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REAL COBOUNDARIES FOR MINIMAL CANTOR SYSTEMS

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In this paper we investigate the role of real-valued coboundaries for classifying of minimal homeomorphisms of the Cantor set. This work follows the work of Giordano, Putnam, and Skau who showed that one can use integer-valued coboundaries to characterize minimal homeomorphisms up to strong orbit equivalence. First, we prove a rigidity result. We show that there is an orbit equivalence between minimal Cantor systems which preserves real-valued coboundaries if and only if the systems are flip conjugate. Second, we investigate a real analogue of the dynamical unital ordered cohomology group studied by Giordano, Putnam and Skau. We show that, in general, isomorphism of our unital ordered vector space determines a weaker relation than strong orbit equivalence and we characterize this relation in a certain finite dimensional case. Finally, we consider isomorphisms of this vector space which preserve the cohomology subgroup. We show that such an isomorphism gives rise to a strictly stronger relation than strong orbit equivalence. In particular, it determines topological discrete spectrum, but does not determine systems up to flip conjugacy.

1. Introduction.

In [**GPS95**], Giordano, Putnam and Skau used C^*-algebraic invariants to characterize minimal homeomorphisms of the Cantor set up to various notions of orbit equivalence. For a minimal homeomorphism $T : X \to X$ of the Cantor set X, their key invariant reduces to the group of continuous integer-valued functions $f : X \to \mathbb{Z}$ modulo the *coboundaries* (functions of the form $f - f \circ T$), along with a positive cone and order unit. In this paper, we examine real-valued coboundaries and look at analogues of their results from three perspectives.

Let S and T be minimal homeomorphisms of the Cantor set. In the main result of Section 2 (Theorem 2.10) we prove that if S and T are orbit equivalent by a homeomorphism which maps the set of real S-coboundaries bijectively onto the set of real T-coboundaries then S is conjugate to T or

T^{-1} (S and T are *flip conjugate*). In fact, we show that any homeomorphism from the Cantor set to itself which identifies real coboundaries of S and T must be an orbit equivalence with a bounded jump function (Theorem 2.11). In contrast, Giordano, Putnam and Skau's work shows that an orbit equivalence induces a bijection between the sets of integer-valued coboundaries if and only if S and T are strongly orbit equivalent. Results in [**BH94**, **Orm97**, **Sug**, **Sug98**] underscore the vast difference between strong orbit equivalence and flip conjugacy for this class of systems. Moreover, an example of Boyle shows that a homeomorphism identifying integer coboundaries need not be a strong orbit equivalence. In appendix A, we present this unpublished example of Boyle in which S and T have the same integer coboundaries, and have the property that $T(x)$ and $T(S^n x)$ are not in the same S-orbit for all x and all $n \neq 0$.

In Section 3, we define and investigate the natural analogue of Giordano, Putnam and Skau's unital ordered group: The vector space of continuous real-valued functions modulo the real coboundaries along with a positive cone and order unit. We show (Theorem 3.10) if the cardinality of the set of ergodic invariant Borel probabilities is finite then this cardinal completely determines our unital ordered vector space $\mathcal{G}_{\mathbb{R}}(T)$. Using a result of Dougherty, Jackson, and Kechris, we see that when the set of ergodic T-invariant Borel probabilities is finite, our unital ordered vector space characterizes Borel orbit equivalence.

In Section 4, we study the dynamical properties which are determined if we consider only isomorphisms of the real unital ordered vector space $\mathcal{G}_{\mathbb{R}}(T)$ which preserve the subgroups of integer-valued functions $\mathcal{G}_{\mathbb{Z}}(T)$. We present results which show that there is some more dynamical information in the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ than in $\mathcal{G}_{\mathbb{Z}}(T)$ alone but not enough to determine T up to flip conjugacy. For example, we show that the isomorphism of the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ determines the topological discrete spectrum of T (Theorem 4.4). The unital ordered group $\mathcal{G}_{\mathbb{Z}}(T)$ already determines the rational discrete spectrum, but does not, in general, determine the irrational spectrum (see [**Orm97**]). We show (Theorem 4.6) that for a minimal Cantor system (X, T) with $G_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ carries no more dynamical information than the unital ordered group $\mathcal{G}_{\mathbb{Z}}(T)$ alone. This shows that one cannot determine flip conjugacy using $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. Taking the previous two results together, we obtain a new result (Corollary 4.7) about minimal Cantor systems and the unital ordered group $\mathcal{G}_{\mathbb{Z}}(T)$. Namely, if $G_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ then T cannot have irrational spectrum.

I thank Mike Boyle for his helpful comments and for allowing me to include his example Appendix A. I thank Bernard Host for allowing me to include his proof of Theorem 2.6.

2. The Same Set of Real Coboundaries.

Throughout this paper we will consider topological dynamical systems (X, T) where $T : X \to X$ is a homeomorphism of X a compact metric space. In particular, we will consider *minimal Cantor systems*. A homeomorphism $T : X \to X$ is called *minimal* if for all $x \in X$ the T-orbit of x, $\{T^n x \ : \ n \in \mathbb{Z}\}$, is dense. We will call the pair (X, T) a *minimal Cantor system* if X is a Cantor set and $T : X \to X$ is minimal. The main properties of minimality that we will make use of are the following: There are no periodic points in a minimal system and for any open set $U \subseteq X$, there is an integer r such that for all $x \in X$, one of $\{x, T(x), \dots, T^r(x)\}$ is in U. Minimal Cantor systems include the odometer systems below.

Example (Odometer systems)**.** Let $\{d_i\}$ be an infinite sequence of positive integers. Let X be the space of infinite sequences $x = x_1x_2x_3\dots$ such that $0 \leq x_i < d_i$ for all i. We put the discrete topology on the sets $\{0, 1, \dots, d_i - 1\}$ and the infinite product of this discrete topology on X. In this way, X becomes a Cantor set. The topology on X is equivalent to the one generated by the metric d where $d(x, y) = 2^{-n}$ if $x_i = y_i$ for all $0 \leq i \leq n$ and $x_{n+1} \neq y_{n+1}$.

