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ON EQUIVALENCE OF TWO CONSTRUCTIONS OF INVARIANTS OF LAGRANGIAN SUBMANIFOLDS

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We give the construction of symplectic invariants which incorporates both the "infinite dimensional" invariants constructed by Oh in 1997 and the "finite dimensional" ones constructed by Viterbo in 1992.

1. Introduction.

Let M be a compact smooth manifold. Its cotangent bundle T^*M carries a natural symplectic structure associated to a Liouville form $\theta = pdq$. For a given compactly supported Hamiltonian function $H : T^*M \to \mathbb{R}$ and a closed submanifold $N \subset M$ Oh [**30**, **27**] defined a symplectic invariants of certain Lagrangian submanifolds in T^*M in a following way. Let $\nu^*N \subset T^*M$ be a conormal bundle of N. Denote by $HF_*^\lambda(H, N; M)$ the Floer homology groups generated by Hamiltonian orbits γ starting at the zero section and ending at ν^*N such that $\mathcal{A}_H(\gamma) := \int_\gamma pdq - Hdt \leq \lambda$ (see, e.g., [**30**]). In particular, for $\lambda = \infty$ we write $HF_*(H, N; M) := HF_*^\infty(H, N; M)$. These groups are known to be isomorphic to $H_*(N)$ [**31**]. We denote the corresponding isomorphism by F. For $a \in H_*(N)$ one defines

$$\rho(a, H : N) := \inf\{\lambda \mid F_H(a) \in \mathrm{Im}(j_*^\lambda) \subset HF_*(H, N; M)\}, \tag{1}$$

where $j_*^\lambda : HF_*^\lambda \to HF_*(H, N; M)$ is a well defined inclusion homomorphism. It is proved in [**30**] that ρ is a well defined invariant which (after a suitable normalization of H) depends only on a Lagrangian submanifold $L := \phi^H(O_M)$ and not on a particular choice of H. We refer the reader to [**26**, **29**, **30**, **27**] for more details.

This construction can be considered as an infinite dimensional version of a construction given earlier by Viterbo [**38**]. Let L be a Hamiltonian deformation of the zero section o_M. It is known [**21**] that L can be realized as

$$L = \left\{ \left(x, \frac{\partial S}{\partial x}\right) \Big| (x, \zeta) \in \left(\frac{\partial S}{\partial \zeta}\right)^{-1}(0) \right\},$$

where $S : M \times \mathbb{R}^m \to \mathbb{R}$ is a smooth function fiberwise quadratic outside a compact set. Using that result, Viterbo [**38**] defined symplectic invariants of L associated to a homology classes of a base M in a following way. For a class $a \in H_*(N)$ denote by Ta its lift to $H_*(S_N^{\infty}, S_N^{-\infty})$, where S_N is the restriction of S to $N \times \mathbb{R}^m$ and $S_N^{\pm\infty} := S_N^{-1}((-\infty, \lambda]) =: S_N^{\lambda}$ for large λ. Note that this makes sense since S is quadratic at infinity. Then one sets

$$c(a, S : N) := \inf\{\lambda | Ta \in Im(j_*^{\lambda}) \subset H_*(S^{\infty}, S^{-\infty})\}, \tag{2}$$

where $j_*^{\lambda} : H_*(S_N^{\lambda}, S_N^{-\infty}) \to H_*(S_N^{\infty}, S_N^{-\infty})$ is an obvious inclusion homomorphism. Viterbo proved that these invariants essentially depend only on L, and not on S. Viterbo carried out the construction for $N = M$ (which generalizes easily to closed $N \subset M$) and for an arbitrary vector bundle $E \to M$. As Viterbo's invariants do not change under a stabilization (i.e., replacing $S : E \to \mathbb{R}$ by $S \oplus Q : E \oplus F \to \mathbb{R}$), it is enough to consider the case $E = M \times \mathbb{R}^m$. We refer the reader to [**38**] for more details. For an alternative construction via Morse homology see [**25**].

The natural question of the equality between the two invariants is raised in [**30**]. In [**26**] we outlined a proof, constructing the invariants which interpolate the above two. The main technical tool, which we omitted in [**26**] was the construction of the interpolated Floer-Morse theory on $T^*(M \times \mathbb{R}^m)$ with an arbitrary coefficient ring. The purpose of this paper is to give the details of this construction. Another way of interpolating Floer and Morse homologies for generating functions, in the case $M = N$ was given by Viterbo in [**39**, **37**].

The dependence of the above invariants on the subset $N \subset M$, in particular the continuity with respect to the C^1-topology of submanifolds is an interesting question, which was further studied by Kasturirangan and Oh [**18**, **19**]. Some applications to wave fronts and Hofer's geometry are given in [**30**].

At the end, we give an application of our result to Hofer's geometry of Lagrangian submanifolds.

2. Preliminaries and notation.

Let M be a compact smooth manifold and $E := M \times \mathbb{R}^m$. The cotangent bundle $T^*E = T^*M \times \mathbb{C}^m$ carries the natural symplectic structure $\omega \oplus \omega_0$.

For a fixed relatively compact open set $K \subset E$ and a Riemannian metric g_M on M we denote

$$\mathcal{G}_{g_M \oplus g_0} := \text{ the set of metrics on } E \text{ which coincide with } g_M \oplus g_0 \text{ outside } K,$$

where g_0 is a standard Euclidean metric on $\mathbb{R}^m$. For a given non-degenerate fiberwise quadratic form Q on E, we denote by $\mathcal{S}_{(E,Q)}$ the set of all smooth functions $S : E \to \mathbb{R}$ such that $S = Q$ outside K and

$$\sum_{k=0}^{\infty} \varepsilon_k \|S - Q\|_{C^k} < \infty \tag{3}$$

for some sequence ε_k of positive real numbers.

Similarly, let $\mathcal{H}(E)$ denote the set of smooth functions $H : T^*E \times [0,1] \to \mathbb{R}$ such that outside K

$$H(x,\xi) = H_1 \oplus H_2(x,\xi) := H_1(x) + H_2(\xi)$$

for some compactly supported functions $H_1 : T^*M \to \mathbb{R}$ and $H_2 : \mathbb{C}^m \to \mathbb{R}$ and

$$\sum_{k=0}^{\infty} \varepsilon_k \|H\|_{C^k} < \infty. \tag{4}$$

Equipped with norms (3) and (4) the spaces

$$\mathcal{S}_{(E,Q)} - Q := \{S - Q \mid S \in \mathcal{S}_{(E,Q)}\}$$

and $\mathcal{H}(E)$ become separable Banach spaces which are (for suitably chosen sequence ε_k) dense in $L^2(E)$ and $L^2(T^*E)$ (see [**11**]).

For a closed submanifold $N \subset M$ and a function $S \in \mathcal{S}_{(E,Q)}$ we define the space of paths

$$\Omega(S;N) := \{\Gamma : [0,1] \to T^*E \mid \Gamma(0) \in \mathrm{Graph}(dS),\ \Gamma(1) \in \nu^*(N \times \mathbb{R}^m)\},$$

and

$$\mathcal{P}^p_{k:loc}(S;N) := \{U : \mathbb{R} \to \Omega(S;N) \mid U \in W^{k,p}_{\mathrm{loc}}(\mathbb{R} \times [0,1], T^*E)\}.$$

After restricting $\mathcal{A}_H$ to $\Omega(S;N)$, for a given path $\Gamma := (\gamma, z) : [0,1] \to T^*E$ and a pair $(H,S) \in \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ the first variation formula gives

$$\begin{aligned} d\mathcal{A}_H(\Gamma)\eta &= \int_0^1 \left[\omega\left(\frac{d\gamma}{dt}, \eta\right) - dH(\gamma(t),t)\eta\right] dt - \theta\eta(0) \\ &= \int_0^1 \left[\omega\left(\frac{d\gamma}{dt}, \eta\right) - dH(\gamma(t),t)\eta\right] dt - dS(\pi(\Gamma(0)))T\pi(\eta(0)), \end{aligned}$$

where $\pi : T^*E \to E$ is the natural projection. Therefore, to get a good variational problem, we set

$$\mathcal{A}_{(H,S)}(\Gamma) = \mathcal{A}_H(\Gamma) + S(\pi(\Gamma(0)))$$

(c.f. [**30**]). Straightforward computation yields:

$$\begin{aligned} &d\mathcal{A}_{(H,S)}(\Gamma)\eta \\ &= \int_0^1 [(\omega \oplus \omega_0)(\dot{\Gamma}, \eta) - dH(\Gamma)\eta] dt \\ &\quad + \langle \Gamma(1), T\pi(\eta(1))\rangle - \langle \Gamma(0), T\pi(\eta(0))\rangle + dS(\pi(\Gamma(0)))T\pi(\eta(0)). \end{aligned}$$

After restricting to $\Omega(S;N)$

$$d\mathcal{A}_{(H,S)}(\Gamma)\eta = \int_0^1 [(\omega \oplus \omega_0)(\dot\Gamma, \eta) - dH(\Gamma)\eta]dt. \tag{5}$$

Hence, the critical points $\Gamma := (\gamma, z) : [0,1] \to T^*E$ of $\mathcal{A}_{(H,S)}$ on $\Omega(S;N)$ are the solutions of

$$\begin{cases} \dot\Gamma = X_H(\Gamma) \\ (\gamma(0), z(0)) \in \mathrm{Graph}(dS) \\ \gamma(1) \in \nu^* N,\ z(1) \in o_{\mathbb{R}^m}. \end{cases} \tag{6}$$

Note that $\Gamma \mapsto \Gamma(1)$ establishes the one-to-one correspondence

$$\begin{aligned} \mathrm{Crit}(\mathcal{A}_H) &= \{\Gamma : [0,1] \to T^*E \mid \Gamma \text{ satisfies } (6)\} \\ &\cong \phi_1^H(\mathrm{Graph}(dS)) \cap \nu^*(N \times \mathbb{R}^m). \end{aligned}$$

For a given Riemannian metric g_M on M, we denote by J_{g_M} the almost complex structure which satisfies the following conditions:

1) J_{g_M} is compatible with the canonical symplectic structure ω on T^*M.
2) J_{g_M} maps the vertical tangent vectors to the horizontal vectors with respect to the Levi-Civita connection of g_M.
3) On the zero section $o_M \subset T^*M$ J_{g_M} assigns to each vector $v \in T_qM$ the cotangent vector $J_{g_M}(v) = g_M(v, .)$ with obvious identifications.

Denote by $j^c_\omega(M)$ the set of ω-compatible almost complex structures which coincide with J_{g_M} outside a compact set in T^*M, and by $\mathcal{J}^c_\omega(M)$ the set of smooth paths $J_t : [0,1] \to j^c_\omega(M)$.

For a path $\{J_t\} \in \mathcal{J}^c_\omega(M)$, the family of product almost complex structures

$$J \oplus i := \{J_t \oplus i\}_{0 \le t \le 1}$$

is compatible with the product symplectic structure $\omega \oplus \omega_0$ on $T^*E = T^*M \times \mathbb{C}^m$. Denote by $\mathcal{J}^c_\omega(E)$ the set of almost complex structures on T^*E which coincide with product structure $J_{g_M} \oplus i$ outside a compact set. Those almost complex structures induce the family of metrics

$$\langle \eta_1, \eta_2 \rangle_{J_t} := \omega \oplus \omega_0(\eta_1, J_t\eta_2)$$

and hence an L^2-type metric

$$\langle\langle \eta_1, \eta_2 \rangle\rangle_J := \int_0^1 \langle \eta_1(t), \eta_2(t) \rangle_{J_t} dt$$

on $\Omega(S;N)$.

In terms of metric $\langle\langle .,. \rangle\rangle_J$ the gradient flow $U := (u,v) \in \mathcal{P}^p_{k:loc}(S;N)$ of $\mathcal{A}_{(H,S)}$ restricted to $\Omega(S;N)$ satisfies

$$\begin{cases} \overline{\partial}_{J,H} U := \frac{\partial U}{\partial \tau} + J\left(\frac{\partial U}{\partial t} - X_H(U)\right) = 0 \\ (u(\tau,0), v(\tau,0)) \in \mathrm{Graph}(dS) \\ u(\tau,1) \in \nu^* N,\ v(\tau,1) \in o_{\mathbb{R}^m}. \end{cases} \tag{7}$$

Denote by $CF(H,S:N)$ the set of critical points of $\mathcal{A}_{(H,S)}|_{\Omega(S;N)}$. Then

$$CF(H,S:N) = \{\Gamma = (\gamma, z) \mid \Gamma \text{ satisfies (6)}\}.$$

The set of critical values of $\mathcal{A}_{(H,S)}$ in $\mathbb{R}$

$$\mathrm{Spec}(H,S:N) := \mathcal{A}_{(H,S)}(CF(H,S:N))$$

is called *the action spectrum of* $\mathcal{A}_{(H,S)}$.

In the construction of Floer homology we will impose on the functions in $\mathcal{S}_{(E,Q)}$ the generic transversality condition

$$\mathrm{Graph}(dS) \pitchfork (\phi_1^H)^{-1}(\nu^* N \times \mathbb{R}^m). \tag{8}$$

Under assumption (8), the sets $CF(H,S:N)$ and $\mathrm{Spec}(H,S:N)$ are finite. In the general case, we have the following lemma, which describes the size of set $\mathrm{Spec}(H,S:N)$. Similar results were established in [**17**, **30**].

Lemma 1. *The action spectrum* $\mathrm{Spec}(H,S:N)$ *is a compact nowhere dense subset of* $\mathbb{R}$.

Proof. For the smooth function

$$f : \nu^* N \times o_{\mathbb{R}^m} \to \mathbb{R}$$
$$f(x) = \mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x))$$

we have, by (5)

$$df(x) = -\theta((\phi_1^H)^{-1}(x)) T(\phi_1^H)^{-1}(x) + dS(\pi(\phi_1^H)^{-1}(x)) T\pi T(\phi_1^H)^{-1}(x)$$

and thus the set $\mathrm{Spec}(H,S:N)$ is contained in the set of critical values of f. The latter is nowhere dense in $\mathbb{R}$ by the classical Sard's theorem.

