Pacific Journal of Mathematics

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Roger Chen and Chiung-Jue Sung

Volume 195 No. 2 October 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 195, No. 2, 2000

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Let (M^n, g) be a smooth compact Riemannian manifold with boundary $\partial M \neq \emptyset$. In this article we discuss the first positive eigenvalue of the Stekloff eigenvalue problem

$$\begin{cases} (-\Delta + q)u(x) = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial M, \end{cases}$$

where $q(x)$ is a C^2 function defined on M, $\partial \nu_g$ is the normal derivative with respect to the unit outward normal vector on the boundary ∂M. In particular, when the boundary ∂M satisfies the "interior rolling $R-$ball" condition, we obtain a positive lower bound for the first nonzero eigenvalue in terms of n, the diameter of M, R, the lower bound of the Ricci curvature, the lower bound of the second fundamental form elements, and the tangential derivatives of the second fundamental form elements.

1. Introduction.

Let (M^n, g) be a smooth compact Riemannian manifold with boundary $\partial M \neq \emptyset$. In local coordinates $(x^1, x^2, \dots, x^n)$, the Riemannian metric is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\ dx^i dx^j,$$

and the the Laplace operator is defined by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. We consider the following Stekloff eigenvalue problem:

$$\begin{cases} (-\Delta + q)u(x) = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial M, \end{cases} \tag{1.1}$$

where $q(x)$ is a C^2 function defined on M, $\partial \nu_g$ is the normal derivative with respect to the unit outward normal vector on the boundary ∂M. More specifically, we shall find a lower estimate for the first eigenvalue of the problem (1.1) in terms of the dimension of M, the geometrical data of M and ∂M, and the potential function q.

Problem (1.1) is known as the Stekloff problem as Stekloff first studied it for bounded plane domains with potential function $q \equiv 0$, and he found applications in physics. Also, it is important because the set of eigenvalues for the Stekloff problem is the same as the set of eigenvalues of the well-known Dirichlet-Neumann map. Also, it is well-known that when metrics on manifolds with boundary are conformally deformed, the sign of the Sobolev quotient $Q(M)$ and Sobolev trace quotient $Q(M, \partial M)$ of the manifold M are important conformal invariants and they can be characterized by the sign of the first eigenvalue of the problems (see [**E**])

$$\begin{cases} Lu + \eta_1 u = 0 & \text{in } M \\ Bu = 0 & \text{on } \partial M, \end{cases}$$

and

$$\begin{cases} Lu = 0 & \text{in } M \\ Bu = \lambda_1 u & \text{on } \partial M, \end{cases}$$

respectively, where $L = \Delta_g - \big[(n-2)/4(n-1)\big] R_g$ is the conformal Laplacian, $B = (\partial/\partial\nu_g) + \big[(n-2)/2\big] h_g$ is the boundary operator, h_g denotes mean curvature of the boundary ∂M with respect to ν_g, and R_g denotes the scalar curvature on M. Hence, it is natural to study the first eigenvalue of the associated equation (1.1) without the functions R_g and h_g. From the analysis viewpoint, this problem closely corresponds to the study of the following Neumann eigenvalue.

$$\begin{cases} (-\Delta + \eta_1) u(x) = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{1.2}$$

It is well known that the first nonzero Neumann eigenvalue of the Laplacian on M will provide an optimal upper estimate for the Poincaré constant and it is important from an analysis viewpoint to prove the Poincaré inequality on the manifold M. Therefore, it is interesting to find a positive lower estimate of the first nonzero eigenvalue, and this has been studied extensively by many authors. We will simply refer the reader to [**B**], [**Ch**], [**C**], [**C-L**], [**L1**], [**L2**], [**L-T**], [**L-Y1**], [**L-Y2**], [**W**] for further references. Analogously, it is also interesting whether one may obtain a positive estimate for the lower bound of the first eigenvalue of the problem (1.1).

In a recent paper [**E**], Escobar generalized problem (1.1) with $q \equiv 0$ to a compact manifold (M^n, g) with boundary ∂M. In the two-dimensional case, if M has nonnegative Gaussian curvature and the geodesic curvature k_g of ∂M satisfies $k_g \geq k_0 > 0$, then he proved that the first nonzero eigenvalue λ_1 of the Stekloff problem satisfies $\lambda_1 \geq k_0$. Also, he proved that the equality holds only for the Euclidean ball of radius k_0^{-1}. In higher dimensional cases, if M has non-negative Ricci curvature then he proved that $\lambda_1 > \frac{k_0}{2}$, where $k_0 > 0$ is a lower bound for the second fundamental form elements of the boundary. However, the lower bounds in the paper [**E**] will become zero if one only assumes nonnegative geodesic curvature on the boundary for dimension two case or nonnegative second fundamental form elements on the boundary for higher dimension case.

From the interests of analysis, we shall try to obtain a positive lower bound for λ_1 of the Stekloff problem on a more general class of manifolds. In particular, we shall follow a similar gradient estimate argument as in [**C**] and [**W**] to prove a quantitative generalization of some results in [**E**].

Theorem 1.1. *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. Suppose that ∂M satisfies the "interior rolling $R-$ball" condition. Let K, H and $\bar{H}$ be nonnegative constants such that the Ricci curvature Ric_M of M is bounded below by $-(n-1)K$, the second fundamental form II of ∂M is bounded below by $-H$ and the absolute value of tangential derivatives of II is bounded by $\bar{H}$. By choosing R small, and for $a = 1$ or $a = 3$, we have the following estimate for the first eigenvalue λ_1 of the Stekloff problem (1.1).*

(1.3)
$$\begin{aligned}
&\frac{2(1+C_{14})^{\frac{1}{2}}}{2d^2(n-1)^2(1+H)^2}\exp\left[-1-\left(1+C_{14}\right)^{\frac{1}{2}}\right]\\
&\leq \lambda_1^a\Bigg[8(n-1)+(n-1)^2(24+12H)+3\beta\max\Bigg\{12n-8+6H,\\
&\qquad 2(n-1)\left(\sqrt{(n-1)K+(n-2)K_R}+\frac{1}{R}+1\right)\Bigg\}\\
&\qquad+\beta\sqrt{2(n-1)}(96+6C_1+2C_4)+\frac{\beta-1}{\beta}\big(36n+28+12(n-1)H\big)\Bigg]\\
&\qquad+\left[3\beta\sup|q|+\beta\sqrt{2(n-1)\delta}\sup|\nabla q|-2(n-1)\inf q\right],
\end{aligned}$$

where $\delta > 0$ is any constant, C_1, C_4, C_5, C_6, are constants depending on n, K, H, $\bar{H}$, R, $d =$ diameter of M,

$$\begin{aligned}
\frac{\beta}{\beta-1} &= \exp\left[1+(1+C_{14})^{\frac{1}{2}}\right],\\
C_{14} &= d^2(1+H)^2\left[3C_5+C_6+(n-1)K+\frac{1}{2\delta}\right],
\end{aligned}$$

and they can be explicitly computed.