Define $T : X \to X$ by adding one with right carry. In other words, for $x \in X$, let n be the smallest positive integer such that $x_n < (d_n - 1)$. If such an n exists, define $T(x)$ to be the sequence $[T(x)]_i = 0$ for $i < n$, $[T(x)]_n = x_n + 1$ and $[T(x)]_i = x_i$ for $i > n$. If $x_n = (d_n - 1)$ for all n, define $T(x)$ to be the sequence $[T(x)]_n = 0$ for all n. The dynamical system (X, T) is minimal since the T-orbit of every point sees all the words of length n in the first n coordinates. The odometer system where $d_i = 2$ for all i is called the *dyadic adding machine*.

Let (X, S) and (Y, T) be minimal Cantor systems. The following are some of the different equivalences we will consider. Of course, the notions make sense for more general topological dynamical systems.

Definition 2.1 (conjugacy)**.** We say (X, S) and (Y,T) are conjugate if there is a homeomorphism $h : X \to Y$ such that $\forall x \in X, hS(x) = Th(x)$.

Definition 2.2 (flip conjugacy)**.** We say (X, S) and (Y, T) are flip conjugate if S is conjugate to T or S is conjugate to T^{-1}.

Definition 2.3 (orbit equivalence)**.** We say (X, S) and (Y, T) are orbit equivalent if there is a homeomorphism $h : X \to Y$ and functions $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ such that

$$\forall x \in X, \ hS(x) = T^{m(x)}h(x) \text{ and } hS^{n(x)}(x) = Th(x).$$

In other words, (X, S) and (Y, T) are conjugate to systems (Z, S') and (Z, T') where

$$\forall x \in Z, \ \{(S')^n(x) \ : \ n \in \mathbb{Z}\} = \{(T')^n(x) \ : \ n \in \mathbb{Z}\}.$$

The theory of orbit equivalence has a long history in the study of measure-theoretic dynamical systems [**KR95**, **KW91**, **Kri69**, **Kri76**, **Rud85**]. It was this work which motivated the study of orbit equivalence in topological systems.

As it turns out, for a given topological orbit equivalence, the continuity properties of the "jump functions" $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ can give information about the extent to which one system is determined by the other. In particular, for minimal Cantor systems Boyle [**Boy83**] proved that the jump functions are bounded if and only if they are continuous, and gave the following characterization (generalized in [**BT98**]) of orbit equivalence with a bounded jump functions.

Theorem 2.4 (Boyle)**.** *Suppose (X, S) and (X, T) are minimal Cantor systems with the same orbits. If there is a bounded function $m : X \to \mathbb{Z}$ such that $S(x) = T^{m(x)}(x)$ for all x then S and T are flip conjugate.*

In [**GPS95**], Giordano, Putnam and Skau used C^*-algebraic invariants to characterize orbit equivalence for minimal Cantor systems, and to give information about the continuity/boundedness properties of the associated jump functions one can achieve. One important notion from their work is the notion of strong orbit equivalence.

Definition 2.5 (strong orbit equivalence)**.** Two minimal Cantor systems (X, S) and (Y, T) are strongly orbit equivalent if they are orbit equivalent by a map $h : X \to Y$ with jump functions $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ such that m and n have at most one point of discontinuity each.

We will say more about strong orbit equivalence in Section 3. For now, we simply point out that strong orbit equivalence is a much weaker relation than flip conjugacy. For example, strongly orbit equivalent systems can have arbitrarily large topological entropy differences and when attached with an ergodic invariant measure, can give rise to vastly different measurable structures (see [**BH94**, **Orm97**, **Sug**, **Sug98**]).

For a minimal Cantor systems (X, T), Giordano, Putnam and Skau's characterization up to orbit equivalence relies upon looking at integer-valued continuous functions of the form $f - fT$ (from here on we use fT to denote $f \circ T$). We will call a continuous function $f : X \to \mathbb{R}$ a *real T-coboundary* if there exists a continuous function $g : X \to \mathbb{R}$ such that $f(x) = g(x) - g(Tx)$ for all $x \in X$. Similarly, we will call a function $f : X \to \mathbb{Z}$ an *integer T-coboundary* if there is a continuous $g : X \to \mathbb{Z}$ such that $f(x) = g(x) - g(Tx)$ for all $x \in X$. The following characterization of coboundaries is well known. With kind permission, we present Bernard Host's proof of this result [**Hos**].

Theorem 2.6. *Let (X, T) be a Cantor minimal system. A continuous function $f : X \to \mathbb{R}$ is a real T-coboundary if and only if sums of the form $\sum_{i=0}^{n} f(T^i x)$ are uniformly bounded over $n \geq 1$ and $x \in X$.*

Proof. If $f = g - gT$ for some g then $\sum_{i=0}^{n} f(T^i x) = g(x) - g(T^n x)$ thus sums of this form are uniformly bounded.

For the other direction, let (X, T) be a minimal system, and f a continuous real-valued function on X. Define

$$f^{(n)}(x) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum_{i=-n}^{-1} f(T^i x) & \text{if } n < 0. \end{cases}$$

Assume that C is a constant such that

$$\forall x \in X, \forall n \in \mathbb{N},\ |f^{(n)}(x)| \leq C.$$

As for all $x \in X$ we have $f^{(0)}(x) = 0$ and $f^{(-n)}(x) = -f^{(n)}T^{-n}(x)$ for $n > 0$ we get:

$$\forall x \in X, \forall n \in \mathbb{Z},\ f^{(n)}(x) \leq C.$$

We write:

$$F(x) = \sup_{n \in \mathbb{Z}} f^{(n)}(x);\ \mathrm{Osc}(x) = \limsup_{y \to x} F(y) - \liminf_{y \to x} F(y).$$

For every $x \in X$ we have $f(x) = F(x) - F(Tx)$ and for all n, we have $f^{(n)}(x) = F(x) - F(Tx)$. We have only to prove that the function $F(x)$ is continuous, i.e., that the function $\mathrm{Osc}(x)$ is identically 0.