Since $H = H_1 \oplus H_2$ and $\mathrm{Graph}(dS) = o_M \times \mathrm{Graph}(dQ)$ outside some compact subset $K \subset T^*E$ and $\mathrm{supp}(H_i) \subset K_i$, $i \in \{1,2\}$ for some compact subsets $K_1 \subset T^*M$ and $K_2 \subset \mathbb{C}^m$, it follows that for $x = (x_1, x_2) \in \nu^* N \times o_{\mathbb{R}^m}$ outside $K_0 := \bigcup_{t\in[0,1]} \phi_t^{H_1 \oplus H_2} \circ (\phi_1^{H_1\oplus H_2})^{-1}(K)$

$$f(x) = g_1(x_1) + g_2(x_2),$$

where

$$g_1 : \nu^* N \to \mathbb{R},\ g_1(x_1) = \mathcal{A}_{H_1}(\phi_t^{H_1} \circ (\phi_1^{H_1})^{-1}(x_1))$$
$$g_2 : o_{\mathbb{R}^m} \to \mathbb{R},\ g_2(x_2) = \mathcal{A}_{H_2}(\phi_t^{H_2} \circ (\phi_1^{H_2})^{-1}(x_2)) + Q(\pi_{\mathbb{C}^m}((\phi_1^{H_2})^{-1}(x_2))).$$

Here $\pi_{\mathbb{C}^m} : \mathbb{C}^m \to \mathbb{R}^m$ denotes the natural projection. Denote $\tilde{K}_0 := K_0 \cap \nu^* N \times o_{\mathbb{R}^m}$, $\tilde{K}_1 := K_1 \cap \nu^* N$, $\tilde{K}_2 := K_2 \cap o_{\mathbb{R}^m}$. Since $g_1 \equiv 0$ outside $\tilde{K}_1$ and $g_2 \equiv 0$ outside $\tilde{K}_2$, all critical points of g are contained in the compact set

$$B = g(\tilde{K}_0) \cup \Big(g_1(\tilde{K}_1) + g_2(\tilde{K}_2)\Big) \cup \{0\}.$$

Hence $\mathrm{Spec}(H, S : N)$ is compact as a closed subset of a compact set B. $\square$

Let $CF_*(H, S : N)$ denote the free abelian group generated by $CF(H, S : N)$ and $CF^*(H, S : N) := \mathrm{Hom}(CF_*(H, S : N), \mathbb{Z})$. Further, denote by $\mathcal{M}_{(J,H,S)}(N : E)$ the set of solutions of (7) with finite energy, i.e., of those which satisfy the condition:

$$E(U) := \int_{-\infty}^{+\infty} \int_0^1 \left(\left| \frac{\partial U}{\partial \tau} \right|_J^2 + \left| \frac{\partial U}{\partial t} - X_H(U) \right|_J^2 \right) dt d\tau < \infty. \tag{9}$$

More generally, consider the τ-dependent families

$$S^{\alpha\beta} := S^{\alpha\beta}_\tau \in \mathcal{S}_{(E,Q)},\ H^{\alpha\beta} := H^{\alpha\beta}_\tau \in \mathcal{H}(E),\ J^{\alpha\beta} := J^{\alpha\beta}_\tau \in \mathcal{J}^c_\omega(E),$$

such that for some $R > 0$ and $\tau < -R$

$$S^{\alpha\beta}_\tau \equiv S^\alpha,\ H^{\alpha\beta}_\tau \equiv H^\alpha,\ J^{\alpha\beta}_\tau \equiv J^\alpha,$$

for some fixed S^α, H^α, J^α and, similarly,

$$S^{\alpha\beta}_\tau \equiv S^\beta,\ H^{\alpha\beta}_\tau \equiv H^\beta,\ J^{\alpha\beta}_\tau \equiv J^\beta,$$

for $\tau > R$ and S^β, H^β, J^β fixed. Denote the sets of all such homotopies by

$$\overline{\mathcal{H}}(E),\ \overline{\mathcal{S}}_{(E,Q)},\ \overline{\mathcal{J}}^c_\omega(E).$$

We define $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(N : E)$ as the set of solutions of

$$\begin{cases} \overline{\partial}_{J^{\alpha\beta}, H_\tau} U := \frac{\partial U}{\partial \tau} + J^{\alpha\beta}\left(\frac{\partial U}{\partial t} - X_{H^{\alpha\beta}}(U)\right) = 0 \\ (u(\tau, 0), v(\tau, 0)) \in \mathrm{Graph}(dS^{\alpha\beta}) \\ u(\tau, 1) \in \nu^* N,\ v(\tau, 1) \in o^m_{\mathbb{R}} \end{cases} \tag{10}$$

which satisfy

$$E(U) := \int_{-\infty}^{+\infty} \int_0^1 \left(\left| \frac{\partial U}{\partial \tau} \right|_{J^{\alpha\beta}}^2 + \left| \frac{\partial U}{\partial t} - X_{H^{\alpha\beta}}(U) \right|_{J^{\alpha\beta}}^2 \right) dt d\tau < \infty. \tag{11}$$

It is a standard result in elliptic regularity theory that the solutions of (10) are smooth.

Finally, for two solutions x, y of (6) we denote by $\mathcal{M}_{(J,H,S)}(x, y)$ the set of solutions U of (7) such that

$$\lim_{\tau \to \infty} U(\tau, t) = x(t),$$

$$\lim_{\tau \to -\infty} U(\tau, t) = y(t).$$

In an analogous way, we define $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$ to be the set of solutions U of Equation (10) such that

$$\lim_{\tau\to-\infty} U(\tau,t)=x^\alpha(t)$$

$$\lim_{\tau\to\infty} U(\tau,t)=x^\beta(t),$$

where

$$\begin{cases} \dot{x^\alpha}=X_{H^\alpha}(x^\alpha) \\ x^\alpha(0)\in \mathrm{Graph}(dS^\alpha) \\ x^\alpha\in \nu^*N\times o_{\mathbb{R}^m} \end{cases} \qquad \begin{cases} \dot{x^\beta}=X_{H^\beta}(x^\beta) \\ x^\beta(0)\in \mathrm{Graph}(dS^\beta) \\ x^\beta(1)\in \nu^*N\times o_{\mathbb{R}^m}. \end{cases} \tag{12}$$

3. C^0-estimates.

In this section we will prove that the solutions of (7) and (10) remain in a compact neighborhood of zero-section. The essential ingredient of the proof is the version of maximum principle which states that a J-holomorphic curve cannot touch certain kind of hypersurfaces.

3.1. Contact type hypersurfaces.

Definition 2 ([**40**])**.** A smooth hypersurface Δ in a symplectic manifold (V,ω) is said to be *of a contact type* if there exists a vector field X defined in a neighborhood U of Δ and transversal to Δ such that $d(X\rfloor\omega)=\omega$ in U. Such vector field is called *conformal.*

It is easy to see that $\varrho := X\rfloor\omega$ defines a contact structure $\zeta :=$ Ker (ϱ) on Δ.

Definition 3 ([**7**])**.** Let Δ be an oriented hypersurface in an almost complex manifold (V,J) and ζ_q the maximal J-invariant subspace of $T_q\Delta$. Then Δ is called *J-convex* if for some (and hence any) defining 1-form ϱ for ζ_q we have $d\varrho(Y,JY)>0$ for all non-zero vectors $Y\in\zeta_q$.

For a contact type hypersurface Δ in symplectic manifold (V,ω) there exist an ω-compatible almost complex structure J such that Δ is J-convex.

Example 4. The sphere $\mathbb{S}^{2n-1}\subset\mathbb{C}^n$ is an i-convex hypersurface.

Example 5. Let J_g be an almost complex structure on T^*M defined in Section 2 and $\|\cdot\|_g$ the fiberwise norm induced by g. Then the hypersurface

$$\Delta := \{p\in T^*M \mid \|p\|_g=1\}$$

is J_g-convex.

For the sake of completeness we give the proof of the following version of the maximum principle for subharmonic functions (c.f., [**23**, **30**]).

Lemma 6. *Let $u : D \to V$ be a J-holomorphic disc in an almost complex manifold V and $\Delta \subset V$ a J-convex hypersurface. Then $u(D)$ cannot be tangent to Δ.*

Proof. Suppose that $Im(D)$ is tangent to Δ at the point $u(z)$, for some $z \in D$. Since u is J-holomorphic, $u_*(T_zD) \subset \zeta_{u(z)}$ for ζ as in Definition 3. Assume that $\Delta = f^{-1}(0)$ for some $f : V \to \mathbb{R}$.

We first prove that $f \circ u : D \to \mathbb{R}$ is subharmonic near z, or, equivalently (see [**7**]) that a two-form $di^*d(f \circ u)$ is positive definite in a neighborhood of z. Here $i^* : T^*D \to T^*D$ is the operator adjoint to $i : TD \to TD$, $\eta \mapsto \sqrt{-1}\eta$.

Choose $Y \in \zeta_{u(z)}$. Then, according to Definition 3, $JY \in \zeta_{u(z)}$ and thus $J^*df(Y) := df(JY) = 0$. Therefore,

$$J^*df|_\Delta = \mu\varrho|_\Delta + \lambda df|_\Delta$$

for ϱ as in Definition 2 and for some $\mu : V \to (0, +\infty)$ and $\lambda : V \to \mathbb{R}$. Hence

$$\begin{aligned} dJ^*df|_{\zeta_{u(z)}} &= d\mu \wedge \varrho|_{\zeta_{u(z)}} + \mu d\varrho|_{\zeta_{u(z)}} + d\lambda \wedge df|_{\zeta_{u(z)}} \\ &= \mu d\varrho|\zeta_{u(z)}. \end{aligned}$$

Since u is J-holomorphic, $i^*u^* = u^*J^*$ and thus

$$\begin{aligned} di^*d(f \circ u) &= u^*dJ^*df \\ &= u^*(\mu d\varrho) \ \text{ at } \ u(z) \\ &= (\mu d\varrho)u_*. \end{aligned}$$

Since $u_*(T_zD) \in \zeta_{u(z)}$, and since $d\varrho|_{\zeta u(z)}$ is positive definite (by Definition 3), two-form $di^*d(f \circ u)$ is positive definite near z. Hence, $f \circ u$ is subharmonic.

Now, we finish the proof arguing indirectly. Suppose that Image (D) is tangent to Δ. Then $f \circ u$ attains its maximum at z. If z is an interior point in D it contradicts the maximum principle for subharmonic functions. If $z \in \partial D$ then

$$\frac{d}{dt}|_{t=1}((f \circ u)(tz)) = 0$$

which contradicts Hopf lemma (see [**32**]). □

3.2. The structure of the space of trajectories.

In this section we prove the following analogue of well-known Floer's theorem (see [**11**, **15**, **35**]).

Proposition 7. *If $U := (u, v)$ is a solution of Equation* (10) *which satisfies the condition* (11), *then there exist the limits*

$$x^\alpha(t) = \lim_{\tau \to -\infty} U(\tau, t)$$

and

$$x^\beta(t) = \lim_{\tau\to\infty} U(\tau,t).$$

Moreover, x^α *and* x^β *are solutions of Equation* (12) *and hence*

$$\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(N:E) = \bigcup_{x^\alpha,x^\beta} \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta).$$

Proof. Choose a sequence $\tau_k \to -\infty$, and consider the sequence $U_k := U(\tau_k,t)$. We claim that U_k is bounded in $W^{1,2}([0,1],T^*E)$.

By assumption (11) we have

$$\int_0^1 \left|\frac{\partial U}{\partial t}(\tau_k,t) - X_{H^\alpha}(U(\tau_k,t))\right|^2 dt \to 0, \text{ as } k\to\infty. \tag{13}$$

Therefore, it remains to prove L^2-estimate. We will prove that $U_k(t)$ is contained in a compact subset of T^*E. We embed T^*E properly in $\mathbb{R}^p$ and denote by $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^p$. Assume first that

$$\lim_{k\to\infty} |U_k(1)| = \infty. \tag{14}$$

Recall that $S^{\alpha\beta} \equiv Q$ outside a compact set $K\subset E$. By compactness of K and (14) we have

$$\lim_{k\to\infty} \operatorname{dist}(U_k(1), \operatorname{Graph}(dS^{\alpha\beta}|_K)) = \infty. \tag{15}$$

Since

$$U_k(1) := (u_k(1), v_k(1)) \in \nu^*N \times o_{\mathbb{R}^m}$$

and

$$\operatorname{Graph}(dS^{\alpha\beta}|_{E\setminus K}) \cong o_M \times \operatorname{Graph}(dQ),$$

(14) implies

$$\begin{aligned}\lim_{k\to\infty}[\operatorname{dist}(U_k(1),\operatorname{Graph}(dS^{\alpha\beta}|_{E\setminus K}))]^2 &= \lim_{k\to\infty}[\operatorname{dist}(u_k(1),o_M)]^2 \\ &+ \lim_{k\to\infty}[\operatorname{dist}(v_k(1),\operatorname{Graph}(dQ))]^2 \\ &= \infty. \end{aligned} \tag{16}$$

Therefore, from (15) and (16) we get

$$\lim_{k\to\infty} \operatorname{dist}(U_k(1), \operatorname{Graph}(dS^{\alpha\beta})) = \infty. \tag{17}$$

Since $U_k(0)\in \operatorname{Graph}(dS^{\alpha\beta})$ (17) gives

$$\lim_{k\to\infty} |U_k(1) - U_k(0)| = \infty. \tag{18}$$

However

$$
\begin{aligned}
|U_k(1) - U_k(0)| &= \left| \int_0^1 \frac{dU_k}{dt} dt \right| \\
&\leq \left(\int_0^1 \left| \frac{dU_k}{dt} \right|^2 dt \right)^{\frac{1}{2}} \\
&< C
\end{aligned}
$$

by (13), which contradicts (18). Therefore, there exists a compact set $K_1 \subset T^*E$ such that

$$U_k(1) \in K_1 \text{ for all } k. \tag{19}$$

Assume now that there exist a sequence $t_k \in [0,1]$ such that $|U_k(t_k)|$ is unbounded. By (19) that means

$$\lim_{k \to \infty} |U_k(1) - U_k(t_k)| = \infty \tag{20}$$

for some subsequence (denoted again by) U_k. Then, by the same argument as above,

$$
\begin{aligned}
|U_k(1) - U_k(0)| &\leq \left(\int_0^{t_k} \left| \frac{dU_k}{dt} \right|^2 dt \right)^{\frac{1}{2}} \\
&< C
\end{aligned}
$$

which contradicts (20). Therefore, U_k is C^0 (and hence L^2) bounded.

Hence we deduce that U_k is bounded in $W^{1,2}([0,1], T^*E)$. Therefore, by Rellich Theorem,

$$U_k(t) \to x^\alpha(t) \text{ as } k \to \infty \text{ (in } L^2).$$

Moreover, since $\frac{dU_k}{dt}$ is L^2-bounded (by (13)), the family U_k is equicontinuous and thus, by Arzelà-Ascoli Theorem

$$U_k(t) \to x^\alpha(t) \text{ as } k \to \infty \text{ (in } C^0).$$

From (13) we conclude that x^α is a (weak) solution of Equation (6). Smoothness of x^α follows from the smoothness of X_{H^α}. Since this is true for every sequence τ_k, it is easy to see that

$$\lim_{\tau \to -\infty} U(\tau, t) = x^\alpha(t).$$

The case $\tau \to \infty$ is treated analogously. □

Remark 8. The converse of previous proposition also holds: If U is a solution of Equation (10) which satisfies (13) then U is bounded in sense of (11).

Indeed, in that case

$$\begin{aligned}\frac{1}{2}E(U) &= \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1\left(\left|\frac{\partial U}{\partial\tau}\right|^2_{J^{\alpha\beta}_\tau}+\left|\frac{\partial U}{\partial t}-X_{H^{\alpha\beta}_\tau}(U)\right|^2_{J^{\alpha\beta}_\tau}\right)dtd\tau\\ &= \int_{-\infty}^{+\infty}\int_0^1\left\langle\frac{\partial U}{\partial\tau},\frac{\partial U}{\partial t}-X_{H^{\alpha\beta}_\tau}\right\rangle_{J^{\alpha\beta}_\tau}dtd\tau\\ &= \mathcal{A}_{(H^\beta,S^\beta)}(x^\beta)-\mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha)-\int_{-\infty}^{+\infty}\int_0^1\frac{\partial H^{\alpha\beta}_\tau}{\partial\tau}dtd\tau\\ &< +\infty.\end{aligned}$$

3.3. The image of the evaluation map.

In this section we prove the C^0 estimate necessary for defining Floer homology on a non-compact manifold (see [**8**, **15**, **30**] for similar propositions). In fact, we will prove that the image of the evaluation map

$$ev:\mathcal{M}_{J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta}}(N:E)\times[0,1]\times\mathbb{R}\to T^*E$$

defined by

$$ev(U,\tau,t):=U(\tau,t)$$

is bounded.