Remark. We shall choose the radius $R < 1$ of the interior rolling ball to satisfy the following inequalities

$$\sqrt{K_R}\tan(R\sqrt{K_R}) \leq \frac{H}{2} + \frac{1}{2},$$

$$\frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R}) \leq \frac{1}{2},$$

$$\frac{\sqrt{(n-1)K + (n-2)K_R}}{e^{2R\sqrt{(n-1)K+(n-2)K_R}} - 1} > H$$

where K_R denotes the upper bound of the radial curvatures in $\partial M(R) = \{x \in M \cup \partial M | \mathrm{dist}(x, \partial M) \leq R\}$.

Corollary 1.2. *Let* (M^n, g) *be as in Theorem* 1.1 *and let* $q \equiv 0$ *in* (1.1). *By choosing* R *small, we have*

$$\lambda_1 \geq C_{15}, \tag{1.4}$$

for some constant C_{15}, *depending on* n, K, H, $\bar{H}$, R, $d =$ *diameter of* M, *and it can be explicitly computed.*

Using the same technique, we may also obtain the following estimate for η_1 in (1.2).

Theorem 1.3. *Let* (M^n, g) *be as in Theorem* 1.1. *By choosing* R *small, we have*

$$\eta_1 \geq C_{16}, \tag{1.5}$$

where C_{16} *is a positive constant depending on* n, K, H, R, $d =$ *diameter of* M *and both can be explicitly computed as in* (3.13).

Corollary 1.4. *Let* (M^n, g) *be as in Theorem* 1.1. *Assume that the Ricci curvature of* M *is nonnegative, the boundary* ∂M *is convex. By choosing* R *small, we have*

$$\eta_1 \geq \frac{C_{17}}{d^2} \tag{1.6}$$

where C_{17} *is a positive constant depending only on* n *and it can be explicitly computed as in* (3.15).

Acknowledgments. We would like to thank Professor Jiaping Wang for his interest in this work and many useful discussions. Part of this work was done while the first author was attending a workshop on analysis on manifolds organized by the IMS at the Chinese University of Hong Kong from 8 July to 29 July 1998. He would like to express his gratitude to the organizers Professors Luen-Fai Tam and Tom Wan, and the IMS at the Chinese University of Hong Kong for their hospitality.

2. Main Lemma.

We recall the following definition from [**C**].

Definition 2.1. ∂M is said to satisfy the "interior rolling R-ball" condition if for each point $p \in \partial M$ there is a geodesic ball $B_q\left(\dfrac{R}{2}\right)$, centered at $q \in M$ with radius $\frac{R}{2}$, such that $p = B_q(\frac{R}{2}) \cap \partial M$ and $B_q(\frac{R}{2}) \subset M$.

We may modify a gradient estimate method as in [**C**] and [**W**] to prove our main lemma for a positive solution of the problem (1.1). In our case, we need to define two functions on M by $\phi(x) = \varphi(r(x))$ and $\psi(x) = \Psi(\frac{r(x)}{R})$, where $r(x)$ denotes the distance from $x \in M$ to ∂M and $\varphi(r)$ and $\Psi(r)$ are nonnegative smooth functions defined on $[0, \infty)$ such that

$$\begin{cases} \varphi(r) \leq \lambda_1 R & \text{if } r \in [0, \frac{R}{2}) \\ \varphi(r) = \lambda_1 R & \text{if } r \in [R, \infty) \end{cases} \tag{2.1}$$

and

$$\begin{cases} \Psi(r) \leq H & \text{if } r \in [0, \frac{1}{2}) \\ \Psi(r) = H & \text{if } r \in [1, \infty) \end{cases} \tag{2.2}$$

with

$$\begin{aligned} &\varphi(0) = 0, \quad 0 \leq \varphi'(r) \leq 2\lambda_1, \quad \varphi'(0) = \lambda_1 \\ &|\varphi''(r)| \leq 2\lambda_1, \quad |\varphi'''(r)| \leq 2\lambda_1, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\Psi(0) = 0, \quad 0 \leq \Psi'(r) \leq 2H, \\ &\Psi'(0) = H, \quad \Psi''(r) \geq -2H. \end{aligned} \tag{2.4}$$

Letting

$$\begin{aligned} &w = (1 + \phi)u, \quad f = \log(1 + \phi), \\ &p = -q + |\nabla f|^2 - \Delta f, \end{aligned}$$

Equation (1.1) for u is transformed into the following equation for w.

$$\begin{cases} \Delta w - 2\langle \nabla f, \nabla w \rangle + pw = 0 & \text{in } M \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{2.5}$$

Lemma 2.2. *Let (M^n, g) be as in Theorem* 1.1*. Normalize w such that $1 = \sup w$. For a constant $\beta > 1$, we consider the function*

$$F(x) = (1 + \psi(x))^2 \frac{|\nabla w|^2}{(\beta - w)^2}. \tag{2.6}$$

Assume that $F(x_0) = \max_{x\in\bar{M}} F(x)$, *and choose* R *small, we have*

$$F(x_0) \le (1+\psi)^2 \left[(2n-1)\theta + \frac{\sqrt{(2n-1)\gamma}}{\beta-1}\right] \le (1+H)^2 \left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right], \tag{2.7}$$