We choose some $\epsilon > 0$ and define:

$$K = \{x \in X\ :\ F(x) \leq \epsilon\}.$$

By construction, for $x \in X$ there is an $n \in \mathbb{Z}$ with

$$T^n x \in K \iff f^{(n)}(x) \geq F(x) - \epsilon.$$

Thus, by definition of $F(x)$, for every $x \in X$ there exists $n \in \mathbb{Z}$ with $T^n x \in K$, and

$$\bigcup_{n \in \mathbb{Z}} T^{-n} K = X.$$

But K is closed. Thus, by Baire's Theorem, the interior U of K is not empty and, by minimality,

$$\bigcup_{n \in \mathbb{Z}} T^{-n} U = X.$$

For $x \in T^{-n}U$ we have $f^{(n)}(x) \leq F(x) \leq f^{(n)}(x) + \epsilon$ thus, by continuity of $f^{(n)}$, $\mathrm{Osc}(x) \leq \epsilon$.

Therefore, $\mathrm{Osc}(x) \leq \epsilon$ for all $x \in X$. □

We will look more closely at Giordano, Putnam and Skau's results in Sections 3 and 4. For the remainder of this section, however, we concentrate on one aspect of their results.

Theorem 2.7 (Giordano, Putnam, Skau)**.** *Suppose (X, S) and (Y, T) are minimal Cantor systems. There is an orbit equivalence $h : X \to Y$ which induces a bijection from the set of integer S-coboundaries to the set of integer T-coboundaries if and only if S and T are strongly orbit equivalent.*

In Theorem 2.10, we prove an analogous result, there is an orbit equivalence which respects real coboundaries if and only if S and T are flip conjugate. To prove the difficult direction of Theorem 2.10, we first note the following.

Lemma 2.8. *Let (X, S) and (X, T) be minimal Cantor systems with the same orbits. For all n, let $E_n = \{x \ : \ S(x) = T^n(x)\}$. Then one of the following holds.*

1. *$X = \bigcup_{|n|\leq N} E_n$ for some N,*
2. *there is an infinite sequence of sets E_{n_k} with $|n_k| < |n_{k+1}|$ such that E_{n_k} contains a clopen set for all k.*

Proof. By the continuity of S and T, the set E_n is closed for all n. Let F_N denote the closed set $F_N = \bigcup_{|n|\leq N} E_n$.

Fix N and assume that $X - F_N$ is nonempty. Then the sets E_n where $|n| > N$ form a countable closed cover of this open set. By the the Baire Category Theorem, one element of this cover must contain an open set, and therefore a clopen set. If condition 1 does not hold then we obtain a sequence of sets as in condition 2. □

From Boyle's result (Theorem 2.4) case 1 implies that S and T are flip conjugate. So to prove Theorem 2.10, it will suffice to show that in case 2, S and T do not have the same real coboundaries. For the remainder of the section, let (X, S) and (X, T) be minimal Cantor systems with the same orbits, and let $E_n = \{x \ : \ S(x) = T^n(x)\}$ for all $n \neq 0$.

Suppose there is an infinite sequence of sets E_{n_k} with $|n_k| < |n_{k+1}|$ such that E_{n_k} contains a clopen set for all k. Then by passing to a monotone increasing or decreasing subsequence of n_k's and possibly exchanging T for T^{-1} we have the hypothesis of the following.

Lemma 2.9. *Suppose there exists an infinite increasing sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ and nonempty clopen sets C_k such that $S(x) = T^{n_k}(x)$ for all $x \in C_k$. Then there exists a continuous real-valued S-coboundary which is not a continuous real-valued T-coboundary.*

Proof. After passing to a subsequence of the C_k, we will define f as

$$f(x) = \sum_{k=1}^{\infty} (1/k) 1_{U_k}(x)$$

where for all k, U_k is a clopen subset of C_k, and 1_{U_k} is the indicator function of U_k. The function $f(x)$ will be continuous as long as there is a point

$x_0 \notin \bigcup_{k\in\mathbb{N}} U_k$ such that if x_k is a sequence of points with $x_k \in U_k$ then $\lim_{k\to\infty} x_k = x_0$.

To get that $f - fS^{-1}$ is not a T-coboundary, we will choose the U_k such that for $x \in U_n$ the T-orbit of x enters each of the sets $U_n, U_{n-1}, \ldots, U_1$ at least once before it enters any set of the form SU_k for $k \in \mathbb{N}$. In this case, for all $x \in U_n$ we will be able to find an integer m such that

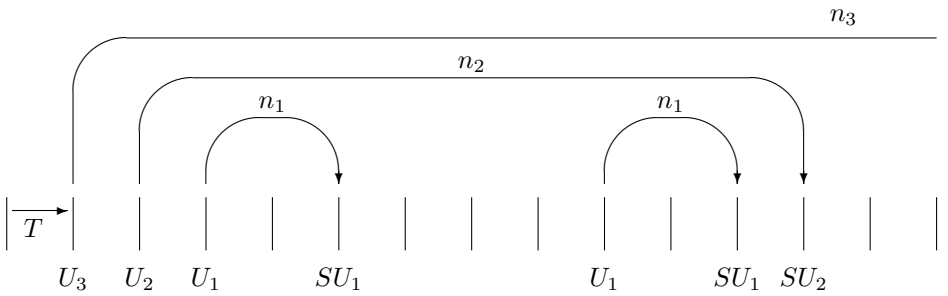
$$\begin{aligned}\sum_{i=0}^{m} \left(f(T^i x) - fS^{-1}(T^i x)\right) &= \sum_{i=0}^{m}\sum_{k\in\mathbb{N}} (1/k)\left(1_{U_k}(T^i x) - 1_{SU_k}(T^i x)\right)\\ &= \sum_{i=0}^{m}\sum_{k\in\mathbb{N}} (1/k) 1_{U_k}(T^i x)\\ &\geq \sum_{k=1}^{n} (1/k)\\ &> \log(n).\end{aligned}$$

If the function $f - fS^{-1}$ were a T-coboundary then by Theorem 2.6 there would be a uniform bound on the functions $\sum_{i=0}^{m}(f - fS^{-1})T^i$.