Proposition 9. *Consider a family of parameters $(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})$ chosen as in Section 2, so that there exist a compact set $K\subset T^*E$ such that*

$$H^{\alpha\beta}\equiv H_1^{\alpha\beta}\oplus H_2^{\alpha\beta},\ J^{\alpha\beta}\equiv J_g\oplus i \text{ outside } K$$

and

$$S^{\alpha\beta}\equiv Q \text{ outside } \pi(K),$$

where $\pi:T^*E\to E$ is the natural projection. Then there exists a compact set $K_0\supset K$, depending on $(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})$, such that

$$ev(\mathcal{M}_{J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta}}(N:E))\subset K_0.$$

Proof. Let $K_0\supset K$ be a compact subset of T^*E such that

$$\mathrm{Graph}(dS^{\alpha\beta}|_{\pi(K)})\subset K_0$$

and

$$\pi_1(K_0)\supset\mathrm{supp}(H_1),\ \pi_2(K_0)\supset\mathrm{supp}(H_2)$$

where

$$\pi_1:T^*E\cong T^*M\times\mathbb{C}^m\to T^*M,\ \pi_2:T^*E\to\mathbb{C}^m$$

are natural projections. It is clear that outside K_0 Equation (10) splits onto

$$\text{(21)}\qquad\begin{cases}\overline{\partial}_{J_g}u-J_gX_{H_1^{\alpha\beta}}(u)=0\\ u(\tau,0)\in o_M\\ u(\tau,1)\in\nu^*N\end{cases}\qquad\begin{cases}\overline{\partial}v-iX_{H_2^{\alpha\beta}}(v)=0\\ v(\tau,0)\in\mathrm{Graph}(dQ)\\ v(\tau,1)\in o_{\mathbb{R}^m}.\end{cases}$$

Let $U := (u, v)$ be a solution of (21) outside K_0. Assume

$$\pi_1(K_0) \subset D_{R_0} := \{(q,p) \in T^*M \mid \|p\|_g < R_0\}.$$

Let

$$R_1 := \sup\left\{\left\|\frac{\partial S^{\alpha\beta}}{\partial q}(e)\right\|_g \mid e \in E\right\}.$$

Note that R_1 is finite since $S^{\alpha\beta}(q,\xi) = Q(\xi)$ $\left(\text{and hence } \frac{\partial S^{\alpha\beta}}{\partial q} \equiv 0\right)$ outside a compact set. Set

$$R_2 := \sup\left\{\sup_{t\in[0,1]} \|y(t)\|_g \mid \ x := (y,z) \text{ solves } (12)\right\}.$$

Since $H_1^{\alpha\beta} \equiv 0$ outside $\pi_1(K_0)$ and $H^{\alpha\beta} = H_1^{\alpha\beta} \oplus H_2^{\alpha\beta}$, $\mathrm{Graph}(dS^{\alpha\beta}) = \mathrm{Graph}(dQ)$ outside K_0 it follows that

$$\max\{R_1, R_2\} \le R_0.$$

We will first prove that

$$R_3(u) := \sup_{(\tau,t)\in\mathbb{R}\times[0,1]} \|u(\tau,t)\|_g < R_0.$$

Arguing indirectly, assume that

$$R_3(u) > R_k \text{ for } i \in \{0,1,2\}. \tag{22}$$

Then u component of Equation (21) outside the set $\{(q,p) \in T^*M \mid \|p\|_g \le R_0\}$ becomes

$$\overline{\partial}_J u = 0,$$

i.e., u is J-holomorphic. Denote

$$\Delta := \{(q,p) \in T^*M \mid \|p\|_g = R_3(u)\}.$$

By Example 5 Δ is J_g-convex. Choose $T \in \mathbb{R}$ such that

$$\sup_{|\tau|>T} \sup_{0\le t\le 1} \|u(\tau,t)\|_g < R_3(u).$$

Since $\max\|u(\tau,0)\| \le R_1$ it follows from (22) that $\max\|u(\tau,t)\|$ is achieved at some point $(\tau_0,t_0) \in [-T,T]\times(0,1]$ and there exists a neighborhood B_ε of (τ_0,t_0) such that $u|_{B_\varepsilon}$ is a J_g-holomorphic disc.

If (τ_0,t_0) is an interior point of $(0,1)\times(-T,T)$, then $u(B_\varepsilon)$ is tangent to the J_g-convex hypersurface Δ, which is, by Lemma 6, a contradiction.

Therefore, assume that $t_0 = 1$. Then $u(\tau,1)$ is a curve tangent to Δ at τ_0. But, since $u(\tau,1) \in \nu^*N$ and ν^*N is Lagrangian, $J\frac{d}{d\tau}u(\tau,1)$ must be perpendicular to ν^*N. In particular, it is perpendicular to the conformal vector field $\frac{\partial}{\partial r} \in T\nu^*N$ (see Definition 2 and Definition 3). Therefore, $J\frac{d}{d\tau}u(\tau,1) \in T\Delta$, and hence $u(B_\varepsilon)$ is tangent to Δ, which is again a contradiction by Lemma 6.

Consider now $\pi_2 : T^*E \cong T^*M \times \mathbb{C}^m \to \mathbb{C}^m$ and assume that

$$\pi_2(K_0) \subset M \times B(0, R_4),$$

where $B(0, R_4)$ is the standard Euclidean ball of radius R_4 in $\mathbb{R}^m$. If

$$R_5 := \sup \left\{ \sup_{t\in[0,1]} |z(t)| \mid x := (y, z) \text{ solves } (12) \right\},$$

where $|\cdot|$ is the standard Euclidean norm on $\mathbb{C}^m$ then $R_5 \leq R_4$. Now

$$\sup_{(\tau,t)\in\mathbb{R}\times[0,1]} |v(\tau, t)| < R_4.$$

Indeed, arguing as above, we rule out the interior points easily. For the boundary points, we use the fact that the radial vector field $\frac{\partial}{\partial\rho} \in T\mathbb{C}^m$ is tangent to both $\mathrm{Graph}(dQ)$ and $o_{\mathbb{R}^m}$ and perpendicular to the standard Euclidean sphere in $\mathbb{R}^m$. Assume that $\sup|v|$ was achieved at some point (τ_0, t_0), for $t_0 = 0$ or 1. Then the curve $v(\tau, t_0)$ is tangent to $\mathbb{S}^{2m-1}$ at τ_0 and perpendicular to the radial vector field $\frac{\partial}{\partial\rho} \in T\mathbb{R}^m$. Since both $\mathrm{Graph}(dQ)$ and $o_{\mathbb{R}^m}$ are Lagrangian, $i\frac{d}{d\tau}v(\tau, t_0)|_{\tau_0}$ is also perpendicular to $\frac{\partial}{\partial\rho}$, i.e., tangent to $\mathbb{S}^{2m-1}$. Since $\mathbb{S}^{2m-1}$ is i-convex (see Example 4), this again contradicts Lemma 6. □

Once we have established C^0 estimates, the standard compactness result follows as in [**12**, **11**, **28**, **35**]:

Proposition 10. *For any sequence $U_k \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha, x^\beta)$ there exist a subsequence (denoted by U_k again), sequences $\tau_k^j \in \mathbb{R}$ $(0 \leq j \leq l)$ and an integer s $(0 \leq s \leq l)$ such that*

1) *for $0 \leq j \leq s-1$ $U_k(\tau + \tau_k^j)$ and all its derivatives converge uniformly on compact sets to $U^j \in \mathcal{M}_{(J^\beta,H^\beta,S^\beta)}(x^j, x^{j-1})$, where x^j are the solutions of Equation (12) and $x^0 = x^\beta$,*
2) *$U_k(\tau + \tau_k^s)$ and all its derivatives converge uniformly on compact sets to $U^j \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^s, x^{s-1})$, where x^s is the solutions of Equation (12),*
3) *for $s+1 \leq j \leq l$ $U_k(\tau+\tau_k^j)$ and all its derivatives converge uniformly on compact sets to $U^j \in \mathcal{M}_{(J^\alpha,H^\alpha,S^\alpha)}(x^j, x^{j-1})$,where x^j are the solutions of Equation (12) and $x^l = x^\alpha$.*

The complementary concept to the compactness property of Proposition 10 is the gluing construction. It is now standard (see [**12**, **22**]) and can be summarized in the following

Proposition 11. *For any pair of trajectories*

$$(U^\alpha, U^{\alpha\beta}) \in \mathcal{M}_{(J^\alpha,H^\alpha,S^\alpha)}(x^\alpha, y^\alpha) \times \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(y^\alpha, z^\beta)$$

there exists a sequence $U_k \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha, z^\beta)$ *converging to* (U^1, U^2) *in the sense of Proposition* 10.

4. Fredholm theory.

Assume that $H \in \mathcal{H}(E)$ and $S \in \mathcal{S}_{(E,Q)}$ are chosen as in (8), i.e., assume that $\mathrm{Graph}(dS)$ intersects $(\phi_1^H)^{-1}(\nu^*N \times o_{\mathbb{R}^m})$ transversely. Then, for each two solutions x, y of Equation (6) there exist a smooth Banach manifold

$$\mathcal{P}_k^p(x,y) \subset \mathcal{P}_{k:loc}^p(x,y) := \{U \in \mathcal{P}_{k:loc}^p(S;N) \mid \lim_{\tau\to-\infty} U = x,\ \lim_{\tau\to\infty} U = y\}$$

such that (7) defines a smooth Fredholm section

$$\overline{\partial}_{J,H} : \mathcal{P}_k^p(x,y) \to \mathcal{L}, \tag{23}$$

where $\mathcal{L}$ is a smooth Banach bundle over $\mathcal{P}_k^p(x,y)$ with fibers

$$\mathcal{L}_U = W^{k-1,p}(\mathbb{R} \times [0,1], U^*T(T^*E)).$$

The linearization of $\overline{\partial}_{J,H}$ at $U \in \mathcal{M}_{(J,H,S)}(x,y)$ is a Fredholm operator

$$E_U := D(\overline{\partial}_{J,H}) : T_U\mathcal{P}_k^p(x,y) \to \mathcal{L}_U, \tag{24}$$

$$E_U\xi = \nabla_\tau\xi + J(U)\nabla_t\xi + \nabla_\xi J(U)\frac{\partial U}{\partial t} + \nabla_\xi\nabla H(t,U)$$

where ∇_τ, ∇_t, ∇_ξ denote the covariant derivative with respect to Levi-Civita connection associated to metric $\omega(\cdot, J\cdot)$ and $T_U\mathcal{P}_k^p(x,y)$ is the set of all $\xi \in W^{k,p}(\mathbb{R}\times[0,1], U^*T(T^*E))$ such that $\xi(\tau,0) \in T(\mathrm{Graph}(dS))$ and $\xi(\tau,1) \in T(\nu^*(N \times o_{\mathbb{R}^m}))$. Furthermore, for fixed J and S,

$$F : (U,H) := (u,v,H) \mapsto \overline{\partial}_{J,H}U$$

defines a smooth section of the Banach bundle

$$\mathcal{L} \to \mathcal{P}_k^p(x,y) \times \mathcal{H}(E)$$

transversal to the zero section. Hence, $F^{-1}(0)$ is a (Banach) manifold. The projection

$$\Pi : F^{-1}(0) \to \mathcal{H}(E)$$
$$(U,H) \mapsto H$$

is a Fredholm map. The point $U \in \mathcal{M}_{(J,H,S)}(x,y)$ is a regular point of Section (23) if and only if $(U,H) \in F^{-1}(0)$ is a regular point of Π. Hence, by Sard-Smale Theorem applied to Π, the set of points in $H \in \mathcal{H}(E)$ for which Section (23) is regular is dense in $\mathcal{H}(E)$. Similarly, one can use $\mathcal{J}_\omega^c(E)$ or $\mathcal{S}_{(E,Q)}$ in place of $\mathcal{H}(E)$.

Indeed, Floer's proof of the above statements in [**11**] (see also [**28**, **35**]) carries over in our situation with slight modifications. Hence we have the following

Proposition 12. *Let N and Q be fixed as in Section 2. Then there exists a dense set*

$$(\mathcal{J}^c_\omega(E)\times\mathcal{S}_{(E,Q)}\times\mathcal{H}(E))_{\rm reg}\subset\mathcal{J}^c_\omega(E)\times\mathcal{S}_{(E,Q)}\times\mathcal{H}(E)$$

such that for every $(J,S,H)\in(\mathcal{J}^c_\omega(E)\times\mathcal{S}_{(E,Q)}\times\mathcal{H}(E))_{\rm reg}$ the linearization of Section (23) at $U\in\mathcal{M}_{(J,H,S)}(N:E)$ is onto. Consequently, for $(J,S,H)\in(\mathcal{J}^c_\omega(E)\times\mathcal{S}_{(E,Q)}\times\mathcal{H}(E))_{\rm reg}$ $\mathcal{M}_{(J,H,S)}(x,y)$ is a smooth finite dimensional manifold.

Similarly, we have the parameterized version of Proposition 12 (see [**9**, **11**]):

Proposition 13. *Let N and Q be fixed as in Proposition 12, and*

$$(J^\alpha,S^\alpha,H^\alpha),\ (J^\beta,S^\beta,H^\beta)\in(\mathcal{J}^c_\omega(E)\times\mathcal{S}_{(E,Q)}\times\mathcal{H}(E))_{\rm reg}.$$

Then there exists a dense subset $(\overline{\mathcal{J}}^c_\omega(E)\times\overline{\mathcal{S}}_{(E,Q)}\times\overline{\mathcal{H}}(E))_{\rm reg}$ in a set of all homotopies $\overline{\mathcal{J}}^c_\omega(E)\times\overline{\mathcal{S}}_{(E,Q)}\times\overline{\mathcal{H}}(E)$ defined in Section 2 such that for

$$(J^{\alpha\beta},S^{\alpha\beta},H^{\alpha\beta})\in(\overline{\mathcal{J}}^c_\omega(E)\times\overline{\mathcal{S}}_{(E,Q)}\times\overline{\mathcal{H}}(E))_{\rm reg}$$

Equation (10) defines a smooth Fredholm section

$$(\overline{\partial}_{J,H},\overline{\partial}):\mathcal{P}^p_k(x^\alpha,x^\beta)\to\mathcal{L},\tag{25}$$

on a smooth Banach bundle $\mathcal{L}$ over $\mathcal{P}^p_k(x^\alpha,x^\beta)\subset\mathcal{P}^p_{k:loc}(x^\alpha,x^\beta)$, where

$$\mathcal{P}^p_{k:loc}(x^\alpha,x^\beta):=\{U\in\mathcal{P}^p_{k:loc}(S;N)\mid\lim_{\tau\to-\infty}U=x^\alpha\ \lim_{\tau\to\infty}U=x^\beta\},$$

which is regular at any $U\in\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$

Example 14. The case $S=S_1\oplus S_2$, $H=H_1\oplus H_2$, $J=J_1\oplus J_2$.
Assume that $S(q,\xi)=S_1(q)+S_2(\xi)$ and $H(x,y)=H_1(x)+H_2(y)$. Then Equations (6) and (7) split onto

$$\begin{cases}\dot\gamma=X_{H_1}(\gamma)\\ \gamma(0)\in\mathrm{Graph}(dS_1)\\ \gamma(1)\in\nu^*N\end{cases}\qquad\begin{cases}\dot z=X_{H_2}(z)\\ z(0)\in\mathrm{Graph}(dS_2)\\ z(1)\in o_{\mathbb{R}^m}\end{cases}$$

and

$$\begin{cases}\frac{\partial u}{\partial\tau}+J_1(\frac{\partial u}{\partial t}-X_{H_1}(u))=0\\ u(\tau,0)\in\mathrm{Graph}(dS_1)\\ u(\tau,1)\in\nu^*N\end{cases}\qquad\begin{cases}\frac{\partial v}{\partial\tau}+J_2(\frac{\partial v}{\partial t}-X_{H_2}(u))=0\\ v(\tau,0)\in\mathrm{Graph}(dS_2)\\ v(\tau,1)\in o_{\mathbb{R}^m},\end{cases}$$

and the linearization (24) splits onto

$$E_u\oplus E_v:T_{(u,v)}\mathcal{P}^p_k((x_1,x_2),(y_1,y_2))=T_u\mathcal{P}^p_k(x_1,y_1)\oplus T_v\mathcal{P}^p_k(x_2,y_2)\to\mathcal{L}_u\oplus\mathcal{L}_v.$$

Hence, if $H_1\in(\mathcal{H}(M))_{\rm reg}$ and $H_2\in(\mathcal{H}(\mathbb{R}^m))_{\rm reg}$ then $H\in(\mathcal{H}(M\times\mathbb{R}^m))_{\rm reg}$.

Example 15. The case $H \equiv 0$.
In this case we have the Morse complex of $S|_N$, which is regular for a dense subset $(\mathcal{S}_{(E,Q)})_{\rm reg} \in \mathcal{S}_{(E,Q)}$ (see Proposition 27).

In this section we will compute the Fredholm index of Sections (23) and (25) in terms of Maslov indices of Hamiltonian paths x^α and x^β. Next, we relate this computation to the Morse index of S and give the groups $CF_*(H,S:N)$ canonical grading. The existence of such grading is established in [**10**] and similar computations to ours are given for the case $S \equiv 0$, $m=0$ in [**30**] and for the periodic orbits problem in [**6**, **36**].

