum_{(a)(b)} b_{(1)}\varepsilon(a_{(1)})1_+ \otimes a_{(2)}\varepsilon(b_{(2)})1_- \\
&= \left(\sum_{(b)} b_{(1)}\varepsilon(b_{(2)})1_+\right) \otimes \left(\sum_{(a)} \varepsilon(a_{(1)})a_{(2)}1_-\right) \\
&= b1_+ \otimes a1_- .
\end{aligned}
$$

It follows that, for $a \in S(\mathfrak{g}_+)$ and $b \in S(\mathfrak{g}_-)$,

$$\varphi(\exp(ab)) = \exp(b \otimes a)(1_+ \otimes 1_-). \tag{A.6}$$

Proposition A.1. *$U_h(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)[[h]]$ as bialgebras.*

Proof. We prove this for $U_h(\mathfrak{g}_+)$. There is a similar proof for $U_h(\mathfrak{g}_-)$.

By Section 11.4, $U_h(\mathfrak{g}_+)$ is the image of the map $f \mapsto f^+$ from $\mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_-)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$. We claim that this image is exactly the submodule $U(\mathfrak{g}_+)[[h]]$ of $U(\mathfrak{d})[[h]]$ consisting of the formal power series with coefficients in $U(\mathfrak{g}_+) \subset U(\mathfrak{d})$. Indeed, an element $f \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_-)$ is of the form $f = \sum_{i \geq 0} f_i\, h^i$ where the maps $f_i : M_+ \otimes M_- \to M_-$ are $U(\mathfrak{d})$-linear. Since $M_+ \otimes M_-$ is a rank-one free module generated by $1_+ \otimes 1_-$, the map f_i is determined by the element $a_i\, 1_- = f_i(1_+ \otimes 1_-) \in M_-$, where a_i is a well-defined element of $U(\mathfrak{g}_+)$. The claim will be proved if we show that $f^+ = \sum_{i \geq 0} a_i\, h^i$.

By (11.5), (A.3) and (A.5) we have

$$
\begin{aligned}
f^+ &= \big(\varphi^{-1} \mu_+(f) \varphi\big)(1) \\
&= \sum_{i \geq 0} \big(\varphi^{-1}(\mathrm{id}_+ \otimes f_i)\, a\, (i_+ \otimes \mathrm{id}_-)\varphi\big)(1)\, h^i \\
&= \sum_{i \geq 0} \big(\varphi^{-1}(\mathrm{id}_+ \otimes f_i)(i_+ \otimes \mathrm{id}_-)\varphi\big)(1)\, h^i \\
&= \sum_{i \geq 0} \big(\varphi^{-1}(\mathrm{id}_+ \otimes f_i)(i_+ \otimes \mathrm{id}_-)\big)(1_+ \otimes 1_-)\, h^i \\
&= \sum_{i \geq 0} \big(\varphi^{-1}(\mathrm{id}_+ \otimes f_i)\big)(1_+ \otimes 1_+ \otimes 1_-)\, h^i \\
&= \sum_{i \geq 0} \varphi^{-1}(1_+ \otimes a_i\, 1_-)\, h^i = \sum_{i \geq 0} a_i\, h^i .
\end{aligned}
$$

The fact that $U_h(\mathfrak{g}_\pm) = U(\mathfrak{g}_\pm)[[h]]$ is a subbialgebra of $U(\mathfrak{d})[[h]]$, hence has the standard product and coproduct, follows from the obvious fact that $U(\mathfrak{g}_\pm)$ is a subbialgebra of $U(\mathfrak{d})$. The Lie algebras $\mathfrak{d}$ and $\mathfrak{g}_\pm$ being abelian, we have $U(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)$. Consequently, $U_h(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)[[h]]$ as bialgebras. □

Corollary A.2. *The bialgebra $\widehat{A}_+$ is the subbialgebra of $S(\mathfrak{g}_+)[[u,v]]$ consisting of the formal power series $\sum_{m,n\geq 0} a_{m,n}\, u^m v^n$ such that $a_{m,n} \in \bigoplus_{k=0}^m S^k(\mathfrak{g}_+)$ for all $m \geq 0$.*

Proof. By (6.1), Proposition A.1 and Lemma 4.7, we have $U_{u,v}(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)[[u,v]]$. We conclude in view of (7.1) and of Proposition 3.8. □

Similarly, the bialgebra $\widehat{A}_-$ of Section 9.1 is the subbialgebra of $S(\mathfrak{g}_-)[[u,v]]$ consisting of the formal power series $\sum_{m,n\geq 0} b_{m,n}\, u^m v^n$ such that $b_{m,n} \in \bigoplus_{k=0}^n S^k(\mathfrak{g}_-)$ for all $n \geq 0$.

In order to determine the subalgebras $A_{u,v}(\mathfrak{g}_+)$ and A_- defined in Sections 6.6 and 9.1, we have to make explicit the element

$$R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$$

of Section 6. Let J_h and R_h be the elements of $(U(\mathfrak{d})\otimes_{\mathbf{C}} U(\mathfrak{d}))[[h]]$ given by (11.4) and (5.6), respectively.

Lemma A.3. *We have $J_h = \exp(hr/2)$ and $R_h = \exp(hr)$.*

Proof. By (11.4), (A.4) and (A.5), we have

$$\begin{aligned}
(\varphi\otimes\varphi)(J_h) &= \chi(1_+\otimes 1_+\otimes 1_-\otimes 1_-)\\
&= \exp(ht_{23}/2)\cdot(1_+\otimes 1_-\otimes 1_+\otimes 1_-)\\
&= \exp(hr_{23}/2)\cdot(1_+\otimes 1_-\otimes 1_+\otimes 1_-)\\
&= \sum_{n\geq 0} \frac{h^n}{2^n n!}\left(\sum_{i=1}^d 1\otimes x_i\otimes y_i\otimes 1\right)^n (1_+\otimes 1_-\otimes 1_+\otimes 1_-)\\
&= 1_+\otimes\left(\sum_{n\geq 0}\frac{h^n}{2^n n!}\sum_{i_1,\dots,i_n=1}^d x_{i_1}\cdots x_{i_n}1_-\otimes y_{i_1}\cdots y_{i_n}1_+\right)\otimes 1_-\\
&= (\varphi\otimes\varphi)\left(\sum_{n\geq 0}\frac{h^n}{2^n n!}\sum_{i_1,\dots,i_n=1}^d x_{i_1}\cdots x_{i_n}\otimes y_{i_1}\cdots y_{i_n}\right)\\
&= (\varphi\otimes\varphi)\big(\exp(hr/2)\big).
\end{aligned}$$

Formula (5.6) implies

$$R_h = (J_h^{-1})_{21}\, \exp\left(\frac{ht}{2}\right)\, J_h = \exp\left((-r_{21}+r+r_{21}+r)\frac{h}{2}\right) = \exp(hr).$$

□

Corollary A.4. *We have*

$$R_{u,v} = \exp(uvr) = \sum_{n\geq 0}\frac{u^n v^n}{n!}\sum_{i_1,\dots,i_n=1}^d x_{i_1}\cdots x_{i_n}\otimes y_{i_1}\cdots y_{i_n}.$$

From $R_{u,v}$ we get maps $\rho_+ : U_{u,v}^*(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_+)$ and $\rho_- : U_{u,v}^*(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_-)$ as in Section 6. Formula (5.10) defines a $\mathbf{C}[[h]]$-linear form $f_x : U_h(\mathfrak{g}_-) = S(\mathfrak{g}_-)[[h]] \to \mathbf{C}[[h]]$, where we may take $\alpha_- = \mathrm{id}$ and $\pi_- : U(\mathfrak{g}_-) = S(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-) = \mathbf{C} \oplus \mathfrak{g}_-$ the natural projection. It follows that the map $\widetilde{f}_x : U_{u,v}(\mathfrak{g}_-) = S(\mathfrak{g}_-)[[u,v]] \to \mathbf{C}[[u,v]]$ of Section 6.4 is given for $b = \sum_{m,n\geq 0} b_{m,n}\, u^m v^n \in S(\mathfrak{g}_-)[[u,v]]$ by

$$\widetilde{f}_x(b) = \sum_{n\geq 0} \langle x, \pi(b_{m,n})\rangle\, u^m v^n. \tag{A.7}$$

Lemma A.5. *We have* $v^{-1}\rho_+(\widetilde{f}_x) = ux$ *for all* $x \in \mathfrak{g}_+$.

Proof. By (6.2), (A.7) and Corollary A.4 we get

$$\begin{aligned}
\rho_+(\widetilde{f}_x) &= (\mathrm{id}\otimes\widetilde{f}_x)(R_{u,v}) \\
&= \sum_{n\geq 0} \frac{u^n v^n}{n!} \sum_{i_1,\dots,i_n=1}^{d} \widetilde{f}_x(y_{i_1}\cdots y_{i_n})\, x_{i_1}\cdots x_{i_n} \\
&= \sum_{n\geq 0} \frac{u^n v^n}{n!} \sum_{i_1,\dots,i_n=1}^{d} \langle x, \pi(y_{i_1}\cdots y_{i_n})\rangle\, x_{i_1}\cdots x_{i_n} \\
&= uv \sum_{i=1}^{d} \langle x, \pi(y_i)\rangle\, x_i = uv \sum_{i=1}^{d} \langle x, y_i\rangle\, x_i = uv\, x.
\end{aligned}$$

□

Corollary A.6. $A_{u,v}(\mathfrak{g}_+)$ *consists of the formal power series* $\sum_{m,n\geq 0} a_{m,n}\, u^m v^n$ *such that* $a_{m,n} \in \bigoplus_{k=0}^m S^k(\mathfrak{g}_+)$ *for all* $m \geq 0$, *and for all* $n \geq 0$ *there exists* N *with* $a_{m,n} = 0$ *for all* $m > N$.

Similarly, the bialgebra A_- consists of the formal power series $\sum_{m,n\geq 0} b_{m,n}\, u^m v^n$ such that $b_{m,n} \in \bigoplus_{k=0}^n S^k(\mathfrak{g}_-)$ for all $n \geq 0$, and for all $m \geq 0$ there exists M with $b_{m,n} = 0$ for all $n > M$. Together with Corollary A.6, this implies that

$$A_- = A_{v,u}(\mathfrak{g}_-).$$

Let us describe the bialgebra pairing $(\ ,\ )_{u,v} : A_{u,v}(\mathfrak{g}_+) \times A_-^{\mathrm{cop}} \to \mathbf{C}[[u,v]]$ defined by (9.9). By (2.11) and Corollary A.6, it suffices to compute $(ux, vy)_{u,v}$ when $x \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$. The following result shows that the pairing $(\ ,\ )_{u,v}$ is the standard one.

Lemma A.7. *We have* $(ux, vy)_{u,v} = \langle x, y\rangle$ *for all* $x \in \mathfrak{g}_+$ *and* $y \in \mathfrak{g}_-$.

Proof. By (9.9), (A.7), and Lemma A.5 we have

$$(ux, vy)_{u,v} = \big(\rho_+^{-1}(ux)\big)(vy) = v^{-1}\,\widetilde{f}_x(vy) = v^{-1}v\,\langle x, \pi(y)\rangle = \langle x, y\rangle.$$

□

A.8. Remark. The reader may check, using (A.4) and (A.6), that the invertible element $\omega \in U_h(\mathfrak{d}) = S(\mathfrak{d})[[h]]$ defined by (11.10) is given by

$$\omega = \exp\left(\frac{h}{2}\sum_{i=1}^{d} x_i y_i\right).$$

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ON EQUIVALENCE OF TWO CONSTRUCTIONS OF INVARIANTS OF LAGRANGIAN SUBMANIFOLDS

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We give the construction of symplectic invariants which incorporates both the "infinite dimensional" invariants constructed by Oh in 1997 and the "finite dimensional" ones constructed by Viterbo in 1992.

1. Introduction.

Let M be a compact smooth manifold. Its cotangent bundle T^*M carries a natural symplectic structure associated to a Liouville form $\theta = pdq$. For a given compactly supported Hamiltonian function $H : T^*M \to \mathbb{R}$ and a closed submanifold $N \subset M$ Oh [**30**, **27**] defined a symplectic invariants of certain Lagrangian submanifolds in T^*M in a following way. Let $\nu^*N \subset T^*M$ be a conormal bundle of N. Denote by $HF_*^\lambda(H, N; M)$ the Floer homology groups generated by Hamiltonian orbits γ starting at the zero section and ending at ν^*N such that $\mathcal{A}_H(\gamma) := \int_\gamma pdq - Hdt \leq \lambda$ (see, e.g., [**30**]). In particular, for $\lambda = \infty$ we write $HF_*(H, N; M) := HF_*^\infty(H, N; M)$. These groups are known to be isomorphic to $H_*(N)$ [**31**]. We denote the corresponding isomorphism by F. For $a \in H_*(N)$ one defines

$$\rho(a, H : N) := \inf\{\lambda \mid F_H(a) \in \operatorname{Im}(j_*^\lambda) \subset HF_*(H, N; M)\}, \tag{1}$$

where $j_*^\lambda : HF_*^\lambda \to HF_*(H, N; M)$ is a well defined inclusion homomorphism. It is proved in [**30**] that ρ is a well defined invariant which (after a suitable normalization of H) depends only on a Lagrangian submanifold $L := \phi^H(O_M)$ and not on a particular choice of H. We refer the reader to [**26**, **29**, **30**, **27**] for more details.

This construction can be considered as an infinite dimensional version of a construction given earlier by Viterbo [**38**]. Let L be a Hamiltonian deformation of the zero section o_M. It is known [**21**] that L can be realized as

$$L = \left\{ \left(x, \frac{\partial S}{\partial x}\right) \Big| (x, \zeta) \in \left(\frac{\partial S}{\partial \zeta}\right)^{-1}(0) \right\},$$

where $S : M \times \mathbb{R}^m \to \mathbb{R}$ is a smooth function fiberwise quadratic outside a compact set. Using that result, Viterbo [**38**] defined symplectic invariants of L associated to a homology classes of a base M in a following way. For a class $a \in H_*(N)$ denote by Ta its lift to $H_*(S_N^\infty, S_N^{-\infty})$, where S_N is the restriction of S to $N \times \mathbb{R}^m$ and $S_N^{\pm\infty} := S_N^{-1}((-\infty, \lambda]) =: S_N^\lambda$ for large λ. Note that this makes sense since S is quadratic at infinity. Then one sets

$$c(a, S : N) := \inf\{\lambda | Ta \in Im(j_*^\lambda) \subset H_*(S^\infty, S^{-\infty})\}, \tag{2}$$

where $j_*^\lambda : H_*(S_N^\lambda, S_N^{-\infty}) \to H_*(S_N^\infty, S_N^{-\infty})$ is an obvious inclusion homomorphism. Viterbo proved that these invariants essentially depend only on L, and not on S. Viterbo carried out the construction for $N = M$ (which generalizes easily to closed $N \subset M$) and for an arbitrary vector bundle $E \to M$. As Viterbo's invariants do not change under a stabilization (i.e., replacing $S : E \to \mathbb{R}$ by $S \oplus Q : E \oplus F \to \mathbb{R}$), it is enough to consider the case $E = M \times \mathbb{R}^m$. We refer the reader to [**38**] for more details. For an alternative construction via Morse homology see [**25**].

The natural question of the equality between the two invariants is raised in [**30**]. In [**26**] we outlined a proof, constructing the invariants which interpolate the above two. The main technical tool, which we omitted in [**26**] was the construction of the interpolated Floer-Morse theory on $T^*(M \times \mathbb{R}^m)$ with an arbitrary coefficient ring. The purpose of this paper is to give the details of this construction. Another way of interpolating Floer and Morse homologies for generating functions, in the case $M = N$ was given by Viterbo in [**39**, **37**].

The dependence of the above invariants on the subset $N \subset M$, in particular the continuity with respect to the C^1-topology of submanifolds is an interesting question, which was further studied by Kasturirangan and Oh [**18**, **19**]. Some applications to wave fronts and Hofer's geometry are given in [**30**].

At the end, we give an application of our result to Hofer's geometry of Lagrangian submanifolds.

2. Preliminaries and notation.

Let M be a compact smooth manifold and $E := M \times \mathbb{R}^m$. The cotangent bundle $T^*E = T^*M \times \mathbb{C}^m$ carries the natural symplectic structure $\omega \oplus \omega_0$.

For a fixed relatively compact open set $K \subset E$ and a Riemannian metric g_M on M we denote

$$\mathcal{G}_{g_M \oplus g_0} := \text{ the set of metrics on } E \text{ which coincide with } g_M \oplus g_0 \text{ outside } K,$$

where g_0 is a standard Euclidean metric on $\mathbb{R}^m$. For a given non-degenerate fiberwise quadratic form Q on E, we denote by $\mathcal{S}_{(E,Q)}$ the set of all smooth functions $S : E \to \mathbb{R}$ such that $S = Q$ outside K and

$$\sum_{k=0}^{\infty} \varepsilon_k \|S - Q\|_{C^k} < \infty \tag{3}$$

for some sequence ε_k of positive real numbers.

Similarly, let $\mathcal{H}(E)$ denote the set of smooth functions $H : T^*E \times [0,1] \to \mathbb{R}$ such that outside K

$$H(x,\xi) = H_1 \oplus H_2(x,\xi) := H_1(x) + H_2(\xi)$$

for some compactly supported functions $H_1 : T^*M \to \mathbb{R}$ and $H_2 : \mathbb{C}^m \to \mathbb{R}$ and

$$\sum_{k=0}^{\infty} \varepsilon_k \|H\|_{C^k} < \infty. \tag{4}$$

Equipped with norms (3) and (4) the spaces

$$\mathcal{S}_{(E,Q)} - Q := \{S - Q \mid S \in \mathcal{S}_{(E,Q)}\}$$

and $\mathcal{H}(E)$ become separable Banach spaces which are (for suitably chosen sequence ε_k) dense in $L^2(E)$ and $L^2(T^*E)$ (see [**11**]).

For a closed submanifold $N \subset M$ and a function $S \in \mathcal{S}_{(E,Q)}$ we define the space of paths

$$\Omega(S;N) := \{\Gamma : [0,1] \to T^*E \mid \Gamma(0) \in \mathrm{Graph}(dS),\ \Gamma(1) \in \nu^*(N \times \mathbb{R}^m)\},$$

and

$$\mathcal{P}^p_{k:loc}(S;N) := \{U : \mathbb{R} \to \Omega(S;N) \mid U \in W^{k,p}_{\mathrm{loc}}(\mathbb{R} \times [0,1], T^*E)\}.$$

After restricting $\mathcal{A}_H$ to $\Omega(S;N)$, for a given path $\Gamma := (\gamma, z) : [0,1] \to T^*E$ and a pair $(H,S) \in \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ the first variation formula gives

$$\begin{aligned} d\mathcal{A}_H(\Gamma)\eta &= \int_0^1 \left[\omega\left(\frac{d\gamma}{dt}, \eta\right) - dH(\gamma(t),t)\eta\right] dt - \theta\eta(0) \\ &= \int_0^1 \left[\omega\left(\frac{d\gamma}{dt}, \eta\right) - dH(\gamma(t),t)\eta\right] dt - dS(\pi(\Gamma(0)))T\pi(\eta(0)), \end{aligned}$$

where $\pi : T^*E \to E$ is the natural projection. Therefore, to get a good variational problem, we set

$$\mathcal{A}_{(H,S)}(\Gamma) = \mathcal{A}_H(\Gamma) + S(\pi(\Gamma(0)))$$

(c.f. [**30**]). Straightforward computation yields:

$$\begin{aligned} &d\mathcal{A}_{(H,S)}(\Gamma)\eta \\ &= \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta]dt \\ &\quad + \langle \Gamma(1), T\pi(\eta(1))\rangle - \langle \Gamma(0), T\pi(\eta(0))\rangle + dS(\pi(\Gamma(0)))T\pi(\eta(0)). \end{aligned}$$

After restricting to $\Omega(S;N)$

$$d\mathcal{A}_{(H,S)}(\Gamma)\eta = \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta]dt. \tag{5}$$

Hence, the critical points $\Gamma := (\gamma, z) : [0,1] \to T^*E$ of $\mathcal{A}_{(H,S)}$ on $\Omega(S;N)$ are the solutions of

$$\begin{cases} \dot{\Gamma} = X_H(\Gamma) \\ (\gamma(0), z(0)) \in \mathrm{Graph}(dS) \\ \gamma(1) \in \nu^* N, \ z(1) \in o_{\mathbb{R}^m}. \end{cases} \tag{6}$$

Note that $\Gamma \mapsto \Gamma(1)$ establishes the one-to-one correspondence

$$\begin{aligned} \mathrm{Crit}(\mathcal{A}_H) &= \{\Gamma : [0,1] \to T^*E \mid \Gamma \text{ satisfies (6)}\} \\ &\cong \phi_1^H(\mathrm{Graph}(dS)) \cap \nu^*(N \times \mathbb{R}^m). \end{aligned}$$

For a given Riemannian metric g_M on M, we denote by J_{g_M} the almost complex structure which satisfies the following conditions:

1) J_{g_M} is compatible with the canonical symplectic structure ω on T^*M.
2) J_{g_M} maps the vertical tangent vectors to the horizontal vectors with respect to the Levi-Civita connection of g_M.
3) On the zero section $o_M \subset T^*M$ J_{g_M} assigns to each vector $v \in T_qM$ the cotangent vector $J_{g_M}(v) = g_M(v, .)$ with obvious identifications.

Denote by $j^c_\omega(M)$ the set of ω-compatible almost complex structures which coincide with J_{g_M} outside a compact set in T^*M, and by $\mathcal{J}^c_\omega(M)$ the set of smooth paths $J_t : [0,1] \to j^c_\omega(M)$.

For a path $\{J_t\} \in \mathcal{J}^c_\omega(M)$, the family of product almost complex structures

$$J \oplus i := \{J_t \oplus i\}_{0 \le t \le 1}$$

is compatible with the product symplectic structure $\omega \oplus \omega_0$ on $T^*E = T^*M \times \mathbb{C}^m$. Denote by $\mathcal{J}^c_\omega(E)$ the set of almost complex structures on T^*E which coincide with product structure $J_{g_M} \oplus i$ outside a compact set. Those almost complex structures induce the family of metrics

$$\langle \eta_1, \eta_2 \rangle_{J_t} := \omega \oplus \omega_0(\eta_1, J_t \eta_2)$$

and hence an L^2-type metric

$$\langle\langle \eta_1, \eta_2 \rangle\rangle_J := \int_0^1 \langle \eta_1(t), \eta_2(t) \rangle_{J_t} dt$$

on $\Omega(S;N)$.

In terms of metric $\langle\langle .,. \rangle\rangle_J$ the gradient flow $U := (u, v) \in \mathcal{P}^p_{k:loc}(S; N)$ of $\mathcal{A}_{(H,S)}$ restricted to $\Omega(S; N)$ satisfies

$$
\begin{cases} \overline{\partial}_{J,H} U := \frac{\partial U}{\partial \tau} + J\left(\frac{\partial U}{\partial t} - X_H(U)\right) = 0 \\ (u(\tau, 0), v(\tau, 0)) \in \mathrm{Graph}(dS) \\ u(\tau, 1) \in \nu^* N,\ v(\tau, 1) \in o_{\mathbb{R}^m}. \end{cases} \tag{7}
$$

Denote by $CF(H, S : N)$ the set of critical points of $\mathcal{A}_{(H,S)}|_{\Omega(S;N)}$. Then

$$
CF(H, S : N) = \{\Gamma = (\gamma, z) \mid \Gamma \text{ satisfies (6)}\}.
$$

The set of critical values of $\mathcal{A}_{(H,S)}$ in $\mathbb{R}$

$$
\mathrm{Spec}(H, S : N) := \mathcal{A}_{(H,S)}(CF(H, S : N))
$$

is called *the action spectrum of* $\mathcal{A}_{(H,S)}$.

In the construction of Floer homology we will impose on the functions in $\mathcal{S}_{(E,Q)}$ the generic transversality condition

$$
\mathrm{Graph}(dS) \pitchfork (\phi_1^H)^{-1}(\nu^* N \times \mathbb{R}^m). \tag{8}
$$

Under assumption (8), the sets $CF(H, S : N)$ and $\mathrm{Spec}(H, S : N)$ are finite. In the general case, we have the following lemma, which describes the size of set $\mathrm{Spec}(H, S : N)$. Similar results were established in [**17**, **30**].

Lemma 1. *The action spectrum* $\mathrm{Spec}(H, S : N)$ *is a compact nowhere dense subset of* $\mathbb{R}$.

Proof. For the smooth function

$$
f : \nu^* N \times o_{\mathbb{R}^m} \to \mathbb{R}
$$

$$
f(x) = \mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x))
$$

we have, by (5)

$$
df(x) = -\theta((\phi_1^H)^{-1}(x)) T(\phi_1^H)^{-1}(x) + dS(\pi(\phi_1^H)^{-1}(x)) T\pi T(\phi_1^H)^{-1}(x)
$$

and thus the set $\mathrm{Spec}(H, S : N)$ is contained in the set of critical values of f. The latter is nowhere dense in $\mathbb{R}$ by the classical Sard's theorem.

Since $H = H_1 \oplus H_2$ and $\mathrm{Graph}(dS) = o_M \times \mathrm{Graph}(dQ)$ outside some compact subset $K \subset T^*E$ and $\mathrm{supp}(H_i) \subset K_i$, $i \in \{1, 2\}$ for some compact subsets $K_1 \subset T^*M$ and $K_2 \subset \mathbb{C}^m$, it follows that for $x = (x_1, x_2) \in \nu^* N \times o_{\mathbb{R}^m}$ outside $K_0 := \bigcup_{t\in[0,1]} \phi_t^{H_1\oplus H_2} \circ (\phi_1^{H_1\oplus H_2})^{-1}(K)$

$$
f(x) = g_1(x_1) + g_2(x_2),
$$

where

$$
g_1 : \nu^* N \to \mathbb{R},\ g_1(x_1) = \mathcal{A}_{H_1}(\phi_t^{H_1} \circ (\phi_1^{H_1})^{-1}(x_1))
$$

$$
g_2 : o_{\mathbb{R}^m} \to \mathbb{R},\ g_2(x_2) = \mathcal{A}_{H_2}(\phi_t^{H_2} \circ (\phi_1^{H_2})^{-1}(x_2)) + Q(\pi_{\mathbb{C}^m}((\phi_1^{H_2})^{-1}(x_2))).
$$

Here $\pi_{\mathbb{C}^m} : \mathbb{C}^m \to \mathbb{R}^m$ denotes the natural projection. Denote $\tilde{K}_0 := K_0 \cap \nu^* N \times o_{\mathbb{R}^m}$, $\tilde{K}_1 := K_1 \cap \nu^* N$, $\tilde{K}_2 := K_2 \cap o_{\mathbb{R}^m}$. Since $g_1 \equiv 0$ outside $\tilde{K}_1$ and $g_2 \equiv 0$ outside $\tilde{K}_2$, all critical points of g are contained in the compact set

$$B = g(\tilde{K}_0) \cup \Big(g_1(\tilde{K}_1) + g_2(\tilde{K}_2)\Big) \cup \{0\}.$$

Hence $\mathrm{Spec}(H, S : N)$ is compact as a closed subset of a compact set B. $\square$

Let $CF_*(H, S : N)$ denote the free abelian group generated by $CF(H, S : N)$ and $CF^*(H, S : N) := \mathrm{Hom}(CF_*(H, S : N), \mathbb{Z})$. Further, denote by $\mathcal{M}_{(J,H,S)}(N : E)$ the set of solutions of (7) with finite energy, i.e., of those which satisfy the condition:

$$E(U) := \int_{-\infty}^{+\infty} \int_0^1 \left(\left| \frac{\partial U}{\partial \tau} \right|_J^2 + \left| \frac{\partial U}{\partial t} - X_H(U) \right|_J^2 \right) dt d\tau < \infty. \tag{9}$$

More generally, consider the τ-dependent families

$$S^{\alpha\beta} := S^{\alpha\beta}_\tau \in \mathcal{S}_{(E,Q)},\ H^{\alpha\beta} := H^{\alpha\beta}_\tau \in \mathcal{H}(E),\ J^{\alpha\beta} := J^{\alpha\beta}_\tau \in \mathcal{J}^c_\omega(E),$$

such that for some $R > 0$ and $\tau < -R$

$$S^{\alpha\beta}_\tau \equiv S^\alpha,\ H^{\alpha\beta}_\tau \equiv H^\alpha,\ J^{\alpha\beta}_\tau \equiv J^\alpha,$$

for some fixed S^α, H^α, J^α and, similarly,

$$S^{\alpha\beta}_\tau \equiv S^\beta,\ H^{\alpha\beta}_\tau \equiv H^\beta,\ J^{\alpha\beta}_\tau \equiv J^\beta,$$

for $\tau > R$ and S^β, H^β, J^β fixed. Denote the sets of all such homotopies by

$$\overline{\mathcal{H}}(E),\ \overline{\mathcal{S}}_{(E,Q)},\ \overline{\mathcal{J}}^c_\omega(E).$$

We define $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(N : E)$ as the set of solutions of

$$\begin{cases} \overline{\partial}_{J^{\alpha\beta}, H_\tau} U := \frac{\partial U}{\partial \tau} + J^{\alpha\beta}\left(\frac{\partial U}{\partial t} - X_{H^{\alpha\beta}}(U)\right) = 0 \\ (u(\tau,0), v(\tau,0)) \in \mathrm{Graph}(dS^{\alpha\beta}) \\ u(\tau,1) \in \nu^* N,\ v(\tau,1) \in o^m_{\mathbb{R}} \end{cases} \tag{10}$$

which satisfy

$$E(U) := \int_{-\infty}^{+\infty} \int_0^1 \left(\left| \frac{\partial U}{\partial \tau} \right|_{J^{\alpha\beta}}^2 + \left| \frac{\partial U}{\partial t} - X_{H^{\alpha\beta}}(U) \right|_{J^{\alpha\beta}}^2 \right) dt d\tau < \infty. \tag{11}$$

It is a standard result in elliptic regularity theory that the solutions of (10) are smooth.

Finally, for two solutions x, y of (6) we denote by $\mathcal{M}_{(J,H,S)}(x, y)$ the set of solutions U of (7) such that

$$\lim_{\tau \to \infty} U(\tau, t) = x(t),$$

$$\lim_{\tau \to -\infty} U(\tau, t) = y(t).$$

In an analogous way, we define $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha, x^\beta)$ to be the set of solutions U of Equation (10) such that

$$\lim_{\tau\to-\infty} U(\tau,t) = x^\alpha(t)$$

$$\lim_{\tau\to\infty} U(\tau,t) = x^\beta(t),$$

where

$$(12) \qquad \begin{cases} \dot{x^\alpha} = X_{H^\alpha}(x^\alpha) \\ x^\alpha(0) \in \mathrm{Graph}(dS^\alpha) \\ x^\alpha \in \nu^*N \times o_{\mathbb{R}^m} \end{cases} \qquad \begin{cases} \dot{x^\beta} = X_{H^\beta}(x^\beta) \\ x^\beta(0) \in \mathrm{Graph}(dS^\beta) \\ x^\beta(1) \in \nu^*N \times o_{\mathbb{R}^m}. \end{cases}$$

3. C^0-estimates.

In this section we will prove that the solutions of (7) and (10) remain in a compact neighborhood of zero-section. The essential ingredient of the proof is the version of maximum principle which states that a J-holomorphic curve cannot touch certain kind of hypersurfaces.

3.1. Contact type hypersurfaces.

Definition 2 ([**40**])**.** A smooth hypersurface Δ in a symplectic manifold (V,ω) is said to be *of a contact type* if there exists a vector field X defined in a neighborhood U of Δ and transversal to Δ such that $d(X\rfloor\omega) = \omega$ in U. Such vector field is called *conformal*.

It is easy to see that $\varrho := X\rfloor\omega$ defines a contact structure $\zeta :=$ Ker (ϱ) on Δ.

Definition 3 ([**7**])**.** Let Δ be an oriented hypersurface in an almost complex manifold (V,J) and ζ_q the maximal J-invariant subspace of $T_q\Delta$. Then Δ is called *J-convex* if for some (and hence any) defining 1-form ϱ for ζ_q we have $d\varrho(Y,JY) > 0$ for all non-zero vectors $Y \in \zeta_q$.

For a contact type hypersurface Δ in symplectic manifold (V,ω) there exist an ω-compatible almost complex structure J such that Δ is J-convex.

Example 4. The sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is an i-convex hypersurface.

Example 5. Let J_g be an almost complex structure on T^*M defined in Section 2 and $\|\cdot\|_g$ the fiberwise norm induced by g. Then the hypersurface

$$\Delta := \{p \in T^*M \mid \|p\|_g = 1\}$$

is J_g-convex.

For the sake of completeness we give the proof of the following version of the maximum principle for subharmonic functions (c.f., [**23**, **30**]).

Lemma 6. *Let $u : D \to V$ be a J-holomorphic disc in an almost complex manifold V and $\Delta \subset V$ a J-convex hypersurface. Then $u(D)$ cannot be tangent to Δ.*

Proof. Suppose that $Im(D)$ is tangent to Δ at the point $u(z)$, for some $z \in D$. Since u is J-holomorphic, $u_*(T_zD) \subset \zeta_{u(z)}$ for ζ as in Definition 3. Assume that $\Delta = f^{-1}(0)$ for some $f : V \to \mathbb{R}$.

We first prove that $f \circ u : D \to \mathbb{R}$ is subharmonic near z, or, equivalently (see [**7**]) that a two-form $di^*d(f \circ u)$ is positive definite in a neighborhood of z. Here $i^* : T^*D \to T^*D$ is the operator adjoint to $i : TD \to TD$, $\eta \mapsto \sqrt{-1}\eta$.

Choose $Y \in \zeta_{u(z)}$. Then, according to Definition 3, $JY \in \zeta_{u(z)}$ and thus $J^*df(Y) := df(JY) = 0$. Therefore,

$$J^*df|_\Delta = \mu\varrho|_\Delta + \lambda df|_\Delta$$

for ϱ as in Definition 2 and for some $\mu : V \to (0, +\infty)$ and $\lambda : V \to \mathbb{R}$. Hence

$$\begin{aligned} dJ^*df|_{\zeta_{u(z)}} &= d\mu \wedge \varrho|_{\zeta_{u(z)}} + \mu d\varrho|_{\zeta_{u(z)}} + d\lambda \wedge df|_{\zeta_{u(z)}} \\ &= \mu d\varrho|\zeta_{u(z)}. \end{aligned}$$

Since u is J-holomorphic, $i^*u^* = u^*J^*$ and thus

$$\begin{aligned} di^*d(f \circ u) &= u^*dJ^*df \\ &= u^*(\mu d\varrho) \text{ at } u(z) \\ &= (\mu d\varrho)u_*. \end{aligned}$$

Since $u_*(T_zD) \in \zeta_{u(z)}$, and since $d\varrho|_{\zeta u(z)}$ is positive definite (by Definition 3), two-form $di^*d(f \circ u)$ is positive definite near z. Hence, $f \circ u$ is subharmonic.

Now, we finish the proof arguing indirectly. Suppose that Image(D) is tangent to Δ. Then $f \circ u$ attains its maximum at z. If z is an interior point in D it contradicts the maximum principle for subharmonic functions. If $z \in \partial D$ then

$$\frac{d}{dt}|_{t=1}((f \circ u)(tz)) = 0$$

which contradicts Hopf lemma (see [**32**]). □

3.2. The structure of the space of trajectories.

In this section we prove the following analogue of well-known Floer's theorem (see [**11**, **15**, **35**]).

Proposition 7. *If $U := (u, v)$ is a solution of Equation* (10) *which satisfies the condition* (11), *then there exist the limits*

$$x^\alpha(t) = \lim_{\tau \to -\infty} U(\tau, t)$$

and

$$x^\beta(t) = \lim_{\tau\to\infty} U(\tau,t).$$

Moreover, x^α *and* x^β *are solutions of Equation* (12) *and hence*

$$\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(N:E) = \bigcup_{x^\alpha,x^\beta} \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta).$$

Proof. Choose a sequence $\tau_k \to -\infty$, and consider the sequence $U_k := U(\tau_k,t)$. We claim that U_k is bounded in $W^{1,2}([0,1],T^*E)$.

By assumption (11) we have

$$\int_0^1 \left|\frac{\partial U}{\partial t}(\tau_k,t) - X_{H^\alpha}(U(\tau_k,t))\right|^2 dt \to 0, \text{ as } k\to\infty. \tag{13}$$

Therefore, it remains to prove L^2-estimate. We will prove that $U_k(t)$ is contained in a compact subset of T^*E. We embed T^*E properly in $\mathbb{R}^p$ and denote by $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^p$. Assume first that

$$\lim_{k\to\infty} |U_k(1)| = \infty. \tag{14}$$

Recall that $S^{\alpha\beta} \equiv Q$ outside a compact set $K \subset E$. By compactness of K and (14) we have

$$\lim_{k\to\infty} \operatorname{dist}(U_k(1), \operatorname{Graph}(dS^{\alpha\beta}|_K)) = \infty. \tag{15}$$

Since

$$U_k(1) := (u_k(1), v_k(1)) \in \nu^*N \times o_{\mathbb{R}^m}$$

and

$$\operatorname{Graph}(dS^{\alpha\beta}|_{E\setminus K}) \cong o_M \times \operatorname{Graph}(dQ),$$

(14) implies

$$\begin{aligned}\lim_{k\to\infty}[\operatorname{dist}(U_k(1),\operatorname{Graph}(dS^{\alpha\beta}|_{E\setminus K}))]^2 &= \lim_{k\to\infty}[\operatorname{dist}(u_k(1),o_M)]^2\\ &+ \lim_{k\to\infty}[\operatorname{dist}(v_k(1),\operatorname{Graph}(dQ))]^2\\ &= \infty.\end{aligned} \tag{16}$$

Therefore, from (15) and (16) we get

$$\lim_{k\to\infty} \operatorname{dist}(U_k(1), \operatorname{Graph}(dS^{\alpha\beta})) = \infty. \tag{17}$$

Since $U_k(0) \in \operatorname{Graph}(dS^{\alpha\beta})$ (17) gives

$$\lim_{k\to\infty} |U_k(1) - U_k(0)| = \infty. \tag{18}$$

However

$$
\begin{aligned}
|U_k(1) - U_k(0)| &= \left| \int_0^1 \frac{dU_k}{dt} dt \right| \\
&\leq \left(\int_0^1 \left| \frac{dU_k}{dt} \right|^2 dt \right)^{\frac{1}{2}} \\
&< C
\end{aligned}
$$

by (13), which contradicts (18). Therefore, there exists a compact set $K_1 \subset T^*E$ such that

$$
U_k(1) \in K_1 \text{ for all } k. \tag{19}
$$

Assume now that there exist a sequence $t_k \in [0,1]$ such that $|U_k(t_k)|$ is unbounded. By (19) that means

$$
\lim_{k \to \infty} |U_k(1) - U_k(t_k)| = \infty \tag{20}
$$

for some subsequence (denoted again by) U_k. Then, by the same argument as above,

$$
\begin{aligned}
|U_k(1) - U_k(0)| &\leq \left(\int_0^{t_k} \left| \frac{dU_k}{dt} \right|^2 dt \right)^{\frac{1}{2}} \\
&< C
\end{aligned}
$$

which contradicts (20). Therefore, U_k is C^0 (and hence L^2) bounded.

Hence we deduce that U_k is bounded in $W^{1,2}([0,1], T^*E)$. Therefore, by Rellich Theorem,

$$
U_k(t) \to x^\alpha(t) \text{ as } k \to \infty \text{ (in } L^2).
$$

Moreover, since $\frac{dU_k}{dt}$ is L^2-bounded (by (13)), the family U_k is equicontinuous and thus, by Arzelà-Ascoli Theorem

$$
U_k(t) \to x^\alpha(t) \text{ as } k \to \infty \text{ (in } C^0).
$$

From (13) we conclude that x^α is a (weak) solution of Equation (6). Smoothness of x^α follows from the smoothness of X_{H^α}. Since this is true for every sequence τ_k, it is easy to see that

$$
\lim_{\tau \to -\infty} U(\tau, t) = x^\alpha(t).
$$

The case $\tau \to \infty$ is treated analogously. □

Remark 8. The converse of previous proposition also holds: If U is a solution of Equation (10) which satisfies (13) then U is bounded in sense of (11).

Indeed, in that case

$$\begin{aligned}\frac{1}{2}E(U) &= \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1\left(\left|\frac{\partial U}{\partial\tau}\right|^2_{J_\tau^{\alpha\beta}}+\left|\frac{\partial U}{\partial t}-X_{H_\tau^{\alpha\beta}}(U)\right|^2_{J_\tau^{\alpha\beta}}\right)dtd\tau\\ &= \int_{-\infty}^{+\infty}\int_0^1\left\langle\frac{\partial U}{\partial\tau},\frac{\partial U}{\partial t}-X_{H_\tau^{\alpha\beta}}\right\rangle_{J_\tau^{\alpha\beta}}dtd\tau\\ &= \mathcal{A}_{(H^\beta,S^\beta)}(x^\beta)-\mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha)-\int_{-\infty}^{+\infty}\int_0^1\frac{\partial H_\tau^{\alpha\beta}}{\partial\tau}dtd\tau\\ &< +\infty.\end{aligned}$$

3.3. The image of the evaluation map.

In this section we prove the C^0 estimate necessary for defining Floer homology on a non-compact manifold (see [**8**, **15**, **30**] for similar propositions). In fact, we will prove that the image of the evaluation map

$$ev:\mathcal{M}_{J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta}}(N:E)\times[0,1]\times\mathbb{R}\to T^*E$$

defined by

$$ev(U,\tau,t):=U(\tau,t)$$

is bounded.

Proposition 9. *Consider a family of parameters* $(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})$ *chosen as in Section* 2*, so that there exist a compact set* $K\subset T^*E$ *such that*

$$H^{\alpha\beta}\equiv H_1^{\alpha\beta}\oplus H_2^{\alpha\beta},\ J^{\alpha\beta}\equiv J_g\oplus i \text{ outside } K$$

and

$$S^{\alpha\beta}\equiv Q \text{ outside } \pi(K),$$

where $\pi:T^*E\to E$ *is the natural projection. Then there exists a compact set* $K_0\supset K$*, depending on* $(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})$*, such that*

$$ev(\mathcal{M}_{J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta}}(N:E))\subset K_0.$$

Proof. Let $K_0\supset K$ be a compact subset of T^*E such that

$$\mathrm{Graph}(dS^{\alpha\beta}|_{\pi(K)})\subset K_0$$

and

$$\pi_1(K_0)\supset\mathrm{supp}(H_1),\ \pi_2(K_0)\supset\mathrm{supp}(H_2)$$

where

$$\pi_1:T^*E\cong T^*M\times\mathbb{C}^m\to T^*M,\ \pi_2:T^*E\to\mathbb{C}^m$$

are natural projections. It is clear that outside K_0 Equation (10) splits onto

$$(21)\qquad \begin{cases}\overline{\partial}_{J_g}u-J_gX_{H_1^{\alpha\beta}}(u)=0\\ u(\tau,0)\in o_M\\ u(\tau,1)\in\nu^*N\end{cases}\qquad \begin{cases}\overline{\partial}v-iX_{H_2^{\alpha\beta}}(v)=0\\ v(\tau,0)\in\mathrm{Graph}(dQ)\\ v(\tau,1)\in o_{\mathbb{R}^m}.\end{cases}$$

Let $U := (u, v)$ be a solution of (21) outside K_0. Assume

$$\pi_1(K_0) \subset D_{R_0} := \{(q,p) \in T^*M \mid \|p\|_g < R_0\}.$$

Let

$$R_1 := \sup\left\{\left\|\frac{\partial S^{\alpha\beta}}{\partial q}(e)\right\|_g \mid e \in E\right\}.$$

Note that R_1 is finite since $S^{\alpha\beta}(q,\xi) = Q(\xi)$ $\left(\text{and hence } \frac{\partial S^{\alpha\beta}}{\partial q} \equiv 0\right)$ outside a compact set. Set

$$R_2 := \sup\left\{\sup_{t\in[0,1]} \|y(t)\|_g \mid \ x := (y,z) \text{ solves (12)}\right\}.$$

Since $H_1^{\alpha\beta} \equiv 0$ outside $\pi_1(K_0)$ and $H^{\alpha\beta} = H_1^{\alpha\beta} \oplus H_2^{\alpha\beta}$, $\mathrm{Graph}(dS^{\alpha\beta}) = \mathrm{Graph}(dQ)$ outside K_0 it follows that

$$\max\{R_1, R_2\} \le R_0.$$

We will first prove that

$$R_3(u) := \sup_{(\tau,t)\in\mathbb{R}\times[0,1]} \|u(\tau,t)\|_g < R_0.$$

Arguing indirectly, assume that

$$R_3(u) > R_k \text{ for } i \in \{0,1,2\}. \tag{22}$$

Then u component of Equation (21) outside the set $\{(q,p) \in T^*M \mid \|p\|_g \le R_0\}$ becomes

$$\overline{\partial}_J u = 0,$$

i.e., u is J-holomorphic. Denote

$$\Delta := \{(q,p) \in T^*M \mid \|p\|_g = R_3(u)\}.$$

By Example 5 Δ is J_g-convex. Choose $T \in \mathbb{R}$ such that

$$\sup_{|\tau|>T}\sup_{0\le t\le 1} \|u(\tau,t)\|_g < R_3(u).$$

Since $\max\|u(\tau,0)\| \le R_1$ it follows from (22) that $\max\|u(\tau,t)\|$ is achieved at some point $(\tau_0,t_0) \in [-T,T]\times(0,1]$ and there exists a neighborhood B_ε of (τ_0,t_0) such that $u|_{B_\varepsilon}$ is a J_g-holomorphic disc.

If (τ_0,t_0) is an interior point of $(0,1)\times(-T,T)$, then $u(B_\varepsilon)$ is tangent to the J_g-convex hypersurface Δ, which is, by Lemma 6, a contradiction.

Therefore, assume that $t_0 = 1$. Then $u(\tau,1)$ is a curve tangent to Δ at τ_0. But, since $u(\tau,1) \in \nu^*N$ and ν^*N is Lagrangian, $J\frac{d}{d\tau}u(\tau,1)$ must be perpendicular to ν^*N. In particular, it is perpendicular to the conformal vector field $\frac{\partial}{\partial r} \in T\nu^*N$ (see Definition 2 and Definition 3). Therefore, $J\frac{d}{d\tau}u(\tau,1) \in T\Delta$, and hence $u(B_\varepsilon)$ is tangent to Δ, which is again a contradiction by Lemma 6.

Consider now $\pi_2 : T^*E \cong T^*M \times \mathbb{C}^m \to \mathbb{C}^m$ and assume that

$$\pi_2(K_0) \subset M \times B(0, R_4),$$

where $B(0, R_4)$ is the standard Euclidean ball of radius R_4 in $\mathbb{R}^m$. If

$$R_5 := \sup \left\{ \sup_{t\in[0,1]} |z(t)| \mid x := (y, z) \text{ solves } (12) \right\},$$

where $|\cdot|$ is the standard Euclidean norm on $\mathbb{C}^m$ then $R_5 \leq R_4$. Now

$$\sup_{(\tau,t)\in\mathbb{R}\times[0,1]} |v(\tau, t)| < R_4.$$

Indeed, arguing as above, we rule out the interior points easily. For the boundary points, we use the fact that the radial vector field $\frac{\partial}{\partial\rho} \in T\mathbb{C}^m$ is tangent to both Graph(dQ) and $o_{\mathbb{R}^m}$ and perpendicular to the standard Euclidean sphere in $\mathbb{R}^m$. Assume that $\sup|v|$ was achieved at some point (τ_0, t_0), for $t_0 = 0$ or 1. Then the curve $v(\tau, t_0)$ is tangent to $\mathbb{S}^{2m-1}$ at τ_0 and perpendicular to the radial vector field $\frac{\partial}{\partial\rho} \in T\mathbb{R}^m$. Since both Graph$(dQ)$ and $o_{\mathbb{R}^m}$ are Lagrangian, $i\frac{d}{d\tau}v(\tau, t_0)|_{\tau_0}$ is also perpendicular to $\frac{\partial}{\partial\rho}$, i.e., tangent to $\mathbb{S}^{2m-1}$. Since $\mathbb{S}^{2m-1}$ is i-convex (see Example 4), this again contradicts Lemma 6. □

Once we have established C^0 estimates, the standard compactness result follows as in [**12**, **11**, **28**, **35**]:

Proposition 10. *For any sequence* $U_k \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha, x^\beta)$ *there exist a subsequence (denoted by* U_k *again), sequences* $\tau_k^j \in \mathbb{R}$ $(0 \leq j \leq l)$ *and an integer* s $(0 \leq s \leq l)$ *such that*

1) *for* $0 \leq j \leq s-1$ $U_k(\tau + \tau_k^j)$ *and all its derivatives converge uniformly on compact sets to* $U^j \in \mathcal{M}_{(J^\beta,H^\beta,S^\beta)}(x^j, x^{j-1})$, *where* x^j *are the solutions of Equation* (12) *and* $x^0 = x^\beta$,
2) $U_k(\tau + \tau_k^s)$ *and all its derivatives converge uniformly on compact sets to* $U^j \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^s, x^{s-1})$, *where* x^s *is the solutions of Equation* (12),
3) *for* $s+1 \leq j \leq l$ $U_k(\tau+\tau_k^j)$ *and all its derivatives converge uniformly on compact sets to* $U^j \in \mathcal{M}_{(J^\alpha,H^\alpha,S^\alpha)}(x^j, x^{j-1})$,*where* x^j *are the solutions of Equation* (12) *and* $x^l = x^\alpha$.

The complementary concept to the compactness property of Proposition 10 is the gluing construction. It is now standard (see [**12**, **22**]) and can be summarized in the following

Proposition 11. *For any pair of trajectories*

$$(U^\alpha, U^{\alpha\beta}) \in \mathcal{M}_{(J^\alpha,H^\alpha,S^\alpha)}(x^\alpha, y^\alpha) \times \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(y^\alpha, z^\beta)$$

there exists a sequence $U_k \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha, z^\beta)$ *converging to* (U^1, U^2) *in the sense of Proposition* 10.