where $\delta > 0$ *is any constant, and*

$$\begin{aligned}
C_1 &= (n-1)\max\{3H+1, \sqrt{(n-1)K + (n-2)K_R}\} \\
C_2 &= \max\{(3H+1)^2, (n-1)K + (n-2)K_R\} + K_R \\
C_3 &= e^{\frac{RC_1}{n-1}} \left(\bar{H} + Re^{\frac{RC_1}{n-1}} \bar{K}_R\right) \\
C_4 &= \max\{C_2, C_3\} \\
C_5 &= \frac{4H^2}{R^2} \\
C_6 &= \frac{2(n-1)H(3H+1)}{R} - \frac{2H}{2R^2} \\
C_7 &= 4\lambda_1^2 \\
C_8 &= \lambda_1 (8\lambda_1 + 6H + 4) \\
C_9 &= 2\lambda_1 \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1\right) \\
C_{10} &= \max\{C_7 + (n-1)C_8, (n-1)C_9\} \\
C_{11} &= 64\lambda_1^3 + 32\lambda_1^2 + 2\lambda_1(3C_1 + C_4) \\
C_{12} &= \frac{[-\inf q + C_7 + (n-1)C_8]\,\beta}{\beta - w} + \frac{(3n+13)C_7}{2(n-1)} \\
&\quad + C_8 + 3C_5 + C_6 + (n-1)K + \frac{1}{2\delta} \\
C_{13} &= \frac{4}{n-1}\left[\sup|q|^2 + C_{10}^2\right] + \delta\left[\sup|\nabla q|^2 + C_{11}^2\right].
\end{aligned}$$

Proof. The proof we give may be divided into the following steps.

(1) In Step (1), we determine the location of the maximum point of x_0 by using a maximum principle.

(2) In Step (2), we apply the maximum principle to obtain an inequality

$$0 \ge aF(x_0)^2 - bF(x_0) - c$$

for $F(x_0)$, where $a > 0$, b, $c \ge 0$ are constants.

(3) In Step (3), we shall find estimates of b and c which lead to an estimate of $F(x_0)$.

Step (1). The point x_0 is either a boundary point or an interior point of M. Suppose that $x_0 \in \partial M$, we let $\{e_i\}$ be a local orthonormal frame field of M^n such that $e_n = \frac{\partial}{\partial \nu}$ on ∂M. If we let h_{ij} denote the second fundamental form elements of ∂M, then

$$\begin{aligned} 0 &\leq \frac{\partial F}{\partial \nu}(x_0) \\ &= \frac{-2\frac{H}{R}|\nabla w|^2 + 2\sum_{i=1}^{n-1} w_i w_{in}}{(\beta - w)^2} \\ &= \frac{-2\frac{H}{R}|\nabla w|^2 - 2\sum_{i,j=1}^{n-1} h_{ij} w_i w_j}{(\beta - w)^2} \\ &\leq \frac{-2\frac{H}{R}|\nabla w|^2 + 2H|\nabla w|^2}{(\beta - w)^2} \\ &< 0, \end{aligned}$$

which is a contradiction, as we may choose R to be smaller than 1. Hence $F(x)$ cannot attain its maximum at the boundary point. Therefore x_0 has to be an interior point of M.

Step (2). Since F attains its maximum value at an point x_0, we have

$$\nabla F(x_0) = 0 \tag{2.8}$$

$$\Delta F(x_0) \leq 0. \tag{2.9}$$

Note that

$$\nabla F \cdot \left(\frac{\beta - w}{1 + \psi}\right)^2 + F \cdot \nabla \left(\frac{\beta - w}{1 + \psi}\right)^2 = \nabla |\nabla w|^2, \tag{2.10}$$

and

$$\Delta F \cdot \left(\frac{\beta - w}{1 + \psi}\right)^2 + 2\nabla F \cdot \nabla \left(\frac{\beta - w}{1 + \psi}\right)^2 + F \cdot \Delta \left(\frac{\beta - w}{1 + \psi}\right)^2 = \Delta |\nabla w|^2$$

which implies that, at x_0,

$$0 \geq -F(x_0)\Delta \left(\frac{\beta - w}{1 + \psi}\right)^2 (x_0) + \Delta |\nabla w|^2 (x_0). \tag{2.11}$$

We may choose an orthonormal frame field $\{e_i\}$ near x_0 such that $w_1(x_0) = |\nabla w|(x_0)$. Note that $|\nabla w|(x_0) \neq 0$, otherwise $F(x) = F(x_0) = 0$ for all $x \in M$ which is a contradiction.

At x_0, Equation (2.8) implies that

$$|\nabla w|^2 \left(-\frac{2w_j}{\beta - w} - \frac{2\psi_j}{1 + \psi}\right) = 2w_1 w_{1j}$$

for each $j = 1, \dots, n$. Therefore, at x_0, we have

$$\begin{cases} w_{11} = -\frac{w_1^2}{\beta - w} - \frac{w_1\psi_1}{1+\psi} \\ w_{1j} = -\frac{w_1\psi_j}{1+\psi} \qquad \text{for } j \neq 1. \end{cases} \tag{2.12}$$

Using the Ricci identity, $w_{ijk} - w_{ikj} = \sum_{l=1}^n w_l R_{lijk}$ and a direct calculation, we get

$$\frac{1}{2}\Delta|\nabla w|^2 = \sum_{i,j=1}^n w_{ij}^2 + \sum_{j=1}^n w_j(\Delta w)_j + \sum_{i,j=1}^n R_{ij} w_i w_j \tag{2.13}$$

$$\frac{1}{2} F \cdot \Delta \left(\frac{\beta - w}{1+\psi}\right)^2 = |\nabla w|^2 \left[\frac{-\Delta w}{\beta - w} + \frac{|\nabla w|^2}{(\beta - w)^2} + \frac{2\langle \nabla w \nabla \psi\rangle}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + \frac{3|\nabla\psi|^2}{(1+\psi)^2}\right].$$

Substituting (2.13) into (2.11) and using a direct calculation, we have

$$0 \geq \sum_{i,j=1}^n w_{ij}^2 + \sum_{j=1}^n w_j(\Delta w)_j + \sum_{i,j=1}^n R_{ij} w_i w_j - |\nabla w|^2 \left[\frac{-\Delta w}{\beta - w} + \frac{|\nabla w|^2}{(\beta - w)^2} + \frac{2\langle \nabla w \nabla \psi\rangle}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + \frac{3|\nabla\psi|^2}{(1+\psi)^2}\right].$$