4.1. The Maslov index.

Maslov index for paths of Lagrangian subspaces has been studied by several authors (see [**1**, **3**, **34**, **33**]). We will follow the notation and terminology of [**34**] and [**33**]. Denote by $\Lambda(k)$ the Lagrangian Grassmanian, i.e., the manifold of Lagrangian subspaces in $\mathbb{C}^k$. The Maslov index assigns to every pair of paths

$$L, L' : [0,1] \to \Lambda(k)$$

a half integer $\mu(L,L') \in \frac{1}{2}\mathbb{Z}$ characterized by:

Naturality: For any path $\Psi : [0,1] \to \text{ Sp } (2k)$

$$\mu(\Psi(t)L(t), \Psi(t)L'(t)) = \mu(L(t), L'(t)).$$

Homotopy: Two paths L, $L' : [0,1] \to \Lambda(k)$ with $L(0) = L'(0)$ and $L(1) = L'(1)$ are homotopic with fixed endpoints if and only if they have the same Maslov index.

Zero: If $L(t) \cap L'(t)$ is of constant dimension, then $\mu(L,L') = 0$.

Direct Sum: $\mu(L_1 \oplus L_1', L_2 \oplus L_2') = \mu(L_1,L_2) + \mu(L_1',L_2')$.

Catenation: For $0 < t_0 < 1$

$$\mu(L,L') = \mu(L|_{[0,t_0]}, L'|_{[0,t_0]}) + \mu(L|_{[t_0,1]}, L'|_{[t_0,1]}).$$

Localization: If $L'(t) \equiv \mathbb{R}^k \times 0$ and $L(t) = \text{Graph}(A(t))$ for a path

$$A : [0,1] \to \text{ End } (\mathbb{R}^k)$$

of symmetric matrices then

$$\mu(L,L') = \frac{1}{2} \text{ sign } A(1) - \frac{1}{2} \text{ sign } A(0).$$

The Maslov index of a symplectic path

$$\Psi : [0,1] \to \text{ Sp } (k)$$

with respect to a fixed Lagrangian submanifold $V \subset \mathbb{C}^k$ (say $V = 0 \times \mathbb{R}^k$) is defined by

$$\mu(\Psi) := \mu(\Psi V, V),$$

or, equivalently, (see [**33**])

$$\mu(\Psi) = \mu(\mathrm{Graph}(\Psi), V \times V).$$

Following [**34**], we consider the perturbed Cauchy-Riemann operator

$$\begin{cases} \overline{\partial}_{J,T,L}\zeta := \frac{\partial\zeta}{\partial\tau} + J\frac{\partial\zeta}{\partial t} + T\zeta \\ \zeta : \mathbb{R}^k \times [0,1] \to \mathbb{C}^k \\ (\zeta(\tau,0), \zeta(\tau,1)) \in R^k \times L(\tau) \subset \mathbb{C}^k \times \mathbb{C}^k. \end{cases} \tag{26}$$

Here we assume that (c.f., [**34**, **30**]):

1) $L : \mathbb{R} \to \Lambda(k)$ is C^1 and $L(\tau) = 0 \times \mathbb{R}^k$ for large $|\tau|$.
2) The almost complex structure $J : \mathbb{R} \times [0,1] \to$ End $(\mathbb{R}^{2k})$ is compatible with symplectic form $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_k \wedge dy_k$ on $\mathbb{C}^k$ and independent of τ for $|\tau|$ large enough; $J(\pm\infty, t) = J^{\pm}(t)$.
3) $T : \mathbb{R} \times [0,1] \to$ End $(\mathbb{R}^{2k})$ is the continuous family of matrices such that

$$\lim_{\tau \to \pm\infty} \sup_{0 \le t \le 1} \|T(\tau,t) - T_{\pm}(t)\| = 0 \tag{27}$$

for some paths $T_{\pm} : [0,1] \to$ End $(\mathbb{R}^k)$ of symmetric matrices.
4) If $\Psi^{\pm} : [0,1] \to$ Sp $(2k)$ is a solution of

$$\frac{\partial \Psi^{\pm}}{\partial t} - J^{\pm}(t) T^{\pm}(t) \Psi^{\pm} = 0, \ \Psi^{\pm}(0) = \mathrm{Id} \tag{28}$$

then $\Psi^{\pm}(\mathbb{R}^k)$ is transverse to $0 \times \mathbb{R}^k$.

We will need the following:

Proposition 16 ([**34**])**.** *The operator*

$$\overline{\partial}_{J,T,L} : W^{1,2}_L \to L^2(\mathbb{R} \times [0,1], \mathbb{C}^k)$$

where

$$W^{1,2}_L := \{\zeta \in W^{1,2}(\mathbb{R} \times [0,1], \mathbb{C}^k) \mid (\zeta(\tau,0)), \zeta(\tau,1) \in \mathbb{R}^k \times L(\tau)\}$$

is Fredholm with the index given by

$$\mathrm{Index}(\overline{\partial}_{J,T,L}) = -\mu(\Psi^-) + \mu(\Psi^+) + \mu(\Delta, \mathbb{R}^k \times L(\tau))$$

where Δ is the diagonal in $\mathbb{C}^k \times \mathbb{C}^k$.

Remark 17. The Proposition above has been proved in [**34**] under the assumption $J \equiv -i$ (i.e., for the operator ∂ instead of $\overline{\partial}$). In index formula in [**34**] the Maslov indices $\mu(\Psi^{\pm})$ appear with the opposite sign. Since the change of variables $t \mapsto -t$ transforms the operator ∂ to $\overline{\partial}$ and changes the sign of Maslov index, these two difference give the index formula in Proposition 16 if $J \equiv i$. The general case is an easy consequence of the contractibility of set $\mathcal{J}^c_{\omega_0}$ of ω_0-compatible almost complex structures in $\mathbb{C}^k$ and the continuity of Fredholm index.

4.2. The dimension of $\mathcal{M}_{(J,H,S)}(N:E)$.

Our goal is to to assign the Maslov index to the Hamiltonian path

$$\begin{cases} \dot{z} = X_H(z) \\ z(0) \in \mathrm{Graph}(dS) \\ z(1) \in \nu^*(N \times \mathbb{R}^m). \end{cases} \tag{29}$$

However, in a manifold instead of a linear space the Maslov index of a Hamiltonian path would depend on the choice of a trivialization of a tangent bundle along that path. Hence, we have to choose some class of admissible trivializations. In the case $S \equiv 0$, $m = 0$ this is done in [**30**] and we will adapt that exposition to our situation. Let

$$\begin{aligned} \psi_t &:= (\phi_t^{H\sharp\pi^*S} \circ (\phi_1^{H\sharp\pi^*S})^{-1}) \circ (\phi_t^H \circ (\phi_1^H)^{-1})^{-1} \\ &= \phi_t^H \circ \phi_t^{\pi^*S} \circ (\phi_1^H \circ \phi_1^{\pi^*S})^{-1} \circ (\phi_t^H \circ (\phi_1^H)^{-1})^{-1} \\ &= \phi_t^H \circ \phi_t^{\pi^*S} \circ (\phi_1^{\pi^*S})^{-1} \circ (\phi_t^H)^{-1}. \end{aligned}$$

The transformation

$$U(\tau, t) \mapsto \tilde{U} := \psi_t(U(\tau, t)) \tag{30}$$

transforms Equation (10) to

$$\begin{cases} \overline{\partial}_{\tilde{J}, H\sharp\pi^*S} \tilde{U} = 0 \\ \tilde{U}(\tau, 0) \in o_M \times \mathbb{R}^m \\ \tilde{U}(\tau, 1) \in \nu^*(N \times \mathbb{R}^m), \end{cases} \tag{31}$$

where $\tilde{J} = \psi_t^* J$, and (29) is equivalent to

$$\begin{cases} \dot{z} = X_{H\sharp\pi^*S}(z) \\ z(0) \in o_{M\times\mathbb{R}^m} \\ z(1) \in \nu^*(N \times \mathbb{R}^m). \end{cases} \tag{32}$$

Hence, we will compute the dimension of $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(N : E)$ by computing the dimension of the space of solutions of (31) and give the grading to $CF_*(H, S)$ by assigning the Maslov index to each solution of (32).

For given z, we choose the class $\mathcal{T}$ of trivializations

$$\varphi : z^* T(T^*(E)) \to [0, 1] \times \mathbb{C}^{n+m}$$

such that

$$\varphi(H_{z(t)}) \equiv \mathbb{R}^{n+m}, \ \varphi(F_{z(t)}) \equiv i\mathbb{R}^{n+m},$$

where H_z and F_z are horizontal and vertical subbundles with respect to Levi-Civita connection on T^*E. Note that $\mathcal{T} \neq \varnothing$ since $[0, 1] \simeq *$.

For $\varphi \in \mathcal{T}$ and a solution z of (32), we define the symplectic path

$$\Psi_\varphi^z(t) := \varphi T\psi_t^{H\sharp\pi^*S}(z(0))(\varphi)^{-1} : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}. \tag{33}$$

Then we have:

Lemma 18 ([**30**]). *If* $\varphi_1, \varphi_2 \in \mathcal{T}$ *then* $\mu(\Psi_{\varphi_1}) = \mu(\Psi_{\varphi_2})$.

We give the groups $CF_*(H, S : N)$ the grading by assigning to each solution of (29) (i.e., the generator of $CF_*(H, S : N)$) the Maslov index of the corresponding solution. More precisely, we have the following:

Definition 19. 1) We call the index of the solution of (32) with respect to some (and thus any) trivialization $\varphi \in \mathcal{T}$ *the Maslov index of a solution* z *of* (29) and denote it by $\mu(z)$.

2) We denote by $CF_p(H, S : N)$ the group generated by solutions z of (29) with $p = \frac{1}{2}\mathrm{dim}(N \times \mathbb{R}^m) - \mu(z)$ and set $CF^p(H, S : N) := \mathrm{Hom}(CF_p(H, S : N), \mathbb{Z})$.

According to Theorem 2.4 in [**33**] p is an integer. We will see later (see Remark 21) that it depends on the rank of the eigenbundle of Q ($= S$ at infinity) but not on the rank of E.

Consider the case $H \equiv 0$. Let $S_N : N \to \mathbb{R}$ be a Morse function and let $S : E \to \mathbb{R}$ be an extension of S_N such that $S_N \circ \pi_N = S$ in a tubular neighborhood $\pi_N : V \to N \times \mathbb{R}^m$ of $N \times \mathbb{R}^m \subset E$. Let $x \in N \times \mathbb{R}^m$ be a critical point of S_N. We identify the neighborhood of x in $N \times \mathbb{R}^m$ with $\mathbb{R}^l$ and the neighborhood of x in E with $\mathbb{R}^l \times \mathbb{R}^{n+m-l}$. In these coordinates

$$\psi_t^{\pi^*S}(q_1, q_2, p_1, p_2) = (q_1, q_2, p_1 + dS_N(q_1), p_2)$$

and

$$T\psi_t^{\pi^*S}(x) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ tD^2S(x) & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Since $T\psi_t^{\pi^*S}(x)(\mathbb{R}^{n+m} \times \{0\}) = \mathrm{Graph}(tD^2S)$, applying the localization property of Maslov index to $A(t) := tD^2S(x)$ we get

$$\begin{aligned} \mu(T\psi_t^{\pi^*S}(x)) &= \frac{1}{2}\mathrm{sign}A(1) - \frac{1}{2}\mathrm{sign}A(0) \\ &= \frac{1}{2}\mathrm{sign}D^2S(x) \\ &= \frac{1}{2}\mathrm{sign}D^2S_N(q_1) \\ &= -m_S^N(q_1) + \frac{1}{2}\mathrm{dim}(N \times \mathbb{R}^m), \end{aligned} \tag{34}$$

where m_S^N is the Morse index of S_N. Therefore, in that case p is the Morse index of S_N.

Now we have the following analogue of Theorem 5.1 in [**30**]:

Proposition 20. *For the regular choice of parameters,*

$$\mathrm{dim}\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta) = -\mu(x^\alpha) + \mu(x^\beta).$$

In particular, for $H \equiv 0$ and S as above

$$\dim\mathcal{M}_{(J,H,S)}(x,y) = m_S^N(x) - m_S^N(y),$$

where m_S^N is the Morse index of $S|_N$.

Proof. Since

$$\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta) = (\overline{\partial}_{J^{\alpha\beta},H^{\alpha\beta}})^{-1}(0),$$

we have

$$\dim\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta) = \mathrm{Index}(E_U),$$

where E_U is the linearization of $\overline{\partial}_{J^{\alpha\beta},H^{\alpha\beta}}$ at $U \in \mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}(x^\alpha,x^\beta)$. Since $\mathrm{Index}(E_U)$ depends only on the homotopy type of U, we can assume that

$$U(-\tau,t) = x^\alpha(t), \text{ and } U(\tau,t) = x^\beta(t) \text{ for } \tau \geq \tau_0.$$

Choose a symplectic trivialization

$$\varphi : U^*T(T^*E) \to \mathbb{R} \times [0,1] \times \mathbb{C}^{n+m}$$

such that

$$\varphi(H_{U(\tau,t)}) \equiv \mathbb{R}^{n+m},\ \varphi(F_{U(\tau,t)}) \equiv i\mathbb{R}^{n+m}.$$

The same computation as in Theorem 5.3 [**36**] shows that

$$\varphi E_U \varphi^{-1} = \overline{\partial}_{J_0,T,L} + \text{compact perturbation},$$

where $\overline{\partial}_{J_0,T,L}$ is the operator (26) with $J_0 = \varphi_* J$, $L(\tau) = \varphi(T(\nu^*N \times o_{\mathbb{R}^m}))$ and T satisfies (27) and (28) with

$$\Psi^+ := \Psi_\varphi^{x^\beta},\ \Psi^- := \Psi_\varphi^{x^\alpha}$$

(see (33)). Since a compact perturbation does not change Fredholm index, we have

$$\begin{aligned}\mathrm{Index}(E_U) &= \mathrm{Index}(\overline{\partial}_{J_0,T,L})\\ &= -\mu(x^\alpha) + \mu(x^\beta) + \mu(\Delta, \mathbb{R}^{n+m} \times L(\tau))\end{aligned}$$

by Proposition 16. Since the trivialization φ is chosen so that $\dim(\Delta \cap \mathbb{R}^{n+m} \times L(\tau))$ is constant, by zero axiom we have

$$\mathrm{Index}(E_U) = -\mu(x^\alpha) + \mu(x^\beta).$$

This proves the first statement. The second statement follows from the first one and (34). □

Remark 21. Let $H = H_1 \oplus 0$ for some compactly supported Hamiltonian $H_1 : T^*M \to \mathbb{R}$ and $S(q,\xi) = Q(\xi)$. Then the grading by $p = \frac{1}{2}\dim(N \times \mathbb{R}^m) - \mu(z)$ does not depend on a fiber dimension m but only on the index of Q. Indeed, consider the stabilization

$$\tilde{Q} : E \times \mathbb{R}^{m_1} \to \mathbb{R},\ \tilde{Q} = Q \oplus Q_0$$

for some quadratic form $Q_0 : \mathbb{R}^{m_1} \to \mathbb{R}$ with zero index. The critical points of $\mathcal{A}_{(H_1 \oplus 0, \tilde{Q})}$ are of the form $(z, 0) : [0,1] \to T^*E \times \mathbb{C}^{m_1}$, where $z : [0,1] \to T^*E$ is the critical point of $\mathcal{A}_{(H_1 \oplus 0, Q)}$. Let $\varphi := (\varphi_1, \varphi_2) \in \mathcal{T}$ be a trivialization of $T^*E \times \mathbb{C}^{m_1}$. By Direct Sum Axiom

$$\begin{aligned}
&\frac{1}{2}\dim(N \times \mathbb{R}^{m+m_1}) - \mu(z,0) \\
&= \frac{1}{2}(\dim(N) + m + m_1) - \mu(\Psi^z_{\varphi_1} \oplus \Psi^0_{\varphi_2}) \\
&= \frac{1}{2}(\dim(N) + m + m_1) - \mu(\Psi^z_{\varphi_1}) - \mu(\Psi^0_{\varphi_2}) \\
&= \frac{1}{2}(\dim(N) + m + m_1) - \mu(z) \\
&\quad - \left(-\mathrm{Index}(Q_0) + \frac{1}{2}m_1\right) \quad \text{(by (34))} \\
&= \frac{1}{2}\dim(N \times \mathbb{R}^m) - \mu(z).
\end{aligned}$$

4.3. Orientation.

In order to define Floer homology for arbitrary coefficients we need the orientation of manifolds $\mathcal{M}_{(J,H,S)}$ and $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}$. Contrary to the case of holomorphic spheres or cylinders (see [**14**], [**24**]), manifolds of holomorphic discs with Lagrangian boundary conditions need not to be orientable in general. However, in case of cotangent bundle such manifold are orientable under the boundary conditions of a conormal type. More precisely, we have the following:

Proposition 22 ([**30**]). *For each* $(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta}) \in (\overline{\mathcal{J}}^c_\omega(E) \times \overline{\mathcal{H}}(E) \times \overline{\mathcal{S}}_{(E,Q)})_{\mathrm{reg}}$ *and each* x^α, x^β *the determinant bundle*

$$\mathbf{Det} \to \mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta)$$

is trivial. Hence, the manifold $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta)$ *is oriented. Moreover, there exists a coherent orientation in the sense of Definition* 11 *in* [**14**] *of all* $\mathcal{M}_{(J,H,S)}$ *and* $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}$ *in each isotopy class of* (J, H, S).