To construct the function $f(x)$ it suffices to construct a sequence of *leap-frogging* sets $\{U_k\}$. Let $B_n(x)$ denote the T-orbit block $B_n(x) = \{x, T(x), \ldots, T^n(x)\}$. We will call a sequence of pairwise disjoint sets $\{U_k\}_{k\in\mathbb{N}}$ leap-frogging for the pair (S, T) if

1) there exists an increasing sequence of integers $\{n_k\}$ such that $S(x) = T^{n_k}(x)$ for all $x \in U_k$,
2) for all $x \in U_j$ and $y \in U_k$ with $1 \leq j \leq k$, the set $B_{n_j}(x) \cap B_{n_k}(y)$ is either empty or equal to $B_{n_j}(x)$,
3) for all $x \in U_k$ the set $B_{n_k}(x) \cap U_{k-1}$ is nonempty.

We call the sets leap-frogging because we imagine the T-orbit of a point laid out along a number line. If a point is in U_k, the S-image of that point leaps forward in the T-orbit.



Condition 3 ensures that for $j < k$, the S-image of a point $x \in U_k$ leap-frogs at least one orbit block $B_{n_j}(y)$ where $y \in U_j$. For $x \in U_n$ let m be the smallest integer such that $x \in U_1$. This condition will give us $\sum_{i=0}^{m} \sum_{k\in\mathbb{N}} (1/k) 1_{U_k}(T^i x)$ is at least $\sum_{k=1}^{n} (1/k)$.

We can think of condition 2 as ensuring that the S-jumps are nested. In other words, for $j \leq k$ if $x \in U_k$ and $y \in U_j$ then $S(x) \notin B_{n_j}(y)$ unless $x = y$. Conditions 2 and 3 imply: If $x \in U_n$ and m is the smallest integer such that $T^m x \in U_1$ then $\sum_{i=0}^{m} \sum_{k\in\mathbb{N}} (1/k) 1_{SU_k}(T^i x) = 0$.

We will construct the leap-frogging sets $U_k \subseteq C_k$ recursively. Before we do so, we will pick a special point x_0 with the property that for large k the set U_k lies within a small clopen subset of x_0. Since X is compact, after passing to a subsequence of the C_k we may assume that there is a sequence of points $\{x_k\}$ with $x_k \in C_k$ such that the limit $\lim_{k\to\infty} x_k$ exists. Let x_0 be the limit of this subsequence. We can replace the C_k with clopen neighborhoods of the x_k's of decreasing diameter. In this way, we may assume that our sets C_k have the property that if $y_k \in C_k$, then $\lim_{k\to\infty} y_k = x_0$. Moreover, we may assume that $x_0 \notin \bigcup_{k\in\mathbb{N}} C_k$.

To construct U_1, pick $y_1 \in C_1$ such that neither x_0 nor Sx_0 are in the T-orbit block $B_{n_1}(y_1)$. Let U_1 be a clopen neighborhood of y_1 such that $x_0, Sx_0 \notin \bigcup_{i=0}^{n_1} T^i U_1$ and $U_1, TU_1, \ldots, T^{n_1}U_1$ are pairwise disjoint. Since the $T^j U_1$ are pairwise disjoint, if x and y are distinct points in U_1 then the intersection of the T-orbit blocks $B_{n_1}(x) \cap B_{n_1}(y)$ is empty.

Now assume that we have sets $U_1, U_2, \ldots, U_k$ satisfying the leap-frogging conditions such that neither x_0 nor Sx_0 are in $\bigcup_{i=0}^{n_k} T^i U_k$. We can find a clopen neighborhood V of x_0 such that $V \cap \bigcup_{i=0}^{n_k} T^i U_k$, $SV \cap \bigcup_{i=0}^{n_k} T^i U_k$ and $V \cap SV$ are all empty.

By the minimality of T there is an integer r_k such that for any $x \in X$ the set $B_{r_k}(x) \cap U_k$ is nonempty. By passing to a subsequence of the C_k, we may assume $n_{k+1} > r_k$ and $C_{k+1} \subseteq V$. Choose $y_{k+1} \in C_{k+1}$. Pick a clopen neighborhood U_{k+1} of y_{k+1} of diameter less than $1/k$ such that $U_{k+1} \subseteq C_{k+1}$ and $U_{k+1}, TU_{k+1}, \ldots, T^{n_{k+1}}U_{k+1}$ are pairwise disjoint.

Since $U_{k+1} \subseteq C_{k+1}$, we have that for all $x \in U_k$, $S(x) = T^{n_k}(x)$ (condition 1). Since $n_{k+1} > r_k$, for all $x \in U_{k+1}$ the set $B_{n_{k+1}}(x) \cap U_k$ is nonempty for all $x \in U_{k+1}$ (condition 3). Since $U_{k+1} \subseteq V$ and $SU_{k+1} \subseteq SV$ we have $U_{k+1} \cap \bigcup_{i=0}^{n_k} T^i U_k$, $SU_{k+1} \cap \bigcup_{i=0}^{n_k} T^i U_k$ are both empty. This gives the nested property of the blocks (condition 2). Since neither x_0 nor Sx_0 are in U_{k+1} we can continue with the recursion. □

The previous two lemmas give us the following theorem.