Using (2.5), we have

$$\begin{aligned}
&= \sum_{i,j=1}^n w_{ij}^2 + 2w_1 \sum_{j=1}^n (f_j w_j)_1 - w_1(pw)_1 + R_{11} w_1 w_1 - w_1^2 \left[\frac{pw}{\beta - w} \right.\\
&\qquad \left. - \frac{2 f_1 w_1}{\beta - w} + \frac{w_1^2}{(\beta - w)^2} + \frac{2 w_1 \psi_1}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + 3\frac{|\nabla\psi|^2}{(1+\psi)^2}\right]\\
&= \sum_{i,j=1}^n w_{ij}^2 + 2w_1^2 f_{11} + 2w_1 \sum_{j=1}^n f_j w_{j1} - w_1^2 p - w_1 w p_1 + R_{11} w_1 w_1\\
&\quad - w_1^2 \left[\frac{pw}{\beta - w} - \frac{2 f_1 w_1}{\beta - w} + \frac{w_1^2}{(\beta - w)^2} \right.\\
&\qquad \left. + \frac{2 w_1 \psi_1}{(1+\psi)(\beta - w)} - \frac{\Delta\psi}{1+\psi} + 3\frac{|\nabla\psi|^2}{(1+\psi)^2}\right]\\
&= \sum_{i,j=1}^n w_{ij}^2 - \frac{w_1^4}{(\beta - w)^2} - \frac{2 w_1^3 \psi_1}{(1+\psi)(\beta - w)}\\
&\quad + 2w_1 \sum_{j=1}^n f_j w_{j1} + \frac{2 f_1 w_1^3}{\beta - w} - w_1 w p_1
\end{aligned}$$

$$
\begin{aligned}
&- w_1^2\left[\frac{pw}{\beta-w}+p-\frac{\Delta\psi}{1+\psi}+3\frac{|\nabla\psi|^2}{(1+\psi)^2}-2f_{11}-R_{11}\right]\\
=&\sum_{i,j=1}^n w_{ij}^2-\frac{w_1^4}{(\beta-w)^2}-\frac{2w_1^3\psi_1}{(1+\psi)(\beta-w)}\\
&+2w_1\sum_{j=1}^n f_j w_{j1}+\frac{2f_1w_1^3}{\beta-w}-w_1wp_1\\
&-w_1^2\left[\frac{p\beta}{\beta-w}-\frac{\Delta\psi}{1+\psi}+3\frac{|\nabla\psi|^2}{(1+\psi)^2}-2f_{11}-R_{11}\right].
\end{aligned}
$$

Using (2.12), and the inequality $x^2+y^2\geq 2xy$, it is easy to see that

$$
\begin{aligned}
w_{11}^2-\frac{w_1^4}{(\beta-w)^2}-\frac{2w_1\psi_1}{(\beta-w)(1+\psi)}&=\frac{w_1^2\psi_1^2}{(1+\psi)^2}\\
\frac{2f_1w_1^3}{\beta-w}+2f_1w_1w_{11}&=-\frac{2w_1^2f_1\psi_1}{1+\psi}\\
2\sum_{j=2}^n w_{1j}^2+2w_1\sum_{j=2}^n f_jw_{1j}&\geq-\frac{w_1^2}{2}\sum_{j=2}^n f_j^2.
\end{aligned}
$$

Putting these into the above, we have

(2.14)

$$
\begin{aligned}
0\geq&\sum_{i,j=2}^n w_{ij}^2-p_1ww_1+\frac{w_1^2\psi_1^2}{(1+\psi)^2}\\
&-w_1^2\left[\frac{p\beta}{\beta-w}+\frac{1}{2}\sum_2^n f_j^2-2f_{11}-\frac{\Delta\psi}{1+\psi}+\frac{2f_1\psi_1}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}\right].
\end{aligned}
$$

To prove our claim, we shall find an inequality for $F(x_0)$ of the form

$$0\geq aF(x_0)^2-bF(x_0)-c$$

where $a>0$, b, c are nonnegative constants. To obtain the quadratic term of $F(x_0)$ with positive coefficient, we observe that Cauchy-Schwarz inequality implies that

$$
\begin{aligned}
\sum_{i,j=2}^n w_{ij}^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}&\geq\sum_{j=2}^n w_{jj}^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}\\
&\geq\frac{1}{n-1}\left(w_{11}-\Delta w\right)^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}.
\end{aligned}
$$

Using (2.5), (2.12), and the inequality $(x+y)^2\geq\frac{1}{2}x^2-y^2$, the above becomes

$$=\frac{1}{n-1}\left(\frac{w_1^2}{\beta-w}+\frac{w_1\psi_1}{1+\psi}+\Delta w\right)^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}$$

$$\geq \frac{1}{2(n-1)}\left(\frac{w_1^2}{\beta-w}+\frac{w_1\psi_1}{1+\psi}\right)^2-\frac{1}{n-1}(\Delta w)^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}.$$

Using the inequality $(x+y)^2 \geq \frac{2(n-1)}{2n-1}x^2 - 2(n-1)y^2$, the above becomes

$$\begin{aligned}&\geq \frac{1}{2n-1}\left(\frac{w_1^4}{(\beta-w)^2}-\frac{w_1^2\psi_1^2}{(1+\psi)^2}\right)-\frac{1}{n-1}(\Delta w)^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2}\\ &= \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{1}{n-1}(\Delta w)^2.\end{aligned}$$

Using (2.5), and the inequality $2(x^2+y^2) \geq (x+y)^2$, we get

$$\sum_{i,j=2}^n w_{ij}^2+\frac{w_1^2\psi_1^2}{(1+\psi)^2} \geq \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{2}{n-1}\left(p^2w^2+4f_1^2w_1^2\right). \tag{2.15}$$

Substituting (2.15) into (2.14), we get

(2.16)

$$\begin{aligned}0 &\geq \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{2}{n-1}p^2w^2-p_1ww_1-w_1^2\left[\frac{p\beta}{\beta-w}+\frac{8}{n-1}f_1^2\right.\\ &\quad\left.+\frac{1}{2}\sum_2^n f_j^2-2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}\right]\\ &\geq \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\frac{2}{n-1}p^2w^2-\frac{\delta}{2}p_1^2w^2-\frac{1}{2\delta}w_1^2-w_1^2\left[\frac{p\beta}{\beta-w}\right.\\ &\quad\left.+\frac{8}{n-1}f_1^2+\frac{1}{2}\sum_2^n f_j^2-2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}\right]\\ &= \frac{1}{2n-1}\frac{w_1^4}{(\beta-w)^2}-\left[\frac{2}{n-1}p^2+\frac{\delta}{2}p_1^2\right]w^2-w_1^2\left[\frac{p\beta}{\beta-w}+\frac{8}{n-1}f_1^2\right.\\ &\quad\left.+\frac{1}{2}\sum_2^n f_j^2-2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}+\frac{1}{2\delta}\right]\end{aligned}$$