Remark 23. In [**30**] the proof Proposition 22 is carried out for the case $S \equiv 0$. The general case follows from the fact that the transformation (30) defines a diffeomorphism

$$\mathcal{M}_{(J,H,S)}(x,y) \xrightarrow{\cong} \mathcal{M}_{(\tilde{J}, H \sharp \pi^* S, 0)}(\tilde{x}, \tilde{y}).$$

Hence the orientation on $\mathcal{M}_{(\tilde{J}, H \sharp \pi^* S, 0)}(\tilde{x}, \tilde{y})$ induces the pull-back orientation on $\mathcal{M}_{(J,H,S)}(x,y)$.

Remark 24. In Section 5.2 we will prove that in the case $H \equiv 0$ for a suitable choice of a J, S, g there exists a diffeomorphism

$$\mathcal{M}^{\mathrm{Floer}}_{(J,0,S)}(x,y) \cong \mathcal{M}^{\mathrm{Morse}}_{(S,g)}(x,y)$$

for $x, y \in \mathrm{Graph}(dS) \cap \nu^*(N \times \mathbb{R}^m) \cong \mathrm{Crit}(S|_{N\times\mathbb{R}^m})$. We will choose the orientations of $\mathcal{M}_{(J,0,S)}(x,y)$ and $\mathcal{M}_{(S,g)}(x,y)$ so that this diffeomorphism is orientation preserving.

The one dimensional components of $\mathcal{M}_{(J,H,S)}$ and $\mathcal{M}_{(J^{\alpha\beta},H^{\alpha\beta},S^{\alpha\beta})}$ carry two orientations: one given in Proposition 22 and another given by orienting each trajectory in the direction of $\frac{\partial U}{\partial \tau}$. Define

$$n(U) = \begin{cases} 1 & \text{if these two orientations coincide} \\ -1 & \text{otherwise.} \end{cases}$$

Coherent (compatible with gluing) definition of orientation in Proposition 22 has the following consequence:

Lemma 25. *If $(U_1,V_1),(U_2,V_2) \in \mathcal{M}_{(J,H,S)}(x,y) \times \mathcal{M}_{(J,H,S)}(y,z)$ are two ends of the component of $\mathcal{M}_{(J,H,S)}(x,z)$ (in sense of Propositions* 10), *then*

$$n(U_1)n(V_1) + n(U_2)n(V_2) = 0.$$

□

Similar statement is true in parameterized version. The proof follows the same lines as the proof of analogous statements in [**12**, **14**, **15**].

5. Floer homology.

5.1. Construction.

For $x \in CF_p(H,S:N)$ and $y \in CF_{p-1}(H,S:N)$ we define $n(x,y)$ to be the number of points in (zero dimensional) manifold

$$\widehat{\mathcal{M}}_{(J,H,S)}(N:E) := \mathcal{M}_{(J,H,S)}(N:E)/\mathbb{R}$$

counted by their orientations, i.e.,

$$n(x,y) = \sum_U n(U),$$

where n(U) is defined in Section 4.3. Here $\mathbb{R}$ acts on $\mathcal{M}_{(J,H,S)}(N:E)$ in a standard way, by the translation in τ-variable.

According to Propositions 12, 13 and 10 for (J,H,S) in a dense subset

$$(\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}} \subset \mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$$

the number $n(x,y)$ is finite.

The following proposition is a reformulation of the result proved in [**12**] and [**28**] for the compact case.

Theorem 26.

(1) *For* $(J,H,S) \in (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ *the homomorphisms*
$$\partial : CF_p(H,S:N) \to CF_{p-1}(H,S:N)$$
$$\partial x = \sum_y n(x,y)y$$
and
$$\delta := \mathrm{Hom}(\partial) : CF^p(H,S:N) \to CF^{p+1}(H,S:N)$$
satisfy
$$\partial \circ \partial = 0, \ \delta \circ \delta = 0.$$
We define
$$HF_p(J,H,S;N:E) := \mathrm{Ker}\partial/\mathrm{Im}\partial$$
and
$$HF^p(J,H,S;N:E) := \mathrm{Ker}\delta/\mathrm{Im}\delta.$$

(2) *For two given parameters*
$$(J^\alpha, H^\alpha, S^\alpha), (J^\beta, H^\beta, S^\beta) \in (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}},$$
there exist canonical isomorphisms
$$h_{\alpha\beta} : HF_p(J^\alpha, H^\alpha, S^\alpha; N:E) \to HF_p(J^\beta, H^\beta, S^\beta; N:E)$$
and
$$h^{\alpha\beta} : HF^p(J^\alpha, H^\alpha, S^\alpha; N:E) \to HF^p(J^\beta, H^\beta, S^\beta; N:E)$$
which satisfy
(i) $h_{\gamma\beta} \circ h_{\beta\alpha} = h_{\gamma\alpha}$
(ii) $h_{\alpha\alpha} = id.$
The analogous equalities hold for $h^{\alpha\beta}$.

Proof. Once we established the C^0 -estimates as in Proposition 9, the proof follows the same lines as in Theorem 4 in [**12**] (see also [**28**]). For the later purpose, we only recall the main points. By definition of ∂, we have

$$\begin{aligned} \partial^2(x) &= \partial\left(\sum_y n(x,y)y\right) \\ &= \sum_z \sum_y n(x,y)n(y,z)z. \end{aligned} \tag{35}$$

According to Propositions 10 and 11, the split trajectories in
$$\mathcal{M}_{(J,H,S)}(x,y) \times \mathcal{M}_{(J,H,S)}(y,z)$$
are the boundaries of one dimensional manifolds contained in $\mathcal{M}_{(J,H,S)}(x,z)$ and oriented as in Section 4.3. Hence, they appear in (35) in pairs with

opposite signs and thus they add to 0. That proves $\partial \circ \partial = 0$. For the proof of the second statement, we define

$$(h_{\alpha\beta})_\sharp : CF_p(H^\alpha, S^\alpha : N) \to CF_p(H^\beta, S^\beta : N)$$

by

$$(h_{\alpha\beta})_\sharp x = \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta,$$

where $n(x^\alpha, x^\beta)$ is the number of points in (zero dimensional by Proposition 20) manifold $\mathcal{M}_{(J^{\alpha\beta}, H^{\alpha\beta}, S^{\alpha\beta})}(x^\alpha, x^\beta)$, counted with their orientations. Set

$$(h^{\alpha\beta})^\sharp := \mathrm{Hom}((h^{\beta\alpha})_\sharp) : CF^p(H^\alpha, S^\alpha : N) \to CF^p(H^\beta, S^\beta : N).$$

Note that the grading is preserved by Proposition 20. Homomorphisms $(h_{\alpha\beta})_\sharp$ and $(h^{\alpha\beta})^\sharp$ commute with ∂ and δ respectively. The proof is based on the same gluing arguments as the proof of $\partial^2 = 0$ (see [**12**, **15**]). Therefore, they define the mappings $h_{\alpha\beta}$ and $h^{\alpha\beta}$ in homology (resp. cohomology).

If $h_{\alpha\beta}$ and $h_{\beta\gamma}$ are defined via regular homotopies

$$(H^{\alpha\beta}, S^{\alpha\beta}, J^{\alpha\beta}) \text{ and } (H^{\beta\gamma}, S^{\beta\gamma}, J^{\beta\gamma})$$

then for large R the regular homotopy $(H^{\alpha\gamma}, S^{\alpha\gamma}, J^{\alpha\gamma})$, where

(36)
$$H_\tau^{\alpha\gamma} = \begin{cases} H_{\tau+R}^{\alpha\beta}, & \tau \le 0 \\ H_{\tau-R}^{\beta\gamma}, & \tau \ge 0, \end{cases} \quad S_\tau^{\alpha\gamma} = \begin{cases} S_{\tau+R}^{\alpha\beta}, & \tau \le 0 \\ S_{\tau-R}^{\beta\gamma}, & \tau \ge 0, \end{cases} \quad J_\tau^{\alpha\gamma} = \begin{cases} J_{\tau+R}^{\alpha\beta}, & \tau \le 0 \\ J_{\tau-R}^{\beta\gamma}, & \tau \ge 0, \end{cases}$$

defines the homomorphism $h_{\alpha\gamma}$ which satisfies property 2 (i). The proof is again based on the same argument as the proof of $\partial^2 = 0$ [**12**].

Finally, homomorphisms $h_{\alpha\beta}$ and $h^{\alpha\beta}$ are independent of the choice of homotopy $H^{\alpha\beta}$. We only sketch the proof of this fact, referring the reader to [**12**, **15**] for the details. Choose two homotopies $H_1^{\alpha\beta}$, $S_1^{\alpha\beta}$, $J_1^{\alpha\beta}$, $H_2^{\alpha\beta}$, $S_2^{\alpha\beta}$, $J_2^{\alpha\beta}$. Let $(h_{\alpha\beta})_\sharp^1$ and $(h_{\alpha\beta})_\sharp^2$ be the corresponding chain homomorphisms. Consider the one-parameter families of homotopies $\{H_\nu^{\alpha\beta}\}_{\nu\in\mathbb{R}}$, $\{S_\nu^{\alpha\beta}\}_{\nu\in\mathbb{R}}$, $\{J_\nu^{\alpha\beta}\}_{\nu\in\mathbb{R}}$ such that

$$H_\nu^{\alpha\beta} \equiv H_1^{\alpha\beta},\ S_\nu^{\alpha\beta} \equiv S_1^{\alpha\beta},\ J_\nu^{\alpha\beta} \equiv J_1^{\alpha\beta} \text{ for } \nu \le 0$$

and

$$H_\nu^{\alpha\beta} \equiv H_2^{\alpha\beta},\ S_\nu^{\alpha\beta} \equiv S_2^{\alpha\beta},\ J_\nu^{\alpha\beta} \equiv J_2^{\alpha\beta} \text{ for } \nu \ge 1.$$

Let $\tilde{n}(x^\alpha, x^\beta)$ denote the algebraic number of the solutions of

$$
\begin{cases}
\frac{\partial U}{\partial \tau} + J_\nu^{\alpha\beta}(\frac{\partial U}{\partial t} - X_{H_\nu^{\alpha\beta}}(U)) = 0 \\
(u(\tau,0), v(\tau,0)) \in \mathrm{Graph}(dS_\nu^{\alpha\beta}) \\
u(\tau,1) \in \nu^* N, \ v(\tau,1) \in o_{\mathbb{R}^m} \\
\lim_{\tau\to-\infty} U(\tau,t) = x^\alpha(t), \\
\lim_{\tau\to\infty} U(\tau,t) = x^\beta(t).
\end{cases} \tag{37}
$$

Define

$$\Phi^{\alpha\beta} : CF_p(H^\alpha, S^\alpha : N) \to CF_{p+1}(H^\beta, S^\beta : N)$$

by

$$\Phi^{\alpha\beta}(x^\alpha) = \sum \tilde{n}(x^\alpha, x^\beta) x^\beta.$$

Then

$$\partial \circ \Phi^{\alpha\beta} - \Phi^{\alpha\beta} \circ \partial = (h_{\alpha\beta})^1_\sharp - (h_{\alpha\beta})^2_\sharp,$$

i.e., $\Phi^{\alpha\beta}$ is a chain homotopy ([**12**, **15**]). Therefore, $h^1_{\alpha\beta} = h^2_{\alpha\beta}$.

Statement 2 (ii) now follows by choosing the constant homotopy $H^{\alpha\alpha} \equiv H^\alpha$. □

5.2. Computation.

In [**13**] Floer proved that if

$$h : M \to \mathbb{R}$$

is a C^2 Morse function, then

$$HF_*(J, h \circ \pi, M) \cong H_*^{\mathrm{Morse}}(h).$$

We incorporate this result and the generalization [**31**] in our framework. Consider the tubular neighborhood $W \cong W_0 \times \mathbb{R}^m$ of $N \times \mathbb{R}^m \subset E$ and the projection

$$\pi_N : W \to N \times \mathbb{R}^m$$

given locally by

$$\pi_N : (x, y, \xi) \mapsto (x, \xi). \tag{38}$$

Following [**31**], assume that the metric g in T^*E is chosen in the following way. Choose a metric g_M on M such that the fibers of π_N are orthogonal to $N \times \mathbb{R}^m$ with respect to the metric $g_E := g_M \oplus g_0$, where g_0 is the standard metric on $\mathbb{R}^m$. The Levi-Civita connection associated with g_E provides the splitting

$$T_\xi(T^*E) = H_\xi \oplus F_\xi$$

into horizontal and vertical subbundles. F_ξ and H_ξ are canonically isomorphic to $T^*_{\pi(\xi)}E$ and $T_{\pi(\xi)}E$. Let g be a metric on T^*E such that H_ξ is orthogonal to F_ξ and that the above isomorphisms are isometries.

Let

$$S_N : N \times \mathbb{R}^m \to \mathbb{R}$$

be a Morse function obtained by restricting $S \in \mathcal{S}_{(E,Q)}$ to $N \times \mathbb{R}^m$. Let $V \subset W$ be another tubular neighborhood of $N \times \mathbb{R}^m$ and let $\kappa : E \to \mathbb{R}$ be a smooth function such that

$$\begin{aligned} \kappa(e) &= 1 \text{ for } e \in V \\ &= 0 \text{ for } e \notin W. \end{aligned}$$

We denote by

$$S_N^V : E \to \mathbb{R}$$

an extension of S_N defined by

$$S_N^V(e) := \begin{cases} \kappa(e) S_N(\pi_N(e)) + (1-\kappa(e))S(e) & \text{for } e \in W \\ S(e) & \text{for } e \notin W. \end{cases}$$

Then $S_N^V : E \to \mathbb{R}$ is smooth and

$$S_N^V(e) = S_N \circ \pi_N(e) \text{ for } e \in V.$$

Note that from (38) and the definition of S_N^V it follows that $S_N^V(x,y,\xi) = Q(\xi)$ whenever $S(x,y,\xi) = Q(\xi)$ and hence S_N^V belongs to the parameter space $\mathcal{S}_{(E,Q)}$. Since we proved in Proposition 9 that images of all solutions of (7) are contained in some relatively compact open submanifold $K_0 \subset T^*E$, we have

$$\sup_{K_0} \left\| \nabla d S_N^V \right\| < \infty,$$

where $\|\nabla d S_N^V\|$ is defined with respect to g_E and the induced Levi-Civita connection on $T^*E|_{\pi(K_0)}$. Hence we can assume, after replacing g_E by χg_E with

$$\begin{aligned} \chi(e) &= \varepsilon_0, \text{ for } e \in K_0 \\ &= 1, \text{ for } e \notin K_1 \supset K_0 \end{aligned}$$

if necessary, that

$$\sup_{K_0} \left\| \nabla d S_N^V \right\| < \varepsilon \tag{39}$$

for small $\varepsilon > 0$. Note that the Levi-Civita connection on $T^*E|_{\pi(K_0)}$ is invariant under the rescaling $g_E \rightsquigarrow \varepsilon_0 g_E$ and thus remains unchanged. Since $\chi \equiv 1$ outside K_1, χg_E remains in parameter space $\mathcal{G}_{g_M \oplus g_0}$.