Theorem 2.10. *Let (X, S) and (Y, T) be minimal Cantor systems. There is an orbit equivalence $h : X \to Y$ which induces a bijection from the set of real S-coboundaries to the set of real T-coboundaries if and only if S and T are flip conjugate.*

Proof. If S and T are not flip conjugate then by Theorem 2.4 and Lemmas 2.8 and 2.9 we can construct an S coboundary which is not a T-coboundary. For the other direction we simply need to see that a homeomorphism $R : X \to X$ and its inverse $R^{-1} : X \to X$ always have the same set of coboundaries. This follows as

$$f - fR^{-1} = (fR^{-1})R - (fR^{-1}).$$

□

Remark. The above is reminiscent of rigidity results of Boyle and Tomiyama [**BT98**, Theorem 3.6] and Giordano, Putnam and Skau [**GPS**]. In the case where S and T are minimal Cantor systems Boyle and Tomiyama show that if the C^*-algebras associated to S and T are related by an isomorphism which identifies the subalgebra of continuous functions, then the systems are flip conjugate. Giordano, Putnam and Skau showed that an algebraic isomorphism of the topological full group must be induced by a flip conjugacy.

Theorem 2.10 can be strengthened. We show below (Theorem 2.11) that we need not require that the homeomorphism which identifies real coboundaries be an orbit equivalence, it is automatic. The analogous statement for integer coboundaries is not true. An example of Boyle (see Appendix A) shows that it is possible for two minimal homeomorphisms S and T of the Cantor set to have the same set of integer coboundaries and have the property that if x and y are in the same S-orbit then Tx and Ty are not in the same S-orbit.

Let $C(X, \mathbb{R})$ denote the set of real-valued continuous functions on a Cantor set X.

Theorem 2.11. *Let (X, S) and (X, T) be minimal Cantor systems. Then (X, S) and (X, T) have the property that for all $f \in C(X, \mathbb{R})$ there exist $g_1, g_2 \in C(X, \mathbb{R})$ such that*

$$f - fT = g_1 - g_1 S$$
$$f - fS = g_2 - g_2 T$$

if and only if S and T have the same orbits and there is a bounded (continuous) function $m : X \to \mathbb{Z}$ such that $S(x) = T^{m(x)}(x)$ for all $x \in X$.

Proof. Let $E_n = \{x \ : \ S(x) = T^n(x)\}$ and $F = X - \cup_{n\in\mathbb{Z}} E_n$.

Suppose such a function $m : X \to \mathbb{Z}$ exits. Then F is empty and there exists an integer M such that E_n is empty for $|n| > M$. For $f \in C(X, \mathbb{R})$ we may write $f - fT = \sum_{n=-M}^{M} 1_{TE_n} f - (1_{TE_n} f)T$. If $x \in E_n$ then $(1_{TE_n} f)Tx = (1_{TE_n} f)S^n x$ and the above function is therefore an S-coboundary.

Suppose that no such function m exists. In other words, assume $X - \cup_{|n|\leq M} E_n$ is nonempty for all M. If infinitely many of the sets E_n have

nonempty interior then by Lemma 2.9 there is a real-valued S-coboundary which is not a real-valued T-coboundary.

If $X - \cup_{|n|\leq M} E_n$ is nonempty for all M and only finitely many of the sets E_n have nonempty interior then by the Baire Category Theorem, $\overline{F}$ contains an open set. It remains to show that S and T cannot have the same set of real coboundaries when $\overline{F}$ contains an open set.

We will construct an S-coboundary which is not a T-coboundary by selecting a nested sequence of clopen sets $U \supseteq U_1 \supseteq U_2 \supseteq \cdots$, a sequence of points $x_k \in U_k$, and an increasing sequence of integers n_k with the following properties.

1) $\sum_{i=0}^{n_k} 1_{U_k}(T^i(x_k)) \geq 2^k$,
2) $\sum_{i=0}^{n_k} 1_{SU_k}(T^i(x_k)) = 0$,
3) $\sum_{i=0}^{n_k} \left(1_{U_j}(T^i(x_k)) - 1_{SU_j}(T^i(x_k))\right) = 0$ for all $1 \leq j < k$.

Assume such a collection of sets exists, and let $f = \sum_{k=1}^{\infty}(3/4)^k 1_{U_k}$. If in addition to the above, the diameters of the U_k's are going to zero then the function $f(x)$ will be continuous. Since the sets SU_k are nested, condition 2 implies that

$$\sum_{j=k}^{\infty}\sum_{i=0}^{n_k} 1_{SU_j}(T^i(x_k)) = 0.$$

Putting this fact together with conditions 1 and 3, we get

$$\begin{aligned}
&\sum_{i=0}^{n_k} f(T^i x_k) - f(S^{-1}(T^i x_k)) \\
&= \sum_{i=0}^{n_k}\sum_{j=1}^{\infty}(3/4)^j \left[1_{U_j}(T^i x_k) - 1_{SU_j}(T^i x_k)\right] \\
&= \sum_{i=0}^{n_k}\sum_{j=1}^{k-1}(3/4)^j \left[1_{U_j}(T^i x_k) - 1_{SU_j}(T^i x_k)\right] \\
&\qquad + \sum_{i=0}^{n_k}\sum_{j=k}^{\infty}(3/4)^j 1_{U_j}(T^i x_k) - \sum_{i=0}^{n_k}\sum_{j=k}^{\infty}(3/4)^j 1_{SU_j}(T^i x_k) \\
&\geq 2^k(3/4)^k \\
&= (3/2)^k.
\end{aligned}$$

Therefore, by Theorem 2.6, $f - fS^{-1}$ cannot be a T-coboundary. It remains then to construct the sets.

We first note that for any point $x \in F$ since $S(x)$ is not in the T-orbit of x, for any positive integers m, n there is a clopen neighborhood U of x such that $T^k U \cap SU = \emptyset$ for all $-m \leq k \leq n$. This implies that for any clopen

set $V \subseteq \overline{F}$ and positive integers m, n, there exists a clopen $U \subseteq V$ such that $T^k U \cap SU = \emptyset$ for $-m \leq k \leq n$.