where δ is any positive constant. In order to see that we have almost obtained the desired inequality for $F(x_0)$, we shall simplify the notations by setting

$$\begin{aligned}\theta = \frac{p\beta}{\beta-w}+\frac{8}{n-1}f_1^2+\frac{1}{2}\sum_2^n f_j^2 \\ -2f_{11}+\frac{2f_1\psi_1}{1+\psi}-\frac{\Delta\psi}{1+\psi}+\frac{2|\nabla\psi|^2}{(1+\psi)^2}-R_{11}+\frac{1}{2\delta},\end{aligned} \tag{2.17}$$

and

$$\gamma = \frac{2}{n-1}p^2 + \frac{\delta}{2}p_1^2. \tag{2.18}$$

Also, note that if we set $\alpha = \frac{w}{\beta - w}$, then we have

$$\alpha = \frac{w}{\beta - w} \leq \frac{1}{\beta - w} \leq \frac{1}{\beta - 1}. \tag{2.19}$$

Multiplying (2.16) through by $\dfrac{(1+\psi)^4}{(\beta - w)^2}$ and using (2.17)-(2.19), we obtain

$$0 \geq \frac{1}{2n-1}F^2 - \theta(1+\psi)^2 F - \gamma(1+\psi)^4\alpha^2. \tag{2.20}$$

Step (3). In this step, we shall give estimates on $\theta(1+\psi)^2$ and $\gamma(1+\psi)^4\alpha^2$ in (2.20). The inequality (2.20) implies that we have

$$\begin{aligned} F(x_0) &\leq (1+\psi)^2 \left[\frac{(2n-1)\theta}{2} + \sqrt{\frac{(2n-1)^2\theta^2}{4} + (2n-1)\gamma\alpha^2}\right] \\ &\leq (1+\psi)^2 \left[(2n-1)\theta + \sqrt{(2n-1)\gamma}\alpha\right]. \end{aligned} \tag{2.21}$$

In order to prove our claim, we shall estimate each term in (2.17) and (2.18) of θ and γ, respectively. Since $p = -q + |\nabla f|^2 - \Delta f$ and $f = \log(1+\phi(r(x)))$, we shall need to estimate Δr and $|\nabla \Delta r|$ near the boundary ∂M if we want to estimate the term $|\nabla p|$ in γ. Here, we shall first derive some estimates for Δr and $|\nabla \Delta r|$. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame fields of M^n in a neighborhood $\partial M(R) = \{x \in M \cup \partial M | r(x) \leq R\}$ of ∂M such that $e_n = -\frac{\partial}{\partial \nu}$, where ν is the unit outward normal vector to ∂M. For any $x \in \partial M \times \{r\}$, where $\partial M \times \{r\} = \{x \in \partial M(R) | r(x) = r\}$, we have $r_n = 1$, and $r_a = 0$ for $a = 1, \dots, n-1$. When R is sufficiently small, we may write each point $x \in \partial M(R)$ as $x = (y, r)$, where $y \in \partial M$ and $\operatorname{dist}(x, y) = r(x)$. A direct calculation shows that

$$r_{na} = 0, \quad r_{nn} = 0, \quad r_{aa} = -h_{aa}$$

for $a = 1, \dots, n-1$, where h_{aa} is a second fundamental form element of $\partial M \times \{r\}$. To estimate $|\nabla \Delta r|$, it suffices to obtain estimates of $|e_n(\Delta r)|$ and $|e_a(\Delta r)|$ for $a = 1, \dots, n-1$. Differentiating r_{aa} in the direction of e_n yields

$$\begin{aligned} e_n(r_{aa}) &= r_{aan} - \sum_{b=1}^{n-1} r_{ab}^2 \\ &= r_{ana} + \sum_{l=1}^{n} r_l R_{laan} - \sum_{b=1}^{n-1} r_{ab}^2 \end{aligned}$$

$$= -\sum_{b=1}^{n-1} r_{ab}^2 - R_{nana},$$

where R_{nana} denotes the curvature tensor of M. Hence, we have

$$e_n(r_{aa}) = -\sum_{b=1}^{n-1} r_{ab}^2 - R_{nana} \tag{2.22}$$

for each $x = (y, r) \in \partial M(R)$. Integrating (2.22) yields

$$r_{aa}(x) = r_{aa}(y) + \int_0^r \left(\sum_{b=1}^{n-1} r_{ab}^2 - R_{nana} \right)(y, t)dt. \tag{2.23}$$

Let K_R and $\bar{K}_R$ denote the upper bounds of the radial curvatures and of the absolute value of covariant derivatives of radial curvatures, respectively, in $\partial M(R)$, i.e., $K_R = \max\ \{R_{nana}(x) | x \in \partial M(R), 1 \leq a \leq n-1\}$ and $\bar{K}_R = \max\ \{|R_{nana,b}(x)|\, | x \in \partial M(R), 1 \leq a, b \leq n\}$. Since the boundary ∂M satisfies the "*interior rolling* $R-$*ball*" condition, its second fundamental form element II is bounded from above by $\frac{1}{R}$ and is bounded from below by hypothesis. We shall follow an index comparison theorem [**Wa**] to obtain estimates on r_{aa} for $a = 1, \dots, n-1$. To apply it, we choose R small as in [**C**] such that

$$\sqrt{K_R}\tan(R\sqrt{K_R}) \leq \frac{H}{2} + \frac{1}{2},$$

$$\frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R}) \leq \frac{1}{2},$$

and

$$\frac{\sqrt{(n-1)K + (n-2)K_R}}{e^{2R\sqrt{(n-1)K+(n-2)K_R}} - 1} > H.$$

Using an index comparison theorem, we have

$$\begin{aligned} r_{aa} &\geq -\frac{H + \sqrt{K_R}\tan(R\sqrt{K_R})}{1 - \frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R})} \\ &\geq -(3H+1), \end{aligned} \tag{2.24}$$

and if we set $\kappa = \sqrt{(n-1)K + (n-2)K_R}$ we have

$$\begin{aligned} r_{aa} &\leq \frac{\kappa\left[\left(e^{2\kappa r(x)} - 1\right)\kappa + \left(e^{2\kappa r(x)} + 1\right)\frac{1}{R}\right]}{\left(e^{2\kappa r(x)} + 1\right)\kappa + \left(e^{2\kappa r(x)} - 1\right)\frac{1}{R}} \\ &\leq \frac{\kappa\left[\left(e^{2\kappa R} + 1\right)\kappa R + \left(e^{2\kappa R} + 1\right)\right]}{\left(e^{2\kappa r(x)} + 1\right)R} \\ &\leq \kappa + \frac{1}{R} = \sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R}, \end{aligned} \tag{2.25}$$

for $a = 1, \dots, n-1$. Hence, we have

$$|\Delta r| \le (n-1)\max\{3H+1, \sqrt{(n-1)K+(n-2)K_R}\} \\ = C_1. \tag{2.26}$$