Proposition 27 (Compare [**31**])**.**

$$HF_p(J, 0, S_N^V; N : E) \cong H_p^{\text{Morse}}(S_N).$$

Proof. Since $H \equiv 0$, Equation (6) becomes

$$\begin{cases} \dot{\Gamma} = 0 \\ \Gamma(0) \in \mathrm{Graph}(dS_N^V) \\ \Gamma(1) \in \nu^* N \times o_{\mathbb{R}^m} \end{cases}$$

i.e.,

$$\Gamma(t) \equiv p \in \nu^*(N \times \mathbb{R}^m) \cap \mathrm{Graph}(dS_N^V) \ \cong \mathrm{Crit}(S_N).$$

Hence, we have one-to-one correspondence

$$CF_p(0, S_N^V; N : E) \cong \mathrm{Crit}(S_N). \tag{40}$$

Since S_N^V is constant along the fibers of π_N and the fibers are orthogonal to $N \times \mathbb{R}^m$, we have, for $e \in N \times \mathbb{R}^m$

$$\nabla^{g_E} S_N^V(e) = \nabla^{g_E^N} S_N(e), \tag{41}$$

where g_E^N is a restriction of g_E to N.

Let γ be a solution of

$$\frac{d\gamma}{d\tau} + \nabla^{g_E^N} S_N(\gamma) = 0. \tag{42}$$

Consider, modifying Lemma 5.1 in [**13**]

$$U(\tau, t) := \psi_{1-t}(\gamma(\tau))$$

and

$$J_t = (\psi_{1-t})_* J_g$$

where $\psi_t := \psi_t^{\pi^* S_N^V}$ and $J_g := J_{g_M} \oplus i$ for J_{g_M} is as in Section 2. Then

$$\begin{aligned} \frac{\partial U}{\partial \tau} + J_t \frac{\partial U}{\partial t} &= T\psi_{1-t} \frac{d\gamma}{d\tau} + (\psi_{1-t})_* J_g T\psi_{1-t}[-X_{\pi^* S_N^V}(\psi_{1-t}(\gamma))] \\ &= T\psi_{1-t} \frac{d\gamma}{d\tau} - (\psi_{1-t})_* J_g T\psi_{1-t} X_{\pi^* S_N^V}(\gamma) \\ &= T\psi_{1-t} \left(\frac{d\gamma}{d\tau} - J_g X_{\pi^* S_N^V}(\gamma) \right) \\ &= T\psi_{1-t} \left(\frac{d\gamma}{d\tau} + \nabla^g \pi^* S_N^V(\gamma) \right). \end{aligned}$$

Since $d\pi^* S_N^V$ vanishes on the vertical subbundle F it follows that $\nabla^g \pi^* S_N^V \subset H$, and since $T\pi|_H : H \to TE$ is an isometry by the choice of g, we have

$$\begin{aligned} \nabla^g \pi^* S_N^V &= \nabla^{g_E} S_N^V \\ &= \nabla^{g_E^N} S_N \ \text{(by (41))}. \end{aligned}$$

Therefore, U satisfies

$$\begin{cases} \frac{\partial U}{\partial \tau} + J_t \frac{\partial U}{\partial t} = 0 \\ U(\tau, 0) = \psi_1(\gamma(\tau)) \in \psi_1(N) \subset \mathrm{Graph}(dS_N^V) \\ U(\tau, 1) = \gamma(\tau) \in N \subset \nu^* N, \end{cases}$$

i.e., $U \in \mathcal{M}_{(J_t, 0, S_N^V)}(N : E)$.

Conversely, for every solution U of

$$\begin{cases} \frac{\partial U}{\partial \tau} + J_t \frac{\partial U}{\partial t} = 0 \\ U(\tau, 0) \in \mathrm{Graph}(dS_N) \\ U(\tau, 1) \in \nu^* N \times o_{\mathbb{R}^m} \end{cases}$$

we define

$$\gamma(\tau, t) := (\psi_{1-t})^{-1}(U(\tau, t)).$$

Note that

$$\begin{aligned} \frac{\partial \gamma}{\partial \tau} + J_g \left(\frac{\partial \gamma}{\partial t} - X_{\pi^* S_N^V}(\gamma) \right) &= (T\psi_{1-t})^{-1} \left(\frac{\partial U}{\partial \tau} + (\psi_{1-t})_* J_g \frac{\partial U}{\partial t} \right) \\ &= 0, \end{aligned}$$

i.e., γ satisfies

$$\begin{cases} \frac{\partial \gamma}{\partial \tau} + J_g \frac{\partial \gamma}{\partial t} + \nabla^g S_N^V(\gamma) = 0 \\ \gamma(\tau, 0) = \psi_1^{-1}(U(\tau, 0)) \in o_E \\ \gamma(\tau, 1) = U(\tau, 1) \in \nu^*(N \times \mathbb{R}^m). \end{cases} \tag{43}$$

We will prove that $\frac{\partial \gamma}{\partial t} \equiv 0$. Let us write $\gamma(\tau, t) = (x(\tau, t), y(\tau, t))$ with $x(\tau, t) \in E$ and $y(\tau, t) \in T^*_{x(\tau,t)} E$. Since J_g maps horizontal vectors to vertical ones, we can write (43) in the form

$$\begin{cases} \frac{\partial x}{\partial \tau} - \nabla_t y + \nabla^{g_E} S_N^V(x) = 0 \\ \nabla_\tau y + \frac{\partial x}{\partial t} = 0 \\ y(\tau, 0) = 0,\ x(\tau, 1) \in N \times \mathbb{R}^m,\ y(\tau, 1) \in \nu^*_{x(\tau,1)}(N \times \mathbb{R}^m). \end{cases} \tag{44}$$

Define

$$f(\tau) := \int_0^1 |y(\tau, t)|^2 dt.$$

Note that, by the construction of S_N^V,

$$\mathrm{Graph}(dS_N^V) \cap \nu^*(N \times \mathbb{R}^m) \subset N \times \mathbb{R}^m.$$

Therefore, we have

$$\lim_{\tau \to \pm\infty} y(\tau, t) \equiv 0$$

and hence

$$\lim_{\tau \to \pm\infty} f(\tau) = 0.$$

Following the same lines as in [**31**] we prove that f is convex, and hence constant. We identify $T_\xi(T^*E) \cong T^*_\xi E \oplus T_\xi E$ and compute

$$\begin{aligned}
\frac{1}{2}f''(\tau) &= \int_0^1 \left(|\nabla_\tau y|^2 + \langle \nabla_\tau^2 y, y\rangle\right) dt \\
&= \int_0^1 \left(|\nabla_\tau y|^2 - \left\langle \nabla_t \frac{\partial x}{\partial \tau}, y\right\rangle\right) dt \\
&= \int_0^1 \left(|\nabla_\tau y|^2 - \langle \nabla_t^2 y, y\rangle + \langle \nabla_{\frac{\partial x}{\partial t}} dS_N^V(x), y\rangle\right) dt.
\end{aligned}$$

Here we used the fact that the Levi-Civita connection is torsion free, and thus $\nabla_\tau \frac{\partial x}{\partial \tau} = \nabla_t \frac{\partial x}{\partial t}$. Since $y(\tau,0)=0$, integrating by parts we compute

$$\begin{aligned}
\int_0^1 \langle \nabla_t^2 y, y\rangle dt &= \langle \nabla_t y(\tau,1), y(\tau,1)\rangle - \int_0^1 |\nabla_t y|^2 dt \\
&= \left\langle \frac{\partial x}{\partial \tau} + \nabla S_N^V(x), y(\tau,1)\right\rangle - \int_0^1 |\nabla_t y|^2 dt \\
&= -\int_0^1 |\nabla y|^2 dt,
\end{aligned}$$

since $\frac{\partial x}{\partial \tau} + \nabla S_N^V(x) \in T(N\times\mathbb{R}^m)$ and $y(\tau,1)\in \nu^*(N\times\mathbb{R}^m)$. Hence

$$\begin{aligned}
\frac{1}{2}f''(\tau) &= \int_0^1 \left(|\nabla_\tau y|^2 + |\nabla_t y|^2 - \langle \nabla_{\nabla_\tau y} dS_N^V(x), y\rangle\right) dt \\
&\geq \|\nabla_\tau y\|_{L^2}^2 + \|\nabla_t y\|_{L^2}^2 - \left\|\nabla dS_N^V\right\|_{L^\infty} \|\nabla_\tau y\|_{L^2} \|y\|_{L^2} \\
&\geq \|\nabla_\tau y\|_{L^2}^2 + \|\nabla_t y\|_{L^2}^2 - \left\|\nabla dS_N^V\right\|_{L^\infty} \|\nabla_t y\|_{L^2}^2
\end{aligned}$$

by Poincaré inequality, since $y(\tau,0)=0$. Hence $f''(\tau)\geq 0$ if ε in (39) is small enough. Therefore $y\equiv 0$ and, by (44) $\frac{\partial x}{\partial t}\equiv 0$. Hence $\frac{\partial \gamma}{\partial t}\equiv 0$. By (43) this means that γ solves

$$\frac{d\gamma}{d\tau} + \nabla^{g_E^N} S_N^V(\gamma) = 0.$$

Therefore, we have one-to-one correspondence

$$\mathcal{M}_{(J,0,S_N^V)} \cong \mathcal{M}_{(S_N,g)}.$$

Together with (40) this finishes the proof. □

Theorem 28. *For regular parameters*

$$(\tilde{J},\tilde{H},S) \in (\mathcal{J}_\omega^c(E)\times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}},$$

and $(J,H) \in (\mathcal{J}^c_\omega(M) \times \mathcal{H}(M))_{\mathrm{reg}}$ *there exist the isomorphisms*

$$HF_{p+k}(\tilde{J},\tilde{H},S;N:E) \cong HF_p(J,H,N:M) \cong H_p(N),$$

where $HF_*(J,H,N:M)$ *is the ordinary Floer homology of the pair* $(o_M,\nu^*(N))$ *of Lagrangian submanifolds in* T^*M *and* $H_*(N)$ *the singular homology of submanifold* N*. Analogously, there exist the isomorphisms*

$$HF^{p+k}(\tilde{J},\tilde{H},S;N:E) \cong HF^p(J,H,N:M) \cong H^p(N).$$

Furthermore, the above isomorphisms commute with isomorphisms $h_{\alpha\beta}$ *(resp.* $h^{\alpha\beta}$*) constructed in Theorem* 26.

Proof. The second isomorphism

$$HF_*(J,H,N:M) \cong H_*(N),$$

follows from Proposition 27, and we will prove only the first one.

According to Theorem 26 we can assume that

$$\tilde{H} = H \oplus 0,\ \tilde{J} = J \oplus i \ \text{ and } S = Q.$$

With such choice of parameters, the critical points $\Gamma := (\gamma, z)$ of $\mathcal{A}_{(H\oplus 0,Q)}$ on $\Omega(Q;N)$ are the solutions of

$$\begin{cases} \dot{\gamma} = X_H(\gamma) \\ \gamma(0) \in o_M,\ \gamma(1) \in \nu^*N \\ \dot{z} = 0 \\ z(0) \in o_{\mathbb{R}^m},\ z(1) \in \mathrm{Graph}(dQ). \end{cases}$$

Hence $z \equiv 0$ and thus

$$CF_*(H\oplus 0, Q:N) \cong CF_*(H,N)$$

where the last group is the usual Floer chain group for the pair (o_M,ν^*N) in T^*M.

The gradient flow of $\mathcal{A}_{(H\oplus 0,Q)}$ satisfies

$$\begin{cases} \overline{\partial}_{J,H}u := \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0 \\ u(\tau,0) \in o_M,\ u(\tau,1) \in \nu^*N \\ \overline{\partial} v = \frac{\partial v}{\partial \tau} + i\frac{\partial v}{\partial t} = 0 \\ v(\tau,0) \in \mathrm{Graph}(dQ),\ v(\tau,1) \in o_{\mathbb{R}^m} \end{cases}$$

and therefore $\mathcal{M}_{(J\oplus i,H\oplus 0,Q)}(N:E)$ is diffeomorphic to $\mathcal{M}_{(J,H)}(N:M)$. Hence, the above isomorphism between Floer chain groups defines the isomorphism between the corresponding Floer homologies, and, consequently, cohomologies. □

6. Invariants.

6.1. Definition.

Observe that, since Equation (7) is the negative gradient flow of $\mathcal{A}_{(H,S)}$, the boundary operator ∂ preserves the level sets of $\mathcal{A}_{(H,S)}$. More precisely, we define

$$CF^{\lambda}(H,S:N) := \{\Gamma \in CF(H,S:N) \mid \mathcal{A}_{(H,S)}(\Gamma) < \lambda\}$$

and

$$CF^{\lambda}_{*}(H,S:N) := \text{ the free abelian group generated by } CF^{\lambda}(H,S:N).$$

Then, the boundary map

$$\partial : CF_{*}(H,S:N) \to CF_{*}(H,S:N)$$

induces the relative boundary map

$$\partial^{\lambda} : CF^{\lambda}_{*}(H,S:N) \to CF^{\lambda}_{*}(H,S:N)$$

which satisfy the obvious identity

$$\partial^{\lambda} \circ \partial^{\lambda} = 0.$$

Therefore, we can define the relative Floer homology groups

$$HF^{\lambda}_{*} := \mathrm{Ker}(\partial^{\lambda})/\mathrm{Im}(\partial^{\lambda}).$$

The natural inclusion

$$j^{\lambda} : CF^{\lambda}(H,S:N) \to CF(H,S:N)$$

induces the group homomorphism

$$j^{\lambda}_{\sharp} : CF^{\lambda}_{*}(H,S:N) \to CF_{*}(H,S:N)$$

which commutes with ∂, i.e.,

$$\partial \circ j^{\lambda}_{\sharp} = j^{\lambda}_{\sharp} \circ \partial^{\lambda}.$$

Hence, $j^{\lambda}_{\sharp}$ induces the natural homomorphism

$$j^{\lambda}_{*} : HF^{\lambda}_{*}(J,H,S:N) \to HF_{*}(J,H,S:N).$$

Furthermore, we define

$$CF^{*}_{\lambda}(H,S:N) := \mathrm{Hom}(CF^{\lambda}_{*}(H,S:N), \mathbb{Z})$$

and denote by δ^{λ} the restriction of δ to $CF^{*}_{\lambda}(H,S:N)$. Now $j^{\lambda}_{\sharp}$ induces dual homomorphism

$$j^{\sharp}_{\lambda} : CF^{*}(H,S:N) \to CF^{*}_{\lambda}(H,S:N)$$

such that

$$j^{\sharp}_{\lambda} \circ \delta = \delta^{\lambda} \circ j^{\sharp}_{\lambda}.$$

Hence, we have the homomorphism

$$j^*_\lambda : HF^*(J,H,S:N) \to HF^*_\lambda(J,H,S:N).$$

Definition 29. (1) For $(a,J,H,S) \in H_*(N)\times(\mathcal{J}^c_\omega(E)\times\mathcal{H}(E)\times\mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ we define

$$\sigma(a,J,H,S:N) := \inf\{\lambda \mid a \in \mathrm{Image}(j^\lambda_* F_*)\}.$$

(2) For $(u,J,H,S) \in H^*(N)\times(\mathcal{J}^c_\omega(E)\times\mathcal{H}(E)\times\mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ we define

$$\sigma(u,J,H,S:N) := \inf\{\lambda \mid j^*_\lambda F^* u \neq 0\}.$$

Here

$$F_* : H_*(N) \to HF_*(J,H,S:N)$$

and

$$F^* : H^*(N) \to HF^*(J,H,S:N)$$

denote the isomorphisms in Theorem 28.

Next lemma shows that the above definition is correct.