4. Fredholm theory.

Assume that $H \in \mathcal{H}(E)$ and $S \in \mathcal{S}_{(E,Q)}$ are chosen as in (8), i.e., assume that $\mathrm{Graph}(dS)$ intersects $(\phi_1^H)^{-1}(\nu^*N \times o_{\mathbb{R}^m})$ transversely. Then, for each two solutions x, y of Equation (6) there exist a smooth Banach manifold

$$\mathcal{P}_k^p(x,y) \subset \mathcal{P}_{k:loc}^p(x,y) := \{U \in \mathcal{P}_{k:loc}^p(S;N) \mid \lim_{\tau\to-\infty} U = x,\ \lim_{\tau\to\infty} U = y\}$$

such that (7) defines a smooth Fredholm section

$$\overline{\partial}_{J,H} : \mathcal{P}_k^p(x,y) \to \mathcal{L}, \tag{23}$$

where $\mathcal{L}$ is a smooth Banach bundle over $\mathcal{P}_k^p(x,y)$ with fibers

$$\mathcal{L}_U = W^{k-1,p}(\mathbb{R}\times[0,1], U^*T(T^*E)).$$

The linearization of $\overline{\partial}_{J,H}$ at $U \in \mathcal{M}_{(J,H,S)}(x,y)$ is a Fredholm operator

$$E_U := D(\overline{\partial}_{J,H}) : T_U\mathcal{P}_k^p(x,y) \to \mathcal{L}_U, \tag{24}$$

$$E_U\xi = \nabla_\tau\xi + J(U)\nabla_t\xi + \nabla_\xi J(U)\frac{\partial U}{\partial t} + \nabla_\xi\nabla H(t,U)$$

where ∇_τ, ∇_t, ∇_ξ denote the covariant derivative with respect to Levi-Civita connection associated to metric $\omega(\cdot, J\cdot)$ and $T_U\mathcal{P}_k^p(x,y)$ is the set of all $\xi \in W^{k,p}(\mathbb{R}\times[0,1], U^*T(T^*E))$ such that $\xi(\tau,0) \in T(\mathrm{Graph}(dS))$ and $\xi(\tau,1) \in T(\nu^*(N \times o_{\mathbb{R}^m}))$. Furthermore, for fixed J and S,

$$F : (U,H) := (u,v,H) \mapsto \overline{\partial}_{J,H}U$$

defines a smooth section of the Banach bundle

$$\mathcal{L} \to \mathcal{P}_k^p(x,y) \times \mathcal{H}(E)$$

transversal to the zero section. Hence, $F^{-1}(0)$ is a (Banach) manifold. The projection

$$\Pi : F^{-1}(0) \to \mathcal{H}(E)$$
$$(U,H) \mapsto H$$

is a Fredholm map. The point $U \in \mathcal{M}_{(J,H,S)}(x,y)$ is a regular point of Section (23) if and only if $(U,H) \in F^{-1}(0)$ is a regular point of Π. Hence, by Sard-Smale Theorem applied to Π, the set of points in $H \in \mathcal{H}(E)$ for which Section (23) is regular is dense in $\mathcal{H}(E)$. Similarly, one can use $\mathcal{J}_\omega^c(E)$ or $\mathcal{S}_{(E,Q)}$ in place of $\mathcal{H}(E)$.

Indeed, Floer's proof of the above statements in [**11**] (see also [**28**, **35**]) carries over in our situation with slight modifications. Hence we have the following

Proposition 12. *Let N and Q be fixed as in Section 2. Then there exists a dense set*

$$(\mathcal{J}^c_\omega(E) \times \mathcal{S}_{(E,Q)} \times \mathcal{H}(E))_{\rm reg} \subset \mathcal{J}^c_\omega(E) \times \mathcal{S}_{(E,Q)} \times \mathcal{H}(E)$$

such that for every $(J,S,H) \in (\mathcal{J}^c_\omega(E)\times\mathcal{S}_{(E,Q)}\times\mathcal{H}(E))_{\rm reg}$ the linearization of Section (23) at $U \in \mathcal{M}_{(J,H,S)}(N:E)$ is onto. Consequently, for $(J,S,H) \in (\mathcal{J}^c_\omega(E) \times \mathcal{S}_{(E,Q)} \times \mathcal{H}(E))_{\rm reg}$ $\mathcal{M}_{(J,H,S)}(x,y)$ is a smooth finite dimensional manifold.

Similarly, we have the parameterized version of Proposition 12 (see [**9**, **11**]):

Proposition 13. *Let N and Q be fixed as in Proposition 12, and*

$$(J^\alpha, S^\alpha, H^\alpha),\ (J^\beta, S^\beta, H^\beta) \in (\mathcal{J}^c_\omega(E) \times \mathcal{S}_{(E,Q)} \times \mathcal{H}(E))_{\rm reg}.$$

Then there exists a dense subset $(\overline{\mathcal{J}}^c_\omega(E) \times \overline{\mathcal{S}}_{(E,Q)} \times \overline{\mathcal{H}}(E))_{\rm reg}$ in a set of all homotopies $\overline{\mathcal{J}}^c_\omega(E) \times \overline{\mathcal{S}}_{(E,Q)} \times \overline{\mathcal{H}}(E)$ defined in Section 2 such that for

$$(J^{\alpha\beta}, S^{\alpha\beta}, H^{\alpha\beta}) \in (\overline{\mathcal{J}}^c_\omega(E) \times \overline{\mathcal{S}}_{(E,Q)} \times \overline{\mathcal{H}}(E))_{\rm reg}$$

Equation (10) defines a smooth Fredholm section

$$(\overline{\partial}_{J,H}, \overline{\partial}) : \mathcal{P}^p_k(x^\alpha, x^\beta) \to \mathcal{L}, \tag{25}$$

on a smooth Banach bundle $\mathcal{L}$ over $\mathcal{P}^p_k(x^\alpha,x^\beta) \subset \mathcal{P}^p_{k:loc}(x^\alpha,x^\beta)$, where

$$\mathcal{P}^p_{k:loc}(x^\alpha,x^\beta) := \{U \in \mathcal{P}^p_{k:loc}(S;N) \mid \lim_{\tau\to-\infty} U = x^\alpha \ \lim_{\tau\to\infty} U = x^\beta\},$$

which is regular at any $U \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$

Example 14. The case $S = S_1 \oplus S_2$, $H = H_1 \oplus H_2$, $J = J_1 \oplus J_2$.
Assume that $S(q,\xi) = S_1(q) + S_2(\xi)$ and $H(x,y) = H_1(x) + H_2(y)$. Then Equations (6) and (7) split onto

$$\begin{cases} \dot\gamma = X_{H_1}(\gamma) \\ \gamma(0) \in \mathrm{Graph}(dS_1) \\ \gamma(1) \in \nu^* N \end{cases} \qquad \begin{cases} \dot z = X_{H_2}(z) \\ z(0) \in \mathrm{Graph}(dS_2) \\ z(1) \in o_{\mathbb{R}^m} \end{cases}$$

and

$$\begin{cases} \frac{\partial u}{\partial\tau} + J_1(\frac{\partial u}{\partial t} - X_{H_1}(u)) = 0 \\ u(\tau,0) \in \mathrm{Graph}(dS_1) \\ u(\tau,1) \in \nu^* N \end{cases} \qquad \begin{cases} \frac{\partial v}{\partial\tau} + J_2(\frac{\partial v}{\partial t} - X_{H_2}(u)) = 0 \\ v(\tau,0) \in \mathrm{Graph}(dS_2) \\ v(\tau,1) \in o_{\mathbb{R}^m}, \end{cases}$$

and the linearization (24) splits onto

$$E_u \oplus E_v : T_{(u,v)}\mathcal{P}^p_k((x_1,x_2),(y_1,y_2)) = T_u\mathcal{P}^p_k(x_1,y_1) \oplus T_v\mathcal{P}^p_k(x_2,y_2) \to \mathcal{L}_u \oplus \mathcal{L}_v.$$

Hence, if $H_1 \in (\mathcal{H}(M))_{\rm reg}$ and $H_2 \in (\mathcal{H}(\mathbb{R}^m))_{\rm reg}$ then $H \in (\mathcal{H}(M \times \mathbb{R}^m))_{\rm reg}$.

Example 15. The case $H \equiv 0$.
In this case we have the Morse complex of $S|_N$, which is regular for a dense subset $(\mathcal{S}_{(E,Q)})_{\text{reg}} \in \mathcal{S}_{(E,Q)}$ (see Proposition 27).

In this section we will compute the Fredholm index of Sections (23) and (25) in terms of Maslov indices of Hamiltonian paths x^α and x^β. Next, we relate this computation to the Morse index of S and give the groups $CF_*(H, S : N)$ canonical grading. The existence of such grading is established in [**10**] and similar computations to ours are given for the case $S \equiv 0$, $m = 0$ in [**30**] and for the periodic orbits problem in [**6**, **36**].

4.1. The Maslov index.

Maslov index for paths of Lagrangian subspaces has been studied by several authors (see [**1**, **3**, **34**, **33**]). We will follow the notation and terminology of [**34**] and [**33**]. Denote by $\Lambda(k)$ the Lagrangian Grassmanian, i.e., the manifold of Lagrangian subspaces in $\mathbb{C}^k$. The Maslov index assigns to every pair of paths

$$L, L' : [0,1] \to \Lambda(k)$$

a half integer $\mu(L, L') \in \frac{1}{2}\mathbb{Z}$ characterized by:

Naturality: For any path $\Psi : [0,1] \to \text{ Sp } (2k)$

$$\mu(\Psi(t)L(t), \Psi(t)L'(t)) = \mu(L(t), L'(t)).$$

Homotopy: Two paths L, $L' : [0,1] \to \Lambda(k)$ with $L(0) = L'(0)$ and $L(1) = L'(1)$ are homotopic with fixed endpoints if and only if they have the same Maslov index.

Zero: If $L(t) \cap L'(t)$ is of constant dimension, then $\mu(L, L') = 0$.

Direct Sum: $\mu(L_1 \oplus L_1', L_2 \oplus L_2') = \mu(L_1, L_2) + \mu(L_1', L_2')$.

Catenation: For $0 < t_0 < 1$

$$\mu(L, L') = \mu(L|_{[0,t_0]}, L'|_{[0,t_0]}) + \mu(L|_{[t_0,1]}, L'|_{[t_0,1]}).$$

Localization: If $L'(t) \equiv \mathbb{R}^k \times 0$ and $L(t) = \text{Graph}(A(t))$ for a path

$$A : [0,1] \to \text{ End } (\mathbb{R}^k)$$

of symmetric matrices then

$$\mu(L, L') = \frac{1}{2} \text{ sign } A(1) - \frac{1}{2} \text{ sign } A(0).$$

The Maslov index of a symplectic path

$$\Psi : [0,1] \to \text{ Sp } (k)$$

with respect to a fixed Lagrangian submanifold $V \subset \mathbb{C}^k$ (say $V = 0 \times \mathbb{R}^k$) is defined by

$$\mu(\Psi) := \mu(\Psi V, V),$$

or, equivalently, (see [**33**])

$$\mu(\Psi) = \mu(\mathrm{Graph}(\Psi), V \times V).$$

Following [**34**], we consider the perturbed Cauchy-Riemann operator

$$\begin{cases} \overline{\partial}_{J,T,L}\zeta := \frac{\partial \zeta}{\partial \tau} + J\frac{\partial \zeta}{\partial t} + T\zeta \\ \zeta : \mathbb{R}^k \times [0,1] \to \mathbb{C}^k \\ (\zeta(\tau,0), \zeta(\tau,1)) \in R^k \times L(\tau) \subset \mathbb{C}^k \times \mathbb{C}^k. \end{cases} \tag{26}$$

Here we assume that (c.f., [**34**, **30**]):

1) $L : \mathbb{R} \to \Lambda(k)$ is C^1 and $L(\tau) = 0 \times \mathbb{R}^k$ for large $|\tau|$.
2) The almost complex structure $J : \mathbb{R}\times[0,1] \to \mathrm{End}\ (\mathbb{R}^{2k})$ is compatible with symplectic form $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_k \wedge dy_k$ on $\mathbb{C}^k$ and independent of τ for $|\tau|$ large enough; $J(\pm\infty, t) = J^{\pm}(t)$.
3) $T : \mathbb{R} \times [0,1] \to \mathrm{End}\ (\mathbb{R}^{2k})$ is the continuous family of matrices such that
$$\lim_{\tau\to\pm\infty} \sup_{0\le t\le 1} \|T(\tau,t) - T_{\pm}(t)\| = 0 \tag{27}$$
for some paths $T_{\pm} : [0,1] \to \mathrm{End}\ (\mathbb{R}^k)$ of symmetric matrices.
4) If $\Psi^{\pm} : [0,1] \to \mathrm{Sp}\ (2k)$ is a solution of
$$\frac{\partial \Psi^{\pm}}{\partial t} - J^{\pm}(t)T^{\pm}(t)\Psi^{\pm} = 0,\ \Psi^{\pm}(0) = \mathrm{Id} \tag{28}$$
then $\Psi^{\pm}(\mathbb{R}^k)$ is transverse to $0 \times \mathbb{R}^k$.

We will need the following:

Proposition 16 ([**34**])**.** *The operator*

$$\overline{\partial}_{J,T,L} : W^{1,2}_L \to L^2(\mathbb{R} \times [0,1], \mathbb{C}^k)$$

where

$$W^{1,2}_L := \{\zeta \in W^{1,2}(\mathbb{R} \times [0,1], \mathbb{C}^k) \mid (\zeta(\tau,0)), \zeta(\tau,1) \in \mathbb{R}^k \times L(\tau)\}$$

is Fredholm with the index given by

$$\mathrm{Index}(\overline{\partial}_{J,T,L}) = -\mu(\Psi^-) + \mu(\Psi^+) + \mu(\Delta, \mathbb{R}^k \times L(\tau))$$

where Δ is the diagonal in $\mathbb{C}^k \times \mathbb{C}^k$.

Remark 17. The Proposition above has been proved in [**34**] under the assumption $J \equiv -i$ (i.e., for the operator ∂ instead of $\overline{\partial}$). In index formula in [**34**] the Maslov indices $\mu(\Psi^{\pm})$ appear with the opposite sign. Since the change of variables $t \mapsto -t$ transforms the operator ∂ to $\overline{\partial}$ and changes the sign of Maslov index, these two difference give the index formula in Proposition 16 if $J \equiv i$. The general case is an easy consequence of the contractibility of set $\mathcal{J}^c_{\omega_0}$ of ω_0-compatible almost complex structures in $\mathbb{C}^k$ and the continuity of Fredholm index.

4.2. The dimension of $\mathcal{M}_{(J,H,S)}(N:E)$.

Our goal is to to assign the Maslov index to the Hamiltonian path

$$\begin{cases} \dot{z} = X_H(z) \\ z(0) \in \mathrm{Graph}(dS) \\ z(1) \in \nu^*(N \times \mathbb{R}^m). \end{cases} \tag{29}$$

However, in a manifold instead of a linear space the Maslov index of a Hamiltonian path would depend on the choice of a trivialization of a tangent bundle along that path. Hence, we have to choose some class of admissible trivializations. In the case $S \equiv 0$, $m = 0$ this is done in [**30**] and we will adapt that exposition to our situation. Let

$$\begin{aligned} \psi_t &:= (\phi_t^{H\sharp\pi^*S} \circ (\phi_1^{H\sharp\pi^*S})^{-1}) \circ (\phi_t^H \circ (\phi_1^H)^{-1})^{-1} \\ &= \phi_t^H \circ \phi_t^{\pi^*S} \circ (\phi_1^H \circ \phi_1^{\pi^*S})^{-1} \circ (\phi_t^H \circ (\phi_1^H)^{-1})^{-1} \\ &= \phi_t^H \circ \phi_t^{\pi^*S} \circ (\phi_1^{\pi^*S})^{-1} \circ (\phi_t^H)^{-1}. \end{aligned}$$

The transformation

$$U(\tau,t) \mapsto \tilde{U} := \psi_t(U(\tau,t)) \tag{30}$$

transforms Equation (10) to

$$\begin{cases} \overline{\partial}_{\tilde{J},H\sharp\pi^*S}\tilde{U} = 0 \\ \tilde{U}(\tau,0) \in o_M \times \mathbb{R}^m \\ \tilde{U}(\tau,1) \in \nu^*(N \times \mathbb{R}^m), \end{cases} \tag{31}$$

where $\tilde{J} = \psi_t^* J$, and (29) is equivalent to

$$\begin{cases} \dot{z} = X_{H\sharp\pi^*S}(z) \\ z(0) \in o_{M\times\mathbb{R}^m} \\ z(1) \in \nu^*(N \times \mathbb{R}^m). \end{cases} \tag{32}$$

Hence, we will compute the dimension of $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(N:E)$ by computing the dimension of the space of solutions of (31) and give the grading to $CF_*(H,S)$ by assigning the Maslov index to each solution of (32).

For given z, we choose the class $\mathcal{T}$ of trivializations

$$\varphi : z^*T(T^*(E)) \to [0,1] \times \mathbb{C}^{n+m}$$

such that

$$\varphi(H_{z(t)}) \equiv \mathbb{R}^{n+m},\ \varphi(F_{z(t)}) \equiv i\mathbb{R}^{n+m},$$

where H_z and F_z are horizontal and vertical subbundles with respect to Levi-Civita connection on T^*E. Note that $\mathcal{T} \neq \varnothing$ since $[0,1] \simeq *$.

For $\varphi \in \mathcal{T}$ and a solution z of (32), we define the symplectic path

$$\Psi_\varphi^z(t) := \varphi T\psi_t^{H\sharp\pi^*S}(z(0))(\varphi)^{-1} : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}. \tag{33}$$

Then we have:

Lemma 18 ([**30**]). *If* $\varphi_1, \varphi_2 \in \mathcal{T}$ *then* $\mu(\Psi_{\varphi_1}) = \mu(\Psi_{\varphi_2})$.

We give the groups $CF_*(H, S : N)$ the grading by assigning to each solution of (29) (i.e., the generator of $CF_*(H, S : N)$) the Maslov index of the corresponding solution. More precisely, we have the following:

Definition 19. 1) We call the index of the solution of (32) with respect to some (and thus any) trivialization $\varphi \in \mathcal{T}$ *the Maslov index of a solution* z *of* (29) and denote it by $\mu(z)$.

2) We denote by $CF_p(H, S : N)$ the group generated by solutions z of (29) with $p = \frac{1}{2}\dim(N \times \mathbb{R}^m) - \mu(z)$ and set $CF^p(H, S : N) := \mathrm{Hom}(CF_p(H, S : N), \mathbb{Z})$.

According to Theorem 2.4 in [**33**] p is an integer. We will see later (see Remark 21) that it depends on the rank of the eigenbundle of Q ($= S$ at infinity) but not on the rank of E.

Consider the case $H \equiv 0$. Let $S_N : N \to \mathbb{R}$ be a Morse function and let $S : E \to \mathbb{R}$ be an extension of S_N such that $S_N \circ \pi_N = S$ in a tubular neighborhood $\pi_N : V \to N \times \mathbb{R}^m$ of $N \times \mathbb{R}^m \subset E$. Let $x \in N \times \mathbb{R}^m$ be a critical point of S_N. We identify the neighborhood of x in $N \times \mathbb{R}^m$ with $\mathbb{R}^l$ and the neighborhood of x in E with $\mathbb{R}^l \times \mathbb{R}^{n+m-l}$. In these coordinates

$$\psi_t^{\pi^* S}(q_1, q_2, p_1, p_2) = (q_1, q_2, p_1 + dS_N(q_1), p_2)$$

and

$$T\psi_t^{\pi^* S}(x) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ tD^2S(x) & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Since $T\psi_t^{\pi^* S}(x)(\mathbb{R}^{n+m} \times \{0\}) = \mathrm{Graph}(tD^2S)$, applying the localization property of Maslov index to $A(t) := tD^2S(x)$ we get

$$\begin{aligned} \mu(T\psi_t^{\pi^* S}(x)) &= \frac{1}{2}\mathrm{sign}A(1) - \frac{1}{2}\mathrm{sign}A(0) \\ &= \frac{1}{2}\mathrm{sign}D^2S(x) \\ &= \frac{1}{2}\mathrm{sign}D^2S_N(q_1) \\ &= -m_S^N(q_1) + \frac{1}{2}\dim(N \times \mathbb{R}^m), \end{aligned} \tag{34}$$

where m_S^N is the Morse index of S_N. Therefore, in that case p is the Morse index of S_N.

Now we have the following analogue of Theorem 5.1 in [**30**]:

Proposition 20. *For the regular choice of parameters,*

$$\dim\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta) = -\mu(x^\alpha) + \mu(x^\beta).$$

In particular, for $H \equiv 0$ and S as above

$$\dim\mathcal{M}_{(J,H,S)}(x,y) = m_S^N(x) - m_S^N(y),$$

where m_S^N is the Morse index of $S|_N$.

Proof. Since

$$\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta) = (\overline{\partial}_{J^{\alpha\beta},H^{\alpha\beta}})^{-1}(0),$$

we have

$$\dim\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta) = \mathrm{Index}(E_U),$$

where E_U is the linearization of $\overline{\partial}_{J^{\alpha\beta},H^{\alpha\beta}}$ at $U \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$. Since $\mathrm{Index}(E_U)$ depends only on the homotopy type of U, we can assume that

$$U(-\tau,t) = x^\alpha(t), \text{ and } U(\tau,t) = x^\beta(t) \text{ for } \tau \geq \tau_0.$$

Choose a symplectic trivialization

$$\varphi : U^*T(T^*E) \to \mathbb{R}\times[0,1]\times\mathbb{C}^{n+m}$$

such that

$$\varphi(H_{U(\tau,t)}) \equiv \mathbb{R}^{n+m},\ \varphi(F_{U(\tau,t)}) \equiv i\mathbb{R}^{n+m}.$$

The same computation as in Theorem 5.3 [**36**] shows that

$$\varphi E_U \varphi^{-1} = \overline{\partial}_{J_0,T,L} + \text{compact perturbation},$$

where $\overline{\partial}_{J_0,T,L}$ is the operator (26) with $J_0 = \varphi_* J$, $L(\tau) = \varphi(T(\nu^*N \times o_{\mathbb{R}^m}))$ and T satisfies (27) and (28) with

$$\Psi^+ := \Psi_\varphi^{x^\beta},\ \Psi^- := \Psi_\varphi^{x^\alpha}$$

(see (33)). Since a compact perturbation does not change Fredholm index, we have

$$\begin{aligned}\mathrm{Index}(E_U) &= \mathrm{Index}(\overline{\partial}_{J_0,T,L})\\ &= -\mu(x^\alpha) + \mu(x^\beta) + \mu(\Delta, \mathbb{R}^{n+m}\times L(\tau))\end{aligned}$$

by Proposition 16. Since the trivialization φ is chosen so that $\dim(\Delta \cap \mathbb{R}^{n+m}\times L(\tau))$ is constant, by zero axiom we have

$$\mathrm{Index}(E_U) = -\mu(x^\alpha) + \mu(x^\beta).$$

This proves the first statement. The second statement follows from the first one and (34). □

Remark 21. Let $H = H_1 \oplus 0$ for some compactly supported Hamiltonian $H_1 : T^*M \to \mathbb{R}$ and $S(q,\xi) = Q(\xi)$. Then the grading by $p = \frac{1}{2}\dim(N \times \mathbb{R}^m) - \mu(z)$ does not depend on a fiber dimension m but only on the index of Q. Indeed, consider the stabilization

$$\tilde{Q} : E \times \mathbb{R}^{m_1} \to \mathbb{R},\ \tilde{Q} = Q \oplus Q_0$$

for some quadratic form $Q_0 : \mathbb{R}^{m_1} \to \mathbb{R}$ with zero index. The critical points of $\mathcal{A}_{(H_1 \oplus 0, \tilde{Q})}$ are of the form $(z, 0) : [0,1] \to T^*E \times \mathbb{C}^{m_1}$, where $z : [0,1] \to T^*E$ is the critical point of $\mathcal{A}_{(H_1 \oplus 0, Q)}$. Let $\varphi := (\varphi_1, \varphi_2) \in \mathcal{T}$ be a trivialization of $T^*E \times \mathbb{C}^{m_1}$. By Direct Sum Axiom

$$\begin{aligned}
&\frac{1}{2}\dim(N \times \mathbb{R}^{m+m_1}) - \mu(z,0) \\
&= \frac{1}{2}(\dim(N) + m + m_1) - \mu(\Psi^z_{\varphi_1} \oplus \Psi^0_{\varphi_2}) \\
&= \frac{1}{2}(\dim(N) + m + m_1) - \mu(\Psi^z_{\varphi_1}) - \mu(\Psi^0_{\varphi_2}) \\
&= \frac{1}{2}(\dim(N) + m + m_1) - \mu(z) \\
&\quad - \left(-\mathrm{Index}(Q_0) + \frac{1}{2}m_1\right) \quad \text{(by (34))} \\
&= \frac{1}{2}\dim(N \times \mathbb{R}^m) - \mu(z).
\end{aligned}$$

4.3. Orientation.

In order to define Floer homology for arbitrary coefficients we need the orientation of manifolds $\mathcal{M}_{(J,H,S)}$ and $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}$. Contrary to the case of holomorphic spheres or cylinders (see [**14**], [**24**]), manifolds of holomorphic discs with Lagrangian boundary conditions need not to be orientable in general. However, in case of cotangent bundle such manifold are orientable under the boundary conditions of a conormal type. More precisely, we have the following:

Proposition 22 ([**30**]). *For each $(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta}) \in (\overline{\mathcal{J}}^c_\omega(E) \times \overline{\mathcal{H}}(E) \times \overline{\mathcal{S}}_{(E,Q)})_{\mathrm{reg}}$ and each x^α, x^β the determinant bundle*

$$\mathbf{Det} \to \mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta)$$

is trivial. Hence, the manifold $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta)$ is oriented. Moreover, there exists a coherent orientation in the sense of Definition 11 *in* [**14**] *of all $\mathcal{M}_{(J,H,S)}$ and $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}$ in each isotopy class of (J, H, S).*

Remark 23. In [**30**] the proof Proposition 22 is carried out for the case $S \equiv 0$. The general case follows from the fact that the transformation (30) defines a diffeomorphism

$$\mathcal{M}_{(J,H,S)}(x, y) \xrightarrow{\cong} \mathcal{M}_{(\tilde{J}, H \sharp \pi^* S, 0)}(\tilde{x}, \tilde{y}).$$

Hence the orientation on $\mathcal{M}_{(\tilde{J}, H \sharp \pi^* S, 0)}(\tilde{x}, \tilde{y})$ induces the pull-back orientation on $\mathcal{M}_{(J,H,S)}(x, y)$.

Remark 24. In Section 5.2 we will prove that in the case $H \equiv 0$ for a suitable choice of a J, S, g there exists a diffeomorphism

$$\mathcal{M}^{\mathrm{Floer}}_{(J,0,S)}(x,y) \cong \mathcal{M}^{\mathrm{Morse}}_{(S,g)}(x,y)$$

for $x, y \in \mathrm{Graph}(dS) \cap \nu^*(N \times \mathbb{R}^m) \cong \mathrm{Crit}(S|_{N\times\mathbb{R}^m})$. We will choose the orientations of $\mathcal{M}_{(J,0,S)}(x,y)$ and $\mathcal{M}_{(S,g)}(x,y)$ so that this diffeomorphism is orientation preserving.

The one dimensional components of $\mathcal{M}_{(J,H,S)}$ and $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}$ carry two orientations: one given in Proposition 22 and another given by orienting each trajectory in the direction of $\frac{\partial U}{\partial \tau}$. Define

$$n(U) = \begin{cases} 1 & \text{if these two orientations coincide} \\ -1 & \text{otherwise.} \end{cases}$$

Coherent (compatible with gluing) definition of orientation in Proposition 22 has the following consequence:

Lemma 25. *If* $(U_1, V_1), (U_2, V_2) \in \mathcal{M}_{(J,H,S)}(x,y) \times \mathcal{M}_{(J,H,S)}(y,z)$ *are two ends of the component of* $\mathcal{M}_{(J,H,S)}(x,z)$ *(in sense of Propositions* 10*), then*

$$n(U_1)n(V_1) + n(U_2)n(V_2) = 0.$$

□

Similar statement is true in parameterized version. The proof follows the same lines as the proof of analogous statements in [**12**, **14**, **15**].

5. Floer homology.

5.1. Construction.

For $x \in CF_p(H, S : N)$ and $y \in CF_{p-1}(H, S : N)$ we define $n(x,y)$ to be the number of points in (zero dimensional) manifold

$$\widehat{\mathcal{M}}_{(J,H,S)}(N : E) := \mathcal{M}_{(J,H,S)}(N : E)/\mathbb{R}$$

counted by their orientations, i.e.,

$$n(x,y) = \sum_U n(U),$$

where n(U) is defined in Section 4.3. Here $\mathbb{R}$ acts on $\mathcal{M}_{(J,H,S)}(N : E)$ in a standard way, by the translation in τ-variable.

According to Propositions 12, 13 and 10 for (J, H, S) in a dense subset

$$(\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}} \subset \mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$$

the number $n(x,y)$ is finite.

The following proposition is a reformulation of the result proved in [**12**] and [**28**] for the compact case.

Theorem 26.

(1) *For* $(J,H,S) \in (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\text{reg}}$ *the homomorphisms*

$$\partial : CF_p(H,S:N) \to CF_{p-1}(H,S:N)$$
$$\partial x = \sum_y n(x,y)y$$

and

$$\delta := \mathrm{Hom}(\partial) : CF^p(H,S:N) \to CF^{p+1}(H,S:N)$$

satisfy

$$\partial \circ \partial = 0,\ \delta \circ \delta = 0.$$

We define

$$HF_p(J,H,S;N:E) := \mathrm{Ker}\partial/\mathrm{Im}\partial$$

and

$$HF^p(J,H,S;N:E) := \mathrm{Ker}\delta/\mathrm{Im}\delta.$$

(2) *For two given parameters*

$$(J^\alpha,H^\alpha,S^\alpha),(J^\beta,H^\beta,S^\beta) \in (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\text{reg}},$$

there exist canonical isomorphisms

$$h_{\alpha\beta} : HF_p(J^\alpha,H^\alpha,S^\alpha;N:E) \to HF_p(J^\beta,H^\beta,S^\beta;N:E)$$

and

$$h^{\alpha\beta} : HF^p(J^\alpha,H^\alpha,S^\alpha;N:E) \to HF^p(J^\beta,H^\beta,S^\beta;N:E)$$

which satisfy

(i) $h_{\gamma\beta} \circ h_{\beta\alpha} = h_{\gamma\alpha}$

(ii) $h_{\alpha\alpha} = id$.

The analogous equalities hold for $h^{\alpha\beta}$.

Proof. Once we established the C^0 -estimates as in Proposition 9, the proof follows the same lines as in Theorem 4 in [**12**] (see also [**28**]). For the later purpose, we only recall the main points. By definition of ∂, we have

$$\partial^2(x) = \partial\left(\sum_y n(x,y)y\right) = \sum_z \sum_y n(x,y)n(y,z)z. \tag{35}$$

According to Propositions 10 and 11, the split trajectories in

$$\mathcal{M}_{(J,H,S)}(x,y) \times \mathcal{M}_{(J,H,S)}(y,z)$$

are the boundaries of one dimensional manifolds contained in $\mathcal{M}_{(J,H,S)}(x,z)$ and oriented as in Section 4.3. Hence, they appear in (35) in pairs with opposite signs and thus they add to 0. That proves $\partial \circ \partial = 0$. For the proof of the second statement, we define

$$(h_{\alpha\beta})_\sharp : CF_p(H^\alpha, S^\alpha : N) \to CF_p(H^\beta, S^\beta : N)$$

by

$$(h_{\alpha\beta})_\sharp x = \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta,$$

where $n(x^\alpha, x^\beta)$ is the number of points in (zero dimensional by Proposition 20) manifold $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta)$, counted with their orientations. Set

$$(h^{\alpha\beta})^\sharp := \mathrm{Hom}((h^{\beta\alpha})_\sharp) : CF^p(H^\alpha, S^\alpha : N) \to CF^p(H^\beta, S^\beta : N).$$

Note that the grading is preserved by Proposition 20. Homomorphisms $(h_{\alpha\beta})_\sharp$ and $(h^{\alpha\beta})^\sharp$ commute with ∂ and δ respectively. The proof is based on the same gluing arguments as the proof of $\partial^2 = 0$ (see [**12**, **15**]). Therefore, they define the mappings $h_{\alpha\beta}$ and $h^{\alpha\beta}$ in homology (resp. cohomology).

If $h_{\alpha\beta}$ and $h_{\beta\gamma}$ are defined via regular homotopies

$$(H^{\alpha\beta}, S^{\alpha\beta}, J^{\alpha\beta}) \text{ and } (H^{\beta\gamma}, S^{\beta\gamma}, J^{\beta\gamma})$$

then for large R the regular homotopy $(H^{\alpha\gamma}, S^{\alpha\gamma}, J^{\alpha\gamma})$, where

(36)

$$H_\tau^{\alpha\gamma} = \begin{cases} H_{\tau+R}^{\alpha\beta}, & \tau \le 0 \\ H_{\tau-R}^{\beta\gamma}, & \tau \ge 0, \end{cases} \quad S_\tau^{\alpha\gamma} = \begin{cases} S_{\tau+R}^{\alpha\beta}, & \tau \le 0 \\ S_{\tau-R}^{\beta\gamma}, & \tau \ge 0, \end{cases} \quad J_\tau^{\alpha\gamma} = \begin{cases} J_{\tau+R}^{\alpha\beta}, & \tau \le 0 \\ J_{\tau-R}^{\beta\gamma}, & \tau \ge 0, \end{cases}$$

defines the homomorphism $h_{\alpha\gamma}$ which satisfies property 2 (i). The proof is again based on the same argument as the proof of $\partial^2 = 0$ [**12**].

Finally, homomorphisms $h_{\alpha\beta}$ and $h^{\alpha\beta}$ are independent of the choice of homotopy $H^{\alpha\beta}$. We only sketch the proof of this fact, referring the reader to [**12**, **15**] for the details. Choose two homotopies $H_1^{\alpha\beta}$, $S_1^{\alpha\beta}$, $J_1^{\alpha\beta}$, $H_2^{\alpha\beta}$, $S_2^{\alpha\beta}$, $J_2^{\alpha\beta}$. Let $(h_{\alpha\beta})_\sharp^1$ and $(h_{\alpha\beta})_\sharp^2$ be the corresponding chain homomorphisms. Consider the one-parameter families of homotopies $\{H_\nu^{\alpha\beta}\}_{\nu\in\mathbb{R}}$, $\{S_\nu^{\alpha\beta}\}_{\nu\in\mathbb{R}}$, $\{J_\nu^{\alpha\beta}\}_{\nu\in\mathbb{R}}$ such that

$$H_\nu^{\alpha\beta} \equiv H_1^{\alpha\beta},\ S_\nu^{\alpha\beta} \equiv S_1^{\alpha\beta},\ J_\nu^{\alpha\beta} \equiv J_1^{\alpha\beta} \text{ for } \nu \le 0$$

and

$$H_\nu^{\alpha\beta} \equiv H_2^{\alpha\beta},\ S_\nu^{\alpha\beta} \equiv S_2^{\alpha\beta},\ J_\nu^{\alpha\beta} \equiv J_2^{\alpha\beta} \text{ for } \nu \ge 1.$$

Let $\tilde{n}(x^\alpha, x^\beta)$ denote the algebraic number of the solutions of

$$
\begin{cases}
\frac{\partial U}{\partial \tau} + J_\nu^{\alpha\beta}(\frac{\partial U}{\partial t} - X_{H_\nu^{\alpha\beta}}(U)) = 0 \\
(u(\tau,0), v(\tau,0)) \in \mathrm{Graph}(dS_\nu^{\alpha\beta}) \\
u(\tau,1) \in \nu^* N, \ v(\tau,1) \in o_{\mathbb{R}^m} \\
\lim_{\tau\to-\infty} U(\tau,t) = x^\alpha(t), \\
\lim_{\tau\to\infty} U(\tau,t) = x^\beta(t).
\end{cases} \tag{37}
$$

Define

$$\Phi^{\alpha\beta} : CF_p(H^\alpha, S^\alpha : N) \to CF_{p+1}(H^\beta, S^\beta : N)$$

by

$$\Phi^{\alpha\beta}(x^\alpha) = \sum \tilde{n}(x^\alpha, x^\beta) x^\beta.$$

Then

$$\partial \circ \Phi^{\alpha\beta} - \Phi^{\alpha\beta} \circ \partial = (h_{\alpha\beta})^1_\sharp - (h_{\alpha\beta})^2_\sharp,$$

i.e., $\Phi^{\alpha\beta}$ is a chain homotopy ([**12**, **15**]). Therefore, $h^1_{\alpha\beta} = h^2_{\alpha\beta}$.

Statement 2 (ii) now follows by choosing the constant homotopy $H^{\alpha\alpha} \equiv H^\alpha$. □

5.2. Computation.

In [**13**] Floer proved that if

$$h : M \to \mathbb{R}$$

is a C^2 Morse function, then

$$HF_*(J, h \circ \pi, M) \cong H_*^{\mathrm{Morse}}(h).$$

We incorporate this result and the generalization [**31**] in our framework. Consider the tubular neighborhood $W \cong W_0 \times \mathbb{R}^m$ of $N \times \mathbb{R}^m \subset E$ and the projection

$$\pi_N : W \to N \times \mathbb{R}^m$$

given locally by

$$\pi_N : (x, y, \xi) \mapsto (x, \xi). \tag{38}$$

Following [**31**], assume that the metric g in T^*E is chosen in the following way. Choose a metric g_M on M such that the fibers of π_N are orthogonal to $N \times \mathbb{R}^m$ with respect to the metric $g_E := g_M \oplus g_0$, where g_0 is the standard metric on $\mathbb{R}^m$. The Levi-Civita connection associated with g_E provides the splitting

$$T_\xi(T^*E) = H_\xi \oplus F_\xi$$

into horizontal and vertical subbundles. F_ξ and H_ξ are canonically isomorphic to $T^*_{\pi(\xi)}E$ and $T_{\pi(\xi)}E$. Let g be a metric on T^*E such that H_ξ is orthogonal to F_ξ and that the above isomorphisms are isometries.

Let

$$S_N : N \times \mathbb{R}^m \to \mathbb{R}$$

be a Morse function obtained by restricting $S \in \mathcal{S}_{(E,Q)}$ to $N \times \mathbb{R}^m$. Let $V \subset W$ be another tubular neighborhood of $N \times \mathbb{R}^m$ and let $\kappa : E \to \mathbb{R}$ be a smooth function such that

$$\begin{aligned} \kappa(e) &= 1 \text{ for } e \in V \\ &= 0 \text{ for } e \notin W. \end{aligned}$$

We denote by

$$S_N^V : E \to \mathbb{R}$$

an extension of S_N defined by

$$S_N^V(e) := \begin{cases} \kappa(e) S_N(\pi_N(e)) + (1-\kappa(e))S(e) & \text{for } e \in W \\ S(e) & \text{for } e \notin W. \end{cases}$$

Then $S_N^V : E \to \mathbb{R}$ is smooth and

$$S_N^V(e) = S_N \circ \pi_N(e) \text{ for } e \in V.$$

Note that from (38) and the definition of S_N^V it follows that $S_N^V(x,y,\xi) = Q(\xi)$ whenever $S(x,y,\xi) = Q(\xi)$ and hence S_N^V belongs to the parameter space $\mathcal{S}_{(E,Q)}$. Since we proved in Proposition 9 that images of all solutions of (7) are contained in some relatively compact open submanifold $K_0 \subset T^*E$, we have

$$\sup_{K_0} \left\| \nabla d S_N^V \right\| < \infty,$$

where $\|\nabla d S_N^V\|$ is defined with respect to g_E and the induced Levi-Civita connection on $T^*E|_{\pi(K_0)}$. Hence we can assume, after replacing g_E by χg_E with

$$\begin{aligned} \chi(e) &= \varepsilon_0, \text{ for } e \in K_0 \\ &= 1, \text{ for } e \notin K_1 \supset K_0 \end{aligned}$$

if necessary, that

$$\sup_{K_0} \left\| \nabla d S_N^V \right\| < \varepsilon \tag{39}$$

for small $\varepsilon > 0$. Note that the Levi-Civita connection on $T^*E|_{\pi(K_0)}$ is invariant under the rescaling $g_E \rightsquigarrow \varepsilon_0 g_E$ and thus remains unchanged. Since $\chi \equiv 1$ outside K_1, χg_E remains in parameter space $\mathcal{G}_{g_M \oplus g_0}$.

Proposition 27 (Compare [**31**])**.**

$$HF_p(J, 0, S_N^V; N : E) \cong H_p^{\mathrm{Morse}}(S_N).$$

Proof. Since $H \equiv 0$, Equation (6) becomes

$$\begin{cases} \dot{\Gamma} = 0 \\ \Gamma(0) \in \mathrm{Graph}(dS_N^V) \\ \Gamma(1) \in \nu^* N \times o_{\mathbb{R}^m} \end{cases}$$

i.e.,

$$\Gamma(t) \equiv p \in \nu^*(N \times \mathbb{R}^m) \cap \mathrm{Graph}(dS_N^V) \;\cong \mathrm{Crit}(S_N).$$

Hence, we have one-to-one correspondence

$$CF_p(0, S_N^V; N : E) \cong \mathrm{Crit}(S_N). \tag{40}$$

Since S_N^V is constant along the fibers of π_N and the fibers are orthogonal to $N \times \mathbb{R}^m$, we have, for $e \in N \times \mathbb{R}^m$

$$\nabla^{g_E} S_N^V(e) = \nabla^{g_E^N} S_N(e), \tag{41}$$

where g_E^N is a restriction of g_E to N.

Let γ be a solution of

$$\frac{d\gamma}{d\tau} + \nabla^{g_E^N} S_N(\gamma) = 0. \tag{42}$$

Consider, modifying Lemma 5.1 in [**13**]

$$U(\tau, t) := \psi_{1-t}(\gamma(\tau))$$

and

$$J_t = (\psi_{1-t})_* J_g$$

where $\psi_t := \psi_t^{\pi^* S_N^V}$ and $J_g := J_{g_M} \oplus i$ for J_{g_M} is as in Section 2. Then

$$\begin{aligned} \frac{\partial U}{\partial \tau} + J_t \frac{\partial U}{\partial t} &= T\psi_{1-t} \frac{d\gamma}{d\tau} + (\psi_{1-t})_* J_g T\psi_{1-t}[-X_{\pi^* S_N^V}(\psi_{1-t}(\gamma))] \\ &= T\psi_{1-t} \frac{d\gamma}{d\tau} - (\psi_{1-t})_* J_g T\psi_{1-t} X_{\pi^* S_N^V}(\gamma) \\ &= T\psi_{1-t} \left(\frac{d\gamma}{d\tau} - J_g X_{\pi^* S_N^V}(\gamma) \right) \\ &= T\psi_{1-t} \left(\frac{d\gamma}{d\tau} + \nabla^g \pi^* S_N^V(\gamma) \right). \end{aligned}$$

Since $d\pi^* S_N^V$ vanishes on the vertical subbundle F it follows that $\nabla^g \pi^* S_N^V \subset H$, and since $T\pi|_H : H \to TE$ is an isometry by the choice of g, we have

$$\begin{aligned} \nabla^g \pi^* S_N^V &= \nabla^{g_E} S_N^V \\ &= \nabla^{g_E^N} S_N \quad \text{(by (41))}. \end{aligned}$$

Therefore, U satisfies

$$\begin{cases} \frac{\partial U}{\partial \tau} + J_t \frac{\partial U}{\partial t} = 0 \\ U(\tau, 0) = \psi_1(\gamma(\tau)) \in \psi_1(N) \subset \mathrm{Graph}(dS_N^V) \\ U(\tau, 1) = \gamma(\tau) \in N \subset \nu^* N, \end{cases}$$

i.e., $U \in \mathcal{M}_{(J_t, 0, S_N^V)}(N : E)$.

Conversely, for every solution U of

$$\begin{cases} \frac{\partial U}{\partial \tau} + J_t \frac{\partial U}{\partial t} = 0 \\ U(\tau, 0) \in \mathrm{Graph}(dS_N) \\ U(\tau, 1) \in \nu^* N \times o_{\mathbb{R}^m} \end{cases}$$

we define

$$\gamma(\tau, t) := (\psi_{1-t})^{-1}(U(\tau, t)).$$

Note that

$$\begin{aligned} \frac{\partial \gamma}{\partial \tau} + J_g \left(\frac{\partial \gamma}{\partial t} - X_{\pi^* S_N^V}(\gamma) \right) &= (T\psi_{1-t})^{-1} \left(\frac{\partial U}{\partial \tau} + (\psi_{1-t})_* J_g \frac{\partial U}{\partial t} \right) \\ &= 0, \end{aligned}$$

i.e., γ satisfies

$$\begin{cases} \frac{\partial \gamma}{\partial \tau} + J_g \frac{\partial \gamma}{\partial t} + \nabla^g S_N^V(\gamma) = 0 \\ \gamma(\tau, 0) = \psi_1^{-1}(U(\tau, 0)) \in o_E \\ \gamma(\tau, 1) = U(\tau, 1) \in \nu^*(N \times \mathbb{R}^m). \end{cases} \tag{43}$$

We will prove that $\frac{\partial \gamma}{\partial t} \equiv 0$. Let us write $\gamma(\tau, t) = (x(\tau, t), y(\tau, t))$ with $x(\tau, t) \in E$ and $y(\tau, t) \in T^*_{x(\tau,t)} E$. Since J_g maps horizontal vectors to vertical ones, we can write (43) in the form

$$\begin{cases} \frac{\partial x}{\partial \tau} - \nabla_t y + \nabla^{g_E} S_N^V(x) = 0 \\ \nabla_\tau y + \frac{\partial x}{\partial t} = 0 \\ y(\tau, 0) = 0, \ x(\tau, 1) \in N \times \mathbb{R}^m, \ y(\tau, 1) \in \nu^*_{x(\tau,1)}(N \times \mathbb{R}^m). \end{cases} \tag{44}$$

Define

$$f(\tau) := \int_0^1 |y(\tau, t)|^2 dt.$$

Note that, by the construction of S_N^V,

$$\mathrm{Graph}(dS_N^V) \cap \nu^*(N \times \mathbb{R}^m) \subset N \times \mathbb{R}^m.$$

Therefore, we have

$$\lim_{\tau \to \pm\infty} y(\tau, t) \equiv 0$$

and hence

$$\lim_{\tau \to \pm\infty} f(\tau) = 0.$$

Following the same lines as in [**31**] we prove that f is convex, and hence constant. We identify $T_\xi(T^*E) \cong T^*_\xi E \oplus T_\xi E$ and compute

$$\begin{aligned}
\frac{1}{2} f''(\tau) &= \int_0^1 \left(|\nabla_\tau y|^2 + \langle \nabla_\tau^2 y, y \rangle \right) dt \\
&= \int_0^1 \left(|\nabla_\tau y|^2 - \left\langle \nabla_t \frac{\partial x}{\partial \tau}, y \right\rangle \right) dt \\
&= \int_0^1 \left(|\nabla_\tau y|^2 - \langle \nabla_t^2 y, y \rangle + \langle \nabla_{\frac{\partial x}{\partial t}} dS_N^V(x), y \rangle \right) dt.
\end{aligned}$$

Here we used the fact that the Levi-Civita connection is torsion free, and thus $\nabla_\tau \frac{\partial x}{\partial \tau} = \nabla_t \frac{\partial x}{\partial t}$. Since $y(\tau, 0) = 0$, integrating by parts we compute

$$\begin{aligned}
\int_0^1 \langle \nabla_t^2 y, y \rangle dt &= \langle \nabla_t y(\tau, 1), y(\tau, 1) \rangle - \int_0^1 |\nabla_t y|^2 dt \\
&= \left\langle \frac{\partial x}{\partial \tau} + \nabla S_N^V(x), y(\tau, 1) \right\rangle - \int_0^1 |\nabla_t y|^2 dt \\
&= - \int_0^1 |\nabla y|^2 dt,
\end{aligned}$$

since $\frac{\partial x}{\partial \tau} + \nabla S_N^V(x) \in T(N \times \mathbb{R}^m)$ and $y(\tau, 1) \in \nu^*(N \times \mathbb{R}^m)$. Hence

$$\begin{aligned}
\frac{1}{2} f''(\tau) &= \int_0^1 \left(|\nabla_\tau y|^2 + |\nabla_t y|^2 - \langle \nabla_{\nabla_\tau y} dS_N^V(x), y \rangle \right) dt \\
&\geq \|\nabla_\tau y\|_{L^2}^2 + \|\nabla_t y\|_{L^2}^2 - \left\| \nabla dS_N^V \right\|_{L^\infty} \|\nabla_\tau y\|_{L^2} \|y\|_{L^2} \\
&\geq \|\nabla_\tau y\|_{L^2}^2 + \|\nabla_t y\|_{L^2}^2 - \left\| \nabla dS_N^V \right\|_{L^\infty} \|\nabla_t y\|_{L^2}^2
\end{aligned}$$

by Poincaré inequality, since $y(\tau, 0) = 0$. Hence $f''(\tau) \geq 0$ if ε in (39) is small enough. Therefore $y \equiv 0$ and, by (44) $\frac{\partial x}{\partial t} \equiv 0$. Hence $\frac{\partial \gamma}{\partial t} \equiv 0$. By (43) this means that γ solves

$$\frac{d\gamma}{d\tau} + \nabla^{g_E^N} S_N^V(\gamma) = 0.$$

Therefore, we have one-to-one correspondence

$$\mathcal{M}_{(J, 0, S_N^V)} \cong \mathcal{M}_{(S_N, g)}.$$

Together with (40) this finishes the proof. □

Theorem 28. *For regular parameters*

$$(\tilde{J}, \tilde{H}, S) \in (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\text{reg}},$$

and $(J,H) \in (\mathcal{J}^c_\omega(M) \times \mathcal{H}(M))_{\mathrm{reg}}$ *there exist the isomorphisms*

$$HF_{p+k}(\tilde{J},\tilde{H},S;N:E) \cong HF_p(J,H,N:M) \cong H_p(N),$$

where $HF_*(J,H,N : M)$ *is the ordinary Floer homology of the pair* $(o_M,\nu^*(N))$ *of Lagrangian submanifolds in* T^*M *and* $H_*(N)$ *the singular homology of submanifold* N*. Analogously, there exist the isomorphisms*

$$HF^{p+k}(\tilde{J},\tilde{H},S;N:E) \cong HF^p(J,H,N:M) \cong H^p(N).$$

Furthermore, the above isomorphisms commute with isomorphisms $h_{\alpha\beta}$ *(resp.* $h^{\alpha\beta}$*) constructed in Theorem* 26.

Proof. The second isomorphism

$$HF_*(J,H,N:M) \cong H_*(N),$$

follows from Proposition 27, and we will prove only the first one.

According to Theorem 26 we can assume that

$$\tilde{H} = H \oplus 0,\ \tilde{J} = J \oplus i\ \text{ and } S = Q.$$

With such choice of parameters, the critical points $\Gamma := (\gamma, z)$ of $\mathcal{A}_{(H\oplus 0,Q)}$ on $\Omega(Q;N)$ are the solutions of

$$\begin{cases} \dot{\gamma} = X_H(\gamma) \\ \gamma(0) \in o_M,\ \gamma(1) \in \nu^* N \\ \dot{z} = 0 \\ z(0) \in o_{\mathbb{R}^m},\ z(1) \in \mathrm{Graph}(dQ). \end{cases}$$

Hence $z \equiv 0$ and thus

$$CF_*(H \oplus 0, Q : N) \cong CF_*(H,N)$$

where the last group is the usual Floer chain group for the pair $(o_M, \nu^* N)$ in T^*M.