Combining (2.22), (2.24), and (2.25), we have

$$|e_n(\Delta r)| \le \max\{(3H+1)^2, (n-1)K+(n-2)K_R\} + K_R \\ = C_2. \tag{2.27}$$

Differentiating (2.23), we get

$$r_{aa,b}(y,r) = r_{aa,b}(y,0) + \int_0^r \left(2\sum_{c=1}^{n-1} r_{ac}r_{ac,b} - R_{nana,b}\right)(y,t)dt \tag{2.28}$$

for $b = 1, \dots, n-1$. We may assume that r_{aa} for $a = 1, \dots, n-1$ denotes an eigenvalue for the Hessian of r. Differentiating (2.28) with respect to r and solving the first order differential equation

$$r_{aa,b}(y,r) = 2r_{aa}r_{aa,b}(y,r) - R_{nana,b}(y,r),$$

we have

$$|e_b(\Delta r)|(x) \le e^{\frac{RC_1}{n-1}}\left(\bar{H} + Re^{\frac{RC_1}{n-1}}\bar{K}_R\right) \\ = C_3. \tag{2.29}$$

Combining (2.26) and (2.29), we have

$$|\nabla\Delta r|(x) \le \max\{C_2, C_3\} \\ = C_4. \tag{2.30}$$

We are now ready to give estimates for θ, γ in (2.17) and (2.18). Note that it suffices to find estimates for terms $|\nabla\psi| = \Psi'$, $\Delta\psi$, $|\nabla f|^2 = |\nabla\log(1+\phi)|^2$, f_{jj}, $p = -q + |\nabla f|^2 - \Delta f$, and $|\nabla p|$. In the following, we shall give an estimate for each of these terms. From the definition of ψ and (2.24), it is easy to see that

$$|\nabla\psi|^2 = \frac{1}{R^2}\Psi'^2 \le \frac{4H^2}{R^2} = C_5 \tag{2.31}$$

and

$$\Delta\psi = \frac{1}{R}\Psi'\Delta r + \frac{1}{R^2}\Psi''|\nabla r|^2 \\ \ge -\frac{2(n-1)H(3H+1)}{R} - \frac{2H}{2R^2} = -C_6. \tag{2.32}$$

For the terms $|\nabla f|^2 = |\nabla\log(1+\phi)|^2$ and f_{jj}, we apply (2.24), (2.25) to obtain

$$|\nabla f|^2 = |\nabla\log(1+\phi)|^2 = \frac{\varphi'^2}{(1+\phi)^2} \le 4\lambda_1^2 = C_7, \tag{2.33}$$

$$(2.34)\qquad \begin{aligned} f_{jj} &= \frac{\varphi' r_{jj}}{(1+\phi)^2} + \frac{\varphi'' r_j^2}{(1+\phi)^2} - \frac{2\varphi'^2 r_j^2}{(1+\phi)^3} \\ &\geq -2\lambda_1(3H+1) - 2\lambda_1 - 8\lambda_1^2 \\ &= -\lambda_1\left(8\lambda_1 + 6H + 4\right) = -C_8, \end{aligned}$$

and

$$(2.35)\qquad \begin{aligned} f_{jj} &\leq 2\lambda_1\left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R}\right) + 2\lambda_1 \\ &= 2\lambda_1\left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1\right) = C_9. \end{aligned}$$

For the term $p = -q + |\nabla f|^2 - \Delta f$, we use (2.32)-(2.35) to get

$$(2.36)\qquad p = -q + |\nabla f| - \Delta f \leq -\inf q + C_7 + (n-1)C_8,$$

$$(2.37)\qquad p = -q + |\nabla f|^2 - \Delta f \geq -\sup q - (n-1)C_9.$$

Combining (2.36), (2.37), we get that

$$(2.38)\qquad |p| \leq \sup|q| + \max\{C_7 + (n-1)C_8, (n-1)C_9\} = \sup|q| + C_{10}.$$

For the term $|\nabla p|$, we note that

$$\begin{aligned} \nabla p &= -\nabla q + \nabla\left[\frac{\varphi'^2}{(1+\phi)^2}\right] - \nabla\left[\frac{\varphi'\Delta r}{(1+\phi)^2} + \frac{\varphi''}{(1+\phi)^2} - \frac{2\varphi'^2}{(1+\phi)^3}\right] \\ &= -\nabla q + \frac{2\varphi'\varphi''\nabla r}{(1+\phi)^2} - \frac{2\varphi'^2\nabla r}{(1+\phi)^3} - \frac{\varphi''\Delta r}{(1+\phi)^2} - \frac{\varphi'\nabla\Delta r}{(1+\phi)^2} + \frac{2\varphi'\Delta r\nabla r}{(1+\phi)^3} \\ &\quad - \frac{\varphi'''\nabla r}{(1+\phi)^2} + \frac{2\varphi'\varphi''\nabla r}{(1+\phi)^3} + \frac{4\varphi'\varphi''\nabla r}{(1+\phi)^3} - \frac{6\varphi'^3\nabla r}{(1+\phi)^4}. \end{aligned}$$

Hence, we use (2.26)and (2.30) to get

$$(2.39)\qquad |\nabla p| \leq 64\lambda_1^3 + 32\lambda_1^2 + 2\lambda_1(3C_1 + C_4) + \sup|\nabla q| = C_{11} + \sup|\nabla q|.$$

Note that each constant $C_7, \ldots, C_{11}$ contains a factor λ_1. Combining estimates (2.24)-(2.26) and (2.30)-(2.39), we have estimates for θ and γ.