Lemma 30. *For $a \neq 0$, $u \neq 0$ and generic (J,H,S), the numbers $\sigma(a,J,H,S:N)$ and $\sigma(u,J,H,S:N)$ are the critical values of $\mathcal{A}_{(H,S)}$. In particular, they are finite numbers.*

Proof. The set of critical points of $\mathcal{A}_{(H,S)}$ is in one-to-one correspondence with

$$\mathrm{Graph}(dS)\cap(\phi^H_1)^{-1}(\nu^*N\times o_{\mathbb{R}^m}). \tag{45}$$

Since $H = H_1 \oplus H_2$ and $S = Q$ at infinity, the set (45) is

$$(o_M \cap (\phi^{H_1}_1)^{-1}(\nu^*N)) \times (\mathrm{Graph}(dQ)\cap(\phi^{H_2}_1)^{-1}(o_{\mathbb{R}^m}))$$

outside a compact set. Since H_1 and H_2 have compact supports, all points in (45) are contained in a compact set. From transversality assumption (8) we conclude that the set (45) is finite. Hence, if λ is not a critical value of $\mathcal{A}_{(H,S)}$, then there exists $\mu < \lambda$ such that there is no critical values of $\mathcal{A}_{(H,S)}$ in closed interval $[\mu,\lambda]$. In that case,

$$CF^\lambda_*(H,S:N) \equiv CF^\mu_*(H,S:N),\ CF^*_\lambda(H,S:N) \equiv CF^*_\mu(H,S:N)$$

and

$$j^\lambda_\sharp \equiv j^\mu_\sharp,\ j^\sharp_\lambda \equiv j^\sharp_\mu.$$

Hence, $z \in \mathrm{Im}(j^\lambda_*)$ (resp. $j^*_\lambda \neq 0$) is equivalent to $z \in \mathrm{Im}(j^\mu_*)$ (resp. $j^*_\mu \neq 0$). It follows that λ cannot be detected by σ.

Finally, since there are only finitely many critical values of $\mathcal{A}_{(H,S)}$, we deduce that both $\sigma(a,J,H,S:N)$ and $\sigma(u,J,H,S:N)$ are finite numbers. $\square$

We next show that the definition of σ does not depend on an almost complex structure J used in construction of Floer homology.

Proposition 31. *The numbers $\sigma(a, J, H, S : N)$ and $\sigma(u, J, H, S : N)$ are independent of the regular choice of J.*

Proof. For $J^\alpha, J^\beta \in \mathcal{J}^c_\omega(E)$ we chose any path $\overline{J}_\tau \in \overline{\mathcal{J}}^c_\omega(E)$. Recall from the proof of Theorem 26 that the isomorphism

$$h_{\alpha\beta} : HF_*(J^\alpha, H, S : N) \to HF_*(J^\beta, H, S : N)$$

is induced by the group homomorphism

$$h_\sharp : CF_*(H, S : N) \to CF_*(H, S : N), \qquad (46)$$
$$h_\sharp(x^\alpha) = \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta,$$

where $n(x^\alpha, x^\beta)$ is the algebraic number of points in zero dimensional manifold $\mathcal{M}_{(J^{\alpha\beta},H,S)}(x^\alpha, x^\beta)$. We compute the difference $\mathcal{A}_{(H,S)}(x^\beta) - \mathcal{A}_{(H,S)}(x^\alpha)$ for every x^β which appears in sum (46). For such x^β, the set $\mathcal{M}_{(J^{\alpha\beta},H,S)}(x^\alpha, x^\beta)$ is nonempty ($n(x^\alpha, x^\beta) \neq 0$). For any $U \in \mathcal{M}_{(J^{\alpha\beta},H,S)}$

$$\begin{aligned}
&\mathcal{A}_{(H,S)}(x^\beta) - \mathcal{A}_{(H,S)}(x^\alpha)\\
&= \int_{-\infty}^{+\infty} \frac{d}{d\tau}\mathcal{A}_{(H,S)}(U) d\tau\\
&= \int_{-\infty}^{+\infty} d\mathcal{A}_{(H,S)}(U)\frac{\partial U}{\partial \tau} d\tau\\
&= \int_{-\infty}^{+\infty}\int_0^1 \left[(\omega \oplus \omega_0)\left(\frac{\partial U}{\partial t}, \frac{\partial U}{\partial \tau}\right) - dH(U)\frac{\partial U}{\partial t}\right] dt d\tau\\
&= \int_{-\infty}^{+\infty}\int_0^1 \left\langle \overline{J}_\tau\left(\frac{\partial U}{\partial t} - X_H(U)\right), \frac{\partial U}{\partial \tau}\right\rangle_{\overline{J}_\tau} dt d\tau\\
&= -\int_{-\infty}^{+\infty}\int_0^1 \left|\frac{\partial U}{\partial \tau}\right|^2_{\overline{J}_\tau} dt d\tau\\
&\leq 0.
\end{aligned}$$

Here we used (5) and (10). Hence, $\mathcal{A}_{(H,S)}(x^\alpha) \geq \mathcal{A}_{(H,S)}(x^\beta)$ and therefore $h_{\alpha\beta}$ is level preserving, i.e.,

$$h_{\alpha\beta} : HF^\lambda_*(J^\alpha, H, S : N) \to HF^\lambda_*(J^\beta, H, S : N), \qquad (47)$$
$$h_{\alpha\beta} \circ j^\lambda_* = j^\lambda_* \circ h_{\alpha\beta}$$

Assume that $F^\alpha a \in \mathrm{Im}(j^\lambda_*)$, where

$$F^\alpha_* : H_*(N) \to HF_*(J^\alpha, H, S : N)$$

is the isomorphism in Theorem 26. Then, by (47), $h_{\alpha\beta}F^\alpha_* a \in \mathrm{Im}(j^\lambda_*)$. Since $h_{\alpha\beta}F^\alpha_* = F^\beta_*$ by Theorem 26, we have $F^\beta_* a \in \mathrm{Im}(j^\lambda_*)$ and hence

$$\sigma(a, J^\alpha, H, S : N) \leq \sigma(a, J^\beta, H, S : N).$$

Since the above argument is valid for any J^α, J^β, the opposite inequality also holds and therefore

$$\sigma(a, J^\alpha, H, S : N) = \sigma(a, J^\beta, H, S : N).$$

$\square$

As a consequence, we can introduce the following notation.

Definition 32. For regular choice of parameters (J, H, S) we denote the numbers $\sigma(a, J, H, S : N)$ and $\sigma(u, J, H, S : N)$ by $\sigma(a, H, S : N)$ and $\sigma(u, H, S : N)$.

6.2. Continuity.

In order to extend Definition 32 from $(\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ to the whole manifold $\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ we need the following continuity result:

Theorem 33. *For $a \in H_*(N)$ the function*

$$\sigma : (\mathcal{J}^c_\omega(E) \times \mathcal{H}(E) \times \mathcal{S}_{(E,Q)})_{\mathrm{reg}} \to \mathbb{R},$$
$$(H, S) \mapsto \sigma(a, H, S : N)$$

is continuous in C^0 topology. The analogous statement is true for $u \in H^*(N)$ and $\sigma(u, H, S : N)$.

Proof. We fix regular parameters (H^α, S^α) and (H^β, S^β) and choose the C^∞ function

$$\rho : \mathbb{R} \to \mathbb{R}$$

such that

$$\rho(\tau) = 1, \text{ for } \tau \geq 1$$
$$\rho(\tau) = 0, \text{ for } \tau \leq 0.$$

Denote by $(\overline{H}_\tau, \overline{S}_\tau)$ a regular homotopy connecting (H^α, S^α) and (H^β, S^β) which is ε-close in C^1-topology to (possibly non-regular) homotopy

$$(\rho(\tau)H^\beta + (1 - \rho(\tau))H^\alpha, \rho(\tau)S^\beta + (1 - \rho(\tau))S^\alpha).$$

Then, as in the proof of Proposition 31 we compute $\mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) - \mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha)$ for a pair x^α, x^β connected by trajectory U satisfying (10). Since

$$\frac{d}{d\tau}\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U(\tau)) = d\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U)\frac{\partial U}{\partial \tau} - \int_0^1 \frac{\partial \overline{H}_\tau}{\partial \tau}dt + \frac{\partial \overline{S}_\tau}{\partial \tau}$$

and since the last two terms are ε-close to

$$-\int_0^1 \rho'(\tau)(H^\beta - H^\alpha)dt + \rho'(\tau)(S^\beta - S^\alpha),$$

we have

$$
\begin{aligned}
&\mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) - \mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha) \\
&= \int_{-\infty}^{+\infty} \frac{d}{d\tau}\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U(\tau))d\tau \\
&\leq \int_{-\infty}^{+\infty}\Bigg\{\int_0^1\left[d\mathcal{A}_{(\overline{H}_\tau,\overline{S}_\tau)}(U)\frac{\partial U}{\partial \tau} - \rho'(\tau)(H^\beta - H^\alpha)\right]dt \\
&\qquad + \rho'(\tau)(S^\beta - S^\alpha)\Bigg\}d\tau + \varepsilon \\
&\leq \int_{-\infty}^{+\infty}\left|\frac{\partial U}{\partial \tau}\right|_J^2 d\tau - \int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha) + \varepsilon.
\end{aligned}
$$

Here, again, we used (5) and (10). Hence, we have the well defined homomorphism

$$h_{\alpha\beta} : HF_*^\lambda(J, H^\alpha : S^\alpha : N) \to HF_*^{\lambda_{\alpha\beta}+\varepsilon}(J, H^\beta; S^\beta : N)$$

where $\lambda_{\alpha\beta} := \lambda - \int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha)$, such that the diagram

$$
\begin{array}{ccc}
HF_*^\lambda(J, H^\alpha : S^\alpha : N) & \xrightarrow{j_*^\lambda} & HF_*(J, H^\alpha; S^\alpha : N) \\
\Big\downarrow h_{\alpha\beta} & & \Big\downarrow h_{\alpha\beta} \\
HF_*^{\lambda_{\alpha\beta}+\varepsilon}(J, H^\beta; S^\beta : N) & \xrightarrow{j_*^{\lambda_{\alpha\beta}+\varepsilon}} & HF_*(J, H^\beta; S^\beta : N)
\end{array}
$$

commutes. By the same argument as in Proposition 31 we deduce, for $a \in H_*(N)$

$$
\begin{aligned}
&\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\beta, S^\beta : N) \\
&\leq -\int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha) + \varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0$ this becomes

$$
\begin{aligned}
&\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\alpha, S^\alpha : N) \\
&\leq -\int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha).
\end{aligned}
$$

By changing the role of α and β we get

$$
\begin{aligned}
&\sigma(a, H^\beta, S^\beta : N) - \sigma(a, H^\alpha, S^\alpha : N) \\
&\geq -\int_0^1 \max(H^\beta - H^\alpha)dt + \min(S^\beta - S^\alpha)
\end{aligned}
$$

and therefore

$$\begin{aligned}&|\sigma(a,H^\beta,S^\beta:N)-\sigma(a,H^\alpha,S^\alpha:N)|\\&\leq\|H^\beta-H^\alpha\|_{C^0}+\|S^\beta-S^\alpha\|_{C^0}.\end{aligned}$$

□

As a consequence we have the following:

Definition 34. For $(a,H,S)\in H_*(N)\times\mathcal{H}(E)\times\mathcal{S}_{(E,Q)}$ we define

$$\sigma(a,H,S:N):=\lim_{k\to\infty}\sigma(a,H_k,S_k:N)$$

where the limit is taken over any sequence

$$(\mathcal{J}^c_\omega(E)\times\mathcal{H}(E)\times\mathcal{S}_{(E,Q)})_{\mathrm{reg}}\ni(J,H_k,S_k)$$

such that

$$C^0-\lim_{k\to\infty}(H_k,S_k)=(H,S).$$

We define $\sigma(u,H,S:N)$ for $u\in H^*(N)$ in the same way.

The following lemma extends Lemma 30:

Lemma 35. *For $a\neq 0$, $u\neq 0$ and arbitrary (not necessarily generic) (J,H,S), the numbers $\sigma(a,J,H,S:N)$ and $\sigma(u,J,H,S:N)$ are the critical values of $\mathcal{A}_{(H,S)}$.*

Proof. For any $(H,S)\in\mathcal{H}(E)\times\mathcal{S}_{(E,Q)}$ there exists a sequence $(H_k,S_k)\in(\mathcal{H}(E)\times\mathcal{S}_{(E,Q)})_{\mathrm{reg}}$ of generic functions such that

$$C^1-\lim_{k\to\infty}(H_k,S_k)=(H,S). \tag{48}$$

According to Lemma 30 there exists a sequence of points

$$x_k\in\phi_1^{H_k}(\mathrm{Graph}(dS_k))\cap\nu^*(N\times\mathbb{R}^m)$$

such that

$$\sigma(a,J,H_k,S_k:N)=\mathcal{A}_{(H_k,S_k)}(\phi_t^{H_k}\circ(\phi_1^{H_k})^{-1}(x_k)).$$

Note that x_k is bounded and hence, after taking a subsequence, we can assume that

$$\lim_{k\to\infty}x_k=x_0. \tag{49}$$

Define

$$\begin{aligned}&f,f_k:\nu^*(N\times\mathbb{R}^m)\to\mathbb{R},\\&f_k(x):=\mathcal{A}_{(H_k,S_k)}(\phi_t^{H_k}\circ(\phi_1^{H_k})^{-1}(x))\\&f(x):=\mathcal{A}_{(H,S)}(\phi_t^{H}\circ(\phi_1^{H})^{-1}(x)).\end{aligned} \tag{50}$$

From (48) it follows that

$$\lim_{k\to\infty} \phi_t^{H_k}(x) = \phi_t^H(x)$$

for all $x \in T^*E$ and hence

$$\lim_{k\to\infty} f_k(x) = f(x). \tag{51}$$

Since f_k are smooth, by Arzelà-Ascoli Theorem the convergence above is uniform on compact subsets of $\nu^*(N\times\mathbb{R}^m)$. Similarly, by (48)

$$\lim_{k\to\infty} df_k = df \tag{52}$$

uniformly on compact subsets of $\nu^*(N\times\mathbb{R}^m)$. According to Definition 34 and by (49) and (51)

$$\begin{aligned} \sigma(a, J, H, S : N) &= \lim_{k\to\infty} \sigma(a, J, H_k, S_k : N) \\ &= \lim_{k\to\infty} f_k(x_k) \\ &= f(x_0) \\ &= \mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x_0)). \end{aligned}$$

By (49) , (50) and (52) we have

$$\begin{aligned} d\mathcal{A}_{(H,S)}(\phi_t^H \circ (\phi_1^H)^{-1}(x_0)) &= df(x_0) \\ &= \lim_{k\to\infty} df_k(x_k) \\ &= 0 \end{aligned}$$

and hence $\phi_t^H \circ (\phi_1^H)^{-1}(x_0) \in \mathrm{Crit}(\mathcal{A}_{(H,S)})$. □

6.3. Normalization.

Consider the Hamiltonian

$$K_t := \chi(t)(H_t + c_0)$$

where $\chi : T^*M \to \mathbb{R}$ is a smooth function with compact support, such that $\chi \equiv 1$ in a neighborhood of $\cup_{t\in[0,1]}\phi_t^H(o_M)$. Then $\phi_1^H(o_M) = \phi_1^K(o_M)$, but

$$\rho(a, K : N) = \rho(a, H : N) + c_0.$$

More generally, it can be shown that for *any* two Hamiltonians H and K such that $\phi_1^H(o_M) = \phi_1^K(o_M)$ we have

$$\rho(a, K : N) = \rho(a, H : N) + c_0$$

for some $c_0 \in \mathbb{R}$ [**30**]. Similar considerations apply to the case of invariants c and σ. Hence, in order to consider the constructed invariants as the invariants of Lagrangian submanifolds, we have to impose certain normalization on the choice of parameters in $(H, S) \in \mathcal{H}(E)\times\mathcal{S}_{(E,Q)}$. Assume that $H = H_1 \oplus 0$

for some compactly supported Hamiltonian function $H : T^*M \to \mathbb{R}$. Denote by $L_S \subset T^*M$ the Lagrangian submanifold having S as a generating function quadratic at infinity. We will need the following result.