The gradient flow of $\mathcal{A}_{(H\oplus 0,Q)}$ satisfies

$$\begin{cases} \overline{\partial}_{J,H} u := \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0 \\ u(\tau,0) \in o_M,\ u(\tau,1) \in \nu^* N \\ \overline{\partial} v = \frac{\partial v}{\partial \tau} + i\frac{\partial v}{\partial t} = 0 \\ v(\tau,0) \in \mathrm{Graph}(dQ),\ v(\tau,1) \in o_{\mathbb{R}^m} \end{cases}$$

and therefore $\mathcal{M}_{(J\oplus i, H\oplus 0,Q)}(N : E)$ is diffeomorphic to $\mathcal{M}_{(J,H)}(N : M)$. Hence, the above isomorphism between Floer chain groups defines the isomorphism between the corresponding Floer homologies, and, consequently, cohomologies. □

6. Invariants.

6.1. Definition.

Observe that, since Equation (7) is the negative gradient flow of $\mathcal{A}_{(H,S)}$, the boundary operator ∂ preserves the level sets of $\mathcal{A}_{(H,S)}$. More precisely, we define

$$CF^{\lambda}(H, S : N) := \{\Gamma \in CF(H, S : N) \mid \mathcal{A}_{(H,S)}(\Gamma) < \lambda\}$$

and

$$CF_*^{\lambda}(H, S : N) := \text{ the free abelian group generated by } CF^{\lambda}(H, S : N).$$

Then, the boundary map

$$\partial : CF_*(H, S : N) \to CF_*(H, S : N)$$

induces the relative boundary map

$$\partial^{\lambda} : CF_*^{\lambda}(H, S : N) \to CF_*^{\lambda}(H, S : N)$$

which satisfy the obvious identity

$$\partial^{\lambda} \circ \partial^{\lambda} = 0.$$

Therefore, we can define the relative Floer homology groups

$$HF_*^{\lambda} := \mathrm{Ker}(\partial^{\lambda})/\mathrm{Im}(\partial^{\lambda}).$$

The natural inclusion

$$j^{\lambda} : CF^{\lambda}(H, S : N) \to CF(H, S : N)$$

induces the group homomorphism

$$j_{\sharp}^{\lambda} : CF_*^{\lambda}(H, S : N) \to CF_*(H, S : N)$$

which commutes with ∂, i.e.,

$$\partial \circ j_{\sharp}^{\lambda} = j_{\sharp}^{\lambda} \circ \partial^{\lambda}.$$

Hence, $j_{\sharp}^{\lambda}$ induces the natural homomorphism

$$j_*^{\lambda} : HF_*^{\lambda}(J, H, S : N) \to HF_*(J, H, S : N).$$

Furthermore, we define

$$CF_{\lambda}^*(H, S : N) := \mathrm{Hom}(CF_*^{\lambda}(H, S : N), \mathbb{Z})$$

and denote by δ^{λ} the restriction of δ to $CF_{\lambda}^*(H, S : N)$. Now $j_{\sharp}^{\lambda}$ induces dual homomorphism

$$j_{\lambda}^{\sharp} : CF^*(H, S : N) \to CF_{\lambda}^*(H, S : N)$$

such that

$$j_{\lambda}^{\sharp} \circ \delta = \delta^{\lambda} \circ j_{\lambda}^{\sharp}.$$

Hence, we have the homomorphism

$$j_\lambda^* : HF^*(J,H,S:N) \to HF_\lambda^*(J,H,S:N).$$

Definition 29. (1) For $(a,J,H,S) \in H_*(N)\times(\mathcal{J}_\omega^c(E)\times\mathcal{H}(E)\times\mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ we define

$$\sigma(a,J,H,S:N) := \inf\{\lambda \mid a \in \mathrm{Image}(j_*^\lambda F_*)\}.$$

(2) For $(u,J,H,S) \in H^*(N)\times(\mathcal{J}_\omega^c(E)\times\mathcal{H}(E)\times\mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ we define

$$\sigma(u,J,H,S:N) := \inf\{\lambda \mid j_\lambda^* F^* u \neq 0\}.$$

Here

$$F_* : H_*(N) \to HF_*(J,H,S:N)$$

and

$$F^* : H^*(N) \to HF^*(J,H,S:N)$$

denote the isomorphisms in Theorem 28.

Next lemma shows that the above definition is correct.

Lemma 30. *For $a \neq 0$, $u \neq 0$ and generic (J,H,S), the numbers $\sigma(a,J,H,S:N)$ and $\sigma(u,J,H,S:N)$ are the critical values of $\mathcal{A}_{(H,S)}$. In particular, they are finite numbers.*

Proof. The set of critical points of $\mathcal{A}_{(H,S)}$ is in one-to-one correspondence with

$$\mathrm{Graph}(dS)\cap(\phi_1^H)^{-1}(\nu^*N\times o_{\mathbb{R}^m}). \tag{45}$$

Since $H = H_1 \oplus H_2$ and $S = Q$ at infinity, the set (45) is

$$(o_M \cap (\phi_1^{H_1})^{-1}(\nu^*N)) \times (\mathrm{Graph}(dQ)\cap(\phi_1^{H_2})^{-1}(o_{\mathbb{R}^m}))$$

outside a compact set. Since H_1 and H_2 have compact supports, all points in (45) are contained in a compact set. From transversality assumption (8) we conclude that the set (45) is finite. Hence, if λ is not a critical value of $\mathcal{A}_{(H,S)}$, then there exists $\mu < \lambda$ such that there is no critical values of $\mathcal{A}_{(H,S)}$ in closed interval $[\mu,\lambda]$. In that case,

$$CF_*^\lambda(H,S:N) \equiv CF_*^\mu(H,S:N),\ CF_\lambda^*(H,S:N) \equiv CF_\mu^*(H,S:N)$$

and

$$j_\sharp^\lambda \equiv j_\sharp^\mu,\ j_\lambda^\sharp \equiv j_\mu^\sharp.$$

Hence, $z \in \mathrm{Im}(j_*^\lambda)$ (resp. $j_\lambda^* \neq 0$) is equivalent to $z \in \mathrm{Im}(j_*^\mu)$ (resp. $j_\mu^* \neq 0$). It follows that λ cannot be detected by σ.

Finally, since there are only finitely many critical values of $\mathcal{A}_{(H,S)}$, we deduce that both $\sigma(a,J,H,S:N)$ and $\sigma(u,J,H,S:N)$ are finite numbers. □

We next show that the definition of σ does not depend on an almost complex structure J used in construction of Floer homology.

Proposition 31. *The numbers* $\sigma(a, J, H, S : N)$ *and* $\sigma(u, J, H, S : N)$ *are independent of the regular choice of* J.

Proof. For $J^\alpha, J^\beta \in \mathcal{J}^c_\omega(E)$ we chose any path $\overline{J}_\tau \in \overline{\mathcal{J}}^c_\omega(E)$. Recall from the proof of Theorem 26 that the isomorphism

$$h_{\alpha\beta} : HF_*(J^\alpha, H, S : N) \to HF_*(J^\beta, H, S : N)$$

is induced by the group homomorphism

$$\begin{aligned} &h_\sharp : CF_*(H, S : N) \to CF_*(H, S : N), \\ &h_\sharp(x^\alpha) = \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta, \end{aligned} \tag{46}$$

where $n(x^\alpha, x^\beta)$ is the algebraic number of points in zero dimensional manifold $\mathcal{M}_{(J^{\alpha\beta}, H, S)}(x^\alpha, x^\beta)$. We compute the difference $\mathcal{A}_{(H,S)}(x^\beta) - \mathcal{A}_{(H,S)}(x^\alpha)$ for every x^β which appears in sum (46). For such x^β, the set $\mathcal{M}_{(J^{\alpha\beta}, H, S)}(x^\alpha, x^\beta)$ is nonempty ($n(x^\alpha, x^\beta) \neq 0$). For any $U \in \mathcal{M}_{(J^{\alpha\beta}, H, S)}$

$$\begin{aligned} &\mathcal{A}_{(H,S)}(x^\beta) - \mathcal{A}_{(H,S)}(x^\alpha) \\ &= \int_{-\infty}^{+\infty} \frac{d}{d\tau} \mathcal{A}_{(H,S)}(U) d\tau \\ &= \int_{-\infty}^{+\infty} d\mathcal{A}_{(H,S)}(U) \frac{\partial U}{\partial \tau} d\tau \\ &= \int_{-\infty}^{+\infty} \int_0^1 \left[(\omega \oplus \omega_0) \left(\frac{\partial U}{\partial t}, \frac{\partial U}{\partial \tau} \right) - dH(U) \frac{\partial U}{\partial t} \right] dt d\tau \\ &= \int_{-\infty}^{+\infty} \int_0^1 \left\langle \overline{J}_\tau \left(\frac{\partial U}{\partial t} - X_H(U) \right), \frac{\partial U}{\partial \tau} \right\rangle_{\overline{J}_\tau} dt d\tau \\ &= - \int_{-\infty}^{+\infty} \int_0^1 \left| \frac{\partial U}{\partial \tau} \right|^2_{\overline{J}_\tau} dt d\tau \\ &\leq 0. \end{aligned}$$

Here we used (5) and (10). Hence, $\mathcal{A}_{(H,S)}(x^\alpha) \geq \mathcal{A}_{(H,S)}(x^\beta)$ and therefore $h_{\alpha\beta}$ is level preserving, i.e.,

$$\begin{aligned} &h_{\alpha\beta} : HF^\lambda_*(J^\alpha, H, S : N) \to HF^\lambda_*(J^\beta, H, S : N), \\ &h_{\alpha\beta} \circ j^\lambda_* = j^\lambda_* \circ h_{\alpha\beta} \end{aligned} \tag{47}$$

Assume that $F^\alpha a \in \mathrm{Im}(j^\lambda_*)$, where

$$F^\alpha_* : H_*(N) \to HF_*(J^\alpha, H, S : N)$$

is the isomorphism in Theorem 26. Then, by (47), $h_{\alpha\beta} F^\alpha_* a \in \mathrm{Im}(j^\lambda_*)$. Since $h_{\alpha\beta} F^\alpha_* = F^\beta_*$ by Theorem 26, we have $F^\beta_* a \in \mathrm{Im}(j^\lambda_*)$ and hence

$$\sigma(a, J^\alpha, H, S : N) \leq \sigma(a, J^\beta, H, S : N).$$

Since the above argument is valid for any J^α, J^β, the opposite inequality also holds and therefore

$$\sigma(a, J^\alpha, H, S : N) = \sigma(a, J^\beta, H, S : N).$$

□

As a consequence, we can introduce the following notation.

Definition 32. For regular choice of parameters (J, H, S) we denote the numbers $\sigma(a, J, H, S : N)$ and $\sigma(u, J, H, S : N)$ by $\sigma(a, H, S : N)$ and $\sigma(u, H, S : N)$.

6.2. Continuity.

In order to extend Definition 32 from $(\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ to the whole manifold $\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ we need the following continuity result:

Theorem 33. *For $a \in H_*(N)$ the function*

$$\sigma : (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}} \to \mathbb{R},$$
$$(H, S) \mapsto \sigma(a, H, S : N)$$

is continuous in C^0 topology. The analogous statement is true for $u \in H^*(N)$ and $\sigma(u, H, S : N)$.

Proof. We fix regular parameters (H^α, S^α) and (H^β, S^β) and choose the C^∞ function

$$\rho : \mathbb{R} \to \mathbb{R}$$

such that

$$\rho(\tau) = 1, \text{ for } \tau \geq 1$$
$$\rho(\tau) = 0, \text{ for } \tau \leq 0.$$

Denote by $(\overline{H}_\tau, \overline{S}_\tau)$ a regular homotopy connecting (H^α, S^α) and (H^β, S^β) which is ε-close in C^1-topology to (possibly non-regular) homotopy

$$(\rho(\tau)H^\beta + (1 - \rho(\tau))H^\alpha, \rho(\tau)S^\beta + (1 - \rho(\tau))S^\alpha).$$

Then, as in the proof of Proposition 31 we compute $\mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) - \mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha)$ for a pair x^α, x^β connected by trajectory U satisfying (10). Since

$$\frac{d}{d\tau}\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U(\tau)) = d\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U)\frac{\partial U}{\partial \tau} - \int_0^1 \frac{\partial \overline{H}_\tau}{\partial \tau}dt + \frac{\partial \overline{S}_\tau}{\partial \tau}$$

and since the last two terms are ε-close to

$$-\int_0^1 \rho'(\tau)(H^\beta - H^\alpha)dt + \rho'(\tau)(S^\beta - S^\alpha),$$

we have

$$\begin{aligned}
&\mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) - \mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha) \\
&= \int_{-\infty}^{+\infty} \frac{d}{d\tau}\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U(\tau))d\tau \\
&\le \int_{-\infty}^{+\infty} \left\{ \int_0^1 \left[d\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U)\frac{\partial U}{\partial \tau} - \rho'(\tau)(H^\beta - H^\alpha)\right] dt \right. \\
&\qquad \left. + \rho'(\tau)(S^\beta - S^\alpha) \right\} d\tau + \varepsilon \\
&\le \int_{-\infty}^{+\infty} \left|\frac{\partial U}{\partial \tau}\right|_J^2 d\tau - \int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha) + \varepsilon.
\end{aligned}$$

Here, again, we used (5) and (10). Hence, we have the well defined homomorphism

$$h_{\alpha\beta} : HF_*^\lambda(J, H^\alpha : S^\alpha : N) \to HF_*^{\lambda_{\alpha\beta}+\varepsilon}(J, H^\beta; S^\beta : N)$$

where $\lambda_{\alpha\beta} := \lambda - \int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha)$, such that the diagram

$$\begin{array}{ccc}
HF_*^\lambda(J, H^\alpha : S^\alpha : N) & \xrightarrow{j_*^\lambda} & HF_*(J, H^\alpha; S^\alpha : N) \\
\Big\downarrow h_{\alpha\beta} & & \Big\downarrow h_{\alpha\beta} \\
HF_*^{\lambda_{\alpha\beta}+\varepsilon}(J, H^\beta; S^\beta : N) & \xrightarrow{j_*^{\lambda_{\alpha\beta}+\varepsilon}} & HF_*(J, H^\beta; S^\beta : N)
\end{array}$$

commutes. By the same argument as in Proposition 31 we deduce, for $a \in H_*(N)$

$$\begin{aligned}
&\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\beta, S^\beta : N) \\
&\le -\int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ this becomes

$$\begin{aligned}
&\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\alpha, S^\alpha : N) \\
&\le -\int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha).
\end{aligned}$$

By changing the role of α and β we get

$$\begin{aligned}
&\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\alpha, S^\alpha : N) \\
&\ge -\int_0^1 \max(H^\beta - H^\alpha)dt + \min(S^\beta - S^\alpha)
\end{aligned}$$

and therefore

$$|\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\alpha, S^\alpha : N)| \\ \leq \|H^\beta - H^\alpha\|_{C^0} + \|S^\beta - S^\alpha\|_{C^0}.$$

□

As a consequence we have the following:

Definition 34. For $(a, H, S) \in H_*(N) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ we define

$$\sigma(a, H, S : N) := \lim_{k\to\infty} \sigma(a, H_k, S_k : N)$$

where the limit is taken over any sequence

$$(\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\rm reg} \ni (J, H_k, S_k)$$

such that

$$C^0 - \lim_{k\to\infty} (H_k, S_k) = (H, S).$$

We define $\sigma(u, H, S : N)$ for $u \in H^*(N)$ in the same way.

The following lemma extends Lemma 30:

Lemma 35. *For $a \neq 0$, $u \neq 0$ and arbitrary (not necessarily generic) (J, H, S), the numbers $\sigma(a, J, H, S : N)$ and $\sigma(u, J, H, S : N)$ are the critical values of $\mathcal{A}_{(H,S)}$.*

Proof. For any $(H, S) \in \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ there exists a sequence $(H_k, S_k) \in (\mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\rm reg}$ of generic functions such that

$$C^1 - \lim_{k\to\infty} (H_k, S_k) = (H, S). \tag{48}$$

According to Lemma 30 there exists a sequence of points

$$x_k \in \phi_1^{H_k}(\mathrm{Graph}(dS_k)) \cap \nu^*(N \times \mathbb{R}^m)$$

such that

$$\sigma(a, J, H_k, S_k : N) = \mathcal{A}_{(H_k,S_k)}(\phi_t^{H_k} \circ (\phi_1^{H_k})^{-1}(x_k)).$$

Note that x_k is bounded and hence, after taking a subsequence, we can assume that

$$\lim_{k\to\infty} x_k = x_0. \tag{49}$$

Define

$$f, f_k : \nu^*(N \times \mathbb{R}^m) \to \mathbb{R}, \tag{50}$$

$$f_k(x) := \mathcal{A}_{(H_k,S_k)}(\phi_t^{H_k} \circ (\phi_1^{H_k})^{-1}(x))$$

$$f(x) := \mathcal{A}_{(H,S)}(\phi_t^{H} \circ (\phi_1^{H})^{-1}(x)).$$

From (48) it follows that

$$\lim_{k\to\infty} \phi_t^{H_k}(x) = \phi_t^H(x)$$

for all $x \in T^*E$ and hence

$$\lim_{k\to\infty} f_k(x) = f(x). \tag{51}$$

Since f_k are smooth, by Arzelà-Ascoli Theorem the convergence above is uniform on compact subsets of $\nu^*(N \times \mathbb{R}^m)$. Similarly, by (48)

$$\lim_{k\to\infty} df_k = df \tag{52}$$

uniformly on compact subsets of $\nu^*(N \times \mathbb{R}^m)$. According to Definition 34 and by (49) and (51)

$$\begin{aligned}\sigma(a, J, H, S : N) &= \lim_{k\to\infty} \sigma(a, J, H_k, S_k : N)\\ &= \lim_{k\to\infty} f_k(x_k)\\ &= f(x_0)\\ &= \mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x_0)).\end{aligned}$$

By (49) , (50) and (52) we have

$$\begin{aligned}d\mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x_0)) &= df(x_0)\\ &= \lim_{k\to\infty} df_k(x_k)\\ &= 0\end{aligned}$$

and hence $\phi_t^H \circ (\phi_1^H)^{-1}(x_0) \in \mathrm{Crit}(\mathcal{A}_{(H,S)})$. □

6.3. Normalization.

Consider the Hamiltonian

$$K_t := \chi(t)(H_t + c_0)$$

where $\chi : T^*M \to \mathbb{R}$ is a smooth function with compact support, such that $\chi \equiv 1$ in a neighborhood of $\cup_{t\in[0,1]}\phi_t^H(o_M)$. Then $\phi_1^H(o_M) = \phi_1^K(o_M)$, but

$$\rho(a, K : N) = \rho(a, H : N) + c_0.$$

More generally, it can be shown that for *any* two Hamiltonians H and K such that $\phi_1^H(o_M) = \phi_1^K(o_M)$ we have

$$\rho(a, K : N) = \rho(a, H : N) + c_0$$

for some $c_0 \in \mathbb{R}$ [**30**]. Similar considerations apply to the case of invariants c and σ. Hence, in order to consider the constructed invariants as the invariants of Lagrangian submanifolds, we have to impose certain normalization on the choice of parameters in $(H, S) \in \mathcal{H}(E)\times\mathcal{S}_{(E,Q)}$. Assume that $H = H_1 \oplus 0$

for some compactly supported Hamiltonian function $H : T^*M \to \mathbb{R}$. Denote by $L_S \subset T^*M$ the Lagrangian submanifold having S as a generating function quadratic at infinity. We will need the following result.

Theorem 36. *If H^α, H^β are two compactly supported Hamiltonians defined on T^*M and S^α, S^β two generating functions quadratic at infinity such that*

$$\phi_1^{H^\alpha}(L_{S^\alpha}) = \phi_1^{H^\beta}(L_{S^\beta}),$$

then there exists a constant $c_0 \in \mathbb{R}$ such that for any $N \subset M$

$$\mathrm{Spec}(H^\alpha \oplus 0, S^\alpha : N) = \mathrm{Spec}(H^\beta \oplus 0, S^\beta : N) + c_0. \tag{53}$$

In particular, if $x_\infty \in M$ is fixed and

$$\begin{aligned}\widetilde{\mathrm{Spec}}(H \oplus 0, S : N) &:= \mathrm{Spec}(H \oplus 0, S : N) - \mathrm{Spec}(H \oplus 0, S : x_\infty) \\ &= \{r - s \mid r \in \mathrm{Spec}(H \oplus 0, S : N),\ s \in \mathrm{Spec}(H \oplus 0, S : x_\infty)\}\end{aligned}$$

then

$$\widetilde{\mathrm{Spec}}(H^\alpha \oplus 0, S^\alpha : N) = \widetilde{\mathrm{Spec}}(H^\beta \oplus 0, S^\beta : N). \tag{54}$$

Proof. The critical points of $\mathcal{A}_{(H^\alpha, S^\alpha)}$ and $\mathcal{A}_{(H^\beta, S^\beta)}$ are in one-to-one correspondence with points of

$$\nu^*N \cap \phi_1^{H^\alpha}(L_S) = \nu^*N \cap \phi_1^{H^\beta}(L_S). \tag{55}$$

More precisely, the solutions of

$$\begin{cases} \frac{d\Gamma}{dt} = X_{H\oplus 0}(\Gamma) \\ \Gamma(0) \in \mathrm{Graph}(dS) \\ \Gamma(1) \in \nu^*N \times o_{\mathbb{R}^m} \end{cases}$$

are of the form $\Gamma = (\gamma, z)$ where

$$\begin{cases} \frac{d\gamma}{dt} = X_H(\gamma) \\ \frac{dz}{dt} = 0 \\ (\gamma(0), z) \in \mathrm{Graph}(dS),\ \gamma(1) \in \nu^*N,\ z \equiv (\xi, 0) \in o_{\mathbb{R}^m}. \end{cases}$$

Denote

$$L := \phi_1^{H^\alpha}(L_{S^\alpha}) = \phi_1^{H^\beta}(L_{S^\beta})$$

and consider the function $f : L \to \mathbb{R}$ defined by

$$\begin{aligned} f(x) &= \mathcal{A}_{(H^\alpha, S^\alpha)}(\phi_t^{H^\alpha \oplus 0}(dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x))))) \\ &- \mathcal{A}_{(H^\beta, S^\beta)}(\phi_t^{H^\beta \oplus 0}(dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x))))). \end{aligned}$$

Since $(\phi_1^{H^\alpha})^{-1}(L) = L_{S^\alpha}$ and and $(\phi_1^{H^\beta})^{-1}(L) = L_{S^\beta}$, for $x \in L$

$$i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x)) = i_{S^\alpha}^{-1}\left(q^\alpha, \frac{\partial S^\alpha}{\partial q}(q^\alpha, \xi^\alpha)\right) = (q^\alpha, \xi^\alpha),$$

$$i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x)) = i_{S^\beta}^{-1}\left(q^\beta, \frac{\partial S^\beta}{\partial q}(q^\beta, \xi^\beta)\right) = (q^\beta, \xi^\beta),$$

where

$$(q^\alpha, \xi^\alpha) \in \Sigma_{S^\alpha}, \quad (q^\beta, \xi^\beta) \in \Sigma_{S^\beta} \tag{56}$$

and

$$\left(q^\alpha, \frac{\partial S^\alpha}{\partial q}\right) = (\phi_1^{H^\alpha})^{-1}(x), \quad \left(q^\beta, \frac{\partial S^\beta}{\partial q}\right) = (\phi_1^{H^\beta})^{-1}(x). \tag{57}$$

Hence

$$\begin{aligned} dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x))) &= dS^\alpha(q^\alpha, \xi^\alpha) \\ &= \left(q^\alpha, \frac{\partial S^\alpha}{\partial q}(q^\alpha, \xi^\alpha), \xi^\alpha, \frac{\partial S^\alpha}{\partial \xi}(q^\alpha, \xi^\alpha)\right) \\ &= \left(q^\alpha, \frac{\partial S^\alpha}{\partial q}(q^\alpha, \xi^\alpha), \xi^\alpha, 0\right) \quad \text{(by (56))} \\ &= (((\phi_1^{H^\alpha})^{-1}(x)), \xi^\alpha, 0) \quad \text{(by (57))}, \end{aligned}$$

and similarly

$$dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x))) = (((\phi_1^{H^\beta})^{-1}(x)), \xi^\beta, 0).$$

Therefore, the paths

$$\begin{aligned} &t \mapsto \phi_t^{H^\alpha \oplus 0}(dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x)))) \quad \text{and} \\ &t \mapsto \phi_t^{H^\beta \oplus 0}(dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x)))) \end{aligned} \tag{58}$$

respectively start at Graph(dS^α) (respectively Graph(dS^β)) and end at the points (in local coordinates)

$$\begin{aligned} \phi_1^{H^\alpha \oplus 0}(dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x)))) &= (\phi_1^{H^\alpha \oplus 0}((\phi_1^{H^\alpha})^{-1}(x)), \xi^\alpha, 0) \\ &= (x, \xi^\alpha, 0) \end{aligned}$$

and

$$\phi_1^{H^\beta \oplus 0}(dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x)))) = (x, \xi^\beta, 0)$$

respectively. Let

$$\chi : \mathbb{R} \to \phi_1^{H^\alpha}(L_S) = \phi_1^{H^\beta}(L_S)$$

be a smooth path connecting two points in (55). Since the paths (58) are Hamiltonian and start at Graph(dS^α) and Graph(dS^β), the same computation as in (5) shows that

$$\frac{d}{ds} f(\chi(s)) = \theta(\eta^\alpha(s)) - \theta(\eta^\beta(s))$$

where

$$\begin{aligned}\eta^\alpha(s) &:= \frac{d}{ds}\phi_1^{H^\alpha\oplus 0}(dS^\alpha(i_{S^\alpha}((\phi_1^{H^\alpha})^{-1}(\chi(s))))) \\ &= \left(\frac{d\chi}{ds}, \frac{d\xi^\alpha(s)}{ds}, 0\right),\end{aligned}$$

with $(\frac{d\xi^\alpha(s)}{ds}, 0) \in T(o_{\mathbb{R}^m})$. Similarly,

$$\eta^\beta(s) = \left(\frac{d\chi}{ds}, \frac{d\xi^\beta(s)}{ds}, 0\right).$$

Since $\theta = \theta_M \oplus \theta_{\mathbb{R}^m}$ and $\theta_{\mathbb{R}^m}(\xi, 0) = 0$ it follows that

$$\frac{d}{ds}f(\chi(s)) = 0.$$

Hence $f \equiv c_0$, for some constant $c_0 \in \mathbb{R}$ independent of N. This proves (53) and (54). □

Definition 37. Fix $x_\infty \in M$. Let S be a generating function quadratic at infinity for the Lagrangian submanifold $L_S = \phi_1^H(o_M) \in T^*M$. We define the normalized parameters $(\tilde{H}, \tilde{S})$ by

$$\tilde{S} = S - \frac{1}{2}\sigma(1, H \oplus 0, S : x_\infty),\ \tilde{H} = H + \frac{1}{2}\sigma(1, H \oplus 0, S : x_\infty).$$

Remark 38. Strictly speaking, $(\tilde{H}, \tilde{S}) \notin \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$. However, it is allowed to add a constant to the parameters in $\mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ since Floer theory depends only on the first derivatives $(\nabla H, \nabla S)$ which remain unchanged.

The normalization described above also gives the normalization of invariants ρ and c defined by (1) and (2). Indeed, these invariants are the special cases of invariant σ, as we show in the following lemma:

Lemma 39. *For $(H, S) \in \mathcal{H}(M) \times \mathcal{S}_{(E,Q)}$ and $a \in H_*(N)$*

$$\sigma(a, H \oplus 0, Q : N) = \rho(a, H : N)$$

and

$$\sigma(a, 0, S : N) = c(a, S : N).$$

Analogous statements hold for any $u \in H^*(N)$.

Proof. The first equality follows from Theorem 28. To prove the second one, we first observe that, if S_N^V is as in Proposition 27 and $S_N^V \equiv S$ outside $U \supset V$, then

$$c(a, S : N) = \sigma(a, 0, S_N^V : N).$$

Since $\|S_N^V - S\|_{C^0} \to 0$ as $U \to N$, the conclusion follows from Theorem 33. □

Definition 40. Fix $x_\infty \in M$. Let S be a generating function quadratic at infinity for the Lagrangian submanifold $L_S = \phi_1^H(o_M) \in T^*M$. For a submanifold $N \subset M$ and $a \in H^*(N)$, $u \in H^*(N)$ we define

$$c(a, L_S : N) := c(a, \tilde{S} : N),\ c(u, L_S : N) := c(u, \tilde{S} : N),$$

where $\tilde{S} = S - c(1, S : x_\infty)$. In a similar way, define

$$\rho(a, L_S : N) := \rho(a, \tilde{H} : N),\ \rho(u, L_S : N) := \rho(u, \tilde{H} : N),$$

with $\tilde{H} = H + \rho(1, H : x_\infty)$.

By Lemma 39 the definition of the parameters $(\tilde{H}, \tilde{S})$ in Definition 37 and Definition 40 agree in a sense that in the cases $H \equiv 0$ and $S \equiv Q$ both definitions give the same functionals

$$\mathcal{A}_{(\tilde{H}\oplus 0, \tilde{Q})} = \mathcal{A}_{(H\oplus 0, Q)} - \sigma(1, H \oplus 0, Q : x_\infty)$$

and

$$\mathcal{A}_{(\tilde{0}, \tilde{S})} = \mathcal{A}_{(0,S)} - \sigma(1, 0, S : x_\infty).$$

Invariants in Definition 40 are well defined invariants of Lagrangian submanifolds of T^*M Hamiltonian isotopic to the zero section.

6.4. Equality between the two invariants.

In this section we will show that the invariants ρ and c give the same invariants of Lagrangian submanifolds of T^*M. The proof below is sketched in [**26**], we present it here for the sake of completeness.

Theorem 41 ([**26**])**.** *Let $L_S = \phi_1^H(o_M) \in T^*M$ be a Lagrangian submanifold generated by generating function S quadratic at infinity. Then for any submanifold $N \subset M$ and any $a \in H_*(N)$*

$$c(a, L_S : N) = \rho(a, L_S : N). \tag{59}$$

The analogous equality holds for any $u \in H^*(N)$.

Proof. Denote by $S_t : E \to \mathbb{R}$ a generating function of $(\phi_t^H)^{-1}(L_S)$, such that $S_0 = S$, $S_1 = S$. Let $H(t)$ denote a Hamiltonian such that $\phi_1^{H(t)} = \phi_t^H$. Note that

$$\begin{aligned} \phi_1^{H(t)}(L_{S_t}) &= \phi_1^{H(t)}(\phi_t^H)^{-1}(L_S) \\ &= \phi_t^H(\phi_t^H)^{-1}(L_S) \\ &= L_S \end{aligned}$$

and therefore, by Theorem 36 the action spectrum $\widetilde{\mathrm{Spec}}(H(t) \oplus 0, S_t : N)$ is fixed. By Theorem 33 the function

$$\tilde{\sigma} : t \mapsto \sigma(a, H(t) \oplus 0, S_t : N) - \sigma(a, H(t) \oplus 0, S_t : x_\infty)$$

is continuous and takes the values in the set $\widetilde{\mathrm{Spec}}(H(t)\oplus 0, S : N)$, which is nowhere dense in $\mathbb{R}$ by Lemma 1. Hence $\tilde{\sigma} \equiv$ constant. In particular,

$$\begin{aligned}(60)\qquad &\sigma(a, 0, \tilde{S} : N)\\ &= \sigma(a, H(0)\oplus 0, S_0 : N) - \sigma(a, H(0)\oplus 0, S_0 : x_\infty)\\ &= \sigma(a, H(1)\oplus 0, S_1 : N) - \sigma(a, H(1)\oplus 0, S_1 : x_\infty)\\ &= \sigma(a, \tilde{H}\oplus 0, Q; N).\end{aligned}$$

According to Lemma 39 and Definition 40

$$\sigma(a, 0, \tilde{S} : N) = c(a, L_S : N) \tag{61}$$

and

$$\sigma(a, \tilde{H}\oplus 0, Q : N) = \rho(a, L_S : N). \tag{62}$$

Now, (59) follows from (60), (61) and (62). □

7. A note on Hofer's geometry.

In [**16**] Hofer introduced a biinvariant metric on a group $\mathcal{D}^c_\omega(P)$ of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold P. For $H \in C^\infty_c(P\times[0,1])$ define *the oscillation* of H_t by

$$\mathrm{osc}(H_t) := \sup_{x\in P} H_t(x) - \inf_{x\in P} H_t(x).$$

That leads to the definition of the length of the curve ϕ^H_t in $\mathcal{D}^c_\omega(P)$ as

$$l(\{\phi^H_t\}_{0\le t\le 1}) := \int_0^1 \mathrm{osc}(H_t)dt.$$

Definition 42 (Hofer's energy)**.** The energy of $\psi \in \mathcal{D}^c_\omega(P)$ is defined by

$$E(\psi) := \inf\{l(\phi^H_t) \mid \phi^H_1 = \psi\}.$$

The non-degeneracy of the energy functional, i.e.,

$$E(\psi) = 0 \text{ iff } \psi = \mathrm{id}$$

has been proved by Hofer [**16**] (see also [**17**]) in the case $P = \mathbb{C}^n$ and by Lalonde and McDuff [**20**] in general.

In the case $P = \mathbb{C}^n$ Bialy and Polterovich [**2**] proved that

$$c(\mu, \Gamma_\psi) - c(1, \Gamma_\psi) \le E(\psi) \tag{63}$$

where $c(\mu, \Gamma_\psi) - c(1, \Gamma_\psi)$ is Viterbo's norm (see [**38**]). Moreover, they proved that Viterbo's and Hofer's metrics coincide locally in the sense of C^1-Whitney topology.

More generally, for a symplectic manifold P, let $\mathcal{L}_M(P)$ be the space of Lagrangian submanifolds $L \subset P$ Hamiltonian isotopic to the Lagrangian submanifold M. In other words,

$$\mathcal{L}_M(P) := \{\phi_1^H(M) \mid \phi_1^H \in \mathcal{D}_\omega^c(P)\}.$$

The group $\mathcal{D}_\omega^c(P)$ acts transitively on $\mathcal{L}_M(P)$ by $(\phi, L) \mapsto \phi(L)$. The manifold $\mathcal{L}_M(P)$ has a natural $\mathcal{D}_\omega^c(P)$-invariant metric defined in the following:

Definition 43. For $L_1, L_2 \in \mathcal{L}_M(P)$ we define

$$d(L_1, L_2) := \inf\{E(\phi) \mid \phi \in \mathcal{D}_\omega^c(P),\ \phi(L_1) = L_2\}. \tag{64}$$

The non-degeneracy of d has been proved by Oh [**30**] for $P = T^*M$ and by Chekanov [**4**, **5**] in general case. Moreover, for $P = T^*M$

$$\rho(\mu, L) - \rho(1, L) \le d(o_M, L) \tag{65}$$

(see [**30**] or apply Lemma 39 to the inequalities at the end of the proof of Theorem 33 with $S^\alpha = S^\beta = Q$, $H^\alpha = 0$, setting first $a = 1$ and then $a = \mu$ and subtracting; then take the infimum over all H^β's such that $\phi_1^{H^\beta}(o_M) = L$).

Theorem 41 together with (65) implies

$$c(\mu, L) - c(1, L) \le d(o_M, L)$$

which is the generalization of (63) to Hofer's and Viterbo's geometries of Lagrangian submanifolds in a cotangent bundle. As $c(\mu, L) = c(1, L)$ if and only if $L = o_M$, it gives another proof of the nondegeneracy of Hofer's metric.

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EXPLICIT REALIZATIONS OF CERTAIN REPRESENTATIONS OF $Sp(n, \mathbb{R})$ VIA THE DOUBLE FIBRATION TRANSFORM

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We consider a family of singular infinite dimensional unitary representations of $G = Sp(n, \mathbb{R})$ which are realized as sheaf cohomology spaces on an open G-orbit D in a generalized flag variety for the complexification of G. By parametrizing an appropriate space, M_D, of maximal compact subvarieties in D, we identify a holomorphic double fibration between D and M_D which we use to define a map P, often referred to as a double fibration or Penrose transform, from the representation into sections of a corresponding sheaf on M_D. Analysis of the construction of P shows that P is injective, the image of P is the kernel of a differential operator on M_D and P is an intertwining map.

1. Introduction.

In this paper, we consider a family of singular infinite dimensional unitary representations of $G = Sp(n, \mathbb{R})$ which are realized on certain sheaf cohomology spaces of D, an open G-orbit in a generalized flag variety for the complexification of G. By parametrizing an appropriate space, M_D, of maximal compact subvarieties in D, we can identify a holomorphic double fibration between D and M_D, a well understood bounded symmetric domain. Using standard constructions from sheaf theory and the fact that M_D is Stein, we define a map P, often referred to as a double fibration or Penrose transform, from the representation into the space of sections of a corresponding sheaf on M_D. By analyzing the spectral sequences involved in the construction of P and applying the Bott-Borel-Weil theorem, we show that P is injective. Further analysis leads to the fact that the image of P is the kernel of a differential operator on M_D and that P is an intertwining map.

More generally, let G be a real semisimple Lie group and let X be a generalized flag manifold for $G_{\mathbb{C}}$, the complexification of G. If D is an open G-orbit in X, then D can be realized as G/H for some subgroup H of G. Associated to each D is a family of representations of G given by the Dolbeault cohomology spaces $H^p(D, \mathcal{L})$ where $\mathcal{L}$ is the sheaf of holomorphic sections of a homogeneous line bundle on D. Under certain conditions, these representations are non-zero, singular, irreducible, unitarizable and infinite dimensional. They provide a construction of an important and mysterious part of the unitary dual of G.

These representations can be studied using a double fibration transform whose purpose is to embed the cohomology space in a space of holomorphic sections of a vector bundle on M_D as the kernel of a differential operator. Although the technique was developed for open orbits G/H where H is compact, some results of Wolf [**Wo2**, **Wo3**] allow the possibility of extending this technique to any open G–orbit in a generalized flag manifold for $G_{\mathbb{C}}$. This technique is related to Schmid's [**S**] construction of discrete series for G associated to an orbit G/H when H is a compact Cartan contained in a maximal compact subgroup K of G.

Wells and Wolf [**WW**] studied G-orbits $D = G/H$ where H is compact. For these orbits, they showed the existence of a holomorphic double fibration where

$$\begin{array}{ccc} & Y_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & M_D. \end{array} \tag{1.1}$$

M_D is the space of $G_{\mathbb{C}}$-translates in D of the maximal compact subvariety $K/H \cap K$ and Y_D is the incidence manifold $Y_D = \{(z, Q) \in D \times M_D : z \in Q\}$. They show that M_D is Stein in this case and use the double fibration to show that $H^s(D, \mathcal{E})$ embeds in $H^0(M_D, R^s_\nu \mu^*(\mathcal{E}))$ where $\mathcal{E}$ is the sheaf of holomorphic sections of a homogeneous bundle on D. This work proves modified versions of conjectures made by Griffiths [**Gr**] while studying automorphic cohomology.

Even if H is not compact, these ideas can be used for any open orbits D if we know that M_D is a Stein manifold. Fortunately, Wolf [**Wo2**, **Wo3**] has shown that M_D is Stein for all open G-orbits D. Eastwood, Penrose, and Wells [**EPW**] used a holomorphic double fibration of this type for an open orbit of $U(2,2)$ with isotropy $U(1) \times U(1,2)$ to study the massless field equations. In this case, M_D is $U(2,2)/(U(2) \times U(2))$. Patton and Rossi [**PR1**, **PR2**], generalizing the work of Eastwood, Penrose and Wells, studied special $SU(p,q)$-orbits.

The key to using the double fibration transform is understanding the structure of M_D. There are two basic cases and, as is expected, the structure of M_D depends on the structure of D. An open orbit D is of holomorphic type if there exists a holomorphic double fibration between D and G/K. In this case M_D is G/K. An open orbit D is of nonholomorphic type if no such holomorphic double fibration exists. In this case M_D is an open submanifold of $G_{\mathbb{C}}/K_{\mathbb{C}}$ ([**WW**]). The $U(2,2)$ example studied by Eastwood, Penrose and Wells is of holomorphic type and further examples and generalizations of the holomorphic type are given in [**BE**]. In fact, open orbits of holomorphic type are well understood. Orbits of holomorphic type correspond to highest weight representations and those of nonholomorphic type correspond to representations which do not have a highest weight. The representations are discrete series if and only if H is compact.

Not as much is known in the nonholomorphic case. This case splits usefully into two subcases: When G/K is Hermitian symmetric and when it is not. When G/K is Hermitian symmetric, the structure of M_D has been computed for two families of examples: For arbitrary $U(p,q)$-orbits [**DZ**, **PR2**] and for the open $Sp(n,\mathbb{R})$-orbits in the flag variety of Lagrangian planes in $\mathbb{C}^{2n}$ [**N**]. In both families M_D is $G/K \times \overline{G/K}$ where $\overline{G/K}$ denotes G/K with the opposite complex structure. More recently, Wolf and Zierau [**WZ**] have shown that M_D is always $G/K \times \overline{G/K}$ in the nonholomorphic Hermitian symmetric case.

When G/K is not Hermitian symmetric, Wells [**We**] and Dunne and Zierau [**DZ**] determined M_D for special $SO(2m,r)$-orbits. Akheizer and Gindikin [**AG**] have also worked out a related example for this case and have suggested that M_D could be described as a particular Stein tubular neighborhood of G/K in $G_{\mathbb{C}}/K_{\mathbb{C}}$. For these examples, it is not clear whether M_D can be realized as a homogeneous space or whether these results can be generalized. No work has been done as yet on defining the transform for these cases.

1.1. Results of Paper. In this paper we will define a double fibration transform for the $Sp(n,\mathbb{R})$-representations $H^s(D_i,\mathcal{L})$. Here, D_i is one of $r-1$ open orbits in the generalized flag variety X of isotropic i-planes in $\mathbb{C}^{2n}$ where $r \leq n$ (see Section 3.2). The dimension of a maximal compact subvariety in D_i is s and $\mathcal{L}$ is the sheaf of holomorphic sections of a sufficiently negative line bundle on D_i. These orbits are in the nonholomorphic Hermitian symmetric case with noncompact H so we are studying representations which are not discrete series and which do not have a highest weight. In this paper, we will construct a double fibration transform for $H^s(D_i,\mathcal{L})$ and show that it is injective (Theorem 4.6 and 4.11). Finally, we will use the transform to realize $H^s(D_i,\mathcal{L})$ as the kernel of a differential operator on $H^0(M_{D_i}, R^s_\nu\mu^*\mathcal{L})$ (Theorem 4.11 and 5.26). Thus these representations are Frechet spaces and are continuous, facts that also follow from work by Wong [**Wg**].

Now we describe the results in more detail. Let $\mathbb{C}^{2n}$ be endowed with a symplectic form and a Hermitian form of signature (n,n). Let X be the set of r-planes in $\mathbb{C}^{2n}$ which are isotropic with respect to the symplectic form where $r \leq n$. For $1 \leq i \leq r-1$, let D_i be planes in X of signature $(i, r-i)$. Then X is a generalized flag variety for the Lie group $Sp(n,\mathbb{C})$ and D_i is the open $Sp(n,\mathbb{R})$-orbit G/H_i in X where H_i is $U(i,r-i) \times Sp(n-r,\mathbb{R})$. Let χ be a unitary character on H_i which determines a homogeneous vector bundle

$\mathbb{L}_\chi$ on D_i. Let $\mathcal{L}_\chi$ be the sheaf of holomorphic sections of $\mathbb{L}_\chi$. When the bundle satisfies a suitable negativity condition and s is the dimension of a maximal compact subvariety of D_i, then $H^s(D_i, \mathcal{L}_\chi)$ is a non-zero irreducible infinite-dimensional singular unitarizable representation of $Sp(n, \mathbb{R})$. In this paper we give another realization of this representation via a double fibration transform.

In Section 2, we outline the construction of the double fibration transform for complex manifolds D, Y and M which are related by the holomorphic double fibration (1.1). When $\mathbb{L}$ is a line bundle on D, the transform is a map from $H^p(D, \mathcal{O}(\mathbb{L}))$ to $H^0(M, R^p_\nu \mathcal{O}(\mu^*\mathbb{L}))$ which is defined using standard constructions from sheaf theory. We establish the conditions necessary for this map to be injective and for the image of $H^p(D, \mathcal{O}(\mathbb{L}))$ to be the kernel of a map from $H^0\big(M, R^p_\nu \mathcal{O}(\mu^*\mathbb{L})\big)$ to $H^0\big(M, R^p_\nu \Omega^1_\mu(\mu^*\mathbb{L})\big)$ where Ω^1_μ is the sheaf of relative holomorphic 1–forms on Y.

In Section 3, we analyze the geometry of the holomorphic double fibration used in the construction of the transform.

In Section 4, we construct the transform for $H^s(D_i, \mathcal{L}_\chi)$. This involves analyzing the sheaves and vector bundles which are in the construction. In particular, we show that each of $R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)$ and $R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of a homogeneous vector bundle. These facts, which are crucial in determining when the transform is injective, are not immediate because μ is a G-equivariant map from a $(G\times G)$-homogeneous manifold to a G-homogeneous manifold.

Next, we show that the transform is injective by analyzing the Leray spectral sequences involved in the construction of the map and by reducing the problem to an application of the Borel-Bott-Weil theorem. An abbreviated version of the main result of Section 4 is the following theorem.

Theorem 4.11. *The double fibration transform*

$$P : H^s(D_i, \mathcal{L}_\chi) \to H^0(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi))$$

is an injection and the image of P is the kernel of a map $\mathcal{D}$ from $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$.

Since $R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)$ and $R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)$ are each the sheaf of sections of a homogeneous bundle, the transform realizes $H^s\big(D_i, \mathcal{L}_\chi\big)$ as a space of functions on M_{D_i} with values in a homogeneous vector bundle.

In Section 5, we analyze the map $\mathcal{D}$ in Theorem 4.11. By construction, $\mathcal{D}$ is determined by the map from $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$ and the kernel of $\mathcal{D}$ is the image of P. The main result of Section 5 is the following theorem.

Theorem 5.26. *$\mathcal{D}$ is a G-equivariant differential operator.*

In Appendix A we consider the situation where the line bundle $\mathbb{L}_\chi$ is replaced with a finite dimensional vector bundle although it is the line bundle case that corresponds to unitarizable representations.

This paper incorporates the results of my thesis which was done at Oklahoma State University. More specifically, my thesis contains these results when $r = n$ along with the contents of [**N**]. The case when $r < n$ is not a part of my thesis. I wish to thank my advisor, Roger Zierau, and Joe Wolf and Anthony Kable for many useful conversations while I was working on these results. Thanks also to the referee for suggesting the extension to the vector bundle case.

2. The general double fibration transform.

Let D, Y, and M be complex manifolds. Then we refer to (2.1) as a holomorphic double fibration for D when μ and ν are holomorphic fibrations.

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\mu}{\swarrow} & & \overset{\nu}{\searrow} & \\ D & & & & M. \end{array} \tag{2.1}$$

Let $\mathbb{L} \to D$ be a holomorphic line bundle on D and $\mathcal{L}$ the sheaf of holomorphic sections of $\mathbb{L}$. In this setting, it is sometimes possible to define a double fibration transform from the Dolbeault cohomology space $H^s\big(D,\mathcal{L}\big)$ to $H^0\big(M, R^s_\nu\mathcal{O}(\mu^*\mathbb{L})\big)$ where $R^s_\nu\mathcal{O}(\mu^*\mathbb{L})$ is the s^{th} higher direct image of $\mathcal{O}(\mu^*\mathbb{L})$ by ν. In this paper, we will define a double fibration transform for a family of open $Sp(n,\mathbb{R})$-orbits D in the generalized flag of isotropic r-planes in $\mathbb{C}^{2n}$ when $r \le n$.

Although the construction of the transform is described in a variety of places [see [**BE**, **EPW**, **PR2**, **WW**], for example], we include a brief discussion here, adapted to our situation, for the convenience of the reader.

The first step in the construction is to determine when $H^s(D,\mathcal{L})$ is isomorphic to $H^s(Y,\mu^{-1}\mathcal{L})$. In the setting of this paper, the fiber of μ is contractible (Proposition 3.13) and this is sufficient to guarantee, by a theorem of Buchdahl [**Bu**], that the isomorphism exists. We note, however, that the contractibility of the fiber of μ is a stronger condition than that required by Buchdahl.

The second step is to construct a resolution of $\mu^{-1}\mathcal{L}$ to which we can apply the following lemma.

Lemma 2.2. *Let*

$$0 \to \mathcal{S} \to \mathcal{S}_0 \to \mathcal{S}_1 \to \cdots \to \mathcal{S}_N \to 0$$

be an exact sequence of sheaves on a manifold Y *and suppose* $H^p(Y,\mathcal{S}_t) = 0$ *for* $p < q$ *and* $1 \le t \le N$. *Then there is an injection from* $H^q(Y,\mathcal{S}) \to$

$H^q(Y,\mathcal{S}_0)$. Furthermore, $H^q(Y,\mathcal{S})$ is the kernel of the induced map from $H^q(Y,\mathcal{S}_0)$ to $H^q(Y,\mathcal{S}_1)$.

To find an appropriate resolution of $\mu^{-1}\mathcal{L}$, we begin by constructing a resolution of $\mu^{-1}\mathcal{O}_D$. We denote by Ω^p_Z the sheaf of holomorphic p-forms on a complex manifold Z.

Definition 2.3.

(1) The sheaf of relative 1-forms on Y, denoted by Ω^1_μ, is defined by the exact sequence

$$\mu^*\Omega^1_D \to \Omega^1_Y \to \Omega^1_\mu \to 0$$

where $\mu^*\Omega^1_D = \mathcal{O}_Y \otimes \mu^{-1}\Omega^1_D$ and we tensor over $\mu^{-1}\mathcal{O}_D$.

(2) The relative p–forms Ω^p_μ are defined by $\wedge^p\Omega^1_\mu$.

We can think of Ω^p_μ as p-forms on Y in the direction of the fiber of μ with coefficients in $\mathcal{O}_Y$ and $d_\mu : \wedge^p\Omega^1_\mu \to \wedge^{p+1}\Omega^1_\mu$ as differentiation along the fiber.

We have the following lemma about relative p-forms.

Lemma 2.4. *Let $m = \dim Y - \dim D$.*

(1) *Then*

$$0 \to \mu^{-1}\mathcal{O}_D \to \mathcal{O}_Y \xrightarrow{d_\mu} \Omega^1_\mu \to \cdots \to \Omega^m_\mu \to 0 \tag{2.5}$$

is an exact sequence of sheaves on Y.

(2) *The sequence (2.6) is a resolution of $\mu^{-1}\mathcal{L}$.*

$$0 \to \mu^{-1}\mathcal{L} \to \mu^*\mathcal{L} \to \Omega^1_\mu(\mu^*\mathbb{L}) \to \cdots \to \Omega^m_\mu(\mu^*\mathbb{L}) \to 0. \tag{2.6}$$

The proof of (1) is the usual Poincaré lemma. To prove (2) we tensor (2.5) by $\mu^{-1}\mathcal{L}$ and observe that $\mu^*\mathcal{L} = \mathcal{O}(\mu^*\mathbb{L})$ and

$$\Omega^p_\mu \otimes_{\mu^{-1}\mathcal{O}_D} \mu^{-1}\mathcal{L} = \Omega^p_\mu \otimes_{\mathcal{O}_Y} \mathcal{O}(\mu^*\mathbb{L}).$$

To simplify notation we denote $\Omega^p_\mu \otimes_{\mu^{-1}\mathcal{O}_D} \mu^{-1}\mathbb{L}$ by let $\Omega^p_\mu(\mu^*\mathbb{L})$.