(2.40)

$$\begin{aligned} \theta &= \frac{p\beta}{\beta - w} + \frac{8}{n-1}f_1^2 + \frac{1}{2}\sum_2^n f_j^2 \\ &\quad - 2f_{11} + \frac{2f_1\psi_1}{1+\psi} - \frac{\Delta\psi}{1+\psi} + \frac{2|\nabla\psi|^2}{(1+\psi)^2} - R_{11} + \frac{1}{2\delta} \\ &\leq \frac{[-\inf q + C_7 + (n-1)C_8]\,\beta}{\beta - 1} + \left(\frac{8}{n-1} + \frac{1}{2}\right)C_7 + C_8 + f_1^2 + \frac{\psi_1^2}{(1+\psi)^2} \\ &\quad + \frac{C_6}{1+\psi} + \frac{2C_5}{(1+\psi)^2} + (n-1)K + \frac{1}{2\delta} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{[-\inf q + C_7 + (n-1)C_8]\,\beta}{\beta - 1} + \frac{(3n+13)C_7}{2(n-1)} \\
&\quad + C_8 + 3C_5 + C_6 + (n-1)K + \frac{1}{2\delta} \\
&= C_{12},
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma &= \frac{2}{n-1}p^2 + \frac{\delta}{2}p_1^2 \\
&\leq \frac{4}{n-1}\left[\sup|q|^2 + C_{10}^2\right] + \delta\left[\sup|\nabla q|^2 + C_{11}^2\right] = C_{13}.
\end{aligned}
\tag{2.41}
$$

Finally, we may substitute (2.40), (2.41) into (2.21) to obtain

$$
\begin{aligned}
F(x_0) &\leq (1+\psi)^2\left[(2n-1)\theta + \frac{\sqrt{(2n-1)\gamma}}{\beta-1}\right] \\
&\leq (1+H)^2\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right].
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

3. Proof.

In this section, we shall utilize Lemma 2.2 to give a proof to Theorem 1.1, 1.3 and Corollary 1.2, 1.4.

Proof of Theorem 1.1. Using (2.7), we have

$$
\begin{aligned}
F(x) &\leq (1+\psi)^2\left[(2n-1)\theta + \frac{\sqrt{(2n-1)\gamma}}{\beta-1}\right] \\
&\leq (1+H)^2\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]
\end{aligned}
$$

for any $x \in M \cup \partial M$. This implies that

$$
\frac{|\nabla w|}{\beta - w} \leq (1+H)\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]^{\frac{1}{2}}.
\tag{3.1}
$$

Let x_1, x_2 be two points in M such that $w(x_1) = 0$, $w(x_2) = \sup w = 1$, and let $\gamma \subset M$ be a minimal geodesic joining from x_1 to x_2. Then we have

$$
\begin{aligned}
\log\frac{\beta}{\beta-1} &\leq \int_\gamma \frac{|\nabla w|}{\beta - w} \\
&\leq (1+H)\left[2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]^{\frac{1}{2}} d
\end{aligned}
\tag{3.2}
$$

where d denotes the diameter of M. Recall that constants $C_1, \dots, C_6$ do not depend on λ_1. We shall group them together. Hence, we have

$$
\begin{aligned}
(3.3)\qquad & \frac{1}{d^2(1+H)^2}\left(\log\frac{\beta}{\beta-1}\right)^2 \\
&\le 2(n-1)C_{12} + \frac{\sqrt{2(n-1)C_{13}}}{\beta-1} \\
&= \frac{2(n-1)\beta\left[-\inf q + C_7 + (n-1)C_8\right]}{\beta-1} + (3n+13)C_7 \\
&\quad + 2(n-1)C_8 + 2(n-1)\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right] \\
&\quad + \frac{8\left[\sup|q|^2 + C_{10}^2\right] + 2(n-1)\delta\left[\sup|\nabla q|^2 + C_{11}^2\right]}{\beta-1} \\
&\le \left[\frac{2(n-1)\beta}{\beta-1} + (3n+13)\right]C_7 + \left[\frac{2(n-1)^2\beta}{\beta-1} + (2n-1)\right]C_8 \\
&\quad + \frac{3C_{10}}{\beta-1} + \frac{\sqrt{2(n-1)\delta}C_{11}}{\beta-1} + \frac{3\sup|q|}{\beta-1} + \frac{\sqrt{2(n-1)\delta}\sup|\nabla q|}{\beta-1} \\
&\quad - \frac{2(n-1)\beta\inf q}{\beta-1} + 2(n-1)\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right].
\end{aligned}
$$

Multiplying (3.3) through by $\frac{\beta-1}{\beta}$, we have

(3.4)

$$
\begin{aligned}
& \frac{\beta-1}{d^2\beta(1+H)^2}\left(\log\frac{\beta}{\beta-1}\right)^2 \\
&\le \left[2(n-1) + \frac{(\beta-1)(3n+13)}{\beta}\right]C_7 + \left[2(n-1)^2 + \frac{2(\beta-1)(n-1)}{\beta}\right]C_8 \\
&\quad + 3\beta C_{10} + \beta\sqrt{2(n-1)\delta}C_{11} + 3\beta\sup|q| + \beta\sqrt{2(n-1)\delta}\sup|\nabla q| \\
&\quad - 2(n-1)\inf q + 2(n-1)\frac{\beta-1}{\beta}\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right] \\
&\le \left[2(n-1)C_7 + 2(n-1)^2C_8 + 3\beta C_{10} + \beta\sqrt{2(n-1)\delta}C_{11}\right] \\
&\quad + \frac{\beta-1}{\beta}\left[(3n+13)C_7 + 2(n-1)C_8\right] \\
&\quad + \left[3\beta\sup|q| + \beta\sqrt{2(n-1)\delta}\sup|\nabla q| - 2(n-1)\inf q\right] \\
&\quad + \frac{2(n-1)(\beta-1)}{\beta}\left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta}\right].
\end{aligned}
$$

To finish the proof, we shall estimate constants $C_7, \dots, C_{11}$ in terms of λ_1. Since we have either $\lambda_1 \ge \lambda_1^3$ or $\lambda_1^3 \ge \lambda_1$, we define a to be the number such that

$$\lambda^a = \max\{\lambda_1, \lambda_1^3\}. \tag{3.5}$$

Using the definitions of the constants $C_7, \dots, C_{11}$ in Lemma 2.2, we have