To construct U_1, we take any clopen set $V \subseteq \overline{F}$. Let x be in V and let n_1 be the smallest positive integer n such that $T^n(x) \in V$. We can choose a clopen neighborhood V_0 of x such that $T^k V_0 \cap SV_0 = \emptyset$ for $0 \leq k \leq n_1$. Let V_1 be a clopen subset of $T^{n_1} V_0 \subseteq V$ such that $T^k V_1 \cap SV_1 = \emptyset$ for $-n_1 \leq k \leq 0$. Now let $U_1 = T^{-n_1} V_1 \cup V_1$ and let x_1 be any point in $T^{-n_1} V_1$. We get $\sum_{i=0}^{n_1} 1_{U_1}(T^i x_1) = 2$, and $\sum_{i=0}^{n_1} 1_{SU_1}(T^i x_1) = 0$.

Now suppose that we have constructed sets $U \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k$, points $x_1, x_2, \ldots, x_k$, and integers $n_1, n_2, \ldots, n_k$ with the desired properties. Consider the function $f_k = \sum_{i=1}^{k} (3/4)^k 1_{U_i}$. If $f_k - f_k S^{-1}$ is not a T-coboundary, then we are done. Assume that $f_k - f_k S^{-1} = g - gT$ for some g. Since $f_k - f_k S^{-1}$ is a locally constant rational valued function, we may assume that g is as well.

Let $y_0 \in U_k \cap F$. By the minimality of T, we can choose an integer N such that $\sum_{i=0}^{N} 1_{U_k}(T^i y_0) > 2^{k+1}$ and $g(y_0) = g(T^{N+1} y_0)$. Choose a clopen neighborhood V around y_0 such that $f_k(T^i y) - f_k(S^{-1} T^i y) = f_k(T^i y_0) - f_k(S^{-1} T^i y_0)$ for all $y \in V$ and all $0 \leq i \leq N$. Then for all $y \in V$,

$$\sum_{i=0}^{N} \big(f_k(T^i y) - f_k(S^{-1} T^i y) \big) = g(y) - g(T^{N+1} y) = 0,$$

which will give condition 3.

Let M denote $\sum_{i=0}^{N} 1_{U_k}(T^i y_0)$ and let $0 = r_0 < r_1 < \cdots < r_M \leq N$ be the integers such that $T^{r_j} V \subseteq U_k$. We know that since $y_0 \in F$ we can choose $V_0 \subseteq V$ such that $T^i V_0 \cap SV_0$ is empty for all $0 \leq i \leq N$. Since $T^{r_1} V_0 \subseteq U_k$, there is a clopen set $V_1 \subseteq T^{r_1} V_0$ such that $T^i V_1 \cap SV_1$ is empty for all $-r_1 \leq i \leq (N - r_1)$. Continuing, for all $0 \leq j \leq M$, we can obtain sets $V_j \subseteq T^{r_j - r_{j-1}} V_{j-1}$ such that $T^i V_j \cap SV_j$ is empty for all $-r_j \leq i \leq (N - r_j)$. Let U_{k+1} be the union over $0 \leq j \leq M$ of the sets $T^{-r_M + r_j} V_M$, and let x_{k+1} be any point in $T^{-r_M} V_M$. Then $\sum_{i=0}^{N} 1_{U_{k+1}}(T^i x_{k+1}) = M \geq 2^{k+1}$ and $\sum_{i=0}^{N} 1_{SU_{k+1}}(T^i x_{k+1}) = 0$, giving conditions 1 and 2. □

3. Real Ordered Group.

The notion of strong orbit equivalence emerged from the study of C^*-algebraic invariants for topological dynamical systems. For minimal homeomorphisms of the Cantor set, Herman, Putnam and Skau showed that these C^*-crossed products are classified by their K-theory [**HPS92**]. The K-theory for these C^*-algebras amounts to the group of continuous integer-valued functions on the Cantor set modulo the coboundaries along with a positive cone and order unit. Giordano, Putnam and Skau showed that

this unital ordered group characterizes strong orbit equivalence for minimal Cantor systems (Theorem 3.3) [**GPS95**, Theorem 2.1].

In this section, we define and investigate the group of continuous real-valued functions modulo the real coboundaries. As in [**GPS95**], our group will be considered along with a natural positive cone and order unit. Unlike the integer case, our space has the structure of a vector space over the reals. With this, the classification problem essentially comes down to counting the dimension of subspaces. When the number of ergodic invariant Borel probabilities is finite, the span of these measures acts as the dual to this space modulo the infinitesimal subgroup (Lemma 3.9). In this finite dimensional case, we are able to show (Theorem 3.10) that the cardinality of the set of ergodic Borel probabilities completely classifies our unital ordered vector space. Interestingly, this leads us back to orbit equivalence. A result of Dougherty, Jackson, and Kechris (Theorem 3.13) [**DJK94**, Theorem 9.1] states that the cardinality of the set of ergodic invariant Borel probabilities characterizes a weaker form of orbit equivalence, Borel orbit equivalence.

3.1. The Unital Ordered Group $\mathcal{G}_{\mathbb{Z}}(T)$. We present the relevant definitions for unital ordered groups. For a more detailed introduction, see [**GPS95**].

Definition 3.1 (unital ordered group)**.** A unital ordered group $\mathcal{G}$ is a triple (G, G_+, u) where:

- G is an abelian group,
- G_+ is subset of G such that
  $$G_+ \cap (-G_+) = \{0\}, \quad G_+ + G_+ \subseteq G_+, \quad \text{and} \quad G_+ - G_+ = G,$$
- u is an element of G_+ such that
  for all $g \in G$ there exists an $n \in \mathbb{Z}_+$ such that $(nu - g) \in G_+$.

Definition 3.2 (isomorphism)**.** Two unital ordered groups (G, G_+, u) and (H, H_+, v) are isomorphic if and only if there is group isomorphism $f : G \to H$ such that $f(G_+) = H_+$ and $f(u) = v$.