Applying Lemma 2.2 to (2.6) yields the following lemma.

Lemma 2.7. *If $H^p\big(Y,\Omega^t_\mu(\mu^*\mathbb{L})\big) = 0$ for all $p < q$ and all t, then $H^q\big(Y,\mu^{-1}\mathcal{L}\big)$ embeds in $H^q(Y,\mathcal{O}(\mu^*\mathbb{L}))$ as the kernel of the induced map from $H^q\big(Y,\mathcal{O}(\mu^*\mathbb{L})\big)$ to $H^q\big(Y,\Omega^1_\mu(\mu^*\mathbb{L})\big)$.*

For the third and final step in the construction of the transform, we must assume that M is Stein, that ν is proper, and that $\mathcal{S}$ is a coherent sheaf on Y. With these assumptions, the following theorem is the key to this final step.

Theorem 2.8. *$H^p(Y,\mathcal{S})$ is isomorphic to $H^0(M,R^p_\nu\mathcal{S})$.*

Proof. There exists a Leray spectral sequence which abuts to $H^*\big(Y,\mathcal{S}\big)$ and whose E_2-term is given by $\mathrm{E}_2^{p,q} = H^p\big(M, R^q_\nu\mathcal{S}\big)$. The direct image theorem [**GR**] implies that $R^q_\nu\mathcal{S}$ is coherent so $\mathrm{E}_2^{p,q} = 0$ for all nonzero p. That is, the spectral sequence collapses and the result follows. □

If $\mathcal{O}(\mu^*\mathbb{L})$ and $\Omega^1_\mu(\mu^*\mathbb{L})$ are coherent, then Theorem 2.8 implies that $H^q\big(Y,\mathcal{O}(\mu^*\mathbb{L})\big)$ is isomorphic to $H^0\big(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})\big)$ and also that $H^q\big(Y,\Omega^1_\mu(\mu^*\mathbb{L})\big)$ is isomorphic to $H^0\big(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L})\big)$. These isomorphisms, along with the isomorphism in Lemma 2.7, determine a map $\mathcal{D}$ from $H^0\big(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})\big)$ to $H^0\big(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L})\big)$.

In the following theorem, we combine these constructions to define the Penrose transform.

Theorem 2.9. *The Penrose tranform is the map*

$$P : H^q(D,\mathcal{L}) \to H^0\big(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})\big).$$

The map P is an injection and the image of P is the kernel of $\mathcal{D}$ which is defined below.

More explicitly, $H^q(D,\mathcal{L})$ is isomorphic to $H^q\big(Y,\mu^{-1}\mathcal{L}\big)$ by Buchdahl's theorem. Then $d_\mu : \mathcal{O}_Y \to \Omega^1_\mu$ determines a map ${d_\mu}^* : \mathcal{O}(\mu^*\mathcal{L}_X) \to \Omega^1_\mu(\mu^*\mathbb{L})$ whose kernel is $\mu^{-1}\mathcal{L}$. By Lemma 2.7, ${d_\mu}^*$ determines an injection $\mathcal{D}_\mu$ from $H^q(Y,\mathcal{O}(\mu^*\mathbb{L}))$ to $H^q\big(Y,\Omega^1_\mu(\mu^*\mathbb{L})\big)$ whose kernel is $H^q\big(Y,\mu^{-1}\mathcal{L}\big)$. Then, Theorem 2.8 gives an isomorphism between $H^q\big(Y,\mathcal{O}(\mu^*\mathbb{L})\big)$ and $H^0\big(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})\big)$ and one between $H^q\big(Y,\Omega^1_\mu(\mu^*\mathbb{L})\big)$ and $H^0\big(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L})\big)$. As a result, $\mathcal{D}_\mu$ determines a differential operator $\mathcal{D}$ such that the following diagram commutes.

$$\begin{array}{ccc} H^q\big(Y,\mathcal{O}(\mu^*\mathbb{L})\big) & \longrightarrow & H^0\big(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})\big) \\ {\scriptstyle\mathcal{D}_\mu}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{D}} \\ H^q\big(Y,\Omega^1_\mu(\mu^*\mathbb{L})\big) & \longrightarrow & H^0\big(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L})\big) \end{array}.$$

In this way, the map P and $\mathcal{D}$ are defined and P embeds $H^q(D,\mathcal{L})$ is $H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}))$ as the kernel of $\mathcal{D}$.

3. The geometry underlying the double fibration transform.

The purpose of this section is to understand the geometry of the holomorphic double fibrations (3.1) and (3.2) which we will use to define a double fibration transform for a family of $Sp(n,\mathbb{R})$-representations. Let D_i be the open $Sp(n,\mathbb{R})$-orbit of isotropic r-planes of signature $(i, r-i)$ in the generalized flag manifold X of isotropic r-planes in $\mathbb{C}^{2n}$. Then D_i is G/H_i where $H_i \simeq U(i, r-i) \times Sp(n-r,\mathbb{R})$ and $K/H_i \cap K$ is a maximal compact subvariety in D_i. Here, K is a maximal compact subgroup of G isomorphic to $U(n)$.

Let M_{X_i} be the $Sp(n,\mathbb{C})$-translates of $K/H_i \cap K$ in X. Let $\widetilde{M}_{D_i}$ be the translates contained in D_i and let M_{D_i} be the connected component of $\widetilde{M}_{D_i}$ containing $K/H_i \cap K$. Let Y_{D_i} and Y_{X_i} be the incidence spaces

$$\begin{aligned} Y_{D_i} &= \{(z,Q) \in D_i \times M_{D_i} : z \in Q\} \\ \text{and} \quad Y_{X_i} &= \{(z,Q) \in X \times M_{X_i} : z \in Q\}. \end{aligned}$$

Then we have the following holomorphic double fibrations

$$\begin{array}{ccc} & Y_{D_i} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D_i & & M_{D_i} \end{array} \tag{3.1}$$

and

$$\begin{array}{ccc} & Y_{X_i} & \\ {}^{\tilde{\mu}}\swarrow & & \searrow^{\tilde{\nu}} \\ X & & M_{X_i} \end{array} \tag{3.2}$$

with the natural projection maps.

3.1. Preliminaries. In this section, we define the bilinear forms and the Lie groups we will use to describe the manifolds in the double fibrations. In addition, we describe various Lie algebras and root systems that will be used later.

Let $\langle \cdot\,,\cdot \rangle_H$ denote the Hermitian form on $\mathbb{C}^{2n}$ corresponding to the matrix $I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and let $\omega(\cdot\,,\cdot)$ denote the symplectic form on $\mathbb{C}^{2n}$ corresponding to $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. We call a subspace y of $\mathbb{C}^{2n}$ *isotropic* if $\omega(u\,,v) = 0$ for all $u, v \in y$ and *Lagrangian* if $y = y^{\perp_\omega}$. We denote the *signature* of a subspace y by $\operatorname{sgn}(y) = (a,b,c)$ if y has a Hermitian orthogonal basis of a positive vectors, b negative vectors and c null vectors. If $c = 0$, we write $\operatorname{sgn}(y) = (a,b)$.

We will use these forms to describe certain subgroups of $GL(2n,\mathbb{C})$. The complex symplectic group $Sp(n,\mathbb{C})$ is the set of matrices that preserve the symplectic form, and $U(n,n)$ is the subgroup that preserves the Hermitian form. Then $Sp(n,\mathbb{C}) \cap U(n,n)$ is a real form of $Sp(n,\mathbb{C})$ which preserves both the symplectic and Hermitian forms. We denote $Sp(n,\mathbb{C})$ by $G_\mathbb{C}$ and the real form by G. We note that $G \simeq Sp(n,\mathbb{R})$.

Let $\mathfrak{g}_\mathbb{C}$ denote the Lie algebra of $G_\mathbb{C}$ and $\mathfrak{g}$ the Lie algebra of G. Fix the Cartan subalgebra

$$\mathfrak{t}_\mathbb{C} = \{\operatorname{diag}(t_1, t_2, \ldots, t_n, -t_1, -t_2, \ldots, -t_n) : t_j \in \mathbb{C}\}$$

of $\mathfrak{g}_\mathbb{C}$ where an element of $\mathfrak{t}_\mathbb{C}$ is a diagonal matrix with the indicated entries. Elements of $\mathfrak{t}_\mathbb{C}^*$ will be identified with points in $\mathbb{C}^n$ as follows. For

$\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{C}^n$, define

$$\gamma\big(\mathrm{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n)\big) = \sum \gamma_j t_j.$$

Let e_j be the element of $\mathfrak{t}^*_{\mathbb{C}}$ which corresponds to the j^{th} standard basis vector in $\mathbb{C}^{2n}$.

The element $\lambda_i = (-1, \ldots, -1 \,|\, 0, \ldots, 0 | \, 1, \ldots, 1)$ in $\mathfrak{t}^*_{\mathbb{C}}$, with i-entries before the first vertical bar, $(n-r)$-entries between the vertical bars, and $(r-i)$-entries after the last vertical bar, will be used to determine a positive system for $\mathfrak{g}_{\mathbb{C}}$. Although these objects depend on i and r, we will only indicate the dependence on i. If $\triangle(\mathfrak{g}_{\mathbb{C}})$ denotes the roots of $\mathfrak{g}_{\mathbb{C}}$, then

$$\triangle(\mathfrak{g}_{\mathbb{C}}) = \triangle(\mathfrak{h}_{i,\mathbb{C}}) \cup \triangle(\mathfrak{q}_{i,+}) \cup \triangle(\mathfrak{q}_{i,-})$$

where $\triangle(\mathfrak{h}_{i,\mathbb{C}})$ is the set of roots of $\mathfrak{g}_{\mathbb{C}}$ whose inner product with λ_i is 0 and $\triangle(\mathfrak{q}_{i,+})$ (respectively, $\triangle(\mathfrak{q}_{i,-})$) is the roots of $\mathfrak{g}_{\mathbb{C}}$ whose inner product with λ_i is positive (respectively, negative).

We fix a positive system

$$\begin{aligned}\triangle^+(\mathfrak{h}_{i,\mathbb{C}}) = &\{(e_j - e_k) : 1 \le k < j \le i \text{ or } i+n-r+1 \le k < j \le n\}\\ &\cup \{(e_j + e_k) : 1 \le j \le i, i+n-r+1 \le k \le n\}\\ &\cup \{(e_j - e_k) : i+1 \le j < k \le i+n-r\}\\ &\cup \{(e_j + e_k) : i+1 \le j \le k \le i+n-r\}.\end{aligned}$$

for $\mathfrak{h}_{i,\mathbb{C}}$ and note that the first two subsets are all the positive roots for $U(i, r-i)$ and the last two for $Sp(n-r,\mathbb{R})$. If $r = n$, the $Sp(n-r,\mathbb{R})$ piece does not appear. The corresponding simple system is

$$\begin{aligned}\Pi_i = &\{e_2 - e_1, e_3 - e_2, \ldots, e_i - e_{i-1}, e_1 + e_{i+n-r+1}\}\\ &\cup \{e_{i+n-r+2} - e_{i+n-r+1}, e_{i+n-r+3} - e_{i+n-r+2}, \ldots, e_n - e_{n-1}\}\\ &\cup \{e_{i+1} - e_{i+2}, e_{i+2} - e_{i+3}, \ldots, e_{i+n-r-1} - e_{i+n-r}, 2e_{i+n-r}\}.\end{aligned}$$

Again we note that the first two subsets are the simple roots for $U(i, r-i)$ and the last for $Sp(n-r,\mathbb{R})$. Now $\triangle^+(\mathfrak{h}_{i,\mathbb{C}}) \cup \triangle(\mathfrak{q}_{i,+})$ forms a positive system for $\triangle(\mathfrak{g}_{\mathbb{C}})$ and this is the system that we shall use throughout.

Let $\mathfrak{h}_{i,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum\limits_{\alpha \in \triangle(\mathfrak{h}_{i,\mathbb{C}})} \mathfrak{g}_\alpha$ and let $H_{i,\mathbb{C}}$ be the analytic subgroup associated to $\mathfrak{h}_{i,\mathbb{C}}$. Let $\mathfrak{h}_i = \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{g}$; then $\mathfrak{h}_i$ is isomorphic to $\mathfrak{u}(i, n-i) \oplus \mathfrak{sp}(n-r,\mathbb{R})$ and $H_i \simeq U(i, n-i) \times Sp(n-r,\mathbb{R})$ is the analytic subgroup for $\mathfrak{h}_i$. For $r = n$, H_i is the fixed points of the involution $\mathrm{Ad}\,\zeta_i$ on G where

$$\zeta_i = \begin{pmatrix} -I_i & 0 & 0 & 0 \\ 0 & I_{n-i} & 0 & 0 \\ 0 & 0 & I_i & 0 \\ 0 & 0 & 0 & -I_{n-i} \end{pmatrix}.$$

Let $\mathfrak{q}_{i,+} = \sum_{\alpha \in \triangle(\mathfrak{q}_{i,+})} \mathfrak{g}_\alpha$ and let $Q_{i,+}$ denote the analytic subgroup of $\mathfrak{q}_{i,+}$ in $G_{\mathbb{C}}$. Let $\mathfrak{q}_{i,-} = \sum_{\alpha \in \triangle(\mathfrak{q}_{i,-})} \mathfrak{g}_\alpha$ and let $Q_{i,-}$ denote the analytic subgroup of $\mathfrak{q}_{i,-}$ in $G_{\mathbb{C}}$.

Let Θ be the Cartan involution on $\mathfrak{g}_{\mathbb{C}}$ given by $\Theta(X) = -{}^t\overline{X}$. Denote by $K_{\mathbb{C}}$ the analytic subgroup of $G_{\mathbb{C}}$ corresponding to the $(+1)$-eigenspace of Θ. The (-1)-eigenspace $\mathfrak{p}_{\mathbb{C}}$ of Θ decomposes into the $K_{\mathbb{C}}$-invariant subspaces $\mathfrak{p}_+$ and $\mathfrak{p}_-$. Let $P_+ = \exp(\mathfrak{p}_+)$ and $P_- = \exp(\mathfrak{p}_-)$. In this case K, the real form of $K_{\mathbb{C}}$, is isomorphic to $U(n)$.

3.2. The Double Fibration for $Sp(n, \mathbb{R})$. The geometry of $\mathbb{C}^{2n}$ induced by the Hermitian and symplectic forms provides a useful tool for describing the spaces D_i, Y_{D_i}, and M_{D_i} in the double fibration (3.1), for realizing them as homogeneous manifolds, and for examining the relationship between the double fibrations (3.1) and (3.2).

We begin by observing that $G_{\mathbb{C}}$ acts transitively on X, the set of isotropic r-planes in $\mathbb{C}^{2n}$, by Witt's theorem (see, for example, [**A**]). If we choose $x_i = \text{span}\{e_1, \ldots, e_i, e_{2n-r+i+1}, \ldots, e_{2n}\}$ as a basepoint in X, then $G_{\mathbb{C}}$ acts with isotropy subgroup $H_{i,\mathbb{C}}Q_{i,-}$. Then X, as a generalized flag manifold for $G_{\mathbb{C}}$, can be realized in several ways; if it is important to specify a realization we will use the convention $X_i = G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}$. We also note that if $r = n$ and $i = 0$ or n, then the isotropy subgroup is $K_{\mathbb{C}}P_+$ or $K_{\mathbb{C}}P_-$, respectively.

The relationship between X and D_i, the set of isotropic $(i, r-i)$-planes, is given in the following proposition.

Proposition 3.3. *D_i is an open G-orbit in X.*

This can be seen in two ways. First, for a fixed r, the open G-orbits in X are parametrized by the signatures $(i, r-i)$ [**Wo1**]. Second, a generalization of Witt's theorem (see [**A**], for example) implies that G acts transitively on D_i. For the basepoint x_i, the stabilizer of this action is H_i and a dimension count shows that D_i is open.

Thus D_i is a complex manifold. If $r = n$ and $i = 0$ or n, then D_i is the Hermitian symmetric space G/K and is of holomorphic type. If $r = n$ and $i \neq 0$ or n, then D_i is the indefinite Kähler symmetric space G/H_i and is of non-holomorphic type. If $r < n$, then D_i is G/H_i which is not a symmetric space. In this case, if $i = 0$ or r, then D_i is of holomorphic type and if $i \neq 0$ or r then D_i is of nonholomorphic type as described in Section 1.

We now define two other members of the double fibrations: M_{D_i} and M_{X_i}.

Definition 3.4. The space M_{X_i} is the set of $G_{\mathbb{C}}$-translates of Kx_i. Let $\widetilde{M}_{D_i}$ be the $G_{\mathbb{C}}$-translates of Kx_i contained in D_i and let M_{D_i} be the connected component of $\widetilde{M}_{D_i}$ containing Kx_i.

To analyze the structure of M_{D_i} and M_{X_i}, we need to understand the structure of the K-orbit of x_i in D_i. First, work of Schmid and Wolf [**SW**] implies that Kx_i is a maximal compact subvariety of D_i.

If $r = n$ and $i = 0$ or n, then $Kx_i = x_i$ and M_{D_i} is D_i. If $r = n$ with $i \neq 0$ or n and if $r < n$ with $i = 0$ or r, then Kx_i is biholomorphic to the Grassmanian of i-planes in $\mathbb{C}^n$ in the first case and to the Grassmanian of r-planes in $\mathbb{C}^n$ in the second. In all cases, Kx_i is realized as the homogeneous space $K/H_i \cap K$. The parametrization of M_{D_i} is given in the following theorem for $r = n$ with $i \neq 0$ or n and for all $r < n$ with $i \neq 0$ or r.

Theorem 3.5. *The manifold M_{D_i} is biholomorphic to $G/K \times \overline{G/K}$, where $\overline{G/K}$ denotes G/K with the opposite complex structure.*

For $r = n$ and $i \neq 0$ or n, the proof of this theorem is the main result of [**N**]. More recently, Wolf and Zierau [**WZ**] have proven this theorem for all open orbits of nonholomorphic type when G/K is a Hermitian symmetric space and G is a classical group. We will outline the idea of the proof in [**N**] so that we can use the explicit description of M_D given there in the proof of the contractibility of the fiber.

First, we describe how to associate a pair of transverse Lagrangian planes in $\mathbb{C}^{2n}$ to a $G_{\mathbb{C}}$-translate of Kx_i. The difficulty here is showing that the association is unique. Once this is complete, we have the parametrization of M_{X_i} given below.

Lemma 3.6. *M_{X_i} is the manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ when $2i \neq r$ and $G_{\mathbb{C}}/L$ when $2i = r$.*

This is another result of [**N**] where L is the subgroup of $G_{\mathbb{C}}$ generated by $K_{\mathbb{C}}$ and the matrix J which defines the symplectic form.

To describe the association, we observe that Kx_i is the set of isotropic r-planes of signature $(i, r-i)$ which meet $y_0 = \text{span}\{e_1, \dots, e_n\}$ in an i-plane and $w_0 = \text{span}\{e_{n+1}, \dots, e_{2n}\}$ in an $(r-i)$-plane. More specifically, each i-plane u in y_0 together with any $(r-i)$-plane u' in $u^{\perp_\omega} \cap w_0$ forms an isotropic $(i, r-i)$-plane in Kx_i and each element of Kx_i can be described in this fashion. We make two observations. First, when $r = n$, the dimension of $u^{\perp_\omega} \cap w_0$ is $n - i$ so for each i-plane in y_0 there exists exactly one $(n-i)$-plane u' in $u^{\perp_\omega} \cap w_0$ such that $u \oplus u'$ is an element of Kx_i. Second, the above description of Kx_i does not depend on the signature of the planes y_0 and w_0, only that the planes are transverse and Lagrangian. In light of this, translating Kx_i by $g \in G_{\mathbb{C}}$ element by element is equivalent to translating y_0 and w_0 by g and creating gKx_i from the translated planes.

To reflect the relationship between Kx_i and the two transverse Lagrangian planes y_0 and w_0, we denote Kx_i by $V^i(y_0,w_0)$. Then gKx_i will be denoted by $V^i(gy_0,gw_0)$ and M_{X_i} is the set of maximal compact subvarieties $V^i(y,w)$ for any pair of transverse Lagrangian planes y and w. The main difficulty in parametrizing M_{X_i} is determining the stabilizer of the action of $G_{\mathbb{C}}$ on M_{X_i}. That is, showing the level of uniqueness of the representation of a maximal compact subvariety by $V^i(y,w)$. When $2i \neq r$, $V^i(y,w) = V^i(y',w')$ if and only if $y = y'$ and $w = w'$. When $2i = r$, it is also the case that $V^i(y,w) = V^i(y',w')$ when $y = w'$ and $w = y'$. This happens because switching the position of y and w in $V^i(y,w)$ does not change the maximal compact subvariety.

To parametrize M_{D_i}, we must identify which pairs of transverse Lagrangian planes are asociated to elements of M_{D_i}. Clearly, if y is positive and w is negative, then $V^i(y,w)$ is in D_i and hence in $\widetilde{M}_{D_i}$. The difficulty lies in showing that such $V^i(y,w)$ are in M_{D_i} and that only $V^i(y,w)$ of this type are in M_{D_i}. See [**N**] for details.

The descriptions of M_{D_i} and M_{X_i} are useful for determining the structure of

$$Y_{D_i} = \{(z, V^i(y,w)) \in D_i \times M_{D_i} : \ z \in V^i(y,w)\} \tag{3.7}$$
$$\text{and} \quad Y_{X_i} = \{(z, V^i(y,w)) \in X \times M_{X_i} : \ z \in V^i(y,w)\}.$$

It is not too difficult to show that $G_{\mathbb{C}}$ acts transitively on Y_{X_i} by $g \cdot (z, V^i(y,w)) = (gz, V^i(gy,gw))$. Making use of the parametrization of M_{X_i} and X_i, we have the following theorem.

Theorem 3.8. *When $2i \neq r$ the manifold Y_{X_i} is ${G_{\mathbb{C}}}/{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}}$ and when $2i = r$ the manifold Y_{X_i} is ${G_{\mathbb{C}}}/{H_{i,\mathbb{C}}Q_{i,-} \cap L}$*

We now turn our attention to analyzing the structure of Y_{D_i}. Although G acts on Y_{D_i} as $G_{\mathbb{C}}$ acts on Y_{X_i}, this action is not transitive. Fortunately, $G \times G$ acts transitively on Y_{D_i}. First, we define the action of $G \times G$ on the basepoint $(x_i, V^i(y_0,w_0))$ by

$$\begin{aligned}(g_1, g_2) \cdot &\big(x_i, V^i(y_0,w_0)\big) \\ &= \big((g_1 \exp(X_+)x_i, V^i(g_1 \exp(X_+)y_0, g_1 \exp(X_+)w_0))\big).\end{aligned}$$

where $\exp(X_+)k\exp(X_-)$ is the Harish-Chandra decomposition (see [**K**], for example) of $g_1^{-1}g_2$. We note that the action in the second factor simplifies to $V^i(g_1y_0, g_2w_0)$. This action of $G \times G$ maps $(x_i, V^i(y_0,w_0))$ onto Y_{D_i} as follows. Since $G \times G$ acts transitively on M_{D_i}, there exists $g_1, g_2 \in G$ such that $(g_1, g_2)V^i(y_0,w_0) = V^i(y,w)$ for any $V^i(y,w)$ in M_{D_i}. Since $K \times K$ fixes $V^i(y_0,w_0)$, as k_1 and k_2 run through K, (g_1k_1, g_2k_2) acting on x_i run through every element of $V^i(g_1y_0,g_2w_0)$. Thus each $(z, V^i(y,w))$ is a translate of $(x_i, V^i(y_0,w_0))$. Then $G \times G$ acts on $(z, V^i(y,w))$ by first writing $(z, V^i(y,w))$ as $(g_3, g_4)(x_i, V^i(y_0,w_0))$ and letting (g_1g_3, g_2g_4) act on $(x_i, V^i(y_0,w_0))$.

We have the following theorem.

Theorem 3.9. *Y_{D_i} is biholomorphic to* $G/_{H_i \cap K} \times \overline{G/_K}$.

Proof. As shown above, $G{\times}G$ acts transitively on Y_{D_i}. Then the stabilizer of $(x_i, V^i(y_0,w_0))$ is $(H_i \cap K){\times}K$ so there is an isomorphism between Y_{D_i} and $G/H_i \cap K \times \overline{G/K}$ which endows Y_{D_i} with a differentiable structure. The complex structure comes from using the Harish-Chandra decomposition to embed $G/H_i \cap K$ into $G_{\mathbb{C}}/(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})P_+$ and G/K in $G_{\mathbb{C}}/K_{\mathbb{C}}P_-$. The opposite complex structure is needed in the second factor since P_+ is replaced with P_-. □

We have the following two observations about the action of $G{\times}G$ on Y_{D_i}. First, if we had decomposed $g_2^{-1}g_1$ as $\exp(X_-)k\exp(X_+)$ instead of decomposing $g_1^{-1}g_2$, then $(g_1, g_2)(x_i, V^i(y_0,w_0)) = (g_2\exp(X_-)x_i, V^i(g_1y_0,g_2w_0))$ determines another action of $G \times G$ on Y_{D_i}. In this case, the space Y_{D_i} would have been realized as $G/K \times \overline{G/H_i \cap K}$. If this action were chosen, the factors would be switched throughout the construction.

Second, when $r = n$ we can describe the action of $G \times G$ on the first component of $(x_i, V^i(y_0,w_0))$ geometrically. This is possible because, as described after Lemma 3.6, each element of Kx_i is of the form $u \oplus u'$ with u an i-plane in y_0 and u' an $(n-i)$-plane in $u^{\perp_\omega} \cap w_0$. When $r = n$, we have $u' = u^{\perp_\omega} \cap w_0$. That is, each element of Kx_i is completely determined by its intersection with y_0. So, if we move $x_i \cap y_0$ by g_1 to $g_1(x_i \cap y_0)$, then $g_1(x_i \cap y_0)$ meets g_1y_0 in an i-plane and the image of x_i under (g_1, g_2) is $z' = g_1(x_i \cap y_o) \oplus [\{g_1(x_i \cap y_o)\}^{\perp_\omega} \cap (g_2w_o)]$ which is an element of $V^i(g_1y_0,g_2w_0)$. Using the Harish-Chandra decompostition of $g_1^{-1}g_2$, we have $z' = g_1\exp(X_+)x_i$. Thus, the action of $G{\times}G$ on Y_{D_i} can be interpreted in terms of planes.

3.3. Relating the two double fibrations. A good understanding of the relationship between the double fibrations (3.1) and (3.2) is crucial for giving an explicit realization of the differential operator in Theorem 2.9. We have already discussed the relationship between D_i and X_i in Proposition 3.3. In this section, we consider the relationship between the other pairs.

From the descriptions of Y_{D_i} and Y_{X_i} in (3.7) as certain pairs of isotropic r-planes and maximal compact subvarieties, it is clear that Y_{D_i} is contained in Y_{X_i}. When these spaces are realized as homogeneous manifolds, the embedding of Y_{D_i} in Y_{X_i} is given by the following theorem.

Theorem 3.10. *The map*

$$\varphi : G/_{H_i \cap K} \times \overline{G/_K} \to G_{\mathbb{C}}/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}}$$

defined by $\varphi(\overline{g_1}, \overline{g_2}) = \overline{g_1\exp(X_+)}$ *is a holomorphic injection where the Harish-Chandra decomposition of* $g_1^{-1}g_2$ *is* $\exp(X_+)k\exp(X_-)$ *with* $X_+ \in \mathfrak{p}_+$, $k \in K_{\mathbb{C}}$ *and* $X_- \in \mathfrak{p}_-$.

Proof. In the following diagram

$$\begin{array}{ccc} G/_{H_i \cap K} \times \overline{G/_K} & \xrightarrow{\varphi} & G_{\mathbb{C}}/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}} \\ \alpha\downarrow & & \downarrow\beta \\ G_{\mathbb{C}}/_{(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})P_+} \times G_{\mathbb{C}}/_{K_{\mathbb{C}}P_-} & \xrightarrow{i} & G_{\mathbb{C}}/_{(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})P_+} \times G_{\mathbb{C}}/_{K_{\mathbb{C}}P_-} \end{array}$$

the embeddings α and β are given by $\alpha(\overline{g_1}, \overline{g_2}) = (\overline{g_1}, \overline{g_2})$ and $\beta(\overline{g}) = (\overline{g}, \overline{g})$ and i is the identity map. The following calculation shows that the image of α is contained in the image of β:

$$\begin{aligned} \alpha(\overline{g_1}, \overline{g_2}) &= (\overline{g_1}, \overline{g_2}) \\ &= g_1 \cdot (\overline{e}, \overline{g_1^{-1} g_2}) \\ &= g_1 \cdot (\overline{e}, \overline{\exp(X_+) k \exp(X_-)}) \\ &= g_1 \cdot (\overline{\exp(X_+)}, \overline{\exp(X_+)}) \\ &= \beta(\overline{g_1 \exp(X_+)}). \end{aligned}$$

Thus this a commutative diagram and the result follows. □

For $2i \neq r$, the map φ embeds Y_{D_i} in Y_{X_i}. For $2i = r$, the realization of Y_{X_i} accounts for the fact that, in this case, $V^i(y,w)$ and $V^i(w,y)$ are the same maximal compact subvariety. The natural projection map

$$\pi : G_{\mathbb{C}}/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}} \to G_{\mathbb{C}}/_{H_{i,\mathbb{C}}Q_{i,-} \cap L}$$

reflects this identification. Since only one of these realizations occurs in the parametrization of Y_{D_i}, the map $\pi \circ \varphi$ is an injection and gives the embedding of Y_{D_i} in Y_{X_i} in this case.

The situation for M_{D_i} and M_{X_i} is similar and we use the following theorem to embed M_{D_i} in M_{X_i}.

Theorem 3.11. *The map*

$$\psi : G/_K \times \overline{G/_K} \to G_{\mathbb{C}}/_{K_{\mathbb{C}}}$$

defined by $\psi(\overline{g_1}, \overline{g_2}) = \overline{g_1 \exp(X_+)}$ *is a holomorphic injection where* $\exp(X_+) k \exp(X_-)$ *is the Harish-Chandra decomposition of* $g_1^{-1} g_2$.

Proof. We embed $G/K \times \overline{G/K}$ and $G_{\mathbb{C}}/K_{\mathbb{C}}$ in $G_{\mathbb{C}}/K_{\mathbb{C}}P_+ \times G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ and proceed as in Theorem 3.10. □

For $2i \neq r$, the map ψ embeds M_{D_i} in M_{X_i}. For $2i = r$, let $\pi : G_{\mathbb{C}}/K_{\mathbb{C}} \to G_{\mathbb{C}}/L$ be the natural projection map. Then, as before, $\pi \circ \psi$ embeds M_{D_i} in M_{X_i}.

3.4. The fiber of μ. The geometry of the fiber of μ plays an important role in the first step of the construction of the Penrose transform. In particular, we need the fiber of μ to be contractible to apply Buchdahl's condition [**Bu**] to conclude that $H^s(D_i, \mathcal{L}_\chi)$ is isomorphic to $H^s(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi)$. Since μ is a G-equivariant map, it suffices to consider the geometry of $\mu^{-1}(x_i)$ where, as before, we have $x_i = \text{span}\{e_1, \dots, e_i, e_{2n-r+i+1}, \dots, e_{2n}\}$. We will show that $\mu^{-1}(x_i)$ is contractible by showing that it fibers over a contractible space with contractible fiber. Let $G(j) = Sp(j,\mathbb{C}) \cap U(j,j)$.

Theorem 3.12. *Let*

$$\pi : \mu^{-1}(x_i) \to$$
$$H_i/U(i) \times G(n-r) \times U(r-i) \times H_i/U(i) \times G(n-r) \times U(r-i)$$

be the map defined by $\pi(x_i, V^i(y,w)) = (\overline{h_1}, \overline{h_2})$ *where* $h_1(x_i \cap y_0) = x_i \cap y$ *and* $h_2(x_i \cap w_0) = x_i \cap w$. *Then* $\mu^{-1}(x_i)$ *fibers over*

$$H_i/U(i) \times G(n-r) \times U(r-i) \times H_i/U(i) \times G(n-r) \times U(r-i)$$

with fiber

$$G(n-i)/U(n-i) \times G(i+n-r)/U(i+n-r).$$

This theorem together with the observation that both the base space and the fiber of π are contractible give us the following proposition.

Proposition 3.13. $\mu^{-1}(x_i)$ *is contractible.*

Proof of Theorem 3.12. To understand the geometry of $\mu^{-1}(x_i)$, we must identify all maximal compact subvarieties $V^i(y,w)$ in M_{D_i} containing x_i. Given the parametrization of M_{D_i}, this is equivalent to finding all positive Lagrangain planes y and all negative Lagrangian planes w such that y meets x_i is an i-plane and w meets x_i is an $(r-i)$-plane.

We begin by looking at a special case: The positive i-plane $u_i = \text{span}\{e_1, \dots, e_i\}$ in x_i. We can extend u_i to a positive Lagrangian plane by any positive $(n-i)$-plane in $u_i^{\perp_\omega} \cap u_i^{\perp_H} = \text{span}\{e_{i+1}, \dots, e_n, e_{n+i+1}, \dots, e_{2n}\}$. One such plane is $v_i = \text{span}\{e_{i+1}, \dots, e_n\}$. To find the others we observe that, in G, the plane $u_i^{\perp_\omega} \cap u_i^{\perp_H}$ is fixed by $G(i) \times G(n-i)$ and the stabilizer of v_i in $G(i) \times G(n-i)$ is $G(i) \times U(n-i)$. Thus, all positive Lagrangian planes containing u_i are of the form $u_i \oplus g v_i$ where $\overline{g} \in G(n-i)/U(n-i)$.

More generally, any positive i-plane u in x_i is an H_i-translate of u_i and the stabilizer of u_i in H_i is $U(i) \times G(n-r) \times U(r-i)$. Thus the positive i-planes in x_i are parametrized by $H_i/(U(i) \times G(n-r) \times U(r-i))$ and for each positive i-plane in x_i the set of positive Lagrangian planes containing it is parametrized by $G(n-i)/U(n-i)$.

In a similar fashion, one can show that the negative $(r-i)$-planes in x_i are parametrized by $H_i/U(i) \times G(n-r) \times U(r-i)$ and for each negative

$(r-i)$-plane in x_i, the set of negative Lagrangain planes containing it is parametrized by $G(i+n-r)/U(i+n-r)$. □

4. Constructing the double fibration transform for $H^s(D_i, \mathcal{L}_\chi)$.

In this section we will define a double fibration transform for the $Sp(n,\mathbb{R})$-representations $H^s(D_i, \mathcal{L}_\chi)$ where s is the dimension of the maximal compact subvariety $K/H_i \cap K$ in D_i and χ is the character on H_i whose differential is given by $\chi = (-a, \ldots, -a \mid 0, \ldots, 0 \mid a, \ldots, a)$. That is, $\chi = \sum_{j=1}^{i} -ae_j + \sum_{p=i+n-r+1}^{n} ae_p$ in $\mathfrak{h}^*_{i,\mathbb{C}}$. In this case $H^s(D_i, \mathcal{L}_\chi)$ is an irreducible, unitarizable nonzero infinite dimensional representation of $Sp(n,\mathbb{R})$ if $a < -2n+r$ [**Wg**]. In the process of defining the transform, it will be necessary to impose additional restrictions on χ so that the transform will be injective.

4.1. Pulling up $H^s(D_i, \mathcal{L}_\chi)$ by μ to Y_{D_i}. The first step in defining the transform is transferring $H^s(D_i, \mathcal{L}_\chi)$ to Y_{D_i}. Since the fiber of μ is contractible (Proposition 3.13), a theorem of Buchdahl [**Bu**] implies the following theorem.

Theorem 4.1. *$H^s\big(D_i, \mathcal{L}_\chi\big)$ is isomorphic to $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi\big)$.*

Now Lemma 2.4 implies that

$$0 \to \mu^{-1}\mathcal{L}_\chi \to \mathcal{O}(\mu^*\mathbb{L}_\chi) \to \Omega^1_\mu(\mu^*\mathbb{L}_\chi) \to \cdots \to \Omega^m_\mu(\mu^*\mathbb{L}_\chi) \to 0 \tag{4.2}$$

is a resolution of $\mu^{-1}\mathcal{L}_\chi$ where $\Omega^p_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of relative p-forms on Y_{D_i} with values in the bundle $\mu^*\mathbb{L}_\chi$ and $m = \dim Y_{D_i} - \dim D_i$.

Upon first inspection, the sheaves in the resolution of $\mu^{-1}\mathcal{L}_\chi$ do not appear to be sheaves of holomorphic sections of homogeneous vector bundles due to the fact that μ is a G–equivariant map, not $G\times G$–equivariant, from the $G\times G$-homogeneous space Y_{D_i} to the G-homogeneous space D_i. We will show, using the natural projection map $\tilde{\mu} : Y_{X_i} \to X_i$, that these sheaves are holomorphic sections of a homogeneous vector bundle. We begin with the sheaf $\mathcal{O}(\mu^*\mathbb{L}_\chi)$.

Theorem 4.3. *The bundle $\mu^*\mathbb{L}_\chi$ on Y_{D_i} is a homogeneous bundle with fiber $\mathbb{C}_\chi$ where $(H_i \cap K) \times K$ acts by $\chi \otimes 1$.*

Proof. Let $\tilde{\chi}$ be the extension of χ to $H_{i,\mathbb{C}}Q_{i,-}$ with $\tilde{\chi}$ trivial on $Q_{i,-}$. Then $\tilde{\mu}^\star\mathbb{L}_{\tilde{\chi}}$ is the homogeneous line bundle on Y_{X_i} with fiber $\mathbb{C}_{\tilde{\chi}}$ and its restriction to $\varphi(Y_{D_i})$ is isomorphic to $\mu^*\mathbb{L}_\chi$ on Y_{D_i} where φ is the embedding of Y_{D_i} in Y_{X_i} in Theorem 3.10. This isomorphism allows us to show that $\mu^*\mathbb{L}_\chi$ is a $G\times G$-homogeneous bundle once we have an explicit expression for the action of $G\times G$ on $\varphi(Y_{D_i})$.

Let $\overline{g} \in \varphi(Y_{D_i})$ and $g_1, g_2 \in G$. Assume, for the moment, that the Harish-Chandra decomposition of $(g_1 g)^{-1} g_2 g$ as $\exp(X_+) k \exp(X_-)$ exists. The key to seeing that $(g_1, g_2) \cdot \overline{g} = \overline{g_1 g \exp(X_+)}$ defines an action of $G{\times}G$ on $\varphi(Y_{D_i})$ is the following computation. Using the identification of $G_{\mathbb{C}}/H_{i,\mathbb{C}} Q_{i,-} \cap K_{\mathbb{C}}$ with Y_{X_i}, we have

$$\begin{aligned}
(g_1, g_2) \cdot \overline{g} &= (g_1, g_2) \cdot \big(g x_i, V^i(g y_0, g w_0)\big) \\
&= (g_1 g, g_2 g)(x_i, V^i(y_0, w_0)) \\
&= \big(g_1 g \exp(X_+) x_i, V^i(g_1 g \exp(X_+) y_0, g_1 g \exp(X_+) w_0)\big) \\
&= g_1 g \exp X_+ (x_i, V^i(y_0, w_0)) \\
&= \overline{g_1 g \exp X_+}.
\end{aligned}$$

Now we address the Harish-Chandra decomposition of $(g_1 g)^{-1} g_2 g$. Since $\overline{g} \in \varphi(Y_{D_i})$, there exist $g_3, g_4 \in G$ such that $\varphi(\overline{g_3}, \overline{g_4}) = \overline{g}$. That is, there exist $X_+' \in \mathfrak{p}_+$ and $h \in H_{i,\mathbb{C}} Q_{i,-} \cap K_{\mathbb{C}}$ such that $g = g_3 \exp(X_+')h$. Using this expression for g in $(g_1 g)^{-1} g_2 g$ and the Harish-Chandra decomposition of $(g_1 g_3)^{-1} g_2 g_4$ yields the decomposition of $(g_1 g)^{-1} g_2 g$.

Let $\mathbb{W}$ denote the restriction of $\tilde{\mu}^* \mathbb{L}_{\tilde{\chi}}$ to $\varphi(Y_{D_i})$ and let $[g, w]$ be in $\mathbb{W}_{\overline{g}}$. For $g_1, g_2 \in G$,

$$(g_1, g_2) \cdot [g, w] = \big[g_1 g \exp(X_+), w\big]$$

defines an action of $G \times G$ on $\mathbb{W}$. Then $\mathbb{W}$ is the $G \times G$-homogeneous line bundle on Y_{D_i} with fiber $\mathbb{C}_\chi \otimes 1$. □

We note that the action of G on $\mathbb{W}$ as a subgroup of $G_{\mathbb{C}}$ is equivalent to the action of G as the diagonal subgroup of $G{\times}G$.

In the remainder of this section, we will show that the sheaves $\Omega^p_\mu(\mu^* \mathbb{L}_\chi)$ in (4.2) are sheaves of sections of homogeneous bundles on Y_{D_i}. Since $\Omega^p_\mu(\mu^* \mathbb{L}_\chi) = \Omega^p_\mu \otimes \mathcal{O}(\mu^* \mathbb{L}_\chi)$, it suffices to show that Ω^p_μ is homogeneous.

First we describe the sheaf of relative differential 1-forms for a general fibration between differentiable manifolds. Let $f : Y \to X$ be a C^∞ fibration. Then $\ker df$, the relative tangent bundle, is a subbundle of the tangent bundle of Y whose stalk at y is the kernel of df_y and $(\ker df)^*$ is the relative cotangent bundle. Let $\mathcal{E}^1_M$ denote the sheaf of smooth differential 1-forms on a manifold M and let $\mathcal{E}_M$ be the sheaf of C^∞ functions on M. Then $f^* \mathcal{E}^1_X = f^{-1} \mathcal{E}^1_X \otimes_{f^{-1}\mathcal{E}_X} \mathcal{E}_Y$ and the sheaf of relative differential 1-forms is $\mathcal{E}^1_f = \mathcal{E}^1_Y / f^* \mathcal{E}^1_X$.

Theorem 4.4. *$\mathcal{E}^1_f$ and $\mathcal{E}((\ker df)^*)$ are isomorphic as sheaves.*

Sketch of Proof. Since it suffices to check this on sufficiently small open sets, we may assume that U, an open subset of Y, is isomorphic to $\mathbb{R}^n \times \mathbb{R}^k$. The map $\gamma_U : \mathcal{E}^1_f(U) \to \mathcal{E}((\ker df)^*)(U)$ defined by $\gamma_U([w]) = \gamma_w$ where $\gamma_w(y) = w(y)|_{\ker df}$ and $y \in Y$ gives the isomorphism. □

We will use this theorem to describe the relative holomorphic (1,0)-forms for the holomorphic fibration from $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}$ to $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}$ given by the natural projection map $\tilde{\mu}$. As is customary, we identify the holomorphic tangent space $T^{1,0}(G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})$ with $T(G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})$; we do likewise for $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}$. Under these identifications, Theorem 4.4 implies that the sheaf of relative holomorphic 1-forms $\Omega^1_{\tilde{\mu}}$ is isomorphic to the sheaf $\mathcal{O}((\ker d\tilde{\mu})^*)$. Now $\ker d\tilde{\mu}$ is the $G_{\mathbb{C}}$–homogeneous bundle with fiber $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}$.

Theorem 4.4 again implies that Ω^1_{μ} is the sheaf $\mathcal{O}((\ker d\mu)^*)$ where here d_μ is the map from $T^{1,0}(Y_{D_i})$ to $T^{1,0}(D_i)$. Since the map μ from $G/H_i \cap K \times \overline{G/K}$ to G/H_i is given in terms of isotropic planes and maximal compact subvarieties and not as a map of homogeneous spaces, we are unable to use the definition of μ to determine $\ker d\mu$. However, we can give an explicit description of Ω^1_{μ} by understanding the relationship between Ω^1_{μ} and $\Omega^1_{\tilde{\mu}}$.

Theorem 4.5.

(1) *The sheaf* Ω^1_{μ} *is isomorphic to* $\mathcal{O}((\ker d\mu)^*)$.

(2) *The vector bundle* $\ker d\mu$ *is* $(G \times G)$*-homogeneous with fiber* $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$ *where* $(H_i \cap K) \times K$ *acts by* $Ad \otimes 1$.

(3) *The vector bundle* $(\ker d\mu)^*$ *is* $(G \times G)$*-homogeneous with fiber* $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}$ *where* $(H_i \cap K) \times K$ *acts by* $Ad \otimes 1$.

Proof of (1). This follows from the discussion before the statement of Theorem 4.5. □

Proof of (2). Recall the map

$$\varphi : {}^{G}\!/_{H_i \cap K} \times \overline{{}^{G}\!/_{K}} \to {}^{G_{\mathbb{C}}}\!/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}}$$

from Section 3.3. Since the image of φ is open in $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}$, the fiber of μ is open in the fiber of $\tilde{\mu}$. Thus, we have $\ker d\mu = (\ker d\tilde{\mu})|_{Im(\varphi)}$. Then, as in Theorem 4.3, we can define an action of $G \times G$ on $\ker d\mu$ and the action of $(H_i \cap K) \times K$ on $(\ker d\mu)_{\overline{e}}$ is determined by its action on $(\ker d\tilde{\mu})|_{Im(\varphi)}$. Thus $(H_i \cap K) \times K$ acts on $\big((\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}\big)$ via $\mathrm{Ad} \otimes 1$. Since $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$ is an $\big[(H_i \cap K) \times K\big]$-invariant complement to $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}$ in $\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}$, the bundle $\ker d\mu$ is $(G \times G)$–homogeneous with fiber $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$. □

Proof of (3). The Killing form can be used to identify $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}$ as the dual of $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$ in $\mathfrak{g}_{\mathbb{C}}$. □

We are now ready to apply Lemma 2.2.

4.2. The Vanishing Condition. We will show in this section that $H^p(Y_{D_i}, \Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ vanishes for all $p < s$ and $1 \le q \le m$ if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$. That is, we obtain the hypothesis of Lemma 2.2 for the resolution of $\mu^{-1}\mathcal{L}_\chi$ given in (4.2). Once this is accomplished, we have:

Theorem 4.6. *If $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$, then there is an injection from $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi\big)$ into $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ whose image is the kernel of the induced map from $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$.*

To obtain the vanishing condition, we make the following observations. First, the manifold M_{D_i} is Stein [**Wo2**, **Wo3**]. Second, since the map ν is a fibration, it is proper because the inverse image of a point in M_{D_i} under ν is isomorphic to the compact submanifold $K/H_i \cap K$. Third, the sheaves $\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ are coherent since each is the sheaf of sections of a homogeneous vector bundle. (See Theorem 4.3 and 4.5.)

Now we can apply Theorem 2.8 to $H^p(Y_{D_i}, \Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ to obtain the following theorem.

Theorem 4.7. *$H^p(Y_{D_i}, \Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ is isomorphic to $H^0(M_{D_i}, R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ for all p and q.*

Now we will show that $H^0(M_{D_i}, R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ vanishes for all $p < s$ and $1 \le q \le m$ if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$. Recall that $\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous bundle $\mathbb{V}^q_\chi = \wedge^q(\ker d\mu)^* \otimes \mu^*\mathbb{L}_\chi$ on Y_{D_i} whose fiber is $\wedge^q\big[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}\big] \otimes \mathbb{C}_\chi$.

Before we look at the structure of $R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ we identify the fiber of ν with $K/H_i \cap K$. Under this identification, the restriction of $\mu^*\mathbb{L}_\chi$ to the fiber of ν is the bundle $K \times_{(H_i\cap K)} \mathbb{C}_\chi$ and the restriction of $(\ker d\mu)^*$ is the K-homogeneous bundle with fiber $\wedge^q\big[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}\big]$ (see Theorem 4.5).

With these identifications, the restriction of $\mathbb{V}^q_\chi$ to the fiber of ν is the bundle

$$K \times_{(H_i\cap K)} [\wedge^q[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}] \otimes \mathbb{C}_\chi]$$

which we also denote by $\mathbb{V}^q_\chi$. With this in mind, a theorem of Bott [**B**] implies that $R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of the $(G\times G)$–homogeneous vector bundle $\mathbb{H}^p\,(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$ on M_{D_i} whose fiber is $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$.

We summarize this discussion with the following lemma.

Lemma 4.8. *The sheaf $R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous vector bundle on M_{D_i} whose fiber is $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$ and the action of $K\times K$ on the fiber is given by $(k_1, k_2)\cdot\omega = \ell^*_{k_1^{-1}}\omega$*

where ℓ_k is the map from $K/H_i \cap K$ to itself given by left translation. We denote $R^p_\nu \Omega^q_\mu(\mu^* \mathbb{L}_\chi)$ by $\mathcal{O}\big[\mathbb{H}^p\big(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi)\big)\big]$.

We now state the vanishing condition.

Theorem 4.9. *$H^p\left({}^{K}\!/_{H_i \cap K}, \mathcal{O}(\mathbb{V}^q_\chi)\right)$ vanishes for $p < s$ and $1 \leq q \leq m$ if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$.*

The proof of this theorem is an application of Bott-Borel-Weil along with the following observations. Since the fiber V^q_χ of $\mathbb{V}^q_\chi$ is reducible, we decompose V^q_χ into irreducible subrepresentations $V_1, \ldots, V_j$. Then Bott-Borel-Weil determines a condition on χ such that $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}_i))$ vanishes for all $p \neq s$. Thus $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$ also vanishes. Since we do not know the V_i's or their highest weights, we choose χ such that $\langle \chi + \gamma + \rho_K, \alpha \rangle < 0$ for all $\alpha \in \triangle(\mathfrak{q}_{i,+} \cap \mathfrak{k}_\mathbb{C})$ and for all weights γ of V^q_χ which guarantees the vanishing of $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}_i))$ for all i and for all $p \neq s$. If χ is chosen such that $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$, then the vanishing is guaranteed. The calculations for this theorem were done with $\triangle^+(\mathfrak{k}_\mathbb{C}) = \{e_j - e_k : 1 \leq k < i$ and $k \leq j \leq n, i+1 \leq j < k \leq i+n-r, i+1 \leq k \leq i+n-r < j \leq n,$ or $i+n-r+1 \leq k < j \leq n\}$. Thus we have obtained the hypothesis for Lemma 2.2 and have proved Theorem 4.6.

4.3. Pushing Down to M_{D_i} by ν.

Now we will push $H^s(Y_{D_i}, \mathcal{O}(\mu^* \mathbb{L}_\chi))$ down to M_{D_i} and construct the double fibration transform.

Theorem 4.10. *$H^s\big(Y_{D_i}, \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ is isomorphic to $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ which is isomorphic to $H^0\big(M_{D_i}, \mathcal{O}\big[\mathbb{H}^s(K/H_i \cap K, \mathcal{L}_\chi)\big]\big)$.*

Proof. This is Theorem 4.3 and Lemma 4.8 applied to the sheaf $\mathcal{O}(\mu^* \mathbb{L}_\chi)$. □

Now we can define the double fibration transform.