(3.6)

$$\begin{aligned}
C_7 &= 4\lambda_1^2 \le 4\lambda_1^a\\
C_8 &= \lambda_1 (8\lambda_1 + 6H + 4) \le (12 + 6H)\, \lambda_1^a\\
C_9 &= 2\lambda_1 \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right)\\
&\le 2\lambda_1^a \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right)\\
C_{10} &= \max\{C_7 + (n-1)C_8, (n-1)C_9\}\\
&\le \lambda_1^a \max\left\{ 12n - 8 + 6H, 2(n-1) \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right) \right\}\\
C_{11} &= 64\lambda_1^3 + 32\lambda_1^2 + 2\lambda_1(3C_1 + C_4) \le \lambda_1^a(96 + 6C_1 + 2C_4).
\end{aligned}$$

Substituting these into (3.4), we get

(3.7)

$$\begin{aligned}
&\frac{\beta - 1}{d^2\beta(1+H)^2} \left(\log \frac{\beta}{\beta - 1} \right)^2\\
&\quad - \frac{2(n-1)(\beta-1)}{\beta} \left[3C_5 + C_6 + (n-1)K + \frac{1}{2\delta} \right]\\
&\le \left[2(n-1)C_7 + 2(n-1)^2 C_8 + 3\beta C_{10} + \beta\sqrt{2(n-1)\delta}\, C_{11} \right]\\
&\quad + \frac{\beta - 1}{\beta} \left[(3n+13)C_7 + 2(n-1)C_8 \right]\\
&\quad + \left[3\beta \sup|q| + \beta\sqrt{2(n-1)\delta} \sup|\nabla q| - 2(n-1) \inf q \right]\\
&\le \lambda_1^a \Bigg[8(n-1) + (n-1)^2(24 + 12H) + 3\beta \max\Bigg\{ 12n - 8 + 6H,\\
&\qquad 2(n-1) \left(\sqrt{(n-1)K + (n-2)K_R} + \frac{1}{R} + 1 \right) \Bigg\}\\
&\qquad + \beta\sqrt{2(n-1)}(96 + 6C_1 + 2C_4) + \frac{\beta - 1}{\beta} \big(36n + 28 + 12(n-1)H \big) \Bigg]\\
&\quad + \left[3\beta \sup|q| + \beta\sqrt{2(n-1)\delta} \sup|\nabla q| - 2(n-1) \inf q \right].
\end{aligned}$$

It is clear that the term

$$\frac{\beta-1}{d^2\beta(1+H)^2}\left(\log\frac{\beta}{\beta-1}\right)^2-\frac{2(n-1)(\beta-1)}{\beta}\left[3C_5+C_6+(n-1)K+\frac{1}{2\delta}\right]$$

in (3.7) can be made to be positive by choosing β sufficiently close to 1. It is easy to see that this term attains maximum value with

$$\frac{\beta}{\beta-1}=\exp\left[1+(1+C_{14})^{\frac{1}{2}}\right], \tag{3.8}$$

where

$$C_{14}=2d^2(n-1)^2(1+H)^2\left[3C_5+C_6+(n-1)K+\frac{1}{2\delta}\right]. \tag{3.9}$$

Putting this into (3.7), we have

(3.10)
$$\begin{aligned}
&\frac{2(1+C_{14})^{\frac{1}{2}}}{d^2(1+H)^2}\exp\left[-1-\left(1+C_{14}\right)^{\frac{1}{2}}\right]\\
&\le\lambda_1^a\Bigg[8(n-1)+(n-1)^2(24+12H)+3\beta\max\Bigg\{12n-8+6H,\\
&\qquad 2(n-1)\left(\sqrt{(n-1)K+(n-2)K_R}+\frac{1}{R}+1\right)\Bigg\}\\
&\qquad+\beta\sqrt{2(n-1)}(96+6C_1+2C_4)+\frac{\beta-1}{\beta}\big(36n+28+12(n-1)H\big)\Bigg]\\
&\quad+\left[3\beta\sup|q|+\beta\sqrt{2(n-1)\delta}\sup|\nabla q|-2(n-1)\inf q\right].
\end{aligned}$$

This completes the proof of Theorem 1.1. □

The proof for Corollary 1.2 is immediate by setting $q=0$ in (3.10).

Proof of Theorem 1.3. In this case $q=-\eta_1\ \lambda_1=0$, then we have $\phi(x)\equiv f(x)\equiv 0$. Hence, the proof of Lemma 2.2 will be simplified by setting constants $C_7,\dots,C_{11}$, and δ to be zero. Then (3.2) will take the form

$$\log\frac{\beta}{\beta-1}\le(1+H)\left[2(n-1)C_{12}+\frac{\sqrt{2(n-1)C_{13}}}{\beta-1}\right]^{\frac{1}{2}}d \tag{3.11}$$

with

$$\begin{aligned}
C_{12}&=\frac{\eta_1\beta}{\beta-1}+2C_5+C_6+(n-1)K\\
C_{13}&=\frac{4\eta_1^2}{n-1}
\end{aligned} \tag{3.12}$$

where C_5, C_6 are constants given in Lemma 2.2. Following the argument as in the proof of Theorem 1.1, we obtain

$$\frac{2(1+C_{14})^{\frac{1}{2}}}{d^2(1+H)^2}\exp\left[-1-\left(1+C_{14}\right)^{\frac{1}{2}}\right]\leq\eta_1\left[2(n-1)+\sqrt{8}\beta\right] \tag{3.13}$$

where

$$C_{14}=2d^2(n-1)(1+H)^2[2C_5+C_6+(n-1)K]$$
$$\beta=\frac{C_{14}}{C_{14}-1}. \tag{3.14}$$

□

When the Ricci curvature is nonnegative, the boundary is convex, $q=-\eta_1$, and $\lambda=0$, it is easy to see that $C_5=C_6=0$ and $K=H=0$. Therefore, one may apply (3.12) to obtain

$$\frac{e^{\frac{1}{2}}}{4d^2}\leq\eta_1\left[2(n-1)+\sqrt{8}\beta\right] \tag{3.15}$$

where

$$\beta=\frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}}-1}. \tag{3.16}$$

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Received December 1, 1998 and revised April 6, 1999. This research was partially supported by a grant from NSC.

Department of Mathematics
National Cheng Kung University
Tainan, Taiwan
E-mail address: rchen@mail.ncku.edu.tw

Department of Mathematics
National Chung Cheng University
Jiayi, Taiwan
E-mail address: cjsung@math.ccu.edu.tw