Suppose (X, T) is a minimal Cantor system. We will use $\mathcal{G}_{\mathbb{Z}}(T)$ to denote the unital ordered group $(G_{\mathbb{Z}}(T), G_{\mathbb{Z}}(T)_+, 1_T)$ defined as follows. Let $G_{\mathbb{Z}}(T)$ be the group of continuous functions from the Cantor set X into the integers modulo the integer coboundaries

$$G_{\mathbb{Z}}(T) = C(X, \mathbb{Z})/\{f - fT \ : \ f \in C(X, \mathbb{Z})\}.$$

Let $G_{\mathbb{Z}}(T)_+$ be the semigroup of equivalence classes of nonnegative functions

$$G_{\mathbb{Z}}(T)_+ = \{[f] \ : \ f(x) \geq 0 \text{ for all } x \in X\}$$

and let 1_T be the equivalence class of the constant function one

$$1_T = [1].$$

The ordered group above is, in fact, a simple dimension group as defined by Elliot [**Ell76**].

Theorem 3.3 (Giordano, Putnam, Skau)**.** *Let (X, S) and (X, T) be minimal Cantor systems. Then $\mathcal{G}_{\mathbb{Z}}(S)$ is isomorphic to $\mathcal{G}_{\mathbb{Z}}(T)$ if and only if S and T are strongly orbit equivalent.*

The above gives strong orbit equivalence a more natural meaning, the equivalence relation which is induced by isomorphism of unital ordered groups. To get a similar statement for orbit equivalence, we must first introduce infinitesimal subgroups and traces of simple dimension groups.

Definition 3.4 (infinitesimals)**.** Let $\mathcal{G} = (G, G_+, u)$ be a unital ordered group. The set

$$\text{Inf}\,(\mathcal{G}) = \{g \in G \ : \ u - ng \in G_+ \text{ for all } n \in \mathbb{Z}\}$$

is the infinitesimal subgroup of $\mathcal{G}$.

Definition 3.5 (trace)**.** A trace σ on a unital ordered group $\mathcal{G} = (G, G_+, u)$ is a homomorphism $\sigma : G \to \mathbb{R}$ such that $\sigma(G_+) \subseteq \mathbb{R}_+$ and $\sigma(u) = 1$.

The order structure of any simple dimension group is determined by the action of the trace space [**Eff81**]. In other words,

$$G_+ = \{g \in G \ : \ \sigma(g) > 0 \text{ for all traces } \sigma\} \cup \{0\}$$

and

$$\text{Inf}\,(\mathcal{G}) = \{g \in G \ : \ \sigma(g) = 0 \text{ for all traces } \sigma\}.$$

If (X, T) a minimal Cantor system then the trace space of $\mathcal{G}_{\mathbb{Z}}(T)$ with the natural topology is a compact, convex metric space which is affinely homeomorphic to the space of T-invariant Borel probabilities $\mathfrak{M}_T$. Moreover,

$$G_{\mathbb{Z}}(T)_+ = \left\{[f] \ : \ \int f d\mu > 0 \text{ for all } \mu \in \mathfrak{M}_T\right\} \cup \{0\}$$

and

$$\text{Inf}\,(T) = \text{Inf}\,(\mathcal{G}_{\mathbb{Z}}(T)) = \left\{[f] \ : \ \int f d\mu = 0 \text{ for all } \mu \in \mathfrak{M}_T\right\}.$$

Theorem 3.6 (Giordano, Putnam, Skau)**.** *Let (X, S) and (X, T) be minimal Cantor systems. Then the unital ordered groups $\mathcal{G}_{\mathbb{Z}}(S)/\text{Inf}\,(S)$ and $\mathcal{G}_{\mathbb{Z}}(T)/\text{Inf}\,(T)$ are isomorphic if and only if S and T are orbit equivalent.*

3.2. A Real Analogue to $\mathcal{G}_{\mathbb{Z}}(T)$. The results of Section 2 and of Giordano, Putnam and Skau motivate our investigation of the triple $\mathcal{G}_{\mathbb{R}}(T) = (G_{\mathbb{R}}(T), G_{\mathbb{R}}(T)_+, 1_T)$ where

$$G_{\mathbb{R}}(T) = C(X, \mathbb{R})/\{f - fT \ : \ f \in C(X, \mathbb{R})\}$$

$$G_{\mathbb{R}}(T)_+ = \{[f] \ : \ f(x) \geq 0 \text{ for all } x \in X\}$$

$$1_T = [1].$$

Remark. Typically, there is an additional assumption that unital ordered groups be countable. However, none of the notions of unital ordered group, isomorphism, infinitesimals and traces depend upon the group being countable.

Suppose that (X, S) and (X, T) are minimal Cantor systems. We first notice that the groups $G_{\mathbb{R}}(S), G_{\mathbb{R}}(T)$ can adopt the structure of a real vector space (with the definition $r[f] := [rf]$). It is this identification which makes the inclusion map of $\mathcal{G}_{\mathbb{Z}}(T) \to \mathcal{G}_{\mathbb{R}}(T)$ worth studying. For example, there can exist locally constant coboundaries $f - fT$ where f cannot be chosen to be locally constant. Thus we are not simply considering the old group $\mathcal{G}_{\mathbb{Z}}(T)$ with real coefficients, there are also new identifications.

Henceforth, we will refer to the triple $(G_{\mathbb{R}}(T), G_{\mathbb{R}}(T)_+, 1_T)$ as a real ordered vector space. The isomorphisms we will consider are $\mathbb{R}$-vector space isomorphisms which preserve classes of nonnegative and constant functions.

For the remainder of this section, we will concentrate on the case where the space of invariant measures $\mathfrak{M}_T$ is finite dimensional. In this case we will characterize Inf $(G_{\mathbb{R}}(T))$ (Theorem 3.10).