Theorem 4.11. *Define the map*

$$P : H^s\big(D_i, \mathcal{L}_\chi\big) \to H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$$

by the composition of the maps in Theorem 4.1, Theorem 4.6, and Theorem 4.10. Then P is the double fibration transform and it is an injection if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and $a < -3n + r$ when $r < n$. Also, the image of P is isomorphic to the kernel of a map $\mathcal{D}$ from $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ to $H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^* \mathbb{L}_\chi)\big)$ where $\mathcal{D}$ is defined in Theorem 2.9.

4.4. Bott-Borel-Weil applied to $H^s(K/H_i \cap K, \mathcal{O}(\mathcal{L}_\chi))$. Before we investigate the map $\mathcal{D}$ in the next section, we will further our understanding of $H^0\big(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ As before, a theorem of Bott [**B**] implies that the fiber of $R^s_\nu(\mu^*\mathbb{L}_\chi)$ is $H^s(K/H_i \cap K, \mathcal{L}_\chi)$.

We have the following lemma.

Lemma 4.12. *For $r = n$, if $a < \frac{1}{2}(1-n)$, the cohomology space $H^p(K/H_i \cap K, \mathcal{L}_\chi)$ vanishes whenever $p < s$ and whenever $p = s$ it is the nonzero irreducible K-representation of highest weight $(a+i, \ldots, a+i\ ;\ -a-n+i, \ldots, -a-n+i)$ where there are $(n-i)$ entries before the semicolon.*

For $r < n$ if $a < -n+1$ then $H^p(K/H_i \cap K, \mathcal{L}_\chi)$ vanishes whenever $p < s$ and whenever $p = s$ it is a nonzero irreducible K-representation. If $r - i \leq i$, the highest weight of the representation is

$$(a+i+n-r, \ldots, a+i+n-r; \\ 2i-r, \ldots 2i-r| - a-n+i, \cdots -a-n+i; \\ 2i-r, \ldots, 2i-r| - a-n+i, \ldots, -a-n+i)$$

where there are $(r-i)$-entries before the first semicolon, a total of i-entries before the first vertical bar, $(2i-r)$-entries between the first vertical bar and the second semicolon, a total of $(n-r)$-entries between the vertical bars, and $(r-i)$-entries after the second vertical bar.

If $r - i > i$, the highest weight of the representation is

$$(a+i+n-r, \ldots, a+i+n-r|2i-r, \ldots 2i-r; \\ a+i+n-r, \ldots a+i+n-r|2i-r, \ldots, 2i-r; \\ -a-n+i, \ldots, -a-n+i)$$

where there are i-entries before the first vertical bar, $(n+2i-2r)$-entries between the first vertical bar and the first semicolon, a total of $(n-r)$-entries between the two vertical bars, $(r-2i)$-entries between the second vertical bar and the second semicolon, a total of $(r-i)$-entries after the second vertical bar.

The proof is an application of Bott-Borel-Weil.

5. The differential operator.

The double fibration transform realizes the representation $H^s\big(D_i, \mathcal{L}_\chi\big)$ as the kernel of the map

$$\mathcal{D} : H^0\big(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\left(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\right) \tag{5.1}$$

defined in Theorem 4.11 and Theorem 2.9. In this section, we will describe $\mathcal{D}$ more explicitly and show that it is a G-invariant differential operator.

Recall that $H^s(D_i, \mathcal{L}_\chi)$ is isomorphic to $H^s(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi)$ and that $H^s(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi)$ is the kernel of the map $\partial_\mu^* : H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)) \to H^s(Y_{D_i}, \Omega_\mu^1(\mu^*\mathbb{L}_\chi))$ where ∂_μ^* is induced from the map $\partial_\mu : \mathcal{O}_{Y_{D_i}} \to \Omega_\mu^1$. Now a Leray spectral sequence argument shows that $H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi))$ is isomorphic to $H^0(M_{D_i}, R_\nu^s\mathcal{O}(\mu^*\mathbb{L}_\chi))$ and that $H^s(Y_{D_i}, \Omega_\mu^1(\mu^*\mathbb{L}_\chi))$ is isomorphic to $H^0(M_{D_i}, R_\nu^s\Omega_\mu^1(\mu^*\mathbb{L}_\chi))$. The map ∂_μ determines a map between the spectral sequences and induces the map $\mathcal{D}$ in (5.1).

To understand $\mathcal{D}$, we need a better understanding of the map $\partial_\mu : \mathcal{O}_{Y_{D_i}} \to \Omega_\mu^1$. By definition $\partial_\mu = \pi \circ \partial$ where $\partial : \mathcal{O}_{Y_{D_i}} \to \Omega^1_{Y_{D_i}}$ is the standard holomorphic deRahm operator on Y_{D_i} and π is the quotient map from $\Omega^1_{Y_{D_i}}$ to $\Omega_\mu^1 = \Omega^1_{Y_{D_i}}/\mu^*\Omega^1_{D_i}$. We note that, in this case, $d = \partial$ since $\overline{\partial} = 0$ on the sheaves of interest. Although we can realize both $\Omega^1_{Y_{D_i}}$ and Ω^1_μ as sheaves of holomorphic sections of a $(G\times G)$–homogeneous bundles, the map π is not determined by a $(G\times G)$-equivariant bundle map. To understand π we will decompose it into $\pi_1 \circ \pi_2$ where $\pi_2 : \Omega^1_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ and $\pi_1 : \nu^*\Omega^1_{M_{D_i}} \to \Omega^1_\mu$. Once we show that π_2 is a $(G\times G)$-equivariant map and π_1 is equivariant for the diagonal embedding of G in $G\times G$, then π will be a G–equivariant map.

Let $\partial_2 = \pi_2 \circ \partial$ and let ∂_2^* be the induced map from the cohomology space $H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi))$ to $H^s(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi))$. We will see in Section 5.1 that the corresponding map

$$\mathcal{D}_2 : H^0(M_{D_i}, R_\nu^s\mathcal{O}(\mu^*\mathbb{L}_\chi)) \to H^0(M_{D_i}, R_\nu^s\left[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\right])$$

is the standard holomorphic d operator on M_{D_i} and that $\mathcal{D}_2$ is a $(G\times G)$-invariant first-order differential operator.

Now π_1 induces a map

$$\pi_1 : H^s\Big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\Big) \to H^s(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)).$$

In Section 5.2, we will show that the corresponding map

$$\mathcal{D}_1 : H^0\Big(M_{D_i}, R_\nu^s[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\Big) \to H^0(M_{D_i}, R_\nu^s\Omega^1_\mu(\mu^*\mathbb{L}_\chi))$$

is a G-invariant zeroth-order differential operator. As $\mathcal{D}_1$ is G-invariant and $\mathcal{D}_2$ is $(G\times G)$-invariant, the map $\mathcal{D} = \mathcal{D}_1 \circ \mathcal{D}_2$ is a G-invariant first-order differential operator for the diagonal embedding of G in $G\times G$.

5.1. The operator $\mathcal{D}_2$. In this section we will define $\pi_2 : \Omega^1_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ and give an explicit realization of the map $\partial_2 : \mathcal{O}_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ which induces the map

$$\partial_2^* : H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)) \to H^s(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)).$$

We will then see how ∂_2^* determines the $(G{\times}G)$-invariant differential operator

$$\mathcal{D}_2 : H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big)$$

by examining the maps between the Leray spectral sequences for $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ and $H^s\big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big)$.

The Leray spectral sequence which defines the isomorphism between the cohomology spaces $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ and $H^0\big(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ is realized from a filtration of the resolution

$$0 \to \mathcal{O}(\mu^*\mathbb{L}_\chi) \to \mathcal{E}^{0,0}(\mu^*\mathbb{L}_\chi) \to \mathcal{E}^{0,1}(\mu^*\mathbb{L}_\chi) \to \cdots \to \mathcal{E}^{0,a}(\mu^*\mathbb{L}_\chi) \to 0$$

with respect to the fiber of ν. (See, for example, [**G**].) Using the homogeneous structure of $\mathcal{E}^{0,c}$ and $\mu^*\mathbb{L}_\chi$, the E_0-term is given by

$$E_0^{p,q} = C^\infty\big(G{\times}G, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes \wedge^p \mathfrak{d}\big)^{(H_i\cap K)\times K} \tag{5.2}$$

where $\mathbb{C}_\chi$ is an $(H_i \cap K)$-representation and where $\mathfrak{c}_i$ and $\mathfrak{d}$ represent the fiber of the antiholomorphic cotangent space of $K/H_i \cap K$ and $G/K \times \overline{G/K}$ respectively. Since $\wedge^p\mathfrak{d}$ is a $(K \times K)$-representation, the following lemma gives another realization of $E_0^{p,q}$.

Lemma 5.3.

$$E_0^{p,q} = C^\infty\Big(G{\times}G, C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q\mathfrak{c}_i\big)^{H_i\cap K} \otimes \wedge^p\mathfrak{d}\Big)^{K\times K}. \tag{5.4}$$

Proof. The isomorphism is given by sending φ to $\tilde\varphi$ where $\tilde\varphi(g_1, g_2)(k) = \varphi(g_1k, g_2)$. A straightforward computation shows that $\tilde\varphi$ has the correct invariance property. ☐

Then $E_1^{p,q} = C^\infty\big(G{\times}G, H^q(K/H_i \cap K, \mathcal{O}(\mathbb{C}_\chi)) \otimes \wedge^p\mathfrak{d}\big)^{K\times K}$ and

$$\begin{aligned} E_2^{p,q} &= H^p\Big(G/K \times \overline{G/K}, \mathcal{O}\big[\mathbb{H}^q(K/H_i \cap K, \mathcal{O}(\mathbb{C}_\chi))\big]\Big) \\ &= H^p\Big(G/K \times \overline{G/K}, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\Big). \end{aligned}$$

Now we turn our attention to the Leray spectral sequence for the cohomology space $H^s\Big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\Big)$. The E_0-term of the Leray spectral sequence which defines the isomorphism between the spaces $H^s\Big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\Big)$ and $H^0\Big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\Big)$ is given by

$$E_{0,M}^{p,q} = C^\infty\big(G{\times}G, \mathbb{C}_\chi \otimes \wedge^q\mathfrak{c}_i \otimes \wedge^p\mathfrak{d} \otimes (\mathfrak{p}_+ \oplus \mathfrak{p}_-)\big)^{(H_i\cap K)\times K} \tag{5.5}$$

where here we are identifying $(\mathfrak{g}_\mathbb{C}/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\pm))^*$ with $\mathfrak{p}_\mp$.

Lemma 5.6.

$$E_{0,M}^{p,q} = C^\infty\Big(G\times G, \big[C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes \mathfrak{p}_+\big)^{H_i\cap K} \otimes \wedge^p \mathfrak{d}\big] \oplus \big[C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes \big(\mathfrak{p}_- \otimes \wedge^p \mathfrak{d}\big)\big]\Big)^{K\times K}. \tag{5.7}$$

Proof. As in Lemma 5.3, we see that (5.5) is isomorphic to

$$C^\infty\big(G\times G, C^\infty\big(K\times K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes (\mathfrak{p}_+ \oplus \mathfrak{p}_-)\big)^{(H_i\cap K)\times K} \otimes \wedge^p \mathfrak{d}\big)^{K\times K}. \tag{5.8}$$

Since $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are K-representations, the inside of (5.8) is isomorphic to

$$C^\infty\big(K\times K, \big(\mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big) \otimes 1\big)^{(H_i\cap K)\times K} \otimes \big((\mathfrak{p}_+ \otimes 1) \oplus (1 \otimes \mathfrak{p}_-)\big)$$

which is isomorphic to

$$C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes \big((\mathfrak{p}_+ \otimes 1) \oplus (1 \otimes \mathfrak{p}_-)\big).$$

The lemma follows from splitting up the direct sum and identifying

$$C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes (\mathfrak{p}_+ \otimes 1)$$

with

$$C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes \mathfrak{p}_+\big)^{H_i\cap K}.$$

□

Now that we have explicit descriptions of the Leray spectral sequences, we look at the map π_2 and $\pi_2 \circ \partial$ in more detail so we can define a map between the spectral sequences. To define the map π_2, we observe that $\nu^*\Omega^1_{M_{D_i}}$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous bundle on Y_{D_i} with fiber

$$\Big(\mathfrak{g}_\mathbb{C}\big/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_+) \quad\oplus\quad \mathfrak{g}_\mathbb{C}\big/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_-)\Big)^* \tag{5.9}$$

and $\Omega^1_{Y_{D_i}}$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous bundle with fiber

$$\Big(\mathfrak{g}_\mathbb{C}\big/[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_\mathbb{C}] \oplus \mathfrak{p}_+ \quad\oplus\quad \mathfrak{g}_\mathbb{C}\big/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_-)\Big)^*. \tag{5.10}$$

The natural map from (5.9) to (5.10) induces the $(G\times G)$–equivariant map $\pi_2 : \Omega^1_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$. Then $\partial_2 = \pi_2 \circ \partial$ is a map from $\mathcal{O}_{Y_{D_i}}$ to $\nu^*\Omega^1_{M_{D_i}}$.

In the following lemma, we give an explicit formula for $\partial_2 : \mathcal{O}_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ which will lead to a formula for $\partial_{2,0} : E_0^{p,q} \to E_{0,M}^{p,q}$.

Lemma 5.11. *The map $\partial_2 : \mathcal{O}_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ is given by*

$$\partial_2(\psi) = \sum_{\alpha\in\mathfrak{p}_-} r_1(X_\alpha)\psi \otimes X_{-\alpha} + \sum_{\beta\in\mathfrak{p}_+} r_2(X_\beta)\psi \otimes X_{-\beta} \tag{5.12}$$

where ψ represents the corresponding element of $\mathcal{O}(G\times G)^{(H_i\cap K)\times K}$. Here

$$\big(r_1(X_\alpha)\psi\big)(g_1,g_2)=\frac{d}{dt}\bigg|_{t=0}\psi(g_1\exp tX_\alpha,g_2)$$

and

$$\big(r_2(X_\beta)\psi\big)(g_1,g_2)=\frac{d}{dt}\bigg|_{t=0}\psi(g_1,g_2\exp tX_\beta).$$

Proof. The manifolds $G/H_i\cap K$ and $\overline{G/K}$ are open orbits in the generalized flags $G_{\mathbb{C}}/(H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}})P_+$ and $G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ respectively. Griffiths and Schmid's [**GS**] formula for the standard $\overline{\partial}$ operator implies that the standard ∂ operator from $\mathcal{O}_{Y_{D_i}}$ to $\Omega^1_{Y_{D_i}}$ is given by

$$\partial(\psi)=\sum_{\alpha\in(\mathfrak{q}_{i,+}\cap\mathfrak{k}_{\mathbb{C}})\oplus\mathfrak{p}_-}r_1(X_\alpha)\psi\otimes X_{-\alpha}+\sum_{\beta\in\mathfrak{p}_+}r_2(X_\beta)\psi\otimes X_{-\beta}.$$

The lemma follows from the fact that $\partial_2=\pi_2\circ\partial$. □

The map $\partial_2:\mathcal{O}_{Y_{D_i}}\to\nu^*\Omega^1_{M_{D_i}}$ determines a map between the resolutions $\mathcal{E}^{0,\bullet}\otimes\mathcal{O}_{Y_{D_i}}(\mu^*\mathcal{L}_\chi)$ and $\mathcal{E}^{0,\bullet}\otimes\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathcal{L}_\chi)$ which respects the filtration along the fibers of ν. This map of resolutions induces a map between the associated Leray spectral sequences (see, for example, [**G**]). Let $\partial_{2,0}$ be the induced map from $E_0^{p,q}$ to $E_{0,M}^{p,q}$ (i.e., from (5.2) to (5.5)). Thus the formula for $\partial_{2,0}$ is the same as the formula for ∂_2.

Lemma 5.13. *The map $\tilde{\partial}_{2,0}$ from* (5.4) *to* (5.7) *is given by* (5.12).

Proof. The isomorphism between (5.2) and (5.4) and the one between (5.5) and (5.7) as defined in Lemma 5.3 and 5.6 respectively imply that

$$\begin{aligned}(\tilde{\partial}_{2,0}\psi)(g_1,g_2)(k)=&\sum_{\alpha\in\mathfrak{p}_-}\big(r_1(X_\alpha)\psi\big)(g_1,g_2)(k)\otimes\mathrm{Ad}(k^{-1})X_{-\alpha}\\&+\sum_{\beta\in\mathfrak{p}_+}\big(r_2(X_\beta)\psi\big)(g_1,g_2)(k)\otimes X_{-\beta}.\end{aligned}$$

Since $C^\infty\big(K,\mathbb{C}_\chi\otimes\wedge^q\mathfrak{c}_i\otimes\mathfrak{p}_+\big)^{H_i\cap K}$ is isomorphic to $C^\infty\big(K,\mathbb{C}_\chi\otimes\wedge^q\mathfrak{c}_i\big)^{H_i\cap K}\otimes\mathfrak{p}_+$, the lemma follows. □

Thus the maps $\tilde{\partial}_{2,0}$ and $\partial_{2,1}:E_1^{p,q}\to E_{1,M}^{p,q}$ where

$$\begin{aligned}E_{1,\mu}^{p,q}=C^\infty\big(G\times G,&\big[\big(H^q(K/H_i\cap K,\mathcal{O}(\mathcal{L}_\chi))\otimes\mathfrak{p}_+\big)\otimes\wedge^p\mathfrak{d}\big]\oplus\\&\big[H^q(K/H_i\cap K,\mathcal{O}(\mathcal{L}_\chi))\otimes(\mathfrak{p}_-\otimes\wedge^p\mathfrak{d})\big]\big)^{K\times K}\end{aligned}$$

are each the standard holomorphic d operator on M_{D_i}. Since both spectral sequences collapse at the E_2–term, the map $\partial_{2,2}:E_2^{p,q}\to E_{2,M}^{p,q}$ is the zero map except when $p = 0$ and $q = s$. In that case, it is the standard holomorphic d operator on M_{D_i}

$$\mathcal{D}_2 : H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big)$$

and it is a $(G\times G)$-equivariant map.

5.2. The operator $\mathcal{D}_1$. In this section, we define $\pi_1 : \nu^*\Omega^1_{M_{D_i}} \to \Omega^1_\mu$ and then see how this determines the operator

$$\mathcal{D}_1 : H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big) \to H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big).$$

To define the map π_1 we observe that the restriction of $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ and $\Omega^1_{\tilde{\mu}}$ to Y_{D_i} is isomorphic to $\nu^*\Omega^1_{M_{D_i}}$ and Ω^1_μ respectively. Because the embedding of Y_{D_i} in Y_{X_i} is G-equivariant, these two isomorphisms are G–equivariant. Now $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ is the sheaf of holomorphic sections of the $G_\mathbb{C}$-homogeneous bundle with fiber

$$\left(\mathfrak{g}_\mathbb{C}/\mathfrak{k}_\mathbb{C}\right)^* \tag{5.14}$$

and $\Omega^1_{\tilde{\mu}}$ is the sheaf of holomorphic sections of the $G_\mathbb{C}$-homogeneous bundle with fiber

$$\left((\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{k}_\mathbb{C}\right)^*. \tag{5.15}$$

Then $\tilde{\pi}_1$ is the $G_\mathbb{C}$–equivariant map induced by the natural restriction map from (5.14) to (5.15) and $\pi_1 = \tilde{\pi}_1|_{\nu^*\Omega^1_{M_{D_i}}}$ is the G-equivariant map from $\nu^*\Omega^1_{M_{D_i}}$ to Ω^1_μ.

As in Section 5.1, the Leray spectral sequences for $H^s\big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big)$ and $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$ together with the map $\pi_1 : \nu^*\Omega^1_{M_{D_i}} \to \Omega^1_\mu$ determine a map

$$\mathcal{D}_1 : H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big) \to H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big).$$

In this case, this process does not yield an explicit description of $\mathcal{D}_1$ because we do not have an explicit description of π_1 in terms of the homogeneous vector bundles for $\nu^*\Omega^1_{M_{D_i}}$ and Ω^1_μ. To resolve this difficulty we use the fact that π_1 is the restriction of the map $\tilde{\pi}_1$ to $\nu^*\Omega^1_{M_{D_i}}$. Since $\tilde{\pi}_1$ can be described explicitly we use the Leray spectral sequences for $H^s\big(Y_{X_i}, \tilde{\nu}^*\Omega^1_{M_{X_i}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\big)$ and $H^s\big(Y_{X_i}, \Omega^1_{\tilde{\mu}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\big)$ to determine the map $\widetilde{\mathcal{D}}_1$ from $H^0\big(M_{X_i}, R^s_{\tilde{\nu}}[\tilde{\nu}^\star\Omega^1_{M_{X_i}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})]\big)$ to $H^0\big(M_{X_i}, R^s_{\tilde{\nu}}\Omega^1_{\tilde{\mu}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\big)$. Then we will show that $\widetilde{\mathcal{D}}_1$ restricts to $\mathcal{D}_1$.

Using the homogeneous structure of $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ we see that the E_0-term of the Leray spectral sequence for $H^s\Big(Y_{X_i}, \tilde{\nu}^*\Omega^1_{M_{X_i}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\Big)$ is given by

$$\widetilde{E}^{p,q}_{0,M} = C^\infty\big(G_{\mathbb{C}}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes \wedge^p \mathfrak{d} \otimes F_2\big)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}}}$$

where $F_2 = (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$. Now $\widetilde{E}^{p,q}_{0,M}$ is isomorphic to

$$C^\infty\big(G_{\mathbb{C}}, C^\infty(K_{\mathbb{C}}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes F_2)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}}} \otimes \wedge^p \mathfrak{d}\big)^{K_{\mathbb{C}}}$$

by sending φ to $\tilde{\varphi}$ where $\tilde{\varphi}(g)(k) = \varphi(gk)$. Likewise the E_0–term of the Leray spectral sequence for $H^s\Big(Y_{X_i}, \Omega^1_{\tilde{\mu}}(\mu^*\mathbb{L}_{\tilde{\chi}})\Big)$ is given by

$$\widetilde{E}^{p,q}_{0,\mu} = C^\infty\big(G_{\mathbb{C}}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes \wedge^p \mathfrak{d} \otimes F_3\big)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}}}$$

where $F_3 = \big((\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}\big)^*$ and $\widetilde{E}^{p,q}_{0,\mu}$ is isomorphic to

$$C^\infty\big(G_{\mathbb{C}}, C^\infty(K_{\mathbb{C}}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes F_3)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}}} \otimes \wedge^p \mathfrak{d}\big)^{K_{\mathbb{C}}}.$$

Since the homogeneous structures of $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ and $\Omega^1_{\tilde{\mu}}$ are compatible, we can give an explicit realization of the map $\tilde{\pi}_{1,0} : \widetilde{E}^{p,q}_{0,M} \to \widetilde{E}^{p,q}_{0,\mu}$ induced by π_1. Let r be the map from F_2 to F_3 given by restriction. Then $(\tilde{\pi}_{1,0}(\varphi))\,(g)(k) = r(\varphi(g)(k))$.

For the E_1-terms we have that

$$\widetilde{E}^{p,q}_{1,M} = C^\infty\big(G_{\mathbb{C}}, H^q(K_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}, \mathcal{O}(\mathbb{F}_{2,\tilde{\chi}})) \otimes \wedge^p \mathfrak{d}\big)^{K_{\mathbb{C}}} \tag{5.16}$$

and

$$\widetilde{E}^{p,q}_{1,\mu} = C^\infty\big(G_{\mathbb{C}}, H^q(K_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}, \mathcal{O}(\mathbb{F}_{3,\tilde{\chi}})) \otimes \wedge^p \mathfrak{d}\big)^{K_{\mathbb{C}}} \tag{5.17}$$

where $\mathbb{F}_{j,\tilde{\chi}}$ is the homogeneous bundle on $K_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}$ with fiber $F_j \otimes \mathbb{C}_{\tilde{\chi}}$.

Since $K/H_i \cap K = K_{\mathbb{C}}/(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})$ we can identify the cohomology space in (5.16) with

$$H^q\big(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})\big) \tag{5.18}$$

and the cohomology space in (5.17) with

$$H^q\big(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})\big) \tag{5.19}$$

where $\mathbb{F}_{j,\chi}$ is the homogeneous bundle on $K/H_i \cap K$ with fiber $F_j \otimes \mathbb{C}_\chi$.

The map $\tilde{\pi}_{1,0}$ induces a map from $\tilde{E}^{p,q}_{1,M}$ to $\tilde{E}^{p,q}_{1,\mu}$ which is determined by the map from (5.18) to (5.19). To determine this map for $q = s$, we let $F_1 = (\mathfrak{g}_{\mathbb{C}}/(\mathfrak{h}_{i,\mathbb{C}} + \mathfrak{q}_{i,-} + \mathfrak{k}_{\mathbb{C}}))^*$.

Lemma 5.20.

$$0 \to F_1 \xrightarrow{\ j\ } F_2 \xrightarrow{\ r\ } F_3 \to 0 \tag{5.21}$$

is a short exact sequence where j is the natural inclusion map.

Proof. For vector spaces $W \subset V$ the dual space $(V/W)^*$ can be identified with the set of $\lambda \in V^*$ such that $\lambda|_W = 0$. Since F_3 can also be written as $\big((\mathfrak{h}_{i,\mathbb{C}} + \mathfrak{q}_{i,-} + \mathfrak{k}_{\mathbb{C}})/\mathfrak{k}_{\mathbb{C}}\big)^*$ the lemma follows. ☐

Now (5.21) induces a short exact sequence in cohomology.

Lemma 5.22.

$$0 \to H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{1,\chi})) \xrightarrow{j^*} H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) \xrightarrow{r^*} H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) \to 0$$

is an exact sequence for $r = n$ if $a < \frac{1}{2} - \frac{3}{2}n$ and for $r < n$ if $a < -3n + r$.

Proof. The short exact sequence (5.21) induces the following short exact sequence of sheaves

$$0 \to \mathcal{O}(\mathbb{F}_{1,\chi}) \xrightarrow{\tilde{j}} \mathcal{O}(\mathbb{F}_{2,\chi}) \xrightarrow{\tilde{r}} \mathcal{O}(\mathbb{F}_{3,\chi}) \to 0 \tag{5.23}$$

since j and r are equivariant for $H_i \cap K$ and $\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}}$. The sequence (5.23) induces a long exact sequence in cohomology. Since the dimension of $K/H_i \cap K$ is s, cohomology vanishes in degree greater than s. The space $H^{s-1}(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) = 0$ by Theorem 4.9 and the lemma follows. ☐

We will now determine r^* explicitly.

Lemma 5.24. *r^* is a linear projection map.*

Proof. Since K is compact, representations of K are semisimple so the short exact sequence splits. Thus, $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{1,\chi}))$ has a complement in $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ which must map isomorphically onto $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$. Thus, r^* is a linear projection map. ☐

In Appendix B, we decompose the K-representations $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ and determine r^* explicitly for the case when $r = n$.

The map from $\widetilde{E}^{p,s}_{1,M} \to \widetilde{E}^{p,s}_{1,\mu}$ is given by sending φ to $r^* \circ \varphi$ where $(r^* \circ \varphi)(g)(k) = r^*(\varphi(g)(k))$. Thus the map $\widetilde{\mathcal{D}}_1$ from $\widetilde{E}^{0,s}_{2,M} \to \widetilde{E}^{0,s}_{2,\mu}$ is the restriction of the map from $\widetilde{E}^{0,s}_{1,M} \to \widetilde{E}^{0,s}_{1,\mu}$ to holomorphic sections of the bundle $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $\widetilde{\mathcal{D}}_1$ is a $G_{\mathbb{C}}$–invariant zeroth-order differential operator. Since M_{D_i} is open in M_{X_i}, the map $\widetilde{\mathcal{D}}_1$ restricts to a differential operator on M_{D_i}. Now

$$R^s_{\tilde{\nu}}[\tilde{\nu}^* \Omega^1_{M_{X_i}}(\tilde{\mu}^* \mathbb{L}_{\tilde{\chi}})] \simeq R^s_{\nu}[\nu^* \Omega^1_{M_{D_i}}(\mu^* \mathbb{L}_{\chi})] \tag{5.25}$$

so $\widetilde{\mathcal{D}}_1$ restricted to M_{D_i} is the map from $H^0\Big(M_{D_i}, R^s_{\nu}[\nu^* \Omega^1_{M_{D_i}}(\mu^* \mathbb{L}_{\chi})]\Big)$ to $H^0\big(M_{D_i}, R^s_{\nu} \Omega^1_{\mu}(\mu^* \mathbb{L}_{\chi})\big)$ which is given by sending φ to $r^* \circ \varphi$. Since

$\tilde{\pi}_1|_{\nu^*\Omega^1_{M_{D_i}}} = \pi_1$ this map is $\mathcal{D}_1$. The G-equivariance of the map π_1 implies that $\mathcal{D}_1$ is also G-invariant. Thus we have proven the following theorem.

Theorem 5.26.

(1) *The map*

$$\mathcal{D}_1 : H^0\Big(M_{D_i}, R^s_\nu\big[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big]\Big) \to H^0\big(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$$

is a G-equivariant zeroth–order differential operator.

(2) *The map*

$$\mathcal{D} : H^0(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)) \to H^0\big(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$$

is given by $\mathcal{D} = \mathcal{D}_1 \circ \mathcal{D}_2$ and $\mathcal{D}$ is a G-equivariant first–order differential operator.

Appendix A.

The main body of the paper considers the double fibration transform for a family of representations of $Sp(n,\mathbb{R})$ which are realized in cohomology with values in a line bundle. The outline for constructing the double fibration transform, as given in Section 2, is valid if we replace the line bundle $\mathbb{L}$ with a finite dimensional vector bundle $\mathbb{V}$. In this appendix, we consider the details of the construction when the line bundle is replaced with a vector bundle.

Let F_λ be a finite-dimensional, irreducible representation of H_i with highest weight λ and $\mathbb{F}_\lambda$ the corresponding homogeneous vector bundle on D_i. When $r = n$ and $\lambda = (a_1, \cdots, a_i \mid a_{i+1}, \cdots, a_n)$, then λ is a highest weight if

$$a_i \geq \cdots \geq a_1 \geq -a_{i+1} \geq \cdots \geq -a_n. \tag{A.1}$$

When $r < n$ and $\lambda = (a_1, \cdots, a_i \mid a_{i+1}, \cdots, a_{i+n-r} \mid a_{i+n-r+1}, \cdots, a_n)$, then λ is a highest weight if

$$a_i \geq \cdots \geq a_1 \geq -a_{i+n-r+1} \geq \cdots \geq -a_n \tag{A.2}$$

and

$$a_{i+1} \geq \cdots \geq a_{i+n-r} \geq 0. \tag{A.3}$$

The representation $H^s(D_i, \mathcal{O}(\mathbb{F}_\lambda))$ is infinite-dimensional, non-zero, and irreducible [**Wg**] under the following circumstances: When $r = n$, in addition to A.1, we require that $-a_n > n$ and when $r < n$, in addition to A.2 and A.3, we require that $-a_n > n$ and $a_{i+1} + a_n < -2n + r$. Unlike the line bundle case, these representations are not unitarizable.

Now we consider the construction of the double fibration transform. The first step, using Buchdahl's theorem [**Bu**] to identify $H^s(D_i, \mathcal{O}(\mathbb{F}_\lambda))$ with

$H^s\big(Y_{D_i}, \mu^{-1}\mathcal{O}(\mathbb{F}_\lambda)\big)$, remains valid because Buchdahl's theorem, which applies to vector bundles, requires only that the fiber of μ be contractible, which we already have.

The second step, embedding $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{O}(\mathbb{F}_\lambda)\big)$ in $H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{F}_\lambda))$, is more complicated. As in Theorem 4.9, this requires a condition on λ which guarantees that $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q \otimes \mathbb{F}_\lambda))$ vanish for all $p < s$ and all $1 \leq q \leq m$. Recall that $\mathbb{V}^q$ is the bundle $\wedge^q(\ker d_\mu)^*$ whose fiber is $V^q = \wedge^q[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}]$. When we consider $V^q \otimes F_\lambda$ as an $K/H_i \cap K$ representation, we see the main difference from the line bundle case. In the line bundle case, when the representation $\mathbb{C}_\chi$ is restricted to $H_i \cap K$, it remains irreducible. This allows us to know explicitly the form of the highest weights of the irreducible components of $V^q \otimes \mathbb{C}_\chi$ (see the discussion after Theorem 4.9) and to compute a specific condition on χ to guarantee vanishing.

Such is not the case for vector bundles. The representation F_λ, when restricted to $H_i \cap K$, may be reducible. If we decompose F_λ as $F_\lambda = \bigoplus_{j=1}^{k} F_{\lambda_j}$ with F_{λ_j} an irreducible representation of $H_i \cap K$ with highest weight λ_j, we can say something about the λ_j's. Since $H_i \cap K = U(i) \times U(r-i) \times Sp(n-r, \mathbb{R})$ is reductive, the highest weight λ_j splits into two pieces: A highest weight λ'_j for the semisimple piece and a character χ_j on the center. Similarly, λ itself is of the form $\lambda = \chi + \lambda'$ when λ is a highest weight of $H_i = U(i, r-i) \times Sp(n-r, \mathbb{R})$. Since the one-dimensional representation remains irreducible under restriction, we have that $\chi_j = \chi$ for all j where $\chi = (-a, \cdots, -a \mid 0, \cdots, 0 \mid a, \cdots, a)$. So, each λ_j is of the form $\chi + \lambda'_j$. Now we replace χ with $\chi + \lambda'_j$ in the proof of Theorem 4.9. Then let

$$C = \max_j \left\{ \left\langle \lambda'_j, e_n - e_1 \right\rangle \right\}$$

and

$$D = \max_{j,t} \left\{ \left\langle \lambda'_j, \alpha_t \right\rangle \right\}$$

with $\alpha_1 = e_{i+1} - e_1, \alpha_2 = e_n - e_i, \alpha_3 = e_n - e_{i+n-r}$. Then the vanishing condition holds for $r = n$ when $a < -3n + 1 - C$ and for $r < n$ when $a < -3n + r - D$.

The third step, pushing $H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{F}_\lambda))$ down to $H^0(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{F}_\lambda))$, is unaffected by changing from a line bundle to a vector bundle.

Likewise, the differential operator is not affected by changing from a line bundle to a vector bundle. Although, as in the line bundle case, when $r < n$, it is difficult to decompose the representations in Lemma 5.22 to give an explicit description of the projection operator r^* as was done when $r = n$ in Appendix B.

Appendix B.

For the case when $r = n$, we will decompose the spaces $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ and determine the map $r^* : H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) \to H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ in Lemma 5.22 explicitly. Since K is compact, each of $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ can be decomposed into a direct sum of irreducible K-representations.

First, we decompose $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$. We cannot apply Bott-Borel-Weil directly to $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ since $\big((\mathfrak{h}_\mathbb{C} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_\mathbb{C} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_\mathbb{C}\big)^*$ $= F_3$ is not an irreducible $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C})$-representation. Although we can decompose F_3 into a direct sum of irreducible $(H_i \cap K)$-representations, in order to decompose $\mathcal{O}(\mathbb{F}_{3,\chi})$ accordingly the decomposition of V must also be as $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C})$-modules (see [**TW**]). If we use the killing form to identify F_3 with $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}$, then on $\mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}$ the action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ is trivial and on $\mathfrak{q}_{i,+} \cap \mathfrak{p}$,the action is by ad. Since we cannot find a decomposition of F_3 which respects the action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$, we will use a composition series for F_3 to determine $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ as indicated in the following theorem.

Theorem B.1. *Let V be a representation of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ and let $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = V$ be a composition series for V. Let W_j denote V_j/V_{j-1} and let $\mathbb{V}$ and $\mathbb{W}_j$ be the associated homogeneous vector bundles on $K/H_i \cap K$. Then there exists a spectral sequence with $E_1^{p,q} = H^{p+q}\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_{N-p})\big)$ which abuts to $H^*\big(K/H_i \cap K, \mathcal{O}(\mathbb{V})\big)$.*

Proof. Since the representations are stable under the action of the antiholomorphic tangent space $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$, the filtration of V induces a filtration in the Dolbeault complex. By the proposition on page 440 of [**GH**] it follows that there exists a spectral sequence with $\mathrm{E}_1^{p,q} = H^{p+q}\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_{N-p})\big)$ which abuts to $H^*\big(K/H_i \cap K, \mathcal{O}(\mathbb{V})\big)$. □

Corollary B.2. *If $H^p\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j)\big) = 0$ for all $p \neq p_0$ and all j, then $H^{p_0}\big(K/H_i \cap K, \mathcal{O}(\mathbb{V})\big) = \sum_{j=1}^{N} H^{p_0}\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j)\big)$.*

Proof. The spectral sequence collapses in Theorem B.1 giving the conclusion. □

Once we find an appropriate decomposition series of F_3 we can use Bott-Borel-Weil to determine when $H^p\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j \otimes \mathbb{L}_\chi)\big)$ vanishes for all j and for all $p \neq s$. Choose the following elements for the composition series:

$$F_3 = V_4 = (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p} \qquad V_2 = (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}_+$$
$$V_3 = [(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}_+] \oplus (\mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_-) \quad V_1 = \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_+.$$

Then each V_j is a representation for $H_i \cap K$ and $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ where the action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ on V_j is the restriction of its action on V. Let $W_j = V_j/V_{j-1}$.

Then the successive quotients are

$$W_4 \simeq \mathfrak{q}_{i,+} \cap \mathfrak{p}_- \qquad W_2 \simeq \mathfrak{q}_{i,+} \cap \mathfrak{p}_+$$
$$W_3 \simeq \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_- \qquad W_1 \simeq \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_+.$$

Each W_j is an irreducible $(H_i \cap K)$–representation (since each is the realization of the holomorphic or anti-holomorphic tangent space of some symmetric space) and each W_j is a $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}})$–representation. The induced action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}}$ on W_j is trivial. Let λ_j denote the highest weight of W_j. Then

$$\lambda_4 = -2e_1 \qquad \lambda_2 = 2e_n$$
$$\lambda_3 = -e_1 - e_{i+1} \qquad \lambda_1 = e_i + e_n.$$

Lemma B.3. *If $a < -\frac{1}{2}(n+1)$ then $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j \otimes \mathbb{L}_\chi)) = 0$ for all j whenever $p < s$ and whenever $p = s$ it is an irreducible K-representation with highest weight $\xi + \lambda'_j$. Here $\xi = (a+i, \ldots, a+i \; ; \; -a-n+i, \ldots, -a-n+i)$ with $(n-i)$ entries before the semicolon and*

$$\lambda'_4 = -2e_{n-i+1} \qquad \lambda'_2 = 2e_{n-i}$$
$$\lambda'_3 = -e_1 - e_{n-i+1} \qquad \lambda'_1 = e_{n-i} + e_n.$$

The proof is an application of Bott-Borel-Weil.

Thus we have proven the following theorem.

Theorem B.4. *If $a < -\frac{1}{2}(n+1)$, then*

$$H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) = \bigoplus_{j=1}^{4} E_{\tau_j}$$

where E_{τ_j} is the irreducible K-representation with highest weight $\tau_j = \xi + \lambda'_j$ where ξ and λ'_j are given in Lemma B.3.

Now we will decompose $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$.

Theorem B.5. *If $a < -\frac{1}{2}(n+1)$, then*

$$H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) = \bigoplus_{j=1}^{6} E_{\tau_j}$$

where E_{τ_j} is the irreducible K–representation of highest weight $\tau_j = \xi + \lambda'_j$. Here ξ and λ'_j are given in Lemma B.3 *for $j = 1, \ldots, 4$. Let $\lambda'_5 = 2e_n$ and $\lambda'_6 = -2e_1$.*

Proof. First we identify $F_2 = (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$ with $\mathfrak{p}$. The decomposition $\mathfrak{p}_+ \oplus \mathfrak{p}_-$ of $\mathfrak{p}$ respects the action of $H_i \cap K$ and $\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}}$. Thus $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ decomposes into the direct sum of the K-representations

$$H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i \cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_+)\big]\big) \qquad \text{and}$$

$$H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i \cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_-)\big]\big).$$

Since $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are indecomposable as $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}})$-representations, the cohomology spaces will be computed using a composition series.

Now $U_3 = \mathfrak{p}_+, U_2 = \mathfrak{p}_+ \cap (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})$ and $U_1 = \mathfrak{p}_+ \cap \mathfrak{q}_{i,-}$ is a composition series for $\mathfrak{p}_+$ and $Z_3 = \mathfrak{p}_-, Z_2 = \mathfrak{p}_- \cap (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})$ and $Z_1 = \mathfrak{p}_- \cap \mathfrak{q}_{i,-}$ is a composition series for $\mathfrak{p}_-$. Lemma B.3 implies that

$$H^p\big(K/H_i \cap K, \mathcal{O}(K \times_{H_i \cap K} (\mathbb{C}_\chi \otimes W))\big) = 0 \tag{B.6}$$

for $p < s$ where $W = U_3/U_2, U_2/U_1, Z_3/Z_2$, and Z_2/Z_1. We will determine the condition necessary for (B.6) to hold when W is U_1 or Z_1.

Let $\lambda_5 = 2e_i$ and $\lambda_6 = -2e_{i+1}$. Then λ_5 (respectively λ_6) is the highest weight of the irreducible K-representation U_1 (respectively Z_1). As in the proof of Lemma B.3, to show (B.6) it suffices to show that $\langle \chi + p_k + \lambda_j, e_n - e_1 \rangle < 0$ for $j = 5, 6$. Since $\langle \chi + p_k + \lambda_j, e_n - e_1 \rangle = 2a + n - 1$ we see that (B.6) is true when $a < -\frac{1}{2}(n+1)$. Thus Theorem B.1 and Corollary B.2 together imply that $H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i \cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_+)\big]\big) = E_{\tau_1} \oplus E_{\tau_2} \oplus E_{\tau_5}$ and that $H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i \cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_-)\big]\big) = E_{\tau_3} \oplus E_{\tau_4} \oplus E_{\tau_6}$ □

We will now determine r^* explicitly.

Lemma B.7. *r^* is a linear projection map.*

Proof. The map r^* is onto by Lemma 5.22. Since $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) = \bigoplus_{j=1}^{4} E_{\tau_j}$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) = \bigoplus_{j=1}^{6} E_{\tau_j}$ and each E_{τ_j} is an irreducible K-representation, r^* is the natural projection map from $\bigoplus_{j=1}^{6} E_{\tau_j}$ to $\bigoplus_{j=1}^{4} E_{\tau_j}$. □

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REAL COBOUNDARIES FOR MINIMAL CANTOR SYSTEMS

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In this paper we investigate the role of real-valued coboundaries for classifying of minimal homeomorphisms of the Cantor set. This work follows the work of Giordano, Putnam, and Skau who showed that one can use integer-valued coboundaries to characterize minimal homeomorphisms up to strong orbit equivalence. First, we prove a rigidity result. We show that there is an orbit equivalence between minimal Cantor systems which preserves real-valued coboundaries if and only if the systems are flip conjugate. Second, we investigate a real analogue of the dynamical unital ordered cohomology group studied by Giordano, Putnam and Skau. We show that, in general, isomorphism of our unital ordered vector space determines a weaker relation than strong orbit equivalence and we characterize this relation in a certain finite dimensional case. Finally, we consider isomorphisms of this vector space which preserve the cohomology subgroup. We show that such an isomorphism gives rise to a strictly stronger relation than strong orbit equivalence. In particular, it determines topological discrete spectrum, but does not determine systems up to flip conjugacy.

1. Introduction.

In [**GPS95**], Giordano, Putnam and Skau used C^*-algebraic invariants to characterize minimal homeomorphisms of the Cantor set up to various notions of orbit equivalence. For a minimal homeomorphism $T : X \to X$ of the Cantor set X, their key invariant reduces to the group of continuous integer-valued functions $f : X \to \mathbb{Z}$ modulo the *coboundaries* (functions of the form $f - f \circ T$), along with a positive cone and order unit. In this paper, we examine real-valued coboundaries and look at analogues of their results from three perspectives.

Let S and T be minimal homeomorphisms of the Cantor set. In the main result of Section 2 (Theorem 2.10) we prove that if S and T are orbit equivalent by a homeomorphism which maps the set of real S-coboundaries bijectively onto the set of real T-coboundaries then S is conjugate to T or

T^{-1} (S and T are *flip conjugate*). In fact, we show that any homeomorphism from the Cantor set to itself which identifies real coboundaries of S and T must be an orbit equivalence with a bounded jump function (Theorem 2.11). In contrast, Giordano, Putnam and Skau's work shows that an orbit equivalence induces a bijection between the sets of integer-valued coboundaries if and only if S and T are strongly orbit equivalent. Results in [**BH94**, **Orm97**, **Sug**, **Sug98**] underscore the vast difference between strong orbit equivalence and flip conjugacy for this class of systems. Moreover, an example of Boyle shows that a homeomorphism identifying integer coboundaries need not be a strong orbit equivalence. In appendix A, we present this unpublished example of Boyle in which S and T have the same integer coboundaries, and have the property that $T(x)$ and $T(S^n x)$ are not in the same S-orbit for all x and all $n \neq 0$.

In Section 3, we define and investigate the natural analogue of Giordano, Putnam and Skau's unital ordered group: The vector space of continuous real-valued functions modulo the real coboundaries along with a positive cone and order unit. We show (Theorem 3.10) if the cardinality of the set of ergodic invariant Borel probabilities is finite then this cardinal completely determines our unital ordered vector space $\mathcal{G}_{\mathbb{R}}(T)$. Using a result of Dougherty, Jackson, and Kechris, we see that when the set of ergodic T-invariant Borel probabilities is finite, our unital ordered vector space characterizes Borel orbit equivalence.

In Section 4, we study the dynamical properties which are determined if we consider only isomorphisms of the real unital ordered vector space $\mathcal{G}_{\mathbb{R}}(T)$ which preserve the subgroups of integer-valued functions $\mathcal{G}_{\mathbb{Z}}(T)$. We present results which show that there is some more dynamical information in the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ than in $\mathcal{G}_{\mathbb{Z}}(T)$ alone but not enough to determine T up to flip conjugacy. For example, we show that the isomorphism of the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ determines the topological discrete spectrum of T (Theorem 4.4). The unital ordered group $\mathcal{G}_{\mathbb{Z}}(T)$ already determines the rational discrete spectrum, but does not, in general, determine the irrational spectrum (see [**Orm97**]). We show (Theorem 4.6) that for a minimal Cantor system (X, T) with $G_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ carries no more dynamical information than the unital ordered group $\mathcal{G}_{\mathbb{Z}}(T)$ alone. This shows that one cannot determine flip conjugacy using $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. Taking the previous two results together, we obtain a new result (Corollary 4.7) about minimal Cantor systems and the unital ordered group $\mathcal{G}_{\mathbb{Z}}(T)$. Namely, if $G_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ then T cannot have irrational spectrum.

I thank Mike Boyle for his helpful comments and for allowing me to include his example Appendix A. I thank Bernard Host for allowing me to include his proof of Theorem 2.6.

2. The Same Set of Real Coboundaries.

Throughout this paper we will consider topological dynamical systems (X, T) where $T : X \to X$ is a homeomorphism of X a compact metric space. In particular, we will consider *minimal Cantor systems*. A homeomorphism $T : X \to X$ is called *minimal* if for all $x \in X$ the T-orbit of x, $\{T^n x \ : \ n \in \mathbb{Z}\}$, is dense. We will call the pair (X, T) a *minimal Cantor system* if X is a Cantor set and $T : X \to X$ is minimal. The main properties of minimality that we will make use of are the following: There are no periodic points in a minimal system and for any open set $U \subseteq X$, there is an integer r such that for all $x \in X$, one of $\{x, T(x), \dots, T^r(x)\}$ is in U. Minimal Cantor systems include the odometer systems below.

Example (Odometer systems)**.** Let $\{d_i\}$ be an infinite sequence of positive integers. Let X be the space of infinite sequences $x = x_1x_2x_3\dots$ such that $0 \le x_i < d_i$ for all i. We put the discrete topology on the sets $\{0, 1, \dots, d_i - 1\}$ and the infinite product of this discrete topology on X. In this way, X becomes a Cantor set. The topology on X is equivalent to the one generated by the metric d where $d(x, y) = 2^{-n}$ if $x_i = y_i$ for all $0 \le i \le n$ and $x_{n+1} \ne y_{n+1}$.

Define $T : X \to X$ by adding one with right carry. In other words, for $x \in X$, let n be the smallest positive integer such that $x_n < (d_n - 1)$. If such an n exists, define $T(x)$ to be the sequence $[T(x)]_i = 0$ for $i < n$, $[T(x)]_n = x_n + 1$ and $[T(x)]_i = x_i$ for $i > n$. If $x_n = (d_n - 1)$ for all n, define $T(x)$ to be the sequence $[T(x)]_n = 0$ for all n. The dynamical system (X, T) is minimal since the T-orbit of every point sees all the words of length n in the first n coordinates. The odometer system where $d_i = 2$ for all i is called the *dyadic adding machine*.

Let (X, S) and (Y, T) be minimal Cantor systems. The following are some of the different equivalences we will consider. Of course, the notions make sense for more general topological dynamical systems.

Definition 2.1 (conjugacy)**.** We say (X, S) and (Y,T) are conjugate if there is a homeomorphism $h : X \to Y$ such that $\forall x \in X, hS(x) = Th(x)$.

Definition 2.2 (flip conjugacy)**.** We say (X, S) and (Y, T) are flip conjugate if S is conjugate to T or S is conjugate to T^{-1}.

Definition 2.3 (orbit equivalence)**.** We say (X, S) and (Y, T) are orbit equivalent if there is a homeomorphism $h : X \to Y$ and functions $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ such that

$$\forall x \in X, \ hS(x) = T^{m(x)}h(x) \text{ and } hS^{n(x)}(x) = Th(x).$$

In other words, (X, S) and (Y, T) are conjugate to systems (Z, S') and (Z, T') where

$$\forall x \in Z, \ \{(S')^n(x) \ : \ n \in \mathbb{Z}\} = \{(T')^n(x) \ : \ n \in \mathbb{Z}\}.$$

The theory of orbit equivalence has a long history in the study of measure-theoretic dynamical systems [**KR95**, **KW91**, **Kri69**, **Kri76**, **Rud85**]. It was this work which motivated the study of orbit equivalence in topological systems.

As it turns out, for a given topological orbit equivalence, the continuity properties of the "jump functions" $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ can give information about the extent to which one system is determined by the other. In particular, for minimal Cantor systems Boyle [**Boy83**] proved that the jump functions are bounded if and only if they are continuous, and gave the following characterization (generalized in [**BT98**]) of orbit equivalence with a bounded jump functions.

Theorem 2.4 (Boyle)**.** *Suppose (X, S) and (X, T) are minimal Cantor systems with the same orbits. If there is a bounded function $m : X \to \mathbb{Z}$ such that $S(x) = T^{m(x)}(x)$ for all x then S and T are flip conjugate.*

In [**GPS95**], Giordano, Putnam and Skau used C^*-algebraic invariants to characterize orbit equivalence for minimal Cantor systems, and to give information about the continuity/boundedness properties of the associated jump functions one can achieve. One important notion from their work is the notion of strong orbit equivalence.

Definition 2.5 (strong orbit equivalence)**.** Two minimal Cantor systems (X, S) and (Y, T) are strongly orbit equivalent if they are orbit equivalent by a map $h : X \to Y$ with jump functions $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ such that m and n have at most one point of discontinuity each.