Theorem 36. *If H^α, H^β are two compactly supported Hamiltonians defined on T^*M and S^α, S^β two generating functions quadratic at infinity such that*

$$\phi_1^{H^\alpha}(L_{S^\alpha}) = \phi_1^{H^\beta}(L_{S^\beta}),$$

then there exists a constant $c_0 \in \mathbb{R}$ such that for any $N \subset M$

$$\operatorname{Spec}(H^\alpha \oplus 0, S^\alpha : N) = \operatorname{Spec}(H^\beta \oplus 0, S^\beta : N) + c_0. \tag{53}$$

In particular, if $x_\infty \in M$ is fixed and

$$\begin{aligned}\widetilde{\operatorname{Spec}}(H \oplus 0, S : N) &:= \operatorname{Spec}(H \oplus 0, S : N) - \operatorname{Spec}(H \oplus 0, S : x_\infty)\\ &= \{r - s \mid r \in \operatorname{Spec}(H \oplus 0, S : N),\ s \in \operatorname{Spec}(H \oplus 0, S : x_\infty)\}\end{aligned}$$

then

$$\widetilde{\operatorname{Spec}}(H^\alpha \oplus 0, S^\alpha : N) = \widetilde{\operatorname{Spec}}(H^\beta \oplus 0, S^\beta : N). \tag{54}$$

Proof. The critical points of $\mathcal{A}_{(H^\alpha, S^\alpha)}$ and $\mathcal{A}_{(H^\beta, S^\beta)}$ are in one-to-one correspondence with points of

$$\nu^*N \cap \phi_1^{H^\alpha}(L_S) = \nu^*N \cap \phi_1^{H^\beta}(L_S). \tag{55}$$

More precisely, the solutions of

$$\begin{cases} \frac{d\Gamma}{dt} = X_{H\oplus 0}(\Gamma) \\ \Gamma(0) \in \operatorname{Graph}(dS) \\ \Gamma(1) \in \nu^*N \times o_{\mathbb{R}^m} \end{cases}$$

are of the form $\Gamma = (\gamma, z)$ where

$$\begin{cases} \frac{d\gamma}{dt} = X_H(\gamma) \\ \frac{dz}{dt} = 0 \\ (\gamma(0), z) \in \operatorname{Graph}(dS),\ \gamma(1) \in \nu^*N,\ z \equiv (\xi, 0) \in o_{\mathbb{R}^m}. \end{cases}$$

Denote

$$L := \phi_1^{H^\alpha}(L_{S^\alpha}) = \phi_1^{H^\beta}(L_{S^\beta})$$

and consider the function $f : L \to \mathbb{R}$ defined by

$$\begin{aligned} f(x) &= \mathcal{A}_{(H^\alpha, S^\alpha)}(\phi_t^{H^\alpha \oplus 0}(dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x))))) \\ &- \mathcal{A}_{(H^\beta, S^\beta)}(\phi_t^{H^\beta \oplus 0}(dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x))))). \end{aligned}$$

Since $(\phi_1^{H^\alpha})^{-1}(L) = L_{S^\alpha}$ and and $(\phi_1^{H^\beta})^{-1}(L) = L_{S^\beta}$, for $x \in L$

$$i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x)) = i_{S^\alpha}^{-1}\left(q^\alpha, \frac{\partial S^\alpha}{\partial q}(q^\alpha, \xi^\alpha)\right) = (q^\alpha, \xi^\alpha),$$

$$i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x)) = i_{S^\beta}^{-1}\left(q^\beta, \frac{\partial S^\beta}{\partial q}(q^\beta, \xi^\beta)\right) = (q^\beta, \xi^\beta),$$

where

$$(q^\alpha, \xi^\alpha) \in \Sigma_{S^\alpha}, \quad (q^\beta, \xi^\beta) \in \Sigma_{S^\beta} \tag{56}$$

and

$$\left(q^\alpha, \frac{\partial S^\alpha}{\partial q}\right) = (\phi_1^{H^\alpha})^{-1}(x), \quad \left(q^\beta, \frac{\partial S^\beta}{\partial q}\right) = (\phi_1^{H^\beta})^{-1}(x). \tag{57}$$

Hence

$$\begin{aligned} dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x))) &= dS^\alpha(q^\alpha, \xi^\alpha) \\ &= \left(q^\alpha, \frac{\partial S^\alpha}{\partial q}(q^\alpha, \xi^\alpha), \xi^\alpha, \frac{\partial S^\alpha}{\partial \xi}(q^\alpha, \xi^\alpha)\right) \\ &= \left(q^\alpha, \frac{\partial S^\alpha}{\partial q}(q^\alpha, \xi^\alpha), \xi^\alpha, 0\right) \quad \text{(by (56))} \\ &= (((\phi_1^{H^\alpha})^{-1}(x)), \xi^\alpha, 0) \quad \text{(by (57))}, \end{aligned}$$

and similarly

$$dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x))) = (((\phi_1^{H^\beta})^{-1}(x)), \xi^\beta, 0).$$

Therefore, the paths

$$\begin{aligned} &t \mapsto \phi_t^{H^\alpha \oplus 0}(dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x)))) \quad \text{and} \\ &t \mapsto \phi_t^{H^\beta \oplus 0}(dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x)))) \end{aligned} \tag{58}$$

respectively start at Graph(dS^α) (respectively Graph(dS^β)) and end at the points (in local coordinates)

$$\begin{aligned} \phi_1^{H^\alpha \oplus 0}(dS^\alpha(i_{S^\alpha}^{-1}((\phi_1^{H^\alpha})^{-1}(x)))) &= (\phi_1^{H^\alpha \oplus 0}((\phi_1^{H^\alpha})^{-1}(x)), \xi^\alpha, 0) \\ &= (x, \xi^\alpha, 0) \end{aligned}$$

and

$$\phi_1^{H^\beta \oplus 0}(dS^\beta(i_{S^\beta}^{-1}((\phi_1^{H^\beta})^{-1}(x)))) = (x, \xi^\beta, 0)$$

respectively. Let

$$\chi : \mathbb{R} \to \phi_1^{H^\alpha}(L_S) = \phi_1^{H^\beta}(L_S)$$

be a smooth path connecting two points in (55). Since the paths (58) are Hamiltonian and start at Graph(dS^α) and Graph(dS^β), the same computation as in (5) shows that

$$\frac{d}{ds} f(\chi(s)) = \theta(\eta^\alpha(s)) - \theta(\eta^\beta(s))$$

where

$$\begin{aligned}\eta^\alpha(s) &:= \frac{d}{ds}\phi_1^{H^\alpha\oplus 0}(dS^\alpha(i_{S^\alpha}((\phi_1^{H^\alpha})^{-1}(\chi(s))))) \\ &= \left(\frac{d\chi}{ds}, \frac{d\xi^\alpha(s)}{ds}, 0\right),\end{aligned}$$

with $(\frac{d\xi^\alpha(s)}{ds}, 0) \in T(o_{\mathbb{R}^m})$. Similarly,

$$\eta^\beta(s) = \left(\frac{d\chi}{ds}, \frac{d\xi^\beta(s)}{ds}, 0\right).$$

Since $\theta = \theta_M \oplus \theta_{\mathbb{R}^m}$ and $\theta_{\mathbb{R}^m}(\xi, 0) = 0$ it follows that

$$\frac{d}{ds}f(\chi(s)) = 0.$$

Hence $f \equiv c_0$, for some constant $c_0 \in \mathbb{R}$ independent of N. This proves (53) and (54). □

Definition 37. Fix $x_\infty \in M$. Let S be a generating function quadratic at infinity for the Lagrangian submanifold $L_S = \phi_1^H(o_M) \in T^*M$. We define the normalized parameters $(\tilde{H}, \tilde{S})$ by

$$\tilde{S} = S - \frac{1}{2}\sigma(1, H \oplus 0, S : x_\infty),\ \tilde{H} = H + \frac{1}{2}\sigma(1, H \oplus 0, S : x_\infty).$$

Remark 38. Strictly speaking, $(\tilde{H}, \tilde{S}) \notin \mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$. However, it is allowed to add a constant to the parameters in $\mathcal{H}(E) \times \mathcal{S}_{(E,Q)}$ since Floer theory depends only on the first derivatives $(\nabla H, \nabla S)$ which remain unchanged.

The normalization described above also gives the normalization of invariants ρ and c defined by (1) and (2). Indeed, these invariants are the special cases of invariant σ, as we show in the following lemma:

Lemma 39. *For $(H, S) \in \mathcal{H}(M) \times \mathcal{S}_{(E,Q)}$ and $a \in H_*(N)$*

$$\sigma(a, H \oplus 0, Q : N) = \rho(a, H : N)$$

and

$$\sigma(a, 0, S : N) = c(a, S : N).$$

Analogous statements hold for any $u \in H^*(N)$.

Proof. The first equality follows from Theorem 28. To prove the second one, we first observe that, if S_N^V is as in Proposition 27 and $S_N^V \equiv S$ outside $U \supset V$, then

$$c(a, S : N) = \sigma(a, 0, S_N^V : N).$$

Since $\|S_N^V - S\|_{C^0} \to 0$ as $U \to N$, the conclusion follows from Theorem 33. □

Definition 40. Fix $x_\infty \in M$. Let S be a generating function quadratic at infinity for the Lagrangian submanifold $L_S = \phi_1^H(o_M) \in T^*M$. For a submanifold $N \subset M$ and $a \in H^*(N)$, $u \in H^*(N)$ we define

$$c(a, L_S : N) := c(a, \tilde{S} : N),\ c(u, L_S : N) := c(u, \tilde{S} : N),$$

where $\tilde{S} = S - c(1, S : x_\infty)$. In a similar way, define

$$\rho(a, L_S : N) := \rho(a, \tilde{H} : N),\ \rho(u, L_S : N) := \rho(u, \tilde{H} : N),$$

with $\tilde{H} = H + \rho(1, H : x_\infty)$.

By Lemma 39 the definition of the parameters $(\tilde{H}, \tilde{S})$ in Definition 37 and Definition 40 agree in a sense that in the cases $H \equiv 0$ and $S \equiv Q$ both definitions give the same functionals

$$\mathcal{A}_{(\tilde{H}\oplus 0, \tilde{Q})} = \mathcal{A}_{(H\oplus 0, Q)} - \sigma(1, H \oplus 0, Q : x_\infty)$$

and

$$\mathcal{A}_{(\tilde{0}, \tilde{S})} = \mathcal{A}_{(0,S)} - \sigma(1, 0, S : x_\infty).$$

Invariants in Definition 40 are well defined invariants of Lagrangian submanifolds of T^*M Hamiltonian isotopic to the zero section.

6.4. Equality between the two invariants.

In this section we will show that the invariants ρ and c give the same invariants of Lagrangian submanifolds of T^*M. The proof below is sketched in [**26**], we present it here for the sake of completeness.

Theorem 41 ([**26**])**.** *Let $L_S = \phi_1^H(o_M) \in T^*M$ be a Lagrangian submanifold generated by generating function S quadratic at infinity. Then for any submanifold $N \subset M$ and any $a \in H_*(N)$*

$$c(a, L_S : N) = \rho(a, L_S : N). \tag{59}$$

The analogous equality holds for any $u \in H^*(N)$.

Proof. Denote by $S_t : E \to \mathbb{R}$ a generating function of $(\phi_t^H)^{-1}(L_S)$, such that $S_0 = S$, $S_1 = S$. Let $H(t)$ denote a Hamiltonian such that $\phi_1^{H(t)} = \phi_t^H$. Note that

$$\begin{aligned} \phi_1^{H(t)}(L_{S_t}) &= \phi_1^{H(t)}(\phi_t^H)^{-1}(L_S) \\ &= \phi_t^H(\phi_t^H)^{-1}(L_S) \\ &= L_S \end{aligned}$$

and therefore, by Theorem 36 the action spectrum $\widetilde{\mathrm{Spec}}(H(t) \oplus 0, S_t : N)$ is fixed. By Theorem 33 the function

$$\tilde{\sigma} : t \mapsto \sigma(a, H(t) \oplus 0, S_t : N) - \sigma(a, H(t) \oplus 0, S_t : x_\infty)$$

is continuous and takes the values in the set $\widetilde{\mathrm{Spec}}(H(t)\oplus 0, S:N)$, which is nowhere dense in $\mathbb{R}$ by Lemma 1. Hence $\tilde{\sigma}\equiv$ constant. In particular,

$$
\begin{aligned}
(60)\qquad & \sigma(a,0,\tilde{S}:N) \\
&= \sigma(a,H(0)\oplus 0,S_0:N)-\sigma(a,H(0)\oplus 0,S_0:x_\infty) \\
&= \sigma(a,H(1)\oplus 0,S_1:N)-\sigma(a,H(1)\oplus 0,S_1:x_\infty) \\
&= \sigma(a,\tilde{H}\oplus 0,Q;N).
\end{aligned}
$$

According to Lemma 39 and Definition 40

$$
\sigma(a,0,\tilde{S}:N)=c(a,L_S:N) \tag{61}
$$

and

$$
\sigma(a,\tilde{H}\oplus 0,Q:N)=\rho(a,L_S:N). \tag{62}
$$

Now, (59) follows from (60), (61) and (62). □

7. A note on Hofer's geometry.

In [**16**] Hofer introduced a biinvariant metric on a group $\mathcal{D}^c_\omega(P)$ of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold P. For $H\in C^\infty_c(P\times[0,1])$ define *the oscillation* of H_t by

$$
\operatorname{osc}(H_t):=\sup_{x\in P}H_t(x)-\inf_{x\in P}H_t(x).
$$

That leads to the definition of the length of the curve ϕ^H_t in $\mathcal{D}^c_\omega(P)$ as

$$
l(\{\phi^H_t\}_{0\le t\le 1}):=\int_0^1\operatorname{osc}(H_t)dt.
$$

Definition 42 (Hofer's energy)**.** The energy of $\psi\in\mathcal{D}^c_\omega(P)$ is defined by

$$
E(\psi):=\inf\{l(\phi^H_t)\mid\phi^H_1=\psi\}.
$$

The non-degeneracy of the energy functional, i.e.,

$$
E(\psi)=0 \text{ iff } \psi=\mathrm{id}
$$

has been proved by Hofer [**16**] (see also [**17**]) in the case $P=\mathbb{C}^n$ and by Lalonde and McDuff [**20**] in general.

In the case $P=\mathbb{C}^n$ Bialy and Polterovich [**2**] proved that

$$
c(\mu,\Gamma_\psi)-c(1,\Gamma_\psi)\le E(\psi) \tag{63}
$$

where $c(\mu,\Gamma_\psi)-c(1,\Gamma_\psi)$ is Viterbo's norm (see [**38**]). Moreover, they proved that Viterbo's and Hofer's metrics coincide locally in the sense of C^1-Whitney topology.

More generally, for a symplectic manifold P, let $\mathcal{L}_M(P)$ be the space of Lagrangian submanifolds $L \subset P$ Hamiltonian isotopic to the Lagrangian submanifold M. In other words,

$$\mathcal{L}_M(P) := \{\phi_1^H(M) \mid \phi_1^H \in \mathcal{D}_\omega^c(P)\}.$$

The group $\mathcal{D}_\omega^c(P)$ acts transitively on $\mathcal{L}_M(P)$ by $(\phi, L) \mapsto \phi(L)$. The manifold $\mathcal{L}_M(P)$ has a natural $\mathcal{D}_\omega^c(P)$-invariant metric defined in the following:

Definition 43. For $L_1, L_2 \in \mathcal{L}_M(P)$ we define

$$d(L_1, L_2) := \inf\{E(\phi) \mid \phi \in \mathcal{D}_\omega^c(P),\ \phi(L_1) = L_2\}. \tag{64}$$

The non-degeneracy of d has been proved by Oh [**30**] for $P = T^*M$ and by Chekanov [**4**, **5**] in general case. Moreover, for $P = T^*M$

$$\rho(\mu, L) - \rho(1, L) \leq d(o_M, L) \tag{65}$$

(see [**30**] or apply Lemma 39 to the inequalities at the end of the proof of Theorem 33 with $S^\alpha = S^\beta = Q$, $H^\alpha = 0$, setting first $a = 1$ and then $a = \mu$ and subtracting; then take the infimum over all H^β's such that $\phi_1^{H^\beta}(o_M) = L$).

Theorem 41 together with (65) implies

$$c(\mu, L) - c(1, L) \leq d(o_M, L)$$

which is the generalization of (63) to Hofer's and Viterbo's geometries of Lagrangian submanifolds in a cotangent bundle. As $c(\mu, L) = c(1, L)$ if and only if $L = o_M$, it gives another proof of the nondegeneracy of Hofer's metric.

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Matematicki fakultet
Studentski trg 16
11000 Belgrade
Serbia
Yugoslavia
E-mail address: milinko@matf.bg.ac.yu