We will say more about strong orbit equivalence in Section 3. For now, we simply point out that strong orbit equivalence is a much weaker relation than flip conjugacy. For example, strongly orbit equivalent systems can have arbitrarily large topological entropy differences and when attached with an ergodic invariant measure, can give rise to vastly different measurable structures (see [**BH94**, **Orm97**, **Sug**, **Sug98**]).

For a minimal Cantor systems (X, T), Giordano, Putnam and Skau's characterization up to orbit equivalence relies upon looking at integer-valued continuous functions of the form $f - fT$ (from here on we use fT to denote $f \circ T$). We will call a continuous function $f : X \to \mathbb{R}$ a *real T-coboundary* if there exists a continuous function $g : X \to \mathbb{R}$ such that $f(x) = g(x) - g(Tx)$ for all $x \in X$. Similarly, we will call a function $f : X \to \mathbb{Z}$ an *integer T-coboundary* if there is a continuous $g : X \to \mathbb{Z}$ such that $f(x) = g(x) - g(Tx)$ for all $x \in X$. The following characterization of coboundaries is well known. With kind permission, we present Bernard Host's proof of this result [**Hos**].

Theorem 2.6. *Let (X, T) be a Cantor minimal system. A continuous function $f : X \to \mathbb{R}$ is a real T-coboundary if and only if sums of the form $\sum_{i=0}^{n} f(T^i x)$ are uniformly bounded over $n \geq 1$ and $x \in X$.*

Proof. If $f = g - gT$ for some g then $\sum_{i=0}^{n} f(T^i x) = g(x) - g(T^n x)$ thus sums of this form are uniformly bounded.

For the other direction, let (X, T) be a minimal system, and f a continuous real-valued function on X. Define

$$f^{(n)}(x) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum_{i=-n}^{-1} f(T^i x) & \text{if } n < 0. \end{cases}$$

Assume that C is a constant such that

$$\forall x \in X, \forall n \in \mathbb{N},\ |f^{(n)}(x)| \leq C.$$

As for all $x \in X$ we have $f^{(0)}(x) = 0$ and $f^{(-n)}(x) = -f^{(n)}T^{-n}(x)$ for $n > 0$ we get:

$$\forall x \in X, \forall n \in \mathbb{Z},\ f^{(n)}(x) \leq C.$$

We write:

$$F(x) = \sup_{n \in \mathbb{Z}} f^{(n)}(x);\ \operatorname{Osc}(x) = \limsup_{y \to x} F(y) - \liminf_{y \to x} F(y).$$

For every $x \in X$ we have $f(x) = F(x) - F(Tx)$ and for all n, we have $f^{(n)}(x) = F(x) - F(Tx)$. We have only to prove that the function $F(x)$ is continuous, i.e., that the function $\operatorname{Osc}(x)$ is identically 0.

We choose some $\epsilon > 0$ and define:

$$K = \{x \in X\ :\ F(x) \leq \epsilon\}.$$

By construction, for $x \in X$ there is an $n \in \mathbb{Z}$ with

$$T^n x \in K \iff f^{(n)}(x) \geq F(x) - \epsilon.$$

Thus, by definition of $F(x)$, for every $x \in X$ there exists $n \in \mathbb{Z}$ with $T^n x \in K$, and

$$\bigcup_{n \in \mathbb{Z}} T^{-n} K = X.$$

But K is closed. Thus, by Baire's Theorem, the interior U of K is not empty and, by minimality,

$$\bigcup_{n \in \mathbb{Z}} T^{-n} U = X.$$

For $x \in T^{-n}U$ we have $f^{(n)}(x) \leq F(x) \leq f^{(n)}(x) + \epsilon$ thus, by continuity of $f^{(n)}$, $\operatorname{Osc}(x) \leq \epsilon$.

Therefore, $\operatorname{Osc}(x) \leq \epsilon$ for all $x \in X$. □

We will look more closely at Giordano, Putnam and Skau's results in Sections 3 and 4. For the remainder of this section, however, we concentrate on one aspect of their results.

Theorem 2.7 (Giordano, Putnam, Skau)**.** *Suppose (X, S) and (Y, T) are minimal Cantor systems. There is an orbit equivalence $h : X \to Y$ which induces a bijection from the set of integer S-coboundaries to the set of integer T-coboundaries if and only if S and T are strongly orbit equivalent.*

In Theorem 2.10, we prove an analogous result, there is an orbit equivalence which respects real coboundaries if and only if S and T are flip conjugate. To prove the difficult direction of Theorem 2.10, we first note the following.

Lemma 2.8. *Let (X, S) and (X, T) be minimal Cantor systems with the same orbits. For all n, let $E_n = \{x : S(x) = T^n(x)\}$. Then one of the following holds.*

1. *$X = \bigcup_{|n| \le N} E_n$ for some N,*
2. *there is an infinite sequence of sets E_{n_k} with $|n_k| < |n_{k+1}|$ such that E_{n_k} contains a clopen set for all k.*

Proof. By the continuity of S and T, the set E_n is closed for all n. Let F_N denote the closed set $F_N = \bigcup_{|n| \le N} E_n$.

Fix N and assume that $X - F_N$ is nonempty. Then the sets E_n where $|n| > N$ form a countable closed cover of this open set. By the the Baire Category Theorem, one element of this cover must contain an open set, and therefore a clopen set. If condition 1 does not hold then we obtain a sequence of sets as in condition 2. □

From Boyle's result (Theorem 2.4) case 1 implies that S and T are flip conjugate. So to prove Theorem 2.10, it will suffice to show that in case 2, S and T do not have the same real coboundaries. For the remainder of the section, let (X, S) and (X, T) be minimal Cantor systems with the same orbits, and let $E_n = \{x : S(x) = T^n(x)\}$ for all $n \neq 0$.

Suppose there is an infinite sequence of sets E_{n_k} with $|n_k| < |n_{k+1}|$ such that E_{n_k} contains a clopen set for all k. Then by passing to a monotone increasing or decreasing subsequence of n_k's and possibly exchanging T for T^{-1} we have the hypothesis of the following.

Lemma 2.9. *Suppose there exists an infinite increasing sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ and nonempty clopen sets C_k such that $S(x) = T^{n_k}(x)$ for all $x \in C_k$. Then there exists a continuous real-valued S-coboundary which is not a continuous real-valued T-coboundary.*

Proof. After passing to a subsequence of the C_k, we will define f as

$$f(x) = \sum_{k=1}^{\infty} (1/k) 1_{U_k}(x)$$

where for all k, U_k is a clopen subset of C_k, and 1_{U_k} is the indicator function of U_k. The function $f(x)$ will be continuous as long as there is a point

$x_0 \notin \bigcup_{k\in\mathbb{N}} U_k$ such that if x_k is a sequence of points with $x_k \in U_k$ then $\lim_{k\to\infty} x_k = x_0$.

To get that $f - fS^{-1}$ is not a T-coboundary, we will choose the U_k such that for $x \in U_n$ the T-orbit of x enters each of the sets $U_n, U_{n-1}, \dots, U_1$ at least once before it enters any set of the form SU_k for $k \in \mathbb{N}$. In this case, for all $x \in U_n$ we will be able to find an integer m such that

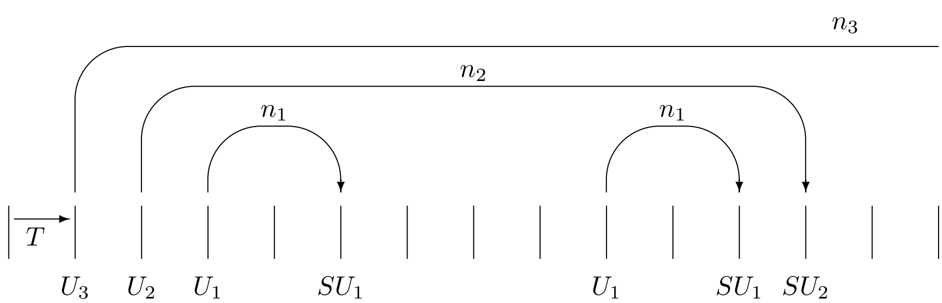
$$\begin{aligned}\sum_{i=0}^{m}\left(f(T^ix) - fS^{-1}(T^ix)\right) &= \sum_{i=0}^{m}\sum_{k\in\mathbb{N}}(1/k)\left(1_{U_k}(T^ix) - 1_{SU_k}(T^ix)\right)\\ &= \sum_{i=0}^{m}\sum_{k\in\mathbb{N}}(1/k)1_{U_k}(T^ix)\\ &\geq \sum_{k=1}^{n}(1/k)\\ &> \log(n).\end{aligned}$$

If the function $f - fS^{-1}$ were a T-coboundary then by Theorem 2.6 there would be a uniform bound on the functions $\sum_{i=0}^{m}(f - fS^{-1})T^i$.

To construct the function $f(x)$ it suffices to construct a sequence of *leap-frogging* sets $\{U_k\}$. Let $B_n(x)$ denote the T-orbit block $B_n(x) = \{x, T(x), \dots, T^n(x)\}$. We will call a sequence of pairwise disjoint sets $\{U_k\}_{k\in\mathbb{N}}$ leap-frogging for the pair (S, T) if

1) there exists an increasing sequence of integers $\{n_k\}$ such that $S(x) = T^{n_k}(x)$ for all $x \in U_k$,
2) for all $x \in U_j$ and $y \in U_k$ with $1 \leq j \leq k$, the set $B_{n_j}(x) \cap B_{n_k}(y)$ is either empty or equal to $B_{n_j}(x)$,
3) for all $x \in U_k$ the set $B_{n_k}(x) \cap U_{k-1}$ is nonempty.

We call the sets leap-frogging because we imagine the T-orbit of a point laid out along a number line. If a point is in U_k, the S-image of that point leaps forward in the T-orbit.



Condition 3 ensures that for $j < k$, the S-image of a point $x \in U_k$ leap-frogs at least one orbit block $B_{n_j}(y)$ where $y \in U_j$. For $x \in U_n$ let m be the smallest integer such that $x \in U_1$. This condition will give us $\sum_{i=0}^{m} \sum_{k\in\mathbb{N}} (1/k)1_{U_k}(T^i x)$ is at least $\sum_{k=1}^{n}(1/k)$.

We can think of condition 2 as ensuring that the S-jumps are nested. In other words, for $j \le k$ if $x \in U_k$ and $y \in U_j$ then $S(x) \notin B_{n_j}(y)$ unless $x = y$. Conditions 2 and 3 imply: If $x \in U_n$ and m is the smallest integer such that $T^m x \in U_1$ then $\sum_{i=0}^{m} \sum_{k\in\mathbb{N}} (1/k)1_{SU_k}(T^i x) = 0$.

We will construct the leap-frogging sets $U_k \subseteq C_k$ recursively. Before we do so, we will pick a special point x_0 with the property that for large k the set U_k lies within a small clopen subset of x_0. Since X is compact, after passing to a subsequence of the C_k we may assume that there is a sequence of points $\{x_k\}$ with $x_k \in C_k$ such that the limit $\lim_{k\to\infty} x_k$ exists. Let x_0 be the limit of this subsequence. We can replace the C_k with clopen neighborhoods of the x_k's of decreasing diameter. In this way, we may assume that our sets C_k have the property that if $y_k \in C_k$, then $\lim_{k\to\infty} y_k = x_0$. Moreover, we may assume that $x_0 \notin \bigcup_{k\in\mathbb{N}} C_k$.

To construct U_1, pick $y_1 \in C_1$ such that neither x_0 nor Sx_0 are in the T-orbit block $B_{n_1}(y_1)$. Let U_1 be a clopen neighborhood of y_1 such that $x_0, Sx_0 \notin \bigcup_{i=0}^{n_1} T^i U_1$ and $U_1, TU_1, \dots, T^{n_1}U_1$ are pairwise disjoint. Since the $T^j U_1$ are pairwise disjoint, if x and y are distinct points in U_1 then the intersection of the T-orbit blocks $B_{n_1}(x) \cap B_{n_1}(y)$ is empty.

Now assume that we have sets $U_1, U_2, \dots, U_k$ satisfying the leap-frogging conditions such that neither x_0 nor Sx_0 are in $\bigcup_{i=0}^{n_k} T^i U_k$. We can find a clopen neighborhood V of x_0 such that $V \cap \bigcup_{i=0}^{n_k} T^i U_k$, $SV \cap \bigcup_{i=0}^{n_k} T^i U_k$ and $V \cap SV$ are all empty.

By the minimality of T there is an integer r_k such that for any $x \in X$ the set $B_{r_k}(x) \cap U_k$ is nonempty. By passing to a subsequence of the C_k, we may assume $n_{k+1} > r_k$ and $C_{k+1} \subseteq V$. Choose $y_{k+1} \in C_{k+1}$. Pick a clopen neighborhood U_{k+1} of y_{k+1} of diameter less than $1/k$ such that $U_{k+1} \subseteq C_{k+1}$ and $U_{k+1}, TU_{k+1}, \dots, T^{n_{k+1}}U_{k+1}$ are pairwise disjoint.

Since $U_{k+1} \subseteq C_{k+1}$, we have that for all $x \in U_k$, $S(x) = T^{n_k}(x)$ (condition 1). Since $n_{k+1} > r_k$, for all $x \in U_{k+1}$ the set $B_{n_{k+1}}(x) \cap U_k$ is nonempty for all $x \in U_{k+1}$ (condition 3). Since $U_{k+1} \subseteq V$ and $SU_{k+1} \subseteq SV$ we have $U_{k+1} \cap \bigcup_{i=0}^{n_k} T^i U_k$, $SU_{k+1} \cap \bigcup_{i=0}^{n_k} T^i U_k$ are both empty. This gives the nested property of the blocks (condition 2). Since neither x_0 nor Sx_0 are in U_{k+1} we can continue with the recursion. □

The previous two lemmas give us the following theorem.

Theorem 2.10. *Let (X, S) and (Y, T) be minimal Cantor systems. There is an orbit equivalence $h : X \to Y$ which induces a bijection from the set of real S-coboundaries to the set of real T-coboundaries if and only if S and T are flip conjugate.*

Proof. If S and T are not flip conjugate then by Theorem 2.4 and Lemmas 2.8 and 2.9 we can construct an S coboundary which is not a T-coboundary. For the other direction we simply need to see that a homeomorphism $R : X \to X$ and its inverse $R^{-1} : X \to X$ always have the same set of coboundaries. This follows as

$$f - fR^{-1} = (fR^{-1})R - (fR^{-1}).$$

□

Remark. The above is reminiscent of rigidity results of Boyle and Tomiyama [**BT98**, Theorem 3.6] and Giordano, Putnam and Skau [**GPS**]. In the case where S and T are minimal Cantor systems Boyle and Tomiyama show that if the C^*-algebras associated to S and T are related by an isomorphism which identifies the subalgebra of continuous functions, then the systems are flip conjugate. Giordano, Putnam and Skau showed that an algebraic isomorphism of the topological full group must be induced by a flip conjugacy.

Theorem 2.10 can be strengthened. We show below (Theorem 2.11) that we need not require that the homeomorphism which identifies real coboundaries be an orbit equivalence, it is automatic. The analogous statement for integer coboundaries is not true. An example of Boyle (see Appendix A) shows that it is possible for two minimal homeomorphisms S and T of the Cantor set to have the same set of integer coboundaries and have the property that if x and y are in the same S-orbit then Tx and Ty are not in the same S-orbit.

Let $C(X, \mathbb{R})$ denote the set of real-valued continuous functions on a Cantor set X.

Theorem 2.11. *Let (X, S) and (X, T) be minimal Cantor systems. Then (X, S) and (X, T) have the property that for all $f \in C(X, \mathbb{R})$ there exist $g_1, g_2 \in C(X, \mathbb{R})$ such that*

$$f - fT = g_1 - g_1 S$$
$$f - fS = g_2 - g_2 T$$

if and only if S and T have the same orbits and there is a bounded (continuous) function $m : X \to \mathbb{Z}$ such that $S(x) = T^{m(x)}(x)$ for all $x \in X$.

Proof. Let $E_n = \{x \ : \ S(x) = T^n(x)\}$ and $F = X - \cup_{n\in\mathbb{Z}} E_n$.

Suppose such a function $m : X \to \mathbb{Z}$ exits. Then F is empty and there exists an integer M such that E_n is empty for $|n| > M$. For $f \in C(X, \mathbb{R})$ we may write $f - fT = \sum_{n=-M}^{M} 1_{TE_n} f - (1_{TE_n} f)T$. If $x \in E_n$ then $(1_{TE_n} f)Tx = (1_{TE_n} f)S^n x$ and the above function is therefore an S-coboundary.

Suppose that no such function m exists. In other words, assume $X - \cup_{|n|\leq M} E_n$ is nonempty for all M. If infinitely many of the sets E_n have nonempty interior then by Lemma 2.9 there is a real-valued S-coboundary which is not a real-valued T-coboundary.

If $X - \cup_{|n|\le M} E_n$ is nonempty for all M and only finitely many of the sets E_n have nonempty interior then by the Baire Category Theorem, $\overline{F}$ contains an open set. It remains to show that S and T cannot have the same set of real coboundaries when $\overline{F}$ contains an open set.

We will construct an S-coboundary which is not a T-coboundary by selecting a nested sequence of clopen sets $U \supseteq U_1 \supseteq U_2 \supseteq \cdots$, a sequence of points $x_k \in U_k$, and an increasing sequence of integers n_k with the following properties.

1) $\sum_{i=0}^{n_k} 1_{U_k}(T^i(x_k)) \ge 2^k$,
2) $\sum_{i=0}^{n_k} 1_{SU_k}(T^i(x_k)) = 0$,
3) $\sum_{i=0}^{n_k} \big(1_{U_j}(T^i(x_k)) - 1_{SU_j}(T^i(x_k))\big) = 0$ for all $1 \le j < k$.

Assume such a collection of sets exists, and let $f = \sum_{k=1}^{\infty}(3/4)^k 1_{U_k}$. If in addition to the above, the diameters of the U_k's are going to zero then the function $f(x)$ will be continuous. Since the sets SU_k are nested, condition 2 implies that

$$\sum_{j=k}^{\infty}\sum_{i=0}^{n_k} 1_{SU_j}(T^i(x_k)) = 0.$$

Putting this fact together with conditions 1 and 3, we get

$$\begin{aligned}
&\sum_{i=0}^{n_k} f(T^i x_k) - f(S^{-1}(T^i x_k)) \\
&= \sum_{i=0}^{n_k}\sum_{j=1}^{\infty}(3/4)^j \left[1_{U_j}(T^i x_k) - 1_{SU_j}(T^i x_k)\right] \\
&= \sum_{i=0}^{n_k}\sum_{j=1}^{k-1}(3/4)^j \left[1_{U_j}(T^i x_k) - 1_{SU_j}(T^i x_k)\right] \\
&\qquad + \sum_{i=0}^{n_k}\sum_{j=k}^{\infty}(3/4)^j 1_{U_j}(T^i x_k) - \sum_{i=0}^{n_k}\sum_{j=k}^{\infty}(3/4)^j 1_{SU_j}(T^i x_k) \\
&\ge 2^k (3/4)^k \\
&= (3/2)^k.
\end{aligned}$$

Therefore, by Theorem 2.6, $f - fS^{-1}$ cannot be a T-coboundary. It remains then to construct the sets.

We first note that for any point $x \in F$ since $S(x)$ is not in the T-orbit of x, for any positive integers m, n there is a clopen neighborhood U of x such that $T^k U \cap SU = \emptyset$ for all $-m \le k \le n$. This implies that for any clopen set $V \subseteq \overline{F}$ and positive integers m, n, there exists a clopen $U \subseteq V$ such that $T^k U \cap SU = \emptyset$ for $-m \le k \le n$.

To construct U_1, we take any clopen set $V \subseteq \overline{F}$. Let x be in V and let n_1 be the smallest positive integer n such that $T^n(x) \in V$. We can choose a clopen neighborhood V_0 of x such that $T^k V_0 \cap SV_0 = \emptyset$ for $0 \le k \le n_1$. Let V_1 be a clopen subset of $T^{n_1} V_0 \subseteq V$ such that $T^k V_1 \cap SV_1 = \emptyset$ for $-n_1 \le k \le 0$. Now let $U_1 = T^{-n_1} V_1 \cup V_1$ and let x_1 be any point in $T^{-n_1} V_1$. We get $\sum_{i=0}^{n_1} 1_{U_1}(T^i x_1) = 2$, and $\sum_{i=0}^{n_1} 1_{SU_1}(T^i x_1) = 0$.

Now suppose that we have constructed sets $U \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k$, points $x_1, x_2, \ldots, x_k$, and integers $n_1, n_2, \ldots, n_k$ with the desired properties. Consider the function $f_k = \sum_{i=1}^{k} (3/4)^k 1_{U_i}$. If $f_k - f_k S^{-1}$ is not a T-coboundary, then we are done. Assume that $f_k - f_k S^{-1} = g - gT$ for some g. Since $f_k - f_k S^{-1}$ is a locally constant rational valued function, we may assume that g is as well.

Let $y_0 \in U_k \cap F$. By the minimality of T, we can choose an integer N such that $\sum_{i=0}^{N} 1_{U_k}(T^i y_0) > 2^{k+1}$ and $g(y_0) = g(T^{N+1} y_0)$. Choose a clopen neighborhood V around y_0 such that $f_k(T^i y) - f_k(S^{-1} T^i y) = f_k(T^i y_0) - f_k(S^{-1} T^i y_0)$ for all $y \in V$ and all $0 \le i \le N$. Then for all $y \in V$,

$$\sum_{i=0}^{N} \left(f_k(T^i y) - f_k(S^{-1} T^i y) \right) = g(y) - g(T^{N+1} y) = 0,$$

which will give condition 3.

Let M denote $\sum_{i=0}^{N} 1_{U_k}(T^i y_0)$ and let $0 = r_0 < r_1 < \cdots < r_M \le N$ be the integers such that $T^{r_j} V \subseteq U_k$. We know that since $y_0 \in F$ we can choose $V_0 \subseteq V$ such that $T^i V_0 \cap SV_0$ is empty for all $0 \le i \le N$. Since $T^{r_1} V_0 \subseteq U_k$, there is a clopen set $V_1 \subseteq T^{r_1} V_0$ such that $T^i V_1 \cap SV_1$ is empty for all $-r_1 \le i \le (N - r_1)$. Continuing, for all $0 \le j \le M$, we can obtain sets $V_j \subseteq T^{r_j - r_{j-1}} V_{j-1}$ such that $T^i V_j \cap SV_j$ is empty for all $-r_j \le i \le (N - r_j)$. Let U_{k+1} be the union over $0 \le j \le M$ of the sets $T^{-r_M + r_j} V_M$, and let x_{k+1} be any point in $T^{-r_M} V_M$. Then $\sum_{i=0}^{N} 1_{U_{k+1}}(T^i x_{k+1}) = M \ge 2^{k+1}$ and $\sum_{i=0}^{N} 1_{SU_{k+1}}(T^i x_{k+1}) = 0$, giving conditions 1 and 2. □

3. Real Ordered Group.

The notion of strong orbit equivalence emerged from the study of C^*-algebraic invariants for topological dynamical systems. For minimal homeomorphisms of the Cantor set, Herman, Putnam and Skau showed that these C^*-crossed products are classified by their K-theory [**HPS92**]. The K-theory for these C^*-algebras amounts to the group of continuous integer-valued functions on the Cantor set modulo the coboundaries along with a positive cone and order unit. Giordano, Putnam and Skau showed that this unital ordered group characterizes strong orbit equivalence for minimal Cantor systems (Theorem 3.3) [**GPS95**, Theorem 2.1].

In this section, we define and investigate the group of continuous real-valued functions modulo the real coboundaries. As in [**GPS95**], our group will be considered along with a natural positive cone and order unit. Unlike the integer case, our space has the structure of a vector space over the reals. With this, the classification problem essentially comes down to counting the dimension of subspaces. When the number of ergodic invariant Borel probabilities is finite, the span of these measures acts as the dual to this space modulo the infinitesimal subgroup (Lemma 3.9). In this finite dimensional case, we are able to show (Theorem 3.10) that the cardinality of the set of ergodic Borel probabilities completely classifies our unital ordered vector space. Interestingly, this leads us back to orbit equivalence. A result of Dougherty, Jackson, and Kechris (Theorem 3.13) [**DJK94**, Theorem 9.1] states that the cardinality of the set of ergodic invariant Borel probabilities characterizes a weaker form of orbit equivalence, Borel orbit equivalence.

3.1. The Unital Ordered Group $\mathcal{G}_{\mathbb{Z}}(T)$. We present the relevant definitions for unital ordered groups. For a more detailed introduction, see [**GPS95**].

Definition 3.1 (unital ordered group)**.** A unital ordered group $\mathcal{G}$ is a triple (G, G_+, u) where:

- G is an abelian group,
- G_+ is subset of G such that
  $G_+ \cap (-G_+) = \{0\}, \quad G_+ + G_+ \subseteq G_+, \quad \text{and} \quad G_+ - G_+ = G,$
- u is an element of G_+ such that
  for all $g \in G$ there exists an $n \in \mathbb{Z}_+$ such that $(nu - g) \in G_+$.

Definition 3.2 (isomorphism)**.** Two unital ordered groups (G, G_+, u) and (H, H_+, v) are isomorphic if and only if there is group isomorphism $f : G \to H$ such that $f(G_+) = H_+$ and $f(u) = v$.

Suppose (X, T) is a minimal Cantor system. We will use $\mathcal{G}_{\mathbb{Z}}(T)$ to denote the unital ordered group $(G_{\mathbb{Z}}(T), G_{\mathbb{Z}}(T)_+, 1_T)$ defined as follows. Let $G_{\mathbb{Z}}(T)$ be the group of continuous functions from the Cantor set X into the integers modulo the integer coboundaries

$$G_{\mathbb{Z}}(T) = C(X, \mathbb{Z})/\{f - fT \ : \ f \in C(X, \mathbb{Z})\}.$$

Let $G_{\mathbb{Z}}(T)_+$ be the semigroup of equivalence classes of nonnegative functions

$$G_{\mathbb{Z}}(T)_+ = \{[f] \ : \ f(x) \geq 0 \text{ for all } x \in X\}$$

and let 1_T be the equivalence class of the constant function one

$$1_T = [1].$$

The ordered group above is, in fact, a simple dimension group as defined by Elliot [**Ell76**].

Theorem 3.3 (Giordano, Putnam, Skau)**.** *Let (X,S) and (X,T) be minimal Cantor systems. Then $\mathcal{G}_{\mathbb{Z}}(S)$ is isomorphic to $\mathcal{G}_{\mathbb{Z}}(T)$ if and only if S and T are strongly orbit equivalent.*

The above gives strong orbit equivalence a more natural meaning, the equivalence relation which is induced by isomorphism of unital ordered groups. To get a similar statement for orbit equivalence, we must first introduce infinitesimal subgroups and traces of simple dimension groups.

Definition 3.4 (infinitesimals)**.** Let $\mathcal{G} = (G, G_+, u)$ be a unital ordered group. The set

$$\text{Inf}\ (\mathcal{G}) = \{g \in G \ : \ u - ng \in G_+ \text{ for all } n \in \mathbb{Z}\}$$

is the infinitesimal subgroup of $\mathcal{G}$.

Definition 3.5 (trace)**.** A trace σ on a unital ordered group $\mathcal{G} = (G, G_+, u)$ is a homomorphism $\sigma : G \to \mathbb{R}$ such that $\sigma(G_+) \subseteq \mathbb{R}_+$ and $\sigma(u) = 1$.

The order structure of any simple dimension group is determined by the action of the trace space [**Eff81**]. In other words,

$$G_+ = \{g \in G \ : \ \sigma(g) > 0 \text{ for all traces } \sigma\} \cup \{0\}$$

and

$$\text{Inf}\ (\mathcal{G}) = \{g \in G \ : \ \sigma(g) = 0 \text{ for all traces } \sigma\}.$$

If (X,T) a minimal Cantor system then the trace space of $\mathcal{G}_{\mathbb{Z}}(T)$ with the natural topology is a compact, convex metric space which is affinely homeomorphic to the space of T-invariant Borel probabilities $\mathfrak{M}_T$. Moreover,

$$G_{\mathbb{Z}}(T)_+ = \left\{[f] \ : \ \int f d\mu > 0 \text{ for all } \mu \in \mathfrak{M}_T\right\} \cup \{0\}$$

and

$$\text{Inf}\ (T) = \text{Inf}\ (\mathcal{G}_{\mathbb{Z}}(T)) = \left\{[f] \ : \ \int f d\mu = 0 \text{ for all } \mu \in \mathfrak{M}_T\right\}.$$

Theorem 3.6 (Giordano, Putnam, Skau)**.** *Let (X,S) and (X,T) be minimal Cantor systems. Then the unital ordered groups $\mathcal{G}_{\mathbb{Z}}(S)/\text{Inf}\ (S)$ and $\mathcal{G}_{\mathbb{Z}}(T)/\text{Inf}\ (T)$ are isomorphic if and only if S and T are orbit equivalent.*

3.2. A Real Analogue to $\mathcal{G}_{\mathbb{Z}}(T)$. The results of Section 2 and of Giordano, Putnam and Skau motivate our investigation of the triple $\mathcal{G}_{\mathbb{R}}(T) = (G_{\mathbb{R}}(T), G_{\mathbb{R}}(T)_+, 1_T)$ where

$$G_{\mathbb{R}}(T) = C(X,\mathbb{R})/\{f - fT \ : \ f \in C(X,\mathbb{R})\}$$

$$G_{\mathbb{R}}(T)_+ = \{[f] \ : \ f(x) \geq 0 \text{ for all } x \in X\}$$

$$1_T = [1].$$

Remark. Typically, there is an additional assumption that unital ordered groups be countable. However, none of the notions of unital ordered group, isomorphism, infinitesimals and traces depend upon the group being countable.

Suppose that (X, S) and (X, T) are minimal Cantor systems. We first notice that the groups $G_{\mathbb{R}}(S), G_{\mathbb{R}}(T)$ can adopt the structure of a real vector space (with the definition $r[f] := [rf]$). It is this identification which makes the inclusion map of $\mathcal{G}_{\mathbb{Z}}(T) \to \mathcal{G}_{\mathbb{R}}(T)$ worth studying. For example, there can exist locally constant coboundaries $f - fT$ where f cannot be chosen to be locally constant. Thus we are not simply considering the old group $\mathcal{G}_{\mathbb{Z}}(T)$ with real coefficients, there are also new identifications.

Henceforth, we will refer to the triple $(G_{\mathbb{R}}(T), G_{\mathbb{R}}(T)_+, 1_T)$ as a real ordered vector space. The isomorphisms we will consider are $\mathbb{R}$-vector space isomorphisms which preserve classes of nonnegative and constant functions.

For the remainder of this section, we will concentrate on the case where the space of invariant measures $\mathfrak{M}_T$ is finite dimensional. In this case we will characterize Inf $(G_{\mathbb{R}}(T))$ (Theorem 3.10).

Proposition 3.7. *Let (X, T) be a minimal Cantor system. If $\mathfrak{M}_T$ is finite dimensional then*

$$G_{\mathbb{R}}(T)_+ = \left\{ [f] \ : \ \int f d\mu > 0 \textit{ for all } \mu \in \mathfrak{M}_T \right\} \cup \{0\}$$

and

$$\mathrm{Inf}_{\,\mathbb{R}}(T) = \mathrm{Inf}\ (G_{\mathbb{R}}(T)) = \left\{ [f] \ : \ \int f d\mu = 0 \textit{ for all } \mu \in \mathfrak{M}_T \right\}.$$

Proof. Suppose $f : X \to \mathbb{R}$ is a continuous function.

If there exists $h \in C(X, \mathbb{R})$ such that $f(x) + h(x) - hT(x) \geq 0$ for all x, then either $f + h - hT \equiv 0$ or $\int f d\mu > 0$ for all $\mu \in \mathfrak{M}_T$.

Now suppose $\int f d\mu > 0$ for all $\mu \in \mathfrak{M}_T$. Then since $\mathfrak{M}_T$ is finite dimensional, there is a $\delta > 0$ such that $\int f d\mu \geq \delta$ for all μ. Select a continuous function $g : X \to \mathbb{Q}$ that takes on finitely many values and $f(x) - \delta/2 < g(x) < f(x)$ for all x. Then there is an integer m such that $mg \in C(X, \mathbb{Z})$ and $\int mg d\mu > 0$ for all $\mu \in \mathfrak{M}_T$. By the properties of $\mathcal{G}_{\mathbb{Z}}(T)$, there is an integer coboundary $h - hT$ such that $mg(x) + h(x) - hT(x) \geq 0$. Therefore, $f + \frac{1}{m}(h - hT)$ is a nonnegative function and $[f] \in G_{\mathbb{R}}(T)_+$.

The second claim now follows easily as $[1] - n[f] \in G_{\mathbb{R}}(T)_+$ iff $n \int f d\mu \leq 1$ for all $\mu \in \mathfrak{M}_T$ iff $\int f d\mu = 0$ for all $\mu \in \mathfrak{M}_T$.

□

Proposition 3.8. *Let (X, T) be a minimal Cantor system. If $\mathfrak{M}_T$ is finite dimensional then the dimension of* $\mathrm{Inf}_{\,\mathbb{R}}(T)$ *is* $|\mathbb{R}|$.

Proof. A continuous function from X to $\mathbb{R}$ is determined by its values on a countable dense subset. Therefore

$$\dim(\mathrm{Inf}_{\mathbb{R}}(T)) \le \dim(C(X,\mathbb{R})) \le |\mathbb{R}|^{|\mathbb{Q}|} = |\mathbb{R}|.$$

To finish the proof, it suffices to construct a family of linearly independent infinitesimals $\{f_\alpha : \alpha \in (0,1)\}$. In other words, we want a collection of functions $\{f_\alpha\}$ which integrate to zero with any T-invariant Borel probability such that no linear combination with nonzero coefficients is a coboundary.

We can use the same techniques as those used in Lemma 2.9 to create infinitesimals which are not T-coboundaries. Recall that in the proof of Lemma 2.9 we had sets U_k and integers n_k such that the function $f(x) = \sum_{k\ge1}(1/k)\left(1_{U_k}(x) - 1_{T^{n_k}U_k}(x)\right)$ was not a T-coboundary (in that proof there was a transformation S such that $SU_k = T^{n_k}U_k$). The reason f failed to be a T-coboundary was that for all n there was a point x and an integer m such that $\sum_{i=0}^{m} f(T^i x) \ge \sum_{k=1}^{n}(1/k)$ and therefore has no uniform upper bound when summed over partial T-orbits. Notice that this function must be an infinitesimal since the integration of f with any T-invariant Borel probability yields zero.

Suppose that we have such an f. (If you like, pick minimal S with the same orbits as T but with an unbounded jump function to create the U_k and n_k.) Now for $\alpha \in (0,1)$, let

$$f_\alpha(x) = \sum_{k\ge1}(1/k)^\alpha\left(1_{U_k}(x) - 1_{T^{n_k}U_k}(x)\right).$$

For any finite collection $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and nonzero real coefficients $\{r_1, r_2, \ldots, r_n\}$ the function $r_1 f_{\alpha_1} + r_2 f_{\alpha_2} + \cdots + r_n f_{\alpha_n}$ is an infinitesimal. It cannot be a T-coboundary since the partial sums $\sum_{k=1}^{N}\sum_{j=1}^{n} r_j (1/k)^{\alpha_j}$ behave like $\sum_{k=1}^{N} r_1 (1/k)^{\alpha_1}$ which is unbounded. □

Since the infinitesimal subgroups have the same dimension and contain no order structure, it remains to characterize $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$.

Let $\mathcal{E}(T)$ denote the set of ergodic T-invariant Borel probability measures.

Lemma 3.9. *Suppose that* (X,T) *is a minimal Cantor system and that* $|\mathcal{E}(T)|$ *is finite. Then* $\dim(G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)) = |\mathcal{E}(T)|$.

Proof. Let V denote the vector space $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$ and V^* the dual of V. The dimension of V is finite if and only if the dimension of V^* is finite. Moreover, if the dimensions are finite, then they are the same.

Let F be an element of V^*, then it is a linear functional on $C(X,\mathbb{R})$. By the Riesz Representation Theorem, there is a finite signed Borel measure μ such that $F(f) = \int f d\mu$. Since F is a linear functional on $C(X,\mathbb{R})/\{f - fT\}$, the measure μ must be T-invariant. Since $|\mathcal{E}(T)|$ is finite, the measure μ is a linear combination of ergodic T-invariant Borel probability measures. Two linear combinations of these ergodic measures are the same as elements of

V^* if and only if they are the same as signed Borel measures. Therefore, $\dim(V) = \dim(V^*) = |\mathcal{E}(T)|$. □

Theorem 3.10. *Let (X, T) be a minimal Cantor systems such that $|\mathcal{E}(T)| = d$ is finite. The unital ordered vector space $\mathcal{G}_{\mathbb{Z}}(T)/\mathrm{Inf}\ (T)$ is isomorphic to $\mathbb{R}^d$ where $(\mathbb{R}^d)_+$ are the elements with strictly positive entries along with the zero vector and $(1, 1, \cdots, 1)$ is the order unit.*

Proof. Since the dimension of $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$ and $\mathbb{R}^d$ are the same, we know that they are isomorphic as vector spaces. It remains to show that we can choose an isomorphism which preserves the order structure and unit.

Let $\{[f_1], [f_2], \dots, [f_d]\}$ be a basis for $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$, and let $\mathcal{E}(T)$ denote the ergodic measures $\mathcal{E}(T) = \{\mu_1, \mu_2, \dots, \mu_d\}$. We define a map from $\mathbb{R}^d$ to $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$ where a vector $\vec{v} \in \mathbb{R}^d$ gets sent to the equivalence class of a linear combination of the basis functions $g = \sum c_i f_i$ which has $\int g d\mu_j = v_j$ for $j = 1, 2, \dots, d$.

Such a map preserves positive cones and order units. □

Corollary 3.11. *Let (X, S) and (X, T) be minimal Cantor systems, such that $|\mathcal{E}(S)|$ and $|\mathcal{E}(T)|$ are finite. The unital ordered vector spaces $\mathcal{G}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)$ are isomorphic if and only if $|\mathcal{E}(S)| = |\mathcal{E}(T)|$.*

Proof. We may write the vector space $G_{\mathbb{R}}(T)$ as $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T) \oplus \mathrm{Inf}_{\mathbb{R}}(T)$. By Theorem 3.10, $\mathcal{G}_{\mathbb{R}}(S)/\mathrm{Inf}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$ are isomorphic as unital ordered vector spaces. By Proposition 3.8 the dimension of the infinitesimal subspaces are the same and therefore there is a vector space isomorphism between them. Since there is no order structure on the infinitesimal subspace, the result follows. □

The work of Dougherty, Jackson, and Kechris gives us a dynamical interpretation for equal cardinality of ergodic invariant Borel probabilities.

Definition 3.12 (Borel orbit equivalence)**.** Let $S : X \to X$ and $T : Y \to Y$ be Borel transformations of compact metric spaces X and Y. A Borel orbit equivalence is a Borel bijection $h : X \to Y$ and functions $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ such that

$$\forall x \in X,\ hS(x) = T^{m(x)}h(x) \text{ and } hS^{n(x)}(x) = Th(x).$$

Theorem 3.13 (Dougherty, Jackson, Kechris)**.** *Let $S : X \to X$ and $T : Y \to Y$ be Borel transformations of compact metric spaces X and Y. Then S and T are Borel orbit equivalent if and only if $|\mathcal{E}(S)| = |\mathcal{E}(T)|$.*

Therefore, we get the following dynamical interpretation of isomorphism of the unital ordered vector space in the case where the spaces of invariant measures are finite dimensional.

Theorem 3.14. *Let (X, S) and (X, T) be minimal Cantor systems such that $|\mathcal{E}(S)|, |\mathcal{E}(T)|$ are finite. Then the following are equivalent:*

1) $\mathcal{G}_{\mathbb{R}}(S) \cong \mathcal{G}_{\mathbb{R}}(T)$ *as unital ordered vector spaces,*
2) S *and* T *are Borel orbit equivalent.*

The result of Dougherty, Jackson and Kechris is true in the case where the cardinality of the ergodic invariant Borel measures is infinite. However, at present the author does not see how to extend the above theorem to include that case.

4. $\mathcal{G}_{\mathbb{R}}(T)$ as an extension of $\mathcal{G}_{\mathbb{Z}}(T)$.

Let (X, S) and (X, T) be a minimal Cantor systems. As we saw in the last section, isomorphism of $\mathcal{G}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)$ induces a weaker relation than strong orbit equivalence. We now consider isomorphisms of the unital ordered vector space $\mathcal{G}_{\mathbb{R}}(S)$ to $\mathcal{G}_{\mathbb{R}}(T)$ which when restricted to $\mathcal{G}_{\mathbb{Z}}(S)$ gives an isomorphism of the integer unital ordered groups. We first notice that $\mathcal{G}_{\mathbb{Z}}(T)$ embeds in $\mathcal{G}_{\mathbb{R}}(T)$.

Proposition 4.1. *The natural inclusion map $i : G_{\mathbb{Z}}(T) \to G_{\mathbb{R}}(T)$ is one-to-one and order-preserving.*

Proof. To show that the map is injective, it suffices to show that if an integer-valued function is a real coboundary, then it is an integer coboundary. Assume we have functions $f \in C(X, \mathbb{Z})$ and $g \in C(X, \mathbb{R})$ with $f = g - gT$.

Let x_0 be any point in X and let $\alpha = g(x_0)$. Then for all $n \in \mathbb{Z}$, $g(T^n x_0)$ is an integer plus α. For example if $n > 0$ then

$$g(x_0) - g(T^n x_0) = \sum_{i=0}^{n-1} g(T^i x_0) - g(T^{i+1} x_0) = \sum_{i=0}^{n-1} f(T^i x_0) \in \mathbb{Z}.$$

Since all T-orbits are dense in X, all values of the function g are an integer plus α. Letting $k = g - \alpha$, we obtain an integer-valued function $k \in C(X, \mathbb{Z})$ where $f = k - kT$.

Clearly, if $[f] \in G_{\mathbb{Z}}(T)_+$ then $i([f]) \in G_{\mathbb{R}}(T)_+$. Now suppose that $f \in C(X, \mathbb{Z})$ and $i([f]) \in G_{\mathbb{R}}(T)_+$. That is, suppose there exists a function $h \in C(X, \mathbb{R})$ such that $f(x) + h(x) - hT(x) \geq 0$ for all x. Then either $f \equiv 0$ or for all invariant probability measures μ, $\int f d\mu > 0$. In either case, $[f] \in G_{\mathbb{Z}}(T)_+$. □

Definition 4.2 (pair isomorphism)**.** Suppose that S and T are minimal homeomorphisms of the Cantor set. We will call H a pair isomorphism of $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ if $H : \mathcal{G}_{\mathbb{R}}(S) \to \mathcal{G}_{\mathbb{R}}(T)$ is a real ordered vector space isomorphism such that $H(G_{\mathbb{Z}}(S)) = G_{\mathbb{Z}}(T)$.

In particular, we are interested in the following questions. Does a pair isomorphism $H : (\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S)) \to (\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ induce a stronger relation than strong orbit equivalence? Does a pair isomorphism imply that S and T are flip conjugate? We will show that the answer to the first question is yes, and the answer to the second is no. To answer these questions, we begin by showing that the isomorphism class of the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ classifies the (topological) *discrete spectrum* of the system (X, T).

Definition 4.3 (discrete spectrum)**.** Let (X, T) be a minimal Cantor system. The discrete spectrum of T is the set of λ such that $FT = \lambda F$ for some continuous function F from X to $\{z \in \mathbb{C}, |z| = 1\}$.

We will call a function F as above an eigenfunction for T and λ an eigenvalue for T.

The strong orbit equivalence class already determines the *rational part* of the discrete spectrum (eigenvalues $\exp(2\pi i\alpha)$ where $\alpha \in \mathbb{Q}$), but strongly orbit equivalent systems may have different irrational spectrum (see [**Orm97**]). The following shows that the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ does indeed carry some additional information beyond strong orbit equivalence.

Theorem 4.4. *Let (X, S) and (X, T) be minimal Cantor systems. Suppose that there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. Then S and T have the same discrete spectrum.*

Proof. We first show that a complex number $\exp(2\pi i\alpha)$ is an eigenvalue for T if and only if there exist $f \in C(X, \mathbb{Z})$ and $k \in C(X, \mathbb{R})$ such that $f = \alpha + k - kT$.

Suppose functions k, f exist as above. Multiplying both sides of $f = \alpha + k - kT$ by $2\pi i$ and exponentiating one obtains

$$\exp(2\pi i kT(x)) \exp(2\pi i f(x)) = \exp(2\pi i\alpha) \exp(2\pi i k(x)).$$

Since $f(x) \in \mathbb{Z}$ for all x, we see that $\exp(2\pi i f(x)) = 1$ and therefore $F(x) = \exp(2\pi i k(x))$ is an eigenfunction for T with eigenvalue $\exp(2\pi i\alpha)$.

Now suppose that $F : X \to S^1$ is an eigenfunction for the eigenvalue $\exp(2\pi i\alpha)$. Let $U_1, U_2, \dots, U_n$ be clopen sets such that a logarithm function L_j can be continuously defined on each $F(U_j)$. For $x \in U_j$ we define $k(x) = L_j(F(x))$. With this definition, $k : X \to \mathbb{R}$ is a continuous function and $k(x) - k(Tx) + \alpha$ is an integer for all $x \in X$. Therefore, there is a $f : X \to \mathbb{Z}$, $k : X \to \mathbb{R}$ such that $f = \alpha + k - kT$.

This completes the proof since if $f = \alpha + k - kS$ as above and there is a pair isomorphism $H : (\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S)) \to (\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ then $H([f]) = \alpha H([1_S]) = \alpha[1_T]$. Taking a representative function g from $H([f])$, we see that there must be a function $k' \in C(X, \mathbb{R})$ such that $g = \alpha + k' - k'T$. Therefore if $\exp(2\pi i\alpha)$ is an eigenvalue for S then $\exp(2\pi i\alpha)$ is an eigenvalue for T as well. □

The above theorem extends to determine the possible discrete spectrum of an *induced system* (A, T_A) of (X, T). An induced system (A, T_A) of (X, T)

is a minimal Cantor system obtained by taking a clopen subset $A \subseteq X$ and the map $T_A : A \to A$. The map T_A is defined to be $T_A(x) = T^n(x)$ where n is the smallest positive integer such that $T^n(x) \in A$.

Theorem 4.5. *Let (X, S) and (X, T) be minimal Cantor systems. Suppose that there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. Then there is an induced system (A, S_A) of (X, S) which has λ in the discrete spectrum if and only if there is an induced system of (B, T_B) of (X, T) which has λ in the discrete spectrum.*

Proof. Suppose that $\lambda = \exp(2\pi i\alpha)$ is in the topological discrete spectrum of an induced system (A, S_A). This occurs if and only if for some $f \in C(A, \mathbb{Z})$ and $h \in C(A, \mathbb{R})$ we have $f - \alpha = h - hS_A$ on the set A. Extend f to $\hat{f} : X \to \mathbb{Z}$ by defining $\hat{f} \equiv 0$ on the complement of A. Then the function $(\hat{f} - \alpha 1_A)$ is an S-coboundary by Theorem 2.6. (Notice that for $x \in A$, $\sum_{i=0}^{n}(\hat{f} - \alpha 1_A)(S^i x) = \sum_{i=0}^{m}(\hat{f} - \alpha 1_A)((S_A)^i x)$ for some $m \leq n$.)

Now since $\hat{f} - \alpha 1_A$ is an S-coboundary and we have a pair isomorphism $H : (\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S)) \to (\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$, we know that $H([\hat{f}]) - \alpha H([1_A]) = 0$ in $\mathcal{G}_{\mathbb{R}}(T)$.

Claim. There is an indicator function $1_B \in H([1_A])$ for some clopen set $B \subseteq X$.

Proof of Claim. Since $[1_A] \in G_{\mathbb{Z}}(S)_+$ there is a $g_1 \in H([1_A])$ such that $g_1(x) \geq 0$ for all x. Since $[1]_S - [1_A] \in G_{\mathbb{Z}}(S)_+$ there is a $g_2 \in H([1_A])$ such that $g_2(x) \leq 1$ for all x. We know that the function $g_1 - g_2$ is an integer T-coboundary, $g_1 - g_2 = k - kT$ for some $k \in C(X, \mathbb{Z})$. Let C be a clopen set on which k is constant. Let $C_1, C_2, \ldots, C_r$ be the clopen subsets of C such that $x \in C_n$ if and only if n is the smallest positive integer such that $T^n(x) \in C$. By minimality, $C = \bigcup_{n=1}^{r} C_n$ for some r.

For $x \in C_n$, we know

$$\begin{aligned}\sum_{i=0}^{n-1} g_1 T^i(x) - g_2 T^i(x) &= \sum_{i=0}^{n-1} kT^i(x) - kT^{i+1}(x)\\ &= k(x) - kT^n(x)\\ &= 0.\end{aligned}$$

Therefore, for all n and $x \in C_n$, we have

$$0 \leq \sum_{i=0}^{n-1} g_1 T^i(x) = \sum_{i=0}^{n-1} g_2 T^i(x) \leq n.$$

Fix n and $x \in C_n$. Let B_n be the union of exactly $\sum_{i=0}^{n-1} g_1 T^i(x)$ of the sets $C_n, TC_n, \ldots, T^{n-1}C_n$. Let $B = \bigcup_{n=1}^{r} B_n$.

The difference between g_1 and 1_B must be a T-coboundary by Theorem 2.6. This follows since for $x \in C_n$, $\sum_{i=0}^{n-1} g_1 T^i(x) = \sum_{i=0}^{n-1} 1_B T^i(x)$.

Thus the difference $g_1 - 1_B$ is bounded along T-orbits. This proves the claim.

Let g be a representative of $H([\hat{f}])$ and let B be a clopen set such that $1_B \in H([1_A])$. Then there exists a function $k \in C(X, \mathbb{R})$ such that $g - \alpha 1_B = k - kT$.

Let $K(x) = \exp(2\pi i k(x))$. The function g is integer-valued so

$$\begin{aligned} K(T(x)) &= \exp(-2\pi i g(x)) \exp(2\pi i \alpha 1_B(x)) K(x) \\ &= \exp(2\pi i \alpha 1_B(x)) K(x). \end{aligned}$$

If $x \in B$, we have $K(T(x)) = \lambda K(x)$. If $x \notin B$, we have $K(T(x)) = K(x)$. Therefore, for $x \in B$, we have $K(T_B(x)) = \lambda K(x)$. Thus, T has an induced system with λ in the spectrum. □

Remark. If $\lambda = \exp(2\pi i \alpha)$ where $\alpha \in \mathbb{Q}$ then the conclusion of the previous theorem is trivially true. For any minimal Cantor system T and any $p \in \mathbb{Z}$ there is an induced system T_A such that a periodic orbit of cardinality p is a factor of T_A. To see this, let $B \subseteq X$ be a clopen set with small enough diameter so that if $x \in B$ and $T^n(x) \in B$ then $n \geq p$. Let $A = \bigcup_{i=0}^{p-1} T^i B$. Then the induced system (A, T_A) has as a factor of a finite orbit of length p.

In the case where $\lambda = \exp(2\pi i \alpha)$, $\alpha \notin \mathbb{Q}$, the statement is nontrivial as we will see in Corollary 4.8.

The following theorem shows some of the limitations on dynamical information that one can get from the pair $(\mathcal{G}_\mathbb{R}(T), \mathcal{G}_\mathbb{Z}(T))$. In particular, it shows that one cannot deduce flip conjugacy from a pair isomorphism between $(\mathcal{G}_\mathbb{R}(S), \mathcal{G}_\mathbb{Z}(S))$ and $(\mathcal{G}_\mathbb{R}(T), \mathcal{G}_\mathbb{Z}(T))$. For a unital ordered group $\mathcal{G}$, when we say $\mathcal{G} \subseteq \mathbb{Q}$ we mean that $\mathcal{G}$ is isomorphic to a subgroup of $(\mathbb{Q}, \mathbb{Q}_+, 1)$ with the induced order.

Theorem 4.6. *Let S and T be minimal homeomorphisms of the Cantor set. Suppose $\mathcal{G}_\mathbb{Z}(T)$ is a subgroup of $\mathbb{Q}$. Then there is a pair isomorphism between $(\mathcal{G}_\mathbb{R}(S), \mathcal{G}_\mathbb{Z}(S))$ and $(\mathcal{G}_\mathbb{R}(T), \mathcal{G}_\mathbb{Z}(T))$ if and only if S and T are strongly orbit equivalent.*

Proof. Since $\mathcal{G}_\mathbb{Z}(T) \subseteq \mathbb{Q}$ every integer-valued function can be written as a constant plus a integer coboundary. The embeddings for $(\mathcal{G}_\mathbb{R}(S), \mathcal{G}_\mathbb{Z}(S))$ and $(\mathcal{G}_\mathbb{R}(T), \mathcal{G}_\mathbb{Z}(T))$ are given by $[f] \mapsto q[1]$ where q is the rational number corresponding to $[f]$. Since the integer unital ordered groups are subsets of $\mathbb{Q}$, the maps S and T are uniquely ergodic. Therefore, the real ordered groups $\mathcal{G}_\mathbb{R}(S)$ and $\mathcal{G}_\mathbb{R}(T)$ are isomorphic by Theorem 3.10. Since the isomorphism maps the constant function one to the constant function one, it must map the subgroup of integer-valued functions onto one another. □

In particular, this shows that systems (X,T) with $\mathcal{G}_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ cannot be strongly orbit equivalent to systems with any irrational discrete spectrum. Systems with $\mathcal{G}_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ include all odometer systems (see example from Section 2). For an odometer system with d_i digits in the ith place the group $G_{\mathbb{Z}}(T)$ is isomorphic to the subgroup of the rationals formed by all rationals whose denominators are products of the d_i's [**HPS92**]. For the dyadic adding machine $G_{\mathbb{Z}}(T)$ is the dyadic rationals $\mathbb{Z}[\frac{1}{2}]$.

Corollary 4.7. *Suppose (X,T) is a minimal Cantor system where $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then T cannot have irrational discrete spectrum.*

Proof. To prove this, one simply needs an S with $\mathcal{G}_{\mathbb{Z}}(S) = \mathcal{G}_{\mathbb{Z}}(T)$ such that S has no irrational spectrum. Then by Theorem 4.6, and Theorem 3.3, there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. But by Theorem 4.4, S and T must have the same discrete spectrum.

To create such an S, make a list of the denominators $\{d_1 < d_2 < \cdots\}$ which appear in elements of $\mathcal{G}_{\mathbb{Z}}(T)$, then construct an odometer system with d_1 digits in the first place, d_2 digits in the second place, and so on.

The odometer systems have no irrational spectrum. This follows from the fact that for every clopen set A in an odometer system (X,T) there is an integer n such that $T^n A = A$. If there were a map $F : X \to S^1$ and a λ such that $FT = \lambda F$, then there would be a clopen set $A \subseteq X$ whose image under F lies within $\{\exp(2\pi i\theta) \ : \ 0 \le \theta \le \pi\}$ such that $FT^n(A) = F(A)$ for some n. If $\lambda = \exp(2\pi i\alpha)$ with α irrational, then $\lambda^n F(A)$ can never equal $F(A)$. □

Corollary 4.8. *Suppose (X,T) is a minimal Cantor system where $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then T cannot have an induced system with irrational discrete spectrum.*

Proof. Suppose that $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then any induced system T_A must also have $\mathcal{G}_{\mathbb{Z}}(T_A) \subseteq \mathbb{Q}$. This follows from results of [**GPS95**], or by the following argument. Since $G_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$, T is uniquely ergodic. Moreover, the integral of any integer-valued continuous function with this measure must be rational. The same holds for an induced system (A, T_A), so $G_{\mathbb{Z}}(T_A) \subseteq \mathbb{Q}$. □

Appendix A. A homeomorphism good on measures and bad on orbits.

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Suppose S and T are minimal homeomorphisms of the Cantor set X. Giordano, Putnam and Skau proved that if $h : X \to X$ is a homeomorphism which identifies integer coboundaries for S and T then S and T are orbit equivalent (Theorem 2.7 of this paper). This result is a spinoff of their beautiful algebraic characterization ("K_0 modulo the infinitesimals") of orbit equivalence of homeomorphisms of the Cantor set. This work developed constructions involving Bratteli diagrams, C^*-algebras and some homological algebra. It is natural to ask whether the theorem above could be proved directly, i.e., without reference to this associated machinery. This seems problematic even at first glance–given h as in the theorem, how could one recover orbit information? Here is an example (circulated informally in 1992) which reinforces this impression. I thank Chris Skau for helpful comments.

Example. Let X be the domain of the dyadic adding machine S. There is a homeomorphism T from X to X such that

- if x and y are any two points in the same S-orbit, then the points $T(x)$ and $T(y)$ are in different S-orbits, and
- for all clopen sets U, there are continuous functions $f, g : X \to \mathbb{Z}$ such that $1_U - 1_{TU} = f - fS$ and $1_U - 1_{SU} = g - gT$.

The dyadic adding machine is defined as an example of an odometer system in Section 2. We recapitulate the definition here. The space X is $\{0,1\}^{\mathbb{N}}$. A point x in X is a one-sided sequence $x_1x_2x_3...$ with each x_i in $\{0,1\}$. The map S sends the sequence $x = 1^\infty$ ($x_i = 1$, for all i) to the sequence 0^∞. Otherwise, x has for some nonnegative k an initial word 1^k0 and Sx is obtained by replacing this word with 0^k1.

Two sequences in X are *cofinal* if they disagree in only finitely many coordinates. Two sequences x, y are in the same S-orbit if and only if either (1) they are cofinal or (2) one is cofinal to 0^∞ and the other is cofinal to 1^∞.

Choose a collection of infinite pairwise disjoint sets A_n, $1 \le n < \infty$, such that $\mathbb{N}$ is the union of the A_n. Enumerate the finite words on $\{0,1\}$ as $W(1), W(2), ...$ such that $n > m$ implies the length $|W(n)|$ of $W(n)$ is at least $|W(m)|$. Define $B_n = \{m \in A_n : m > |W(n)|\}$, an infinite subset of $\mathbb{N}$. For each $n > 0$, we define a homeomorphism $\phi_n : X \to X$ by

$$(\phi_n x)_i = \begin{cases} x_i + 1 \pmod 2 & \text{if } i \in B_n \text{ and } x_1 ... x_{|W(n)|} = W(n) \\ x_i & \text{otherwise.} \end{cases}$$

Now define $\psi_n = \phi_n \circ \phi_{n-1} \circ ... \circ \phi_1$. In other words, $\psi_1 = \phi_1$ and $\psi_n(x) = \phi_n(\psi_{n-1}(x))$. Finally, let $\psi = \lim \psi_n$. Apart from a technical detail, ψ will be the homeomorphism T of the example.

For each n and x,

$$(\psi_n x)_i = (\psi x)_i, \qquad 1 \le i \le |W(n)|.$$

Therefore the maps ψ_n are converging uniformly and ψ is a homeomorphism. Also, for every k and x, the word $(\psi x)_1 ... (\psi x)_k$ is determined by the word $x_1 ... x_k$. So, for every k, ψ induces a permutation of the initial cylinders of length k. This means that for any cylinder set U, there is a unique integer l

such that $\psi U = S^l U$ and $1_U - 1_{\psi U} = 1_U - 1_{S^l U}$. In particular, we may deduce that any integer ψ-coboundary is an integer S-coboundary and vice-versa.

If x and y are distinct points in X, then let $N = N(x,y)$ denote the largest integer such that $x_i = y_i$ if $i < N$. Notice, if ϕ_i corresponds to a word $W(i)$ such that $|W(i)| < N(x,y)$ (equivalently, $i \leq 2^{N-1}$) then $N(x,y) = N(\phi_i x, \phi_i y)$. On the other hand, if ϕ_i corresponds to a word $W(i)$ of length N, then ϕ_i fixes the initial word of length N in every point, and ϕ_i changes a point x (by flipping symbols in the coordinates indexed by B_i) if and only if $x_1 ... x_N = W(i)$.

Now given distinct points x and y, set $\gamma = \psi_{2^{N-1}}$, where $N = N(x,y)$. Let ϕ_n be the map corresponding to the initial word $W(n)$ of γx of length N. Our discussion above gives the following implications:

$$\begin{aligned} (\psi x)_i &\neq x_i \qquad \text{if } i \in B_n, \\ (\psi y)_i &= y_i \qquad \text{if } i \in B_n. \end{aligned}$$

It follows immediately that if x and y are distinct cofinal points, the ψx and ψy are not cofinal.

Next note that if x and y are distinct points, then for some B_n,

$$x_i = (\psi x)_i \qquad \text{and} \qquad y_i = (\psi y)_i, \qquad \text{for} \qquad i \in B_n.$$

(In fact we can use n such that B_n corresponds to a word $W(n)$ of length 2 which begins neither $\psi_2 x$ nor $\psi_2 y$.) Consequently, if x is cofinal to 0^∞ and y is cofinal to 1^∞, then $\psi x, \psi y$ are not cofinal.

This finishes the proof for the example, except for a technical detail: It might be the case that there are points x, y in the same S-orbit such that ψx is cofinal to 0^∞ and ψy is cofinal to 1^∞. To take care of this, choose points u and v such that the preimage under ψ of the S-orbit of u does not intersect any S-orbit containing a point in the preimage under ψ of the S-orbit of v. Let β be a cofinal homeomorphism of X (x, y are cofinal iff $\beta x, \beta y$ are cofinal) which exchanges 0^∞ with u and which exchanges 1^∞ with v. Let T be the composition, ψ followed by β. Now we have for all distinct points x and y: If x, y are cofinal then Tx, Ty are not cofinal; if x is cofinal to 0^∞ and y is cofinal to 1^∞, then Tx,Ty are not cofinal; if Tx is cofinal to 0^∞ and Ty is cofinal to 1^∞, then x, y are not in the same S-orbit.

This finishes the proof.

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APPLICATION TO GLOBAL BERTINI THEOREMS

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Let k be an infinite field of arbitrary characteristic, (A, M, K) a k-algebra of essentially finite type, with K/k separable and P a local property. We say that $LB_k(\mathrm{P})$ holds if: For the generic $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \Rightarrow \mathrm{P}(Ax_\alpha A) \subseteq \mathrm{P}(A) \cap V(x_\alpha) \cap U_P$ ($x_\alpha = \sum \alpha_i x_i$, $\langle x_1, \dots, x_n \rightarrow= M$, U_P non-empty open subset of Spec A and $\mathrm{P}(A) = \{P \in \mathrm{Spec}\, A | A_p$ is P$\}$). We show that: $LB_K(\mathrm{P})$ holds $\Rightarrow LB_K(\mathrm{GP})$ holds for the corresponding geometric property (in particular, for P = regular, normal, reduced, R_s, $LB_K(\mathrm{GP})$ holds). As an appliance we obtain a Bertini Theorem for hypersurgace setions of a variety $X \subseteq P_k^n$ concerning the geometric properties.

1. Introduction.

Bertini showed that, given a smooth projective variety x contained in P_k^n with $k = \mathbf{C}$, the generic hypersurface section of x is also smooth (see [**B**, Chap. 10, n. 25]; for a modern approach, see [**H**, Th. 8.18] or [**J**, Th. 6.3]).

There have been many generalizations of this Theorem: We recall the recent algebraic studies on trasversality made by Kleiman in [**K**] and Speiser in [**S**] where they introduced a fully modern point of view of schemes over an algebraically closed field of arbitrary characteristic.

Another approach to this problem has been proposed by Flenner in [**F**] (following Grothendieck, see [**G**]).

He shows that, given a field k of arbitrary characteristic and given a local k-algebra $(A < M < K)$ with K/k separable, then, for the generic $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \Rightarrow$

$$\mathbf{P}(A/x_\alpha A) \subseteq \mathbf{P}(A) \cap U_P \tag{1}$$

where $x_\alpha = \sum \alpha_i x_i$, $\{x_1, \dots, x_n\}$ is a generator system of M and U_P is a non-empty open subset of Spec A depending on $\mathbf{P}$, being $\mathbf{P}$ one of the following local properties: Regular, normal, reduced, R_s and S_r.

These results applied to the local ring of the vertex of the affine cone corresponding to a projective variety X, imply, by standard techniques, the corresponding global Bertini Theorem for the variety X.

In this work we want to show that every time we have a result like (1) for a property $\mathbf{P}$ we have the same result for the corresponding geometric property **GP** and that the coresponding global results hold (these are known only for geometrically regular, see [**J**, Chap. 1, §6]).

In Section 3 we introduce some topological remarks that we use in next section: We show that if k is a subfield of K, (k infinite), every non-empty open set of K^n can be constructed to a non-empty open set of K^n.

In Section 4, that is main section, we give a local Bertini Theorem for the properties **GP** in an axiomatic form and we show that there are properties **GP** (for example **GP** $= S_r$, geom. R_s, geom. regular, geom. normal, geom. reduced) to which we can apply the Theorem. In these cases we show that the **GP**-locus is open.

In Section 5 we deduce a global Bertini's Theorem for the hypersurface sections of a variety X in a projective space over a field of arbitrary characteristic and for the above cited **GP** (we extend for many geometrically properties Th. 6.3 in [**J**] concerning the only geometrical regular property).

2. Preliminaries and notation.

In this section we fix the standard notation to be used in the following.

The rings considered are always commutative with an identity element.

If A is a ring, $\Omega(A)$ is the set of maximal ideals of A.

We recall here the definition of essentially finite type algebra and some properties of this algebra that we shall have to use in Section 4.

Definition 2.1 ([**EGA**, Chap. IV, 1.3.8])**.** Let T be a ring. A T-algebra S is of essentially finite type (e.f.t. for short) if S is T-isomorphic to $S^{-1}C$ where C is a T-algebra of finite type and S is a multiplicatively closed subset of C.

Properties 2.2 ([**EGA**, Chap. IV, 1.3.9 (ii)]; [**M**, 34.A])**.**

(i) If S is a T-algebra of e.f.t. and T' is a T-algebra then $S' = S_{\oplus_T}T'$ is a T'-algebra of e.f.t..

(ii) If S is a T-algebra of e.f.t. and T is an excellent ring then S is an excellent ring.

In the following, all topological spaces are considered with their Zariski topology. If A is a ring we put $V(x_1, \dots, x_n)$ to closed subset of $\operatorname{Spec} A$ corresponding to the ideal generated to the elements $x_1, \dots, x_n$ of A.

Let $F[\underline{T}] = F[T_1, \dots, T_n]$ be the polynomial ring with coefficients in the field F. We identify $F^n\{(\alpha_1, \alpha_n) | \alpha_1 \in F\}$ with the topological subspace $S = \{(T_1 - \alpha_1, \dots, T_n - \alpha_n) | \alpha_1 \in F\} \operatorname{MaxSpec} F[\underline{T}]$. (We observe that $\overline{F}^n = \operatorname{MaxSpec} \overline{F}[\underline{T}]$ where $\overline{F}$ denotes the algebraic closure of the field F.)

The expression "x generic in X", where X is a topological space, means that x is in a dense open subset of X.

We recall here the definition of geometric property.

Definition 2.3. Let $\mathbf{P}$ be a local property and A a local ring containing a field k. We say that A is geometrically $\mathbf{P}$ if $A_{\oplus_k} k$ is $\mathbf{P}$.

(See also [**EGA**, Chap. IV, 6.7.7] for equivalent definitions.)

Finally we put $\mathbf{P}(A) = \{P \in \operatorname{Spec} A | A_{\mathcal{P}}$ verifies the local property $\mathbf{P}\}$.

3. Some topological remarks.

For our aim we have to prove that, given an infinite field k, if K/k is a field extension and $\Im$ is an open dense subset K^n then $\Im \cap k^n$ is an open dense subset of k^n (Prop. 3.3). We prove this fact in two steps (the first one for the 'open' property, the second one for the 'dense' property).

We consider the following commutative diagram:

$$\begin{array}{ccc} K^n & \xrightarrow[i]{} & \operatorname{Spec} K[T_1, \dots, T_n] \\ j\big\uparrow & & f\big\downarrow \\ k^n & \xrightarrow[h]{} & \operatorname{Spec} k[T_1, \dots, T_n] \end{array}$$

where i, h are the inclusions of canonical maps and, as well known, K^n (resp. k^n) is a topological subspace of $\operatorname{Spec} K[\underline{T}]$ (resp. $\operatorname{Spec}[\underline{T}]$).

Lemma 3.1. *Let K/k be a field extension, then k^n is a subspace of K^n.*

Proof.

Case 1. *K/k algebraic extension.*

One can suppose that $\mathfrak{I} = V(g)$ is a fundamental closed set.

Consider a representation $g = \sum x_i g_i$ with $x_i \in K$ linearly independent over k and $g_i \in k[X_1, \dots, X_n]$. Then $\mathfrak{I} \cap k^n = V(g_1, \dots, g_r)$. The inclusion $\mathfrak{I} \cap k^n \subseteq (C) \cap k^n$ is trivial. The other one is easy if we remark that $f^{-1}(k^n) = k^n$.

Case 2. *K/k purely transcendental extension.*

Let $\Im = \{(x_1, \dots, x_n) \in K^n | g(x_1, \dots, x_n) = 0$ with $g \in K[\underline{T}]\}$ be a fundamental closed set of K^n. Among the coefficients of g there are only a finite number t of elements of K transcendental over k and so we can reduce to the transcendental extension of finite type. Using induction on t we can consider that there is only one transcendental element Z (i.e., $t = 1$).

So $g(T_1, \dots, T_n) = a_{i_1 \cdots i_n}(z) T^{i_1} \cdots T^{i_n}$ with $a_{i_1 \cdots i_n}(z) \in k(z)$.

$$(k_1, \dots, k_n) \in k^n \cap \Im \Leftrightarrow g(k_1, \dots, k_n) = 0 \Leftrightarrow b_{i_1 \cdots i_n}(z) k^{i_1} \cdots k^{i_n} = 0$$

with $b_{i_1 \cdots i_n}(Z) \in k[Z]$ (obtained by clearing denominators and simplifying) $\Leftrightarrow$ $g_r(k_1, \dots, k_n) Z^r + \cdots + g_0(k_1, \dots, k_n) = 0$ (obtaining ordering $b_{i_1 \cdots i_n}(z) k^{i_1} \cdots k^{i_n}$ like a polynomial in z) where $g_r(T_1, \dots, T_n) \in k[\underline{T}]$.

But Z is transcendental over k and so $(k_1, \dots, k_n) = k^n \cap \Im \Leftrightarrow g(k_1, \dots, k_n) = 0 \; \forall \; i \;\; 0 \le i \le r$. Then we have $\Im \cap k^n = V(g_1, \dots, g_r)$.

General case.

It is well known that every field extension can be written as $k \subseteq K' \subseteq K$ with K'/k purely transcendental and K/K' algebraic. So we can apply subsequently Case 2 and Case 1.

Lemma 3.2. *Let k be an infinite field, then k^n is irreducible.*

Proof. We want to show that the intersection of two non-empty open sets is still non-empty.

For this it is clearly sufficient to show that if $f, g \in k[T_1, \dots, T_n]$ and $V(f) \neq k^n$, $V(g) \neq k^n$ then $V(fg) \neq k^n$. We use induction on n. If $n = 1$ we consider the polynomial: $fg = (f_0 + \cdots + f_i T^l)(g_0 + \cdots + g_h T^h)$. $Fg = 0$ has at most $i + h$ solutions in $\overline{k}$ (and so in k) and this proves that $V(fg) = k$ because k is infinite.

Suppose now that the conclusion is true for any number of variables smaller than n.

We have $fg = (f_0 + f_1 T_n + \cdots + f_i T_n^l)(g_0 + g_1 T_n + \cdots + g_n T_n^h) = f_0 g_0 + \cdots + f_1 g_h T^{l+h}$.

With $f_j g_l \in k[T_1, \dots, T_{n-1}]$ for $0 \leq j \leq i$ and $0 \leq l \leq h$ observe that $f_1 g_h$ is a polynomial in $n-1$ variables $\Rightarrow$ by the induction hypothesis, there exists an element $w = (k_1, \dots, k_n) \in k^{n-1}$ such that $f_1(k_1, \dots, k_n) g_h(k_1, \dots, k_n) \neq 0$. For this w we can find an element $a \in k$ such that $f(k_1, \dots, k_n, a) g(k_1, ..., k_n, a) \neq 0$ because the polynomial in a single variable $f(k_1, \dots, k_n, T_n) g(k_1, \dots, k_n, T_n)$ has at most $i + h$ solutions in k and k is infinite.

Then there exists k_n such that $y = (k_1, \dots, k_{n-1}, k_n) \in V(fg)$.

From the above lemmas we get:

Proposition 3.3. *Let K be an extension of infinite field k. If $\mathfrak{I}$ is an open dense subset of K^n then $\mathfrak{I} \cap k^n$ is an open dense subset of K^n.*

Proof. By Lemma 3.1 we know that $\mathfrak{I} \cap k^n$ is open in k^n. By Lemma 3.2 it is enough to show that $\mathfrak{I} \cap k^n$ is non-empty. It is sufficient to prove this fact for $\mathfrak{I} = k^n - V(f)$ with $f \in K[T_1, \dots, T_n]$, by induction on n.

If $n = 1$, $f(T) = K_0 + \cdots + K_r T^r$ has at most r solutions in K and so in k.

Suppose that it is true for any integer $m < n$. Put $f(T_1, \dots, T_n) = f_0 + f_1 T_n + \cdots + f_i T_n^l$ where $f_j \in K[T_1, \dots, T_{n-1}]$ for $0 \leq j \leq i$. By induction hypothesis there exists $(k_1, \dots, k_{n-1}) \in k^{n-1}$ such that $f_i(k_1, \dots, k_{n-1}) \neq 0$. Considering $f_i(k_1, \dots, k_{n-1}, T^n)$ we observe that f has at most j solutions in k. Let $a \in k$ be a non-solution for $f_i(k_1, \dots, k_{n-1}, T^n)$, then $(k_1, \dots, k_{n-1}, a) \in \mathfrak{I} \cap k^n$.

4. Main result.

The main purpose of this paragraph is to give a local Bertini thorem for the geometric properties. We need some definitions.

Definition 4.1. A local ring (A, M, K) is a Flenner k-algebra if A is a northerian k-algebra, k is an infinite field and K is separable over k.

Definition 4.2. Let $\mathbf{P}$ be a local property of commutative rings. We say that $\mathbf{P}$ is a local Bertini property if, for every local Flenner k-algebra (A, M, K) e.f.t. and every set of generators $\langle x_1, \dots, x_n \rangle$ of M, the following condition holds:

$LB_k(\mathbf{P})$ for generic $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \Rightarrow$

$$\mathbf{P}(Ax_\alpha A) \subseteq (A) \cap V(x_\alpha) \cap U_{\mathbf{P}}$$

where $x_\alpha = \Sigma\alpha_1 x_1$, and $U_{\mathbf{P}}$ is either $\operatorname{Spec} A$ or $\operatorname{Spec} A - \{M\}$, depending on $\mathbf{P}$.

We say briefly that $LB_k(\mathbf{P})$ holds.

Remark 4.3. We observe that $LB_k(\mathbf{P})$ holds for $\mathbf{P}$ = regular, normal, reduced. Serre's properties R_s and S_r (in fact more general statements holds: See [**F**] Theorem 4.1 and Corollaries 4.2 and 4.3).

We want to prove that if A is a Flenner K-algebra of e.f.t. and $LB_k(\mathbf{P})$ holds for some property $\mathbf{P}$ then $LB_k(\mathbf{GP})$ holds too for the corresponding geometric property.

We need some lemmas.

Lemma 4.4. *Let (A, M, K) be a Flenner k-algebra of e.f.t. and $B = A \otimes_k \overline{k}$. Then, for every $M \in \Omega(B)$, (B_M, NB_M, K_M) is a Flenner $\overline{k}$-algebra of e.f.t.*

Proof. Recall that $\varphi : A \longrightarrow B$ is a flat homomorphism.

Case 1. B is a semilocal $\overline{k}$-algebra and $\mathcal{M}B_M = MB_M$ $\forall M \in \Omega(B)$. Clearly B is a $\overline{k}$-algebra of e.f.t. and, being integral over A, we have $\mathcal{M}B \subseteq \operatorname{Rad}(B)$.

$B/\mathcal{M}B = K \otimes_A (A \otimes_k \overline{k}) = K \otimes_k \overline{k}$ and $\dim K \otimes_k \overline{k} = 0$. In fact $K \otimes_k \overline{k}$ is noetherian (because B is a $\overline{K}$-algebra of e.f.t. by Prop. 2.2 (i) and so it is noetherian) and integral over K and we can apply Theorem 20 in [**M**]. So $K \otimes_k \overline{k}$ is an artinian ring (Theorem 8.5 in [**A-M**]) and this proves that B is semilocal.

$K \otimes_k \overline{k}$ is also reduced (because K/k is separable and we can apply (27.1) Lemma 1 in [**M**]) and $\dim(K \otimes_k \overline{k})_M = \dim(B/\mathcal{M}B)_M = 0$. This proves that $(B/\mathcal{M}B)_M = B_M/\mathcal{M}B_M$ is a field, that is $\mathcal{M}B_M = MB_M$.

Case 2. K_M is separable over $\overline{k}$ for every $M \in \Omega(B)$ because every extension of an algebraically closed field is separable.

Lemma 4.5. *Let $(A, \mathcal{M}, K)$ be a Flenner k-algebra of e.f.t., $\{x_1, \dots, x_n\}$ a generator system of $\mathcal{M}$ and $B = A \otimes_k \overline{k}$. If $LB_k(\mathbf{P})$ holds then:*

a) *for the generic $\alpha = (\alpha_1, \dots, \alpha_n) \in \overline{k}^n$: $\mathbf{P}(B/x_\alpha B) \subseteq \mathbf{P}(B) \cap V(x_\alpha B) \cap U_{\mathbf{P}}$,*

b) *for the generic* $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$: $\mathbf{P}(B/x_\alpha B) \subseteq \mathbf{P}(B) \cap V(x_\alpha B) \cap U_{\mathbf{P}}$,

where $X_\alpha = \Sigma \alpha_i X_i$ *and* $U_P = \operatorname{Spec} B - \Omega(B)$.

Proof. a) In fact the condition $LB_k(\mathbf{P})$ holds for $(B_M, MB_M, K_M)\, \forall\, M \in \Omega(B)$ by Lemma 4.4. So we can find an open dense subset $\mathfrak{I}_M$ of $\overline{k}^n$ such that $\forall\, \alpha \in \mathfrak{I}_M$. $\mathbf{P}(B_M/X_\alpha B_M) \subseteq \mathbf{P}(B_M) \cap V(X_\alpha B_M) \cap U_{\mathbf{P}}$. But B is semilocal by 4.4 so it has a finite number of maximal ideals: $M_1, \ldots, M_d$. Putting $\mathfrak{I} = \mathfrak{I}_{M_1} \cap \ldots \cap \mathfrak{I}_{M_d}$. This is an open dense subset of $\overline{k}^n$ (by Lemma 3.2), independent from M_1 and so $\forall\, \alpha \in \mathfrak{I}$ we have $\mathbf{P}(B/X_\alpha B) \subseteq \mathbf{P}(B) \cap V(X_\alpha B) \cap U_{\mathbf{P}}$.

b) Use a) and Proposition 3.3.

Theorem 4.6. *If* $LB_k(\mathbf{P})$ *holds for some local property* $\mathbf{P}$ *then* $LB_k(\mathbf{GP})$ *holds for the corresponding geometric property* $\mathbf{GP}$.

Proof. If (A, M, K) is a Flenner k-algebra of e.f.t. and $\mathcal{P} \in \mathbf{GP}(A) \cap V(X_\alpha) \cap U_{\mathbf{GP}}$ we have to prove that $\mathcal{P} \in \mathbf{GP}(A/X_\alpha A)$.

Clearly we have: $\mathcal{P} \in \mathbf{GP}(A/X_\alpha A) \Leftrightarrow (A_{\mathcal{P}}/X_\alpha A_{\mathcal{P}}) \otimes_k \overline{k}$ is $\mathbf{P} \Leftrightarrow (A/X_\alpha A) \otimes_A (A_{\mathcal{P}} \otimes_k \overline{k})$ is $\mathbf{P}$.

Considering $\varphi : A \longrightarrow B = A \otimes_k \overline{k}$ and $S = A - \mathcal{P} \Rightarrow A_{\mathcal{P}} \otimes_k \overline{k} \cong S^{-1}B$ by Prop. 3.5 in [**A-M**]. If $\mathcal{Q} \in \operatorname{Spec}(A_{\mathcal{P}} \otimes_k \overline{k})$, let Q be its image in $S^{-1}B$. Then, $\forall\, \mathcal{Q} \in \operatorname{Spec}(A\mathfrak{p} \otimes_k \overline{k})$, $(A\mathfrak{p} \otimes_k \overline{k})_{\mathcal{Q}} \cong B_Q$ is $\mathbf{P}$, i.e., $Q \in \mathbf{P}(B)$ It is also $Q \subset (X_\alpha)^e$ and $Q \in U_{\mathbf{P}}$ (because $\mathfrak{p} \neq \mathcal{M} \Rightarrow Q \notin \Omega(B)$). Applying Lemma 4.5 to B we have: $(B_Q)/(X_\alpha)B_Q \cong (A\mathfrak{p} \otimes_k \overline{k})_{\mathcal{Q}}/(X_\alpha)(A\mathfrak{p} \otimes_k \overline{k})_{\mathcal{Q}}$ is $\mathbf{P}\, \forall\, \mathcal{Q} \in \operatorname{Spec}(A\mathfrak{p} \otimes_k \overline{k}) \Rightarrow (A/x_\alpha A\mathfrak{p}) \otimes_k \overline{k}$ is $\mathbf{P} \Rightarrow \mathfrak{p} \in \mathbf{GP}(A/x_\alpha A)$.

Corollary 4.7. $LB_k(\mathbf{GP})$ *holds for Flenner* K *algebra of e.f.t.* (A, M, K) *if:*

i) $\mathbf{GP} =$ *geom. Regular and* $U_{\mathbf{GP}}(A) = \operatorname{Spec} A$;

ii) $\mathbf{GP} = S_r$, *geom. Serre's property* R_s, *geom. normal, geom. reduced and* $U_{\mathbf{GP}}(A) = \operatorname{Spec} A - \{M\}$;

(*with the notation given in Def.* 4.2)

Proof. By Remark 4.3 and Theorem 4.6.

In connection with Theorem 4.6 it is important know that the $\mathbf{GP}$-locus of an e.f.t. K-algebra is open, at least for the properties $\mathbf{P}$ cited above. This will be shown in Theorem 4.8 below.

Theorem 4.8. *Let* A *be a* K*-algebra of finite type, then* $\mathbf{GP}(A)$ *is an open subset of* $\operatorname{Spec} A$ *for* $\mathbf{GP} = S_r$, *geom. Serre's property* R_s, *geom. Regular, geom. Normal, geom. Reduced.*

Proof. We may assume that A is a K-algebra of finite type. Indeed if A is a K-algebra of e.f.t. then (Def. 2.1) $A = S^{-1}C$ where C is a K-algebra of finite type and S is a multiplicatively closed subset of C. If U is an open subset of $\operatorname{Spec} C$ and if we call φ the continuous map defined from $\operatorname{Spec}(S^{-1}, C)$ to $\operatorname{Spec} C$ induced by the canonical homomorphism $\varphi^* : C \to S^{-1}C$, then $\varphi^{-1}(U)$ is an open subset of $\operatorname{Spec}(S^{-1}C) = \operatorname{Spec} A$. Moreover the properties **GP** are preserved by localization.

(a) Case $\mathbf{GP}$ = geom. Normal, geom. R_n.

We use a proof that looks like Zariski's Theorem in [**EGA**, Chap. IV, 6.12.5].

We consider $A \otimes_k K'$ where $K' = K^{P^{-\infty}}$. The morphism $\operatorname{Spec}(K') \to \operatorname{Spec}(k)$ is a universal homomorphism and so the morphism $\operatorname{Spec}(A \otimes_k K') \to \operatorname{Spec} A$ is a homomorphism.

Then the projection of $\mathbf{P}(A \otimes_k K')$ in $\operatorname{Spec} A$ is just the set $\mathbf{GP}(A)$ (by [**EGA**, Theorem 6.7.7 Chap. IV].

We have only to show that $\mathbf{P}(A \otimes_k K')$ is open in $\operatorname{Spec}(A \otimes_k K')$. But this is true:

i) for $\mathbf{P}$ = regular by [**EGA**, Chap. IV 6.12.5];
ii) for $\mathbf{P} = R_n$ by i) and [**EGA**, Chap. IV 6.12.9];
iii) for $\mathbf{P}$ = normal by i) and [**EGA**, Chap. IV 6.13.5].

(b) Case $\mathbf{GP} = S_n$ and geom. Reduced.

A is a K-algebra of finite type and so it is excellent by Prop. 2.2 (ii). So we can apply consideration [**EGA**, 7.9.7 Chap. IV] for $\mathbf{P} = S_n$ and Prop. 4.6.13 Chap. IV [**EGA**] for $\mathbf{P}$ = reduced.

Using Theorem 4.8 we have:

Corollary 4.9. *If (A, M, K) is a Flenner K-algebra of e.f.t. then $\mathbf{GP}(A)$ is an open subset of* $\operatorname{Spec} A$ *for $\mathbf{GP} = S_r$, geom. Serre's property R_s, geom. regular, geom. normal, geom. reduced.*

5. Application to Global Bertini Theorems.

We want now to deduce from Theorem 4.6 a global Bertini Theorem for geometric properties of hypersurface sections of a projective variety over an arbitrary field.

For this we use a standard technique involving the vertex of the affine cone (see also [**F**, §5]).

We give some notation: Let k be a field, $X \subseteq \mathbf{P}^n_k$ a projective variety over the field k and $Y \subseteq X$ a closed subset of X. Let $Y^+ \subset X^+ \subseteq \mathbf{A}^{n+1}_k$ be the corresponding affine cones; put $A = 0_{x^+,v}$ (where v is the vertex) and let I be the ideal of Y^+ in A. Let $X(\overline{k})$, $Y(\overline{k})$ be the varieties obtained from X and Y by making the base extension field $\to \overline{k}$.

Proposition 5.1. *Let $\mathbf{P}$ be a local property which is preserved by polynomials and fractions and which descends by faithful flatness. With the notation given above, the following are equivalent:*

(i) *$X - Y$ is* **GP** *over k;*
(ii) *$X^+ - Y^+$ is* **GP** *over k;*
(iii) Spec $A - V(I)$ *is* **GP** *over k.*

Proof. $X - y$ is **GP** over $k \Leftrightarrow X(\overline{k}) - Y(\overline{k})$ is **P** $\underset{(1)}{\Leftrightarrow} X^+(\overline{k}) - Y^+(\overline{k})$ is **P**($\Leftrightarrow X^+ - Y^+$ is **GP** over k) $\underset{(2)}{\Leftrightarrow}$ Spec $A(\overline{k}) - V(I(\overline{k}))$ is **P** $\Leftrightarrow$ Spec $A - V(I)$ is **GP** over k, where the equivalencies (1) and (2) are due to Proposition 2.1 in [**CGM**].

In the following let $S = \otimes S_d$ be graded k-algebra of finite type so that $S_0 \cong k$ and $S = k[S_1]$.

Theorem 5.2. *$S = k[S_1]$ a graded k-algebra, k a field with infinitely many elements and $\{f_0, \dots, f_{n(q)}\}$ a generator system of S_q as a k-vector space. Let* **P** *be as in* 5.1.

*If LB_k(**GP**) holds for some geometrical property* **GP** *then, for the generic $\alpha = (\alpha_0, \dots, \alpha_{n(q)}) \in k^{n(q)+1}$ we have that,*

$$\mathbf{GP}(\mathrm{Proj}\,(S/f_\alpha S)) \subseteq \mathbf{GP}(\mathrm{Proj}\,(S)) \cap V^+(f_\alpha)$$

where $f_\alpha = \Sigma \alpha_1 f_1$.

Proof. For $q = 1$ we can apply Prop. 5.1 and Th. 4.6. (Observe that K, the residue field of A, coincides with k and so it is separable over k.) For $q > 1$ we can reduce to the hyperplane case using the Veronese map of degree q.

Corollary 5.3. *With the hypothesis and notation as in Theorem* 5.2 *we have* $\mathbf{GP}(\mathrm{Proj}\,(S/x_\alpha S)) \subseteq \mathbf{GP}(\mathrm{Proj}\,(S)) \cap V^+(X_\alpha)$ *for* $\mathbf{GP} = S_r$, *geom. Serre's property R_s, geom. regular, geom. normal, geom. reduced, regular, etc.*

Proof. Apply Theorem 5.2 and Corollary 4.7.

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ON FINITE PRESENTABILITY OF MONOIDS AND THEIR SCHÜTZENBERGER GROUPS

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The main result of this paper asserts that a monoid with finitely many left and right ideals is finitely presented if and only if all its Schützenberger groups are finitely presented. The most important part of the proof is a rewriting theorem, giving a presentation for a Schützenberger group, which is similar to the Reidemeister-Schreier rewriting theorem for groups.

1. Introduction.

In [**24**, Theorem 4.1] it was proved that a regular monoid S with finitely many left and right ideals is finitely presented if and only if all its maximal subgroups are finitely presented. Recall that the maximal subgroups of S are precisely the $\mathcal{H}$-classes of S containing idempotents. Schützenberger [**25**, **26**] showed how one can assign to an arbitrary $\mathcal{H}$-class H a group $\Gamma(H)$, called the Schützenberger group of H. Schützenberger groups have many features in common with maximal subgroups; in particular, if the $\mathcal{H}$-class H contains an idempotent (and hence is a maximal subgroup) then H and $\Gamma(H)$ are isomorphic. Since their discovery, they have been used in the structure theory of semigroups (see, for example, [**9**, **10**, **15**, **16**, **20**]), but perhaps, as argued in [**13**], not as much as they deserve.

In this paper we consider connections between presentations for a monoid and for its Schützenberger groups, and we prove the following:

Theorem 1.1. *A monoid with finitely many left and right ideals is finitely presented if and only if all its Schützenberger groups are finitely presented.*

The theorem follows from Corollaries 3.3 and 4.4.

In proving the above theorem we show how one can combine presentations of Schützenberger groups to obtain a presentation for the monoid; see Theorem 3.2. More importantly, we prove a rewriting theorem (Theorem 4.2), in many ways similar to the Reidemeister-Schreier theorem for subgroups of groups, which gives a presentation for a Schützenberger group from a presentation for the monoid. In fact, this presentation is effectively computable, provided that the monoid has finitely many left and right ideals.

This opens the way for our rewriting theorem to be used as a tool in computing with finitely presented monoids, along the similar lines to the use of the Reidemeister-Schreier theorem in computational group theory.

The best known consequence of the Reidemeister-Schreier theorem is that a subgroup of finite index in a finitely presented group is itself finitely presented. Paralleling this are Corollary 2.11 and Proposition 2.16 of [**24**], which combined give that a maximal subgroup H (i.e., an $\mathcal{H}$-class containing an idempotent) of a finitely presented monoid is finitely presented, provided the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes. Given the similarity between maximal subgroups and Schützenberger groups, one could reasonably hope that this last condition would be sufficient to guarantee finite presentability of the Schützenberger group $\Gamma(H)$ of an arbitrary $\mathcal{H}$-class (not necessarily containing an idempotent). However, in Section 6 we use our rewriting theorem to construct a finitely presented monoid which contains an $\mathcal{H}$-class H such that H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class, but the Schützenberger group $\Gamma(H)$ is not finitely presented.

2. Preliminaries.

Green's equivalences. Green's equivalences were introduced in [**8**]. They describe the ideal structure of a monoid (or a semigroup). Since their discovery they have become the principal tool in describing the structure and properties of monoids and semigroups; see [**11**]. We give definitions of the relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$, and some of their basic properties that we need in the sequel. For a more complete treatment we refer the reader to [**11**] or [**12**].

Let S be a monoid. Two elements $s, t \in S$ are said to be $\mathcal{R}$-equivalent (respectively, $\mathcal{L}$-equivalent) if they generate the same right (respectively, left) ideal, i.e., if $sS = tS$ (respectively, $Ss = St$); we write $s\mathcal{R}t$ (respectively, $s\mathcal{L}t$). Two elements are $\mathcal{H}$-equivalent if they are both $\mathcal{R}$-equivalent and $\mathcal{L}$-equivalent.

In the following proposition we list some properties of these relations that we will require later.

Proposition 2.1. *Let S be a monoid.*

(i) *Let $s, t \in S$ be such that $s\mathcal{R}t$, and let $p, q \in S$ be such that $sp = t$ and $tq = s$. Then the mapping $x \mapsto xp$ is a bijection from the $\mathcal{H}$-class of s onto the $\mathcal{H}$-class of t; its inverse is the mapping $x \mapsto xq$. In particular, any two $\mathcal{H}$-classes within the same $\mathcal{R}$-class have the same size.*

(ii) *If $s, p_1, p_2 \in S$ are such that $sp_1p_2\mathcal{R}s$ then $sp_1\mathcal{R}s$.*

(iii) *The relation $\mathcal{R}$ is a left congruence, i.e., for all $s, t_1, t_2 \in S$, if $t_1\mathcal{R}t_2$ then $st_1\mathcal{R}st_2$.*

(iv) *Let $s, t, p \in S$. If $s\mathcal{R}t$ and $ps\mathcal{H}s$ then $pt\mathcal{H}t$.*

(v) *For every $s \in S$ the set sS is a union of $\mathcal{R}$-classes.*

The left-right dual statements hold for $\mathcal{L}$-classes.

Proof. Part (i) is [**11**, Lemma 2.2.1, Lemma 2.2.3]. Parts (ii), (iii) and (v) follow immediately from the definitions. For (iv), if $q \in S$ is such that $sq = t$, then the mapping $x \mapsto xq$ is a bijection from the $\mathcal{H}$-class of s onto the $\mathcal{H}$-class of t by (i), and hence $pt = psq\mathcal{H}t$. □

We remark that, unlike $\mathcal{R}$ and $\mathcal{L}$, the relation $\mathcal{H}$ is not, in general, a one sided congruence.

Theorem 1.1 concerns monoids S with finitely many left and right ideals. Since the $\mathcal{R}$-classes of S are in a one-one correspondence with the principal right ideals of S, and since every right ideal of S is a union of principal right ideals of S, we have:

Proposition 2.2. *A monoid has finitely many right ideals if and only if it has finitely many $\mathcal{R}$-classes. Dually, a monoid has finitely many left ideals if and only if it has finitely many $\mathcal{L}$-classes. A monoid has finitely many left and right ideals if and only if it has finitely many $\mathcal{H}$-classes.*

Schützenberger groups. The $\mathcal{H}$-classes in a monoid S exhibit many properties of subgroups of groups. For example, Proposition 2.1 (i) shows that the $\mathcal{H}$-classes within a single $\mathcal{R}$-class behave very much like cosets of a subgroup in a group – a parallel that will be explored in more depth in Sections 4-6. Also it is known that an $\mathcal{H}$-class which contains an idempotent is a maximal subgroup of S, and that all maximal subgroups of S arise in this way; see [**12**, Corollary 2.6].

Schützenberger [**25**, **26**] showed how to assign a group to an arbitrary $\mathcal{H}$-class, so as to reflect the group-like properties of that class. Here we give his construction and some of its basic properties. For more details we refer the reader to [**12**].

Let S be a monoid, and let H be an $\mathcal{H}$-class of S. Denote by $\mathrm{Stab}(H)$ the (right) stabiliser of H in S, i.e., $\mathrm{Stab}(H) = \{s \in S \ : \ Hs = H\}$. On this set define a relation $\sigma(H) = \{(s,t) \in \mathrm{Stab}(H) \times \mathrm{Stab}(H) \ : \ (\forall h \in H)(hs = ht)\}$. It is easy to see that $\sigma(H)$ is a congruence; we call it the *Schützenberger congruence* of H. It is also relatively easy to see that the quotient $\Gamma(H) = \mathrm{Stab}(H)/\sigma(H)$ is a group; it is called the *Schützenberger group* of H. It turns out that $\Gamma(H)$ has the following properties:

- $\Gamma(H)$ acts regularly on H; in particular $|H| = |\Gamma(H)|$;
- if H_1 is an $\mathcal{H}$-class of S belonging to the same $\mathcal{R}$-class, or the same $\mathcal{L}$-class, as H then $\Gamma(H_1) \cong \Gamma(H)$;
- if H contains an idempotent then $\Gamma(H) \cong H$.

For proofs see [**12**, Section 2.3]. Of course, by left-right duality, one may define the *left* Schützenberger group. It turns out, however, that the two are isomorphic.

In the following proposition we list some properties that we will use later. For proofs the reader is again referred to [**12**, Section 2.3].

Proposition 2.3. *Let S be a monoid, let H be an $\mathcal{H}$-class of S, and let $h_0 \in H$ be an arbitrary element. Then:*

(i) $\mathrm{Stab}(H) = \{s \in S \ : \ h_0 s \mathcal{H} h_0\}$*;*

(ii) $\sigma(H) = \{(s,t) \in \mathrm{Stab}(H) \times \mathrm{Stab}(H) \ : \ h_0 s = h_0 t\}$*;*

(iii) $H = h_0 \mathrm{Stab}(H)$*.*

Presentations. Along with transformations, presentations are the most general means of constructing monoids. Throughout the development of the theory of monoid presentations, one of the leitmotivs has been the connection with group presentations; see, for example, [**1**, **4**, **18**, **19**, **21**, **24**]. The results of this paper continue and deepen this theme.

A (monoid) *presentation* is a pair $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$, where A is an alphabet, and $\mathfrak{R} \subseteq A^* \times A^*$ is a set of pairs of words over A. A typical pair $(u,v) \in \mathfrak{R}$ is usually written as $u = v$ and is called a *defining relation.* A monoid S is said to be *defined* by $\mathfrak{P}$ if $S \cong A^*/\rho$, where ρ is the smallest congruence on the free monoid A^* containing $\mathfrak{R}$. Thus every word $w \in A^*$ represents an element of S. As is customary, we identify a word and the element of S it represents. To lessen the likelihood of confusion in doing so, for two words $w_1, w_2 \in A^*$ we write $w_1 \equiv w_2$ if they are identical, and $w_1 = w_2$ if they represent the same element of S, i.e., if $w_1/\rho = w_2/\rho$.

For two words $w_1, w_2 \in A^*$ we say that w_2 is *obtained from w_1 by one application of a relation* from $\mathfrak{R}$ if $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$, where $\alpha, \beta \in A^*$ and $(u = v) \in \mathfrak{R}$ or $(v = u) \in \mathfrak{R}$. We shall often use the following standard fact without explicit mention:

Proposition 2.4. *Let $\langle A \mid \mathfrak{R} \rangle$ be a presentation, let S be the monoid defined by it, and let $w_1, w_2 \in A^*$ be two arbitrary words. Then the relation $w_1 = w_2$ holds in S if and only if $w_1 \equiv w_2$ or there exists a sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_m \equiv w_2$ of words in which each α_{i+1} $(1 \leq i \leq m-1)$ is obtained from α_i by one application of a relation from $\mathfrak{R}$.*

3. From the Schützenberger groups to the monoid.

In this section we show how one can combine presentations of the Schützenberger groups of a monoid to obtain a presentation for the whole monoid. An immediate corollary of this result is the converse part of Theorem 1.1.

Let S be an arbitrary monoid, and let $S/\mathcal{H} = \{H_i \ : \ i \in I\}$ be the collection of all $\mathcal{H}$-classes of S. For each $i \in I$ fix an element

$$h_i \in H_i. \tag{1}$$

Without loss of generality assume that I contains a distinguished element 1 and that

$$1_S = h_1 \in H_1. \tag{2}$$

(In other words, H_1 is the group of units of S.) For $i \in I$ let $\Gamma_i = \Gamma(H_i)$ be the Schützenberger group of H_i. Recall that $\Gamma_i = T_i/\sigma_i$, where $T_i = \mathrm{Stab}(H_i)$ is the stabiliser of H_i, and $\sigma_i = \sigma(H_i)$ is the corresponding Schützenberger congruence.

Proposition 3.1. *With the above notation, if each H_i is generated by a set A_i/σ_i ($i \in I$, $A_i \subseteq S$) then S is generated by the set $B = \{h_i : i \in I\} \cup (\bigcup_{i\in I} A_i)$.*

Proof. Let $s \in S$ be arbitrary. Then there is a unique $i \in I$ such that $s \in H_i$, i.e., $s = h_i t$ for some $t \in T_i$ by Proposition 2.3 (iii). Hence we have $t/\sigma_i \in \Gamma_i = \langle A_i/\sigma_i\rangle = \langle A_i\rangle/\sigma_i$, so that there exists $w \in \langle A_i\rangle$ such that $(t, w) \in \sigma_i$. Now we have $s = h_i t = h_i w \in \langle B\rangle$, completing the proof. □

Our aim now is to find a presentation for S in terms of the generating set B given above. The idea is to note from the above proof that every element of S can be written in the form $h_i w$ ($i \in I$, $w \in A_i^*$), and to find defining relations which allow one to transform any word from B^* into this form. To do this we consider the results of multiplying representatives h_i ($i \in I$) of $\mathcal{H}$-classes by arbitrary generators from B both from left and right.

First for each $i \in I$ and each $x \in B$ we let $\zeta(i, x) \in I$ be the unique element such that

$$h_i x \in H_{\zeta(i,x)}. \tag{3a}$$

By Proposition 2.3 (iii) it follows that $h_i x = h_{\zeta(i,x)} s$ for some $s \in T_{\zeta(i,x)}$. From $s/\sigma_{\zeta(i,x)} \in \Gamma_{\zeta(i,x)} = \langle A_{\zeta(i,x)}\rangle/\sigma_{\zeta(i,x)}$ it follows that there exists $w \in A^*_{\zeta(i,x)}$ such that $s/\sigma_{\zeta(i,x)} = w/\sigma_{\zeta(i,x)}$. For each choice of i and x we choose (arbitrarily) and fix one such word $w = \mu(i, x)$. Thus we have

$$\mu(i, x) \in A^*_{\zeta(i,x)} \tag{3b}$$

and the relation

$$h_i x = h_{\zeta(i,x)}\mu(i, x) \ \ (i \in I,\ x \in B) \tag{3}$$

holds in S. In a similar way, for any $i \in I$, $x \in B$ we let

$$\eta(i, x) \in I,\ \nu(i, x) \in A^*_{\eta(i,x)} \tag{4a}$$

be such that the relation

$$x h_i = h_{\eta(i,x)}\nu(i, x) \ \ (i \in I,\ x \in B) \tag{4}$$

holds in S, and we also let

$$\theta(i, x) \in I,\ \pi(i, x) \in B^* \tag{5a}$$

be such that

$$h_i x \in H_{\theta(i,x)} \tag{5b}$$

and the relation

$$h_i x = \pi(i,x) h_{\theta(i,x)} \quad (i \in I,\ x \in B) \tag{5}$$

holds in S.

Theorem 3.2. *If, with the above notation, each Schützenberger group* Γ_i $(i \in I)$ *is defined by a presentation* $\langle A_i \mid \mathfrak{R}_i \rangle$ *in terms of generators* A_i/σ_i*, then* S *is defined by the presentation with generators* $B = \{h_i \ :\ i \in I\} \cup (\bigcup_{i\in I} A_i)$ *and relations* (3), (4), (5) *and*

$$h_i u = h_i v \quad (i \in I,\ (u = v) \in \mathfrak{R}_i), \tag{6}$$

$$h_1 = 1. \tag{7}$$

Proof. First we note that all the relations obviously hold in S. So to prove the theorem it is sufficient to show that every relation holding in S is a consequence of the relations (3)-(7).

To this end we let $w \in B^*$ be arbitrary. Via a series of claims we show that w can be transformed to a particular form using (3)-(7).

Claim 1. There exist $i_1 \in I$ and $w_1 \in B^*$ such that the following two conditions are satisfied:

(i) the relation $w = w_1 h_{i_1}$ is a consequence of the relations (3)-(7); and

(ii) for every suffix w_1' of w we have $w_1' h_{i_1} \mathcal{L} h_{i_1}$ in S.

Proof. Write $w \equiv x_1 x_2 \ldots x_m$ $(x_i \in B)$. From (7) we have

$$w = h_1 x_1 x_2 \ldots x_m.$$

By successively applying relations (5) we obtain

$$h_1 x_1 x_2 \ldots x_m = \pi(j_1, x_1)\pi(j_2, x_2) \ldots \pi(j_m, x_m) h_{j_{m+1}},$$

where

$$j_1 = 1,\ j_{k+1} = \theta(j_k, x_k) \quad (k = 1, \ldots, m).$$

So if we let $w_1 \equiv \pi(j_1, x_1) \ldots \pi(j_m, x_m)$ and $i_1 = j_{m+1}$, the condition (i) is satisfied. From (5b) we have

$$x_1 = h_{j_1} x_1 \mathcal{L} h_{j_2},\ h_{j_2} x_2 \mathcal{L} h_{j_3}, \ldots, h_{j_m} x_m \mathcal{L} h_{j_{m+1}} = h_{i_1}.$$

Since $\mathcal{L}$ is a right congruence (the dual of Proposition 2.1 (iii)) it follows that

$$w_1 h_{i_1} = w \equiv x_1 x_2 \ldots x_m \mathcal{L} h_{j_2} x_2 \ldots x_m \mathcal{L} \ldots \mathcal{L} h_{j_m} x_m \mathcal{L} h_{j_{m+1}} = h_{i_1}.$$

Therefore, by the dual of Proposition 2.1 (ii), it follows that the condition (ii) is satisfied as well. □

Claim 2. There exist $i_2 \in I$ and $w_2 \in B^*$ such that the following two conditions are satisfied:

(i) the relation $w_1 h_{i_1} = h_{i_2} w_2$ is a consequence of the relations (3)-(7); and
(ii) for every letter x of w_2 we have $h_{i_2} x \in H_{i_2}$.

Proof. Write $w_1 \equiv x_1 \dots x_m$ and apply (4) successively to obtain

$$w_1 h_{i_1} = h_{j_0} \nu(j_1, x_1) \dots \nu(j_m, x_m),$$

where

$$j_m = i_1,\ j_{k-1} = \eta(j_k, x_k) \ \ (k = m, \dots, 1).$$

So, if we let $i_2 = j_0$ and $w_2 \equiv \nu(j_1, x_1) \dots \nu(j_m, x_m)$, the condition (i) is satisfied.

We next claim that for every k $(1 \le k \le m)$ we have

$$x_k x_{k+1} \dots x_m h_{i_1} \in H_{j_{k-1}}. \tag{8}$$

For $k = m$ this follows from (4a) and (4). Assume inductively that (8) holds for some k. Since, by Claim 1 (ii), we have

$$x_{k-1} x_k \dots x_m h_{i_1} \mathcal{L} h_{i_1} \mathcal{L} x_k \dots x_m h_{i_1},$$

it follows by (the dual of) Proposition 2.1 (i) that the mapping $t \mapsto x_{k-1} t$ is a bijection from the $\mathcal{H}$-class of $x_k \dots x_m h_{i_1}$ onto that of $x_{k-1} x_k \dots x_m h_{i_1}$. In particular, we have

$$x_{k-1} x_k \dots x_m h_{i_1} \mathcal{H} x_{k-1} h_{j_{k-1}} \mathcal{H} h_{j_{k-2}}$$

by (4a) and (4), thus completing the inductive proof of (8).

By Claim 1 (ii) we now conclude that

$$h_{i_2} = h_{j_0} \mathcal{L} h_{j_1} \mathcal{L} \dots \mathcal{L} h_{j_m} = h_{i_1}. \tag{9}$$

By (4a) we have $\nu(j_k, x_k) \in A^*_{j_{k-1}}$ $(k = 1, \dots, m)$. By the choice of $A_{j_{k-1}}$ every letter from it stabilises $H_{j_{k-1}}$. Therefore by the dual of Proposition 2.1 (iv), Proposition 2.3 (i) and (9) it follows that every letter x of $\nu(j_k, x_k)$ stabilises H_{i_2}; in particular, $h_{i_2} x \in H_{i_2}$, as required. □

Claim 3. There exist $i_3 \in I$ and $w_3 \in A^*_{i_3}$ such that the relation $h_{i_2} w_2 = h_{i_3} w_3$ is a consequence of the relations (3)-(7).

Proof. Write $w_2 \equiv x_1 \dots x_m$, and note that

$$\zeta(i_2, x_j) = \theta(i_2, x_j) = i_2 \ \ (j = 1, \dots, m)$$

by (3a), (5b) and Claim 2 (ii). Therefore we have

$$\begin{array}{rcll} h_{i_2}w_2 & \equiv & h_{i_2}x_1\dots x_m & \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-1})h_{i_2}x_m & \text{(by (5))} \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-1})h_{i_2}\mu(i_2,x_m) & \text{(by (3))} \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-2})h_{i_2}x_{m-1}\mu(i_2,x_m) & \text{(by (5))} \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-2})h_{i_2}\mu(i_2,x_{m-1})\mu(i_2,x_m) & \text{(by (3))} \\ & = & \dots & \\ & = & h_{i_2}\mu(i_2,x_1)\mu(i_2,x_2)\dots\mu(i_2,x_m). & \end{array}$$

Therefore it is sufficient to let $i_3 = i_2$ and $w_3 \equiv \mu(i_2,x_1)\dots\mu(i_2,x_m)$. □

Now let $w' \in B^*$ be any word, and assume that the relation $w = w'$ holds in S. Write w' as $w' = h_{i'_3}w'_3$ as above. Then, since $h_{i_3}\mathcal{H}w = w'\mathcal{H}h_{i'_3}$, we must have $i_3 = i'_3 = i$ and also $w_3/\sigma_i = w'_3/\sigma_i$ in Γ_i by Proposition 2.3 (ii). Since $\langle A_i \mid \mathfrak{R}_i\rangle$ is a presentation for Γ_i, it follows that w'_3 can be obtained from w_3 by a sequence of applications of relations from $\mathfrak{R}_i$. We are now going to show that this implies that $h_iw'_3$ can be obtained from h_iw_3 by a sequence of applications of relations (3)-(7), which will complete the proof that we indeed have a presentation for S.

Without loss of generality we may assume that w'_3 is obtained from w_3 by one application of a relation from $\mathfrak{R}_i$:

$$w_3 \equiv \alpha u\beta,\ w'_3 \equiv \alpha v\beta\ \ (\alpha,\beta \in A_i^*,\ (u = v) \in \mathfrak{R}_i).$$

Writing $\alpha \equiv x_1\dots x_m$, we have

$$\begin{array}{rcll} h_iw_3 & \equiv & h_ix_1\dots x_mu\beta & \\ & = & \pi(i,x_1)\dots\pi(i,x_m)h_iu\beta & \text{(by (5))} \\ & = & \pi(i,x_1)\dots\pi(i,x_m)h_iv\beta & \text{(by (6))} \\ & = & h_ix_1\dots x_mv\beta \equiv h_iw'_3, & \text{(by (5))} \end{array}$$

as required. □

If the set I is finite (which, by Proposition 2.2 is the case precisely when S has finitely many left and right ideals), and if all the presentations $\langle A_i \mid \mathfrak{R}_i\rangle$ $(i \in I)$ are finite, then so is the above presentation for S. Therefore we have the converse part of Theorem 1.1:

Corollary 3.3. *Let S be a monoid with finitely many left and right ideals. If all the Schützenberger groups of S are finitely presented then S is finitely presented as well.*

4. A rewriting theorem for the Schützenberger group.

The aim of this section is to state a theorem (Theorem 4.2) giving a presentation for the Schützenberger group of an $\mathcal{H}$-class in a monoid defined by a presentation, and to deduce some immediate corollaries, including the direct part of Theorem 1.1. The theorem is proved in the next section.

Let S be a monoid, and let $\langle A \mid \mathfrak{R}\rangle$ be a presentation for S. Let H be an arbitrary $\mathcal{H}$-class of S, and fix a word $h \in A^*$ representing an element of H. Denote by $\Gamma = \Gamma(H)$ the Schützenberger group of H; so $\Gamma = T/\sigma$, where $T = \mathrm{Stab}(H)$ and $\sigma = \sigma(H)$ is the Schützenberger congruence on T.

Let R be the $\mathcal{R}$-class of h, and let $\{H_\lambda \ : \ \lambda \in \Lambda\}$ be the collection of all $\mathcal{H}$-classes of S contained in R. For each $\lambda \in \Lambda$ choose words $p_\lambda, p'_\lambda \in A^*$ such that

$$Hp_\lambda = H_\lambda,\ h_1 p_\lambda p'_\lambda = h_1,\ h_2 p'_\lambda p_\lambda = h_2\ \ (\lambda \in \Lambda,\ h_1 \in H,\ h_2 \in H_\lambda); \tag{10}$$

such words exist by Proposition 2.1 (i). Without loss of generality assume that Λ contains a distinguished element 1, and that

$$H_1 = H,\ p_1 \equiv p'_1 \equiv \epsilon, \tag{11}$$

where ϵ denotes the empty word.

By Proposition 2.1 (i), (ii), for any $s \in S$ and any $\lambda \in \Lambda$, either $H_\lambda s = H_\mu$ for some $\mu \in \Lambda$, or $H_\lambda s s_1 \cap R = \emptyset$ for all $s_1 \in S$. Therefore we can define an action $(\lambda, s) \mapsto \lambda \cdot s$ of S on the set $\Lambda \cup \{0\}$ (assuming $0 \notin \Lambda$) by

$$\lambda \cdot s = \begin{cases} \mu & \text{if } \lambda, \mu \in \Lambda \text{ and } H_\lambda s = H_\mu, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

In the following theorem we give a generating set for Γ, resembling the Schreier generating set for a subgroup of a group (see [**17**, Theorem 2.7]). This result is not new – it is an immediate consequence of Schützenberger's original results [**25**, **26**], and can also be found, in a slightly different notation from ours, in [**14**, Corollary 2.3]. Nevertheless, we will give a proof of this result, because it motivates the definition of a rewriting mapping to follow.

Proposition 4.1. *With the above notation the Schützenberger group Γ of H is generated by the set*

$$X = \{(p_\lambda a p'_{\lambda\cdot a})/\sigma \ : \ \lambda \in \Lambda,\ a \in A,\ \lambda \cdot a \neq 0\}.$$

Proof. First we claim that

$$\Gamma = \{(p_\lambda s p'_{\lambda\cdot s})/\sigma \ : \ \lambda \in \Lambda,\ s \in S,\ \lambda \cdot s \neq 0\}.$$

Denote the right hand side by Γ'. By using (10) and (12) we have

$$Hp_\lambda s p'_{\lambda\cdot s} = H_\lambda s p'_{\lambda\cdot s} = H_{\lambda\cdot s} p'_{\lambda\cdot s} = H.$$

Hence $p_\lambda s p'_{\lambda\cdot s} \in T$, so that Γ' is well defined and $\Gamma' \subseteq \Gamma$. Conversely, if $s/\sigma \in \Gamma$, then from $Hs = H$ it follows that $1 \cdot s = 1$, and hence $s/\sigma = (p_1 s p'_1)/\sigma \in \Gamma'$.

To complete the proof of the proposition we show that an arbitrary element $(p_\lambda s p'_{\lambda\cdot s})/\sigma$ of Γ' can be written as a product of elements of X. We write $s \equiv a_1 \dots a_m$ $(a_i \in A)$ and proceed by induction on m. For $m = 0$

there is nothing to prove, and for $m = 1$ we have an element of X. For $m > 1$ write $a = a_1$, $t \equiv a_2 \dots a_m$; we have

$$\begin{array}{lll} & (p_\lambda s p'_{\lambda\cdot s})/\sigma & \\ = & (p_\lambda a t p'_{\lambda\cdot at})/\sigma & \\ = & (p_\lambda a p'_{\lambda\cdot a} p_{\lambda\cdot a} t p'_{(\lambda\cdot a)\cdot t})/\sigma & \text{(by (10))} \\ = & ((p_\lambda a p'_{\lambda\cdot a})/\sigma)((p_{\lambda\cdot a} t p'_{(\lambda\cdot a)\cdot t})/\sigma) & (\text{since } p_\lambda a p'_{\lambda\cdot a}, p_{\lambda\cdot a} t p'_{(\lambda\cdot a)\cdot t} \in T) \\ \in & \langle X\rangle, & \text{(by induction)} \end{array}$$

completing the proof. □

We are going to find a presentation for Γ in terms of the above generating set X. To this end we introduce a new alphabet

$$B = \{b(\lambda, a) \ : \ \lambda \in \Lambda,\ a \in A,\ \lambda \cdot a \neq 0\}. \tag{13}$$

The letter $b(\lambda, a)$ is thought of as representing the generator $(p_\lambda a p'_{\lambda\cdot a})/\sigma$. To make this more formal we introduce a homomorphism

$$\psi \ : \ B^* \longrightarrow A^*,\ b(\lambda, a) \mapsto p_\lambda a p'_{\lambda\cdot a}; \tag{14}$$

we refer to ψ as the *representation mapping.*

Motivated by the proof of Proposition 4.1 we also define a mapping

$$\phi \ : \ \{(\lambda, w) \in \Lambda \times A^* \ : \ \lambda \cdot w \neq 0\} \longrightarrow B^*,$$

called the *rewriting mapping*, inductively by

$$\phi(\lambda, \epsilon) = \epsilon,\ \phi(\lambda, aw) = b(\lambda, a)\phi(\lambda \cdot a, w). \tag{15}$$

The idea behind this definition is that ϕ simulates the process of rewriting an element $p_\lambda s p'_{\lambda\cdot s}$ into a product of generators from X as in the proof of Proposition 4.1.

Since for each $\lambda \in \Lambda$, $a \in A$ satisfying $\lambda \cdot a \neq 0$ we have $hp_\lambda a p'_{\lambda\cdot a} \in H$, it follows that there is a word $\pi(b(\lambda, a)) \in A^*$ such that

$$hp_\lambda a p'_{\lambda\cdot a} = \pi(b(\lambda, a))h. \tag{16}$$

We extend the mapping $b(\lambda, a) \mapsto \pi(b(\lambda, a))$ to a homomorphism $\pi \ : B^* \longrightarrow A^*$.

Recall that the relation $\mathcal{R}$ is a left congruence on S. Therefore there is a natural left action $(s, R') \mapsto s * R'$ of S on the set $S/\mathcal{R}$ of all $\mathcal{R}$-classes given by

$$s * R' = R'' \Longleftrightarrow sR' \subseteq R'' \ \ (s \in S,\ R', R'' \in S/\mathcal{R}). \tag{17}$$

Let $\{R_i \ : \ i \in I\}$ be the inverse orbit of R under this action, i.e., let it be the set $\{R' \in S/\mathcal{R} \ : \ (\exists s \in S)(s * R' = R)\}$. Then the action of S on $S/\mathcal{R}$

induces a partial action on $\{R_i \ : \ i \in I\}$, which, in turn, translates into an action $(s,i) \mapsto s * i$ of S on the set $I \cup \{0\}$ (assuming $0 \notin I$) given by

$$(18) \qquad s * i = \begin{cases} j & \text{if } i,j \in I \text{ and } s * R_i = R_j, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in I$ choose a word $r_i \in A^*$ representing an element of R_i. Without loss of generality assume that I contains two distinguished elements 1 and ω, and that

$$(19) \qquad 1_S \in R_1, \ r_1 \equiv \epsilon, \ R = R_\omega, \ r_\omega \equiv h.$$

For each $a \in A$ and each $i \in I$ such that $a * i \neq 0$ we have $ar_i \in a * R_i = R_{a*i}$ by (17) and (18). Therefore we can choose words $\tau(a,i) \in A^*$ such that the relations

$$(20) \qquad ar_i = r_{a*i}\tau(a,i) \ \ (a \in A, \ i \in I)$$

hold in S. We extend the mapping $(a,i) \mapsto \tau(a,i)$ to a mapping

$$\tau \ : \ \{(w,i) \in A^* \times I \ : \ w * i \neq 0\} \longrightarrow A^*$$

inductively by

$$(21) \qquad \tau(\epsilon,i) = \epsilon, \ \tau(wa,i) = \tau(w,a*i)\tau(a,i).$$

We can now formulate the result giving a presentation for Γ.

Theorem 4.2. *With the above notation the Schützenberger group Γ of H is defined by the presentation with generators B and relations*

(22) $\phi(\lambda,u) = \phi(\lambda,v) \ \ (\lambda \in \Lambda, \ (u=v) \in \mathfrak{R}, \ \lambda \cdot u \neq 0)$,

(23) $\phi(\lambda,\tau(u,i)) = \phi(\lambda,\tau(v,i)) \ \ (\lambda \in \Lambda, \ i \in I, \ (u=v) \in \mathfrak{R}, \ H_\lambda \subseteq Sr_{u*i})$,

(24) $\phi(1,\tau(\pi(b(\lambda,a)),\omega)) = b(\lambda,a) \ \ (\lambda \in \Lambda, \ a \in A, \ \lambda \cdot a \neq 0)$,

(25) $\phi(1,\tau(h,1)) = 1$.

The above presentation is finite, provided that A, $\mathfrak{R}$, Λ and I are all finite. Therefore we have:

Corollary 4.3. *Let S be a finitely presented monoid, and let H be an $\mathcal{H}$-class of S such that the following two conditions are satisfied:*

(i) *the $\mathcal{R}$-class R of H has only finitely many $\mathcal{H}$-classes; and*

(ii) *the inverse orbit of R under the left action of S on its $\mathcal{R}$-classes has only finitely many elements.*

Then the Schützenberger group of H is finitely presented.

Bearing in mind Proposition 2.2, we obtain the direct part of Theorem 1.1 as a special case of Corollary 4.3:

Corollary 4.4. *If S is a finitely presented monoid with finitely many left and right ideals, then all Schützenberger groups of S are finitely presented.*

In conclusion to this section, we emphasise again the significant role of the rewriting mapping ϕ in the above results. This notion is standard in combinatorial group theory, in the context of Reidemeister-Schreier theory; see [**17**]. It has proved equally important in the theory of monoid and semigroup presentations, in which it occurs in a variety of contexts and takes on a number of different forms; see [**2**, **5**, **6**, **7**, **22**, **23**].

5. Proof of the rewriting theorem.

We prove Theorem 4.2 by showing that all the relations (22)-(25) hold in the Schützenberger group Γ (Lemmas 5.5-5.8) and that any other relation which holds in Γ is a consequence of these relations (Lemma 5.11). Since the argument contains a considerable amount of technical detail, we break it up into a number of lemmas. We use the notation introduced in the previous section.

We begin by giving some properties of the rewriting mapping ϕ.

Lemma 5.1. (i) *For every $w_1, w_2 \in A^*$ and every $\lambda \in \Lambda$ such that $\lambda \cdot w_1 w_2 \neq 0$ we have*

$$\phi(\lambda, w_1 w_2) \equiv \phi(\lambda, w_1)\phi(\lambda \cdot w_1, w_2).$$

(ii) *For every $w \in A^*$ and every $\lambda \in \Lambda$ such that $\lambda \cdot w \neq 0$ the relation*

$$h\psi\phi(\lambda, w) = hp_\lambda w p'_{\lambda \cdot w}$$

holds in S.

(iii) *If $w_1, w_2 \in A^*$ are such that the relation $w_1 = w_2$ holds in S, and if $\lambda \in \Lambda$ is such that $\lambda \cdot w_1 \neq 0$, then the relation $\phi(\lambda, w_1) = \phi(\lambda, w_2)$ is a consequence of the relations* (22)-(25).

Proof. (i) The assertion is proved by a straightforward induction on the length of w_1, using (15).

(ii) This is proved by induction on the length of w, essentially repeating the proof of Proposition 4.1.

(iii) If $w_1 = w_2$ in S then there is a sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_n \equiv w_2$ of words from A^* in which each α_{i+1} is obtained from α_i by one application of a relation from $\mathfrak{R}$. If

$$\alpha_i \equiv \beta_i u_i \gamma_i,\ \alpha_{i+1} \equiv \beta_i v_i \gamma_i \ \ (i \in I,\ \beta_i, \gamma_i \in A^*,\ (u_i = v_i) \in \mathfrak{R})$$

then

$$\begin{aligned}
\phi(\lambda, \alpha_i) &\equiv \phi(\lambda, \beta_i)\phi(\lambda \cdot \beta_i, u_i)\phi(\lambda \cdot \beta_i u_i, \gamma_i) && \text{(part (i))} \\
&= \phi(\lambda, \beta_i)\phi(\lambda \cdot \beta_i, v_i)\phi(\lambda \cdot \beta_i u_i, \gamma_i) && \text{(by (22))} \\
&\equiv \phi(\lambda, \beta_i)\phi(\lambda \cdot \beta_i, v_i)\phi(\lambda \cdot \beta_i v_i, \gamma_i) && \text{(since } u_i = v_i \text{ in } S) \\
&\equiv \phi(\lambda, \alpha_{i+1}), && \text{(part (i))}
\end{aligned}$$

a consequence of (22). Therefore $\phi(\lambda, w_2)$ can be obtained from $\phi(\lambda, w_1)$ by using relations (22). □

Next we prove two similar properties of the mapping τ.

Lemma 5.2. (i) *For every $w_1, w_2 \in A^*$ and $i \in I$ such that $w_1w_2 * i \neq 0$ we have*

$$\tau(w_1w_2, i) \equiv \tau(w_1, w_2 * i)\tau(w_2, i).$$

(ii) *For every $w \in A^*$ and $i \in I$ such that $w * i \neq 0$ the relation*

$$wr_i = r_{w*i}\tau(w, i)$$

holds in S.

Proof. (i) The assertion can be proved by a straightforward induction on the length of w_2, using (21).

(ii) We prove the statement by induction on the length of w. For $|w| = 0$ there is nothing to prove, and for $|w| = 1$ the statement is (20). Let $|w| > 1$ and write $w \equiv w_1w_2$ with $|w_1|, |w_2| > 0$. By using (i) and induction we have

$$wr_i \equiv w_1w_2r_i = w_1r_{w_2*i}\tau(w_2, i) = r_{w_1w_2*i}\tau(w_1, w_2 * i)\tau(w_2, i) \equiv r_{w*i}\tau(w, i),$$

as required. ☐

The following lemma describes the connection between the mappings ψ and π.

Lemma 5.3. *For every $w \in B^*$ the relation*

$$h\psi(w) = \pi(w)h$$

holds in S.

Proof. The assertion follows from (14), (16) and the fact that both ψ and π are homomorphisms. ☐

Next we give two facts relating the mappings τ and π and the actions of S on the sets $I \cup \{0\}$ and $\Lambda \cup \{0\}$.

Lemma 5.4. *For every $w \in B^*$ we have*

(i) *$\pi(w) * \omega = \omega$; and*

(ii) *$1 \cdot \tau(\pi(w), \omega) = 1$.*

Proof. By Proposition 4.1, Lemma 5.3, (14) and (19) we have

$$\pi(w)r_\omega \equiv \pi(w)h = h\psi(w) \in H \subseteq R = R_\omega,$$

proving (i). Now, by Lemma 5.2 (ii) and (19), we have

$$h\tau(\pi(w), \omega) \equiv r_{\pi(w)*\omega}\tau(\pi(w), \omega) = \pi(w)r_\omega \equiv \pi(w)h = h\psi(w) \in H = H_1,$$

and (ii) follows. ☐

We can now proceed to prove that the relations (22)-(25) hold in Γ. Recall that a generator $b(\lambda, a)$ from B represents the element $(p_\lambda a p'_{\lambda\cdot a})/\sigma = \psi(b(\lambda, a))/\sigma$ of Γ. Therefore in order to verify that a relation $\alpha = \beta$ ($\alpha, \beta \in B^*$) holds in Γ one needs to verify that $\psi(\alpha)/\sigma = \psi(\beta)/\sigma$, or, equivalently, that $h\psi(\alpha) = h\psi(\beta)$ holds in S.

Lemma 5.5. *For every relation* $(u = v) \in \mathfrak{R}$ *and every* $\lambda \in \Lambda$ *such that* $\lambda \cdot u \neq 0$ *the relation*

$$\phi(\lambda, u) = \phi(\lambda, v)$$

holds in Γ.

Proof. Using Lemma 5.1 (ii) and the fact that $u = v$ in S we have

$$h\psi\phi(\lambda, u) = hp_\lambda u p'_{\lambda\cdot u} = hp_\lambda v p'_{\lambda\cdot v} \equiv h\psi\phi(\lambda, v),$$

as required. □

Lemma 5.6. *For every relation* $(u = v) \in \mathfrak{R}$ *and every* $\lambda \in \Lambda$ *and* $i \in I$ *such that* $H_\lambda \subseteq Sr_{u*i}$ *the relation*

$$\phi(\lambda, \tau(u, i)) = \phi(\lambda, \tau(v, i))$$

holds in Γ.

Proof. Since $u = v$ holds in S it follows that $ur_i = vr_i$ also holds in S and that $u * i = v * i$. Therefore, by Lemma 5.2 (ii), the relation

$$r_{u*i}\tau(u, i) = r_{v*i}\tau(v, i) \tag{26}$$

holds in S. Let $q \in A^*$ be such that $hp_\lambda = qr_{u*i}$. Premultiplying (26) by q yields

$$hp_\lambda\tau(u, i) = hp_\lambda\tau(v, i); \tag{27}$$

in particular $\lambda \cdot \tau(u, i) = \lambda \cdot \tau(v, i)$. Now, using Lemma 5.1 and (27), we have

$$h\psi\phi(\lambda, \tau(u, i)) = hp_\lambda\tau(u, i)p'_{\lambda\cdot\tau(u,i)} = hp_\lambda\tau(v, i)p'_{\lambda\cdot\tau(v,i)} = h\psi\phi(\lambda, \tau(v, i)),$$

as required. □

Lemma 5.7. *For every* $a \in A$ *and every* $\lambda \in \Lambda$ *such that* $\lambda \cdot a \neq 0$ *the relation*

$$\phi(1, \tau(\pi(b(\lambda, a)), \omega)) = b(\lambda, a)$$

holds in Γ.

Proof. This time we have

$$\begin{aligned}
& h\psi\phi(1, \tau(\pi(b(\lambda, a)), \omega)) \\
= \; & h\tau(\pi(b(\lambda, a)), \omega) && \text{(Lemmas 5.4 and 5.1 (ii) and (11))} \\
= \; & \pi(b(\lambda, a))r_\omega && \text{(Lemma 5.2 (ii) and (19))} \\
= \; & h\psi(b(\lambda, a)), && \text{(Lemma 5.3 and (19))}
\end{aligned}$$

as required. □

Lemma 5.8. *The relation*

$$\phi(1, \tau(h, 1)) = 1$$

holds in Γ.

Proof. From Lemma 5.2 (ii) and (19) we have

$$h \equiv hr_1 = r_{h*1}\tau(h, 1) \equiv r_\omega \tau(h, 1) \equiv h\tau(h, 1),$$

and hence $1 \cdot \tau(h, 1) = 1$. By Lemma 5.1 (ii) and (11) we now have

$$h\psi\phi(1, \tau(h, 1)) = h\tau(h, 1) = h \equiv h\psi(\epsilon),$$

completing the proof of the lemma. □

We now turn to the second part of the proof of Theorem 4.2, that is to show that every relation which holds in Γ is a consequence of the relations (22)-(25). The technical part of the argument is contained in the following two lemmas.

Lemma 5.9. *For any word* $w \in B^*$ *the relation*

$$\phi(1, \tau(\pi(w)h, 1)) = w$$

is a consequence of the relations (22)-(25).

Proof. We prove the lemma by induction on the length of w. If $|w| = 0$ then this is the relation (25), and if $|w| = 1$ this is one of the relations (24). Assume that $|w| > 1$ and write $w \equiv w_1w_2$ with $|w_1|, |w_2| > 0$. Recall that π is a homomorphism, so that $\pi(w) \equiv \pi(w_1)\pi(w_2)$. Now we have

$$\begin{aligned}
& \phi(1, \tau(\pi(w)h, 1)) \\
\equiv\ & \phi(1, \tau(\pi(w_1), \omega)\tau(\pi(w_2), \omega)\tau(h, 1)) && \text{(Lemmas 5.2 (i) and 5.4)} \\
\equiv\ & \phi(1, \tau(\pi(w_1), \omega))\phi(1, \tau(\pi(w_2), \omega))\phi(1, \tau(h, 1)) \\
& && \text{(Lemmas 5.1 (i) and 5.4)} \\
=\ & \phi(1, \tau(\pi(w_1), \omega))\phi(1, \tau(h, 1))\phi(1, \tau(\pi(w_2), \omega))\phi(1, \tau(h, 1)) \\
& && \text{(by (25))} \\
\equiv\ & \phi(1, \tau(\pi(w_1)h, 1))\phi(1, \tau(\pi(w_2)h, 1)) && \text{(Lemmas 5.1 (i), 5.2 (i), 5.4)} \\
=\ & w_1w_2 \equiv w, && \text{(induction)}
\end{aligned}$$

as required. □

Lemma 5.10. *Let* $\alpha, \beta \in A^*$ *and* $(u = v) \in \mathfrak{R}$ *be such that* $\alpha u\beta$ *represents an element of* R. *Then the relation*

$$\phi(1, \tau(\alpha u\beta, 1)) = \phi(1, \tau(\alpha v\beta, 1))$$

is a consequence of the relations (22)-(25).

Proof. Since $u = v$ in S it follows that $u\beta * 1 = v\beta * 1$, and hence

$$\phi(1, \tau(\alpha, u\beta * 1)) \equiv \phi(1, \tau(\alpha, v\beta * 1)). \tag{28}$$

Next we claim that

$$\phi(1 \cdot \tau(\alpha, u\beta * 1), \tau(u, \beta * 1)) = \phi(1 \cdot \tau(\alpha, v\beta * 1), \tau(v, \beta * 1)) \tag{29}$$

is one of the relations (23). Indeed, from $\alpha u\beta \in R$, $u\beta \in R_{u\beta*1}$ and Proposition 2.1 (iii), it follows that $\alpha r_{u\beta*1} \in R$. Next, by Lemma 5.2 (ii) and (19), in S we have

$$\alpha r_{u\beta*1} = r_{\alpha u\beta*1}\tau(\alpha, u\beta * 1) \equiv r_\omega \tau(\alpha, u\beta * 1) = h\tau(\alpha, u\beta * 1).$$

We conclude that $1 \cdot \tau(\alpha, u\beta * 1) \neq 0$ and $\alpha r_{u\beta*1} \in H_{1\cdot\tau(\alpha,u\beta*1)} \cap Sr_{u\beta*1}$. By the dual of Proposition 2.1 (v) we conclude that $H_{1\cdot\tau(\alpha,u\beta*1)} \subseteq Sr_{u\beta*1}$, which is precisely the condition for (29) to be one of the relations (23).

Finally, by Lemma 5.2 (ii) and (19) in S we have

$$h\tau(\alpha u, \beta * 1) \equiv r_{\alpha u\beta*1}\tau(\alpha u, \beta * 1) = \alpha u r_{\beta*1} = \alpha v r_{\beta*1} = h\tau(\alpha v, \beta * 1).$$

Therefore $1 \cdot \tau(\alpha u, \beta * 1) = 1 \cdot \tau(\alpha v, \beta * 1)$, and hence

$$\phi(1 \cdot \tau(\alpha u, \beta * 1), \tau(\beta, 1)) \equiv \phi(1 \cdot \tau(\alpha v, \beta * 1), \tau(\beta, 1)). \tag{30}$$

Using Lemmas 5.1 (i) and 5.2 (i) and (28), (29), (30) we obtain

$$\begin{aligned}
& \phi(1, \tau(\alpha u\beta, 1)) \\
\equiv\ & \phi(1, \tau(\alpha, u\beta * 1)\tau(u, \beta * 1)\tau(\beta, 1)) \\
\equiv\ & \phi(1, \tau(\alpha, u\beta * 1))\phi(1 \cdot \tau(\alpha, u\beta * 1), \tau(u, \beta * 1)) \\
& \quad \cdot\phi(1 \cdot \tau(\alpha, u\beta * 1)\tau(u, \beta * 1), \tau(\beta, 1)) \\
\equiv\ & \phi(1, \tau(\alpha, u\beta * 1))\phi(1 \cdot \tau(\alpha, u\beta * 1), \tau(u, \beta * 1)) \\
& \quad \cdot\phi(1 \cdot \tau(\alpha u, \beta * 1), \tau(\beta, 1)) \\
=\ & \phi(1, \tau(\alpha, v\beta * 1))\phi(1 \cdot \tau(\alpha, v\beta * 1), \tau(v, \beta * 1)) \\
& \quad \cdot\phi(1 \cdot \tau(\alpha v, \beta * 1), \tau(\beta, 1)) \\
\equiv\ & \phi(1, \tau(\alpha v\beta, 1)),
\end{aligned}$$

as a consequence of (22)-(25). □

Lemma 5.11. *If $w_1, w_2 \in B^*$ are any two words such that $w_1 = w_2$ holds in Γ then the relation $w_1 = w_2$ is a consequence of the relations* (22)-(25).

Proof. The assumption that $w_1 = w_2$ holds in Γ is equivalent to the relation $h\psi(w_1) = h\psi(w_2)$ holding in S, which is, in turn, equivalent to the relation $\pi(w_1)h = \pi(w_2)h$ by Lemma 5.3. Therefore there is a sequence of words $\pi(w_1)h \equiv \gamma_1, \gamma_2, \ldots, \gamma_n \equiv \pi(w_2)h$ from A^* such that each γ_{i+1} is obtained from γ_i by one application of a relation from $\mathfrak{R}$. By Lemma 5.10 we have that each relation $\phi(1, \tau(\gamma_i, 1)) = \phi(1, \tau(\gamma_{i+1}, 1))$ is a consequence of the relations (22)-(25). Therefore the relation

$$\phi(1, \tau(\pi(w_1)h, 1)) = \phi(1, \tau(\pi(w_2)h, 1)) \tag{31}$$

is a consequence of the relations (22)-(25). By Lemma 5.9 the relations

$$w_j = \phi(1, \tau(\pi(w_j)h, 1)) \ \ (j = 1, 2) \tag{32}$$

are also consequences of the relations (22)-(25). Combining (31) and (32) we conclude that $w_1 = w_2$ is a consequence of the relations (22)-(25). □

The proof of Theorem 4.2 is now complete.

6. Example: A non-finitely presented Schützenberger group.

Recall that if an $\mathcal{H}$-class of a monoid S contains an idempotent then H is a maximal subgroup of S and $H \cong \Gamma(H)$. Therefore, in this case, a presentation for $\Gamma(H)$ can be obtained from [**24**, Theorem 2.9]. In particular, by [**24**, Corollary 2.11], we have that $\Gamma(H)$ is finitely presented provided that S is finitely presented and the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes. Comparing this to Corollary 4.3, we see that, in this case, condition (ii) is not needed. So one may ask whether the same is true in general. In this section we present an example which answers this question in negative. More precisely, we are going to construct a finitely presented monoid which contains an $\mathcal{H}$-class H which is the only $\mathcal{H}$-class in its $\mathcal{R}$-class, but $\Gamma(H)$ is not finitely presented. This difference in behaviour between the group and non-group $\mathcal{H}$-classes is relatively surprising, because Schützenberger groups usually have the same properties as maximal subgroups. For instance, the generation theorems for the two are essentially identical (compare Proposition 4.1 and [**24**, Theorem 2.7]), leading to the same rewriting mapping (compare (15) with [**24**, Equality (2)]), and also they satisfy the same global results with respect to finite presentability (compare Theorem 1.1 and [**24**, Theorem 4.1]). It is also worth pointing out that the presentation for $\Gamma(H)$ we have obtained here essentially contains the presentation for H from [**24**, Theorem 2.9] – the relations (22), (24) and (25) correspond to [**24**, (3), (4), (5)] respectively.

Let A be the alphabet

$$A = \{a_1, a_2, a_3, a_4, a_1', a_2', a_3', a_4', b, c, d\},$$

and consider the presentation

$$\mathfrak{P} = \langle A \mid a_j a_j' = a_j' a_j = \epsilon,\ a_1 a_2 = a_3 a_4,\ a_j b = b a_j^2,\ cb^2 = cb,\ a_j d = d a_j,\\ cbda_j = a_j cbd \ \ (j = 1, 2, 3, 4)\rangle.$$

Let S be the monoid defined by $\mathfrak{P}$, and let H be the $\mathcal{H}$-class of $h \equiv cbd$. First we are going to show that H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class, and then we use our rewriting theorem (Theorem 4.2) to find a presentation for $\Gamma(H)$ and show that it is not finitely presented.

We begin by finding some properties of equal words in the semigroup S. We denote by A_0 the alphabet $\{a_j, a_j' \ : \ j = 1, 2, 3, 4\}$.

Lemma 6.1. *Let $w_1, w_2 \in A^*$ be arbitrary two words such that the relation $w_1 = w_2$ holds in S. Then the following statements are true.*

(i) *If w_1 contains a letter $x \in \{b, c, d\}$ then w_2 contains x as well.*
(ii) *The number of occurrences of each of the letters c and d is the same for w_1 and w_2.*
(iii) *If $w_1 \equiv \alpha x \beta$ and $w_2 \equiv \gamma y \delta$ with $\alpha, \gamma \in (A_0 \cup \{b\})^*$, $x, y \in \{c, d\}$ and $\beta, \delta \in A^*$ then the number of occurrences of b in α is equal to the number of occurrences of b in γ.*
(iv) *If w_1 has an occurrence of b preceding an occurrence of c (i.e., if w_1 has the form $w_1 \equiv \alpha b \beta c \gamma$, $\alpha, \beta, \gamma \in A^*$) then so does w_2.*
(v) *If w_1 has an occurrence of d preceding an occurrence of c then so does w_2.*
(vi) *If w_1 has an occurrence of d preceding an occurrence of b then so does w_2.*

Proof. Each part can be proved by noting that the property in question is invariant under the defining relations of S. □

Lemma 6.2. *If $w \in A^*$ is such that $cbdw\mathcal{R}cbd$ in S, then $w \in A_0^*$.*

Proof. Let $w_1 \in A^*$ be such that $cbdww_1 = cbd$ in S. By Lemma 6.1 (ii) and (vi) it follows that the word ww_1 contains no occurrence of either b, c or d. Therefore $w \in A_0^*$, as required. □

Lemma 6.3. *H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class.*

Proof. Let $s \in S$ be an arbitrary element which is $\mathcal{R}$-equivalent to cbd. Then s can be written as $s = cbdw$. By Lemma 6.2, it follows that $w \in A_0^*$. By using relations $a_j cbd = cbda_j$, we see that $s = wcbd$, and, since all a_j are invertible, we conclude that $s\mathcal{H}cbd$. □

In the notation of Sections 4 and 5 we have $\Lambda = \{1\}$ and $R = H$. The action of S on $\Lambda \cup \{0\}$ is given by

$\cdot$	a_j	a'_j	b	c	d
1	1	1	0	0	0
0	0	0	0	0	0

which follows immediately from Lemmas 6.2 and 6.3. The formal generators (13) for $\Gamma(H)$ are $b(1, a_j)$, $b(1, a'_j)$ $(j = 1, 2, 3, 4)$. If we identify these symbols with a_j and a'_j respectively, we obtain $B = A_0$. The definitions (15) and (16) of the mappings ϕ and π now simplify to

$$\phi(w) \equiv \pi(w) \equiv w \quad (w \in A_0^*). \tag{33}$$

Next we find the set I and the action of S on it.

Lemma 6.4. *The words ϵ, $b^i d$ $(i \geq 0)$, cbd, form a system of representatives of $\mathcal{R}$-classes in the inverse orbit of H.*

Proof. Recall that an $\mathcal{R}$-class R' is in the inverse orbit of H if and only if there exists $s \in S$ such that $sR' \subseteq H$. Therefore, from $cbd\epsilon = cb(b^i d) = \epsilon(cbd) = cbd$, it follows that the $\mathcal{R}$-class of each of the given words is indeed in the inverse orbit of H. It remains to be proved that these are all the $\mathcal{R}$-classes in the inverse orbit, and that they are all distinct.

First of all, by Lemma 6.1 (i) there does not exist a word $w \in A^*$ such that $b^i dw = \epsilon$ or $cbdw = \epsilon$ in S. Therefore ϵ is not $\mathcal{R}$-equivalent to either $b^i d$ or cbd. Similarly, $b^i d$ is not $\mathcal{R}$-equivalent to cbd. Finally, if $k \neq i$, then $b^i d$ is not $\mathcal{R}$-equivalent to $b^k d$ by Lemma 6.1 (iii).

Let $w \in A^*$ be any word whose $\mathcal{R}$-class is in the inverse orbit of H. By the dual of Proposition 2.1 (v) this means that there exists a word $w_1 \in A^*$ such that $w_1 w = cbd$ in S. By Lemma 6.1 (ii), w contains at most one occurrence of the letter c, as well as at most one occurrence of the letter d. Thus we can distinguish the following four cases:

Case 1: *w contains no occurrences of c or d.* By Lemma 6.1 (i), w_1 must contain occurrences of both c and d. Hence, by Lemma 6.1 (vi), w does not contain any occurrences of b. In other words, $w \in A_0^*$, and hence $w\mathcal{R}\epsilon$.

Case 2: *w contains one occurrence of c and no occurrences of d.* By Lemma 6.1 (i) w_1 must contain an occurrence of d, but then we obtain a contradiction with Lemma 6.1 (v). Therefore, this case never occurs.

Case 3: *w contains one occurrence of d and no occurrences of c.* By Lemma 6.1 (vi), w cannot contain any occurrences of b after the only occurrence of d. Hence w can be written as

$$w \equiv \alpha_1 b \alpha_2 b \ldots \alpha_m b \alpha_{m+1} d \alpha_{m+2},$$

where $m \geq 0$ and $\alpha_k \in A_0^*$, $k = 1, \ldots, m+2$. By using relations $a_j b = ba_j^2$ and $a_j d = da_j$ we see that w is equal in S to a word of the form $b^m d\alpha$ with $\alpha \in A_0^*$, and hence $w\mathcal{R}b^m d$.

Case 4: *w contains one occurrence of c and one occurrence of d.* By Lemma 6.1 (i), (iv), (v) and (vi) we have that w must have occurrences of b, that all these occurrences must supercede the occurrence of c and also must precede the occurrence of d. In other words, w has the form

$$w \equiv \alpha_1 c \alpha_2 b \alpha_3 b \ldots \alpha_m b \alpha_{m+1} d \alpha_{m+2},$$

with $m \geq 2$ and $\alpha_k \in A_0^*$, $k = 1, \ldots, m+2$. By applying relations $a_j b = ba_j^2$, $a_j d = da_j$, $cb^2 = cb$ and $a_j cbd = cbda_j$, we see that w is equal in S to a word of the form $cbd\alpha$, $\alpha \in A_0^*$, and hence $w\mathcal{R}cbd$.

This completes the proof of the lemma. □

Following the notation from Sections 4 and 5, we let $I = \{1, 2, \ldots\} \cup \{\omega\}$, and then we denote by R_1 the $\mathcal{R}$-class of $r_1 \equiv \epsilon$, by R_i ($i = 2, 3, \ldots$) the $\mathcal{R}$-class of $r_i \equiv b^{i-2}d$, and by R_ω the $\mathcal{R}$-class of $r_\omega \equiv cbd$ (i.e., H).

Lemma 6.5. *The left action of S on $I \cup \{0\}$ is as given in Table* 1.

a_j	a'_j	b	c	d	$*$
1	1	0	0	2	1
2	2	3	0	0	2
3	3	4	ω	0	3
4	4	5	ω	0	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	i	$i+1$	ω	0	i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ω	ω	0	0	0	ω

Table 1. The left action of S on $I \cup \{0\}$.

Proof. Since a_j is a unit it follows that $a_j \epsilon \mathcal{R} \epsilon$, and hence $a_j * 1 = 1$. A multiple application of the relation $a_j b = b a_j^2$ yields $a_j b^i = b^i a_j^{2^i}$. Since we also have $a_j d = d a_j$ it follows that $a_j(b^{i-2}d) = b^{i-2} d a_j^{2^{i-2}} \in R_i$, and so $a_j * i = i$. Finally, from the relation $a_j cbd = cbda_j$ it follows that $a_j * \omega = \omega$. This completes the proof for the a_j column of the table. The proof for the a'_j column is analogous.

Assume that $b \in R_i$ for some $i \in I$. Since $\{R_i \ : \ i \in I\}$ is the inverse orbit of H, it follows that there exists a word $w \in A^*$ such that $wb = cbd$. Now w must contain an occurrence of the letter d by Lemma 6.1 (i), and this yields a contradiction with Lemma 6.1 (vi). Therefore $b * 1 = 0$. Similarly from Lemma 6.1 (iv) it follows that $b(cbd) \notin R_i$ for all $i \in I$, and hence $b * \omega = 0$. Finally, we have $b(b^{i-2}d) \equiv b^{i-1}d \in R_{i+1}$, so that $b * i = i+1$, and this completes the proof for the b column of the table.

For the c column we have $c*1 = c*2 = c*\omega = 0$ by Lemma 6.1 (iii) and (ii). We also have $c * i = \omega$ $(i \geq 3)$ because $cb^{i-2}d = cbd \in R_\omega$. Finally, for the d column we have $d * 1 = 2$, since $d\epsilon \in R_2$, and $d * i = 0$ for $i \geq 2$ and $i = \omega$, by Lemma 6.1 (i). □

The mapping τ is defined by (21), once the values $\tau(x, i)$ $(x \in A,\ i \in I,\ x * i \neq 0)$ are chosen in accord with (20). One possible choice is given in Table 2. All the entries are easily verified by direct computation. For example, the entry $a_j^{2^{i-2}}$ in the position (a_j, i) follows from

$$a_j r_i \equiv a_j b^{i-2} d = b^{i-2} a_j^{2^{i-2}} d = b^{i-2} d a_j^{2^{i-2}} \equiv r_i a_j^{2^{i-2}}.$$

a_j	a'_j	b	c	d	τ
a_j	a'_j	$-$	$-$	ϵ	1
a_j	a'_j	ϵ	$-$	$-$	2
a_j^2	$(a'_j)^2$	ϵ	ϵ	$-$	3
a_j^4	$(a'_j)^4$	ϵ	ϵ	$-$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_j^{2^{i-2}}$	$(a'_j)^{2^{i-2}}$	ϵ	ϵ	$-$	i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_j	a'_j	$-$	$-$	$-$	ω

Table 2. Values for $\tau(x, i)$ ($x \in A$, $i \in I$, $x * i \neq 0$).

With (33) in mind, the presentation for $\Gamma(H)$ given in Theorem 4.2 has generators A_0 and the relations

(34) $$u = v \ \ ((u = v) \in \mathfrak{R},\ 1 \cdot u \neq 0),$$

(35) $$\tau(u, i) = \tau(v, i) \ \ ((u = v) \in \mathfrak{R},\ i \in I,\ u * i \neq 0),$$

(36) $$\tau(x, \omega) = x \ \ (x \in A_0),$$

(37) $$\tau(h, 1) = 1,$$

where, as usual, $\mathfrak{R}$ denotes the defining relations of S. The group (34) clearly consists of the relations

(38) $$a_j a'_j = a'_j a_j = 1 \ (j = 1, 2, 3, 4),\ a_1 a_2 = a_3 a_4.$$

Consider now the relations (35). Let $u = v$ be the relation $a_1 a_2 = a_3 a_4$, and let $i \geq 2$ be arbitrary. By using (21) and Table 2 we have

$$\tau(a_1 a_2, i) \equiv \tau(a_1, a_2 * i)\tau(a_2, i) \equiv \tau(a_1, i)\tau(a_2, i) \equiv a_1^{2^{i-2}} a_2^{2^{i-2}}$$

and, similarly, $\tau(a_3 a_4, i) \equiv a_3^{2^{i-2}} a_4^{2^{i-2}}$. Therefore we obtain the relations

(39) $$a_1^{2^{i-2}} a_2^{2^{i-2}} = a_3^{2^{i-2}} a_4^{2^{i-2}} \ \ (i \geq 2).$$

In a similar way we may check that all the remaining defining relations are identical. Therefore, as a group, $\Gamma(H)$ is defined by (39). In particular, $\Gamma(H)$ is an amalgamated product of two free groups of rank two with a free group of infinite rank amalgamated (see [**17**, Section 4.2]) and is not finitely presented by [**3**]. To summarise:

Proposition 6.6. *Let S be the semigroup defined by the presentation*

$$\mathfrak{P} = \langle A \mid a_j a'_j = a'_j a_j = \epsilon,\ a_1 a_2 = a_3 a_4,\ a_j b = b a_j^2,\ a_j d = d a_j,\ cb^2 = cb,$$
$$cbda_j = a_j cbd \ \ (j = 1, 2, 3, 4)\rangle,$$

and let H be the $\mathcal{H}$-class of the element cbd. Then H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class. However, the Schützenberger group $\Gamma(H)$ of H is defined by the presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i} \quad (i = 0, 1, 2, \dots) \rangle,$$

and is not finitely presented.

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