Proposition 3.7. *Let (X, T) be a minimal Cantor system. If $\mathfrak{M}_T$ is finite dimensional then*

$$G_{\mathbb{R}}(T)_+ = \left\{ [f] \ : \ \int f d\mu > 0 \text{ for all } \mu \in \mathfrak{M}_T \right\} \cup \{0\}$$

and

$$\text{Inf}_{\mathbb{R}}(T) = \text{Inf}\,(G_{\mathbb{R}}(T)) = \left\{ [f] \ : \ \int f d\mu = 0 \text{ for all } \mu \in \mathfrak{M}_T \right\}.$$

Proof. Suppose $f : X \to \mathbb{R}$ is a continuous function.

If there exists $h \in C(X, \mathbb{R})$ such that $f(x) + h(x) - hT(x) \geq 0$ for all x, then either $f + h - hT \equiv 0$ or $\int f d\mu > 0$ for all $\mu \in \mathfrak{M}_T$.

Now suppose $\int f d\mu > 0$ for all $\mu \in \mathfrak{M}_T$. Then since $\mathfrak{M}_T$ is finite dimensional, there is a $\delta > 0$ such that $\int f d\mu \geq \delta$ for all μ. Select a continuous function $g : X \to \mathbb{Q}$ that takes on finitely many values and $f(x) - \delta/2 < g(x) < f(x)$ for all x. Then there is an integer m such that $mg \in C(X, \mathbb{Z})$ and $\int mg d\mu > 0$ for all $\mu \in \mathfrak{M}_T$. By the properties of $\mathcal{G}_{\mathbb{Z}}(T)$, there is an integer coboundary $h - hT$ such that $mg(x) + h(x) - hT(x) \geq 0$. Therefore, $f + \frac{1}{m}(h - hT)$ is a nonnegative function and $[f] \in G_{\mathbb{R}}(T)_+$.

The second claim now follows easily as $[1] - n[f] \in G_{\mathbb{R}}(T)_+$ iff $n \int f d\mu \leq 1$ for all $\mu \in \mathfrak{M}_T$ iff $\int f d\mu = 0$ for all $\mu \in \mathfrak{M}_T$.

□

Proposition 3.8. *Let (X, T) be a minimal Cantor system. If $\mathfrak{M}_T$ is finite dimensional then the dimension of* Inf $_{\mathbb{R}}(T)$ *is $|\mathbb{R}|$.*

Proof. A continuous function from X to $\mathbb{R}$ is determined by its values on a countable dense subset. Therefore

$$\dim(\mathrm{Inf}_{\mathbb{R}}(T)) \leq \dim(C(X,\mathbb{R})) \leq |\mathbb{R}|^{|\mathbb{Q}|} = |\mathbb{R}|.$$

To finish the proof, it suffices to construct a family of linearly independent infinitesimals $\{f_\alpha \ : \ \alpha \in (0,1)\}$. In other words, we want a collection of functions $\{f_\alpha\}$ which integrate to zero with any T-invariant Borel probability such that no linear combination with nonzero coefficients is a coboundary.

We can use the same techniques as those used in Lemma 2.9 to create infinitesimals which are not T-coboundaries. Recall that in the proof of Lemma 2.9 we had sets U_k and integers n_k such that the function $f(x) = \sum_{k\geq 1}(1/k)\left(1_{U_k}(x) - 1_{T^{n_k}U_k}(x)\right)$ was not a T-coboundary (in that proof there was a transformation S such that $SU_k = T^{n_k}U_k$). The reason f failed to be a T-coboundary was that for all n there was a point x and an integer m such that $\sum_{i=0}^{m} f(T^i x) \geq \sum_{k=1}^{n}(1/k)$ and therefore has no uniform upper bound when summed over partial T-orbits. Notice that this function must be an infinitesimal since the integration of f with any T-invariant Borel probability yields zero.

Suppose that we have such an f. (If you like, pick minimal S with the same orbits as T but with an unbounded jump function to create the U_k and n_k.) Now for $\alpha \in (0,1)$, let

$$f_\alpha(x) = \sum_{k\geq 1}(1/k)^\alpha \left(1_{U_k}(x) - 1_{T^{n_k}U_k}(x)\right).$$

For any finite collection $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and nonzero real coefficients $\{r_1, r_2, \ldots, r_n\}$ the function $r_1 f_{\alpha_1} + r_2 f_{\alpha_2} + \cdots + r_n f_{\alpha_n}$ is an infinitesimal. It cannot be a T-coboundary since the partial sums $\sum_{k=1}^{N}\sum_{j=1}^{n} r_j (1/k)^{\alpha_j}$ behave like $\sum_{k=1}^{N} r_1 (1/k)^{\alpha_1}$ which is unbounded. □

Since the infinitesimal subgroups have the same dimension and contain no order structure, it remains to characterize $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$.

Let $\mathcal{E}(T)$ denote the set of ergodic T-invariant Borel probability measures.

Lemma 3.9. *Suppose that (X,T) is a minimal Cantor system and that $|\mathcal{E}(T)|$ is finite. Then* $\dim(G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)) = |\mathcal{E}(T)|$.

Proof. Let V denote the vector space $G_{\mathbb{R}}(T)/\mathrm{Inf}_{\mathbb{R}}(T)$ and V^* the dual of V. The dimension of V is finite if and only if the dimension of V^* is finite. Moreover, if the dimensions are finite, then they are the same.

Let F be an element of V^*, then it is a linear functional on $C(X,\mathbb{R})$. By the Riesz Representation Theorem, there is a finite signed Borel measure μ such that $F(f) = \int f d\mu$. Since F is a linear functional on $C(X,\mathbb{R})/\{f - fT\}$, the measure μ must be T-invariant. Since $|\mathcal{E}(T)|$ is finite, the measure μ is a linear combination of ergodic T-invariant Borel probability measures. Two linear combinations of these ergodic measures are the same as elements of

V^* if and only if they are the same as signed Borel measures. Therefore, $\dim(V) = \dim(V^*) = |\mathcal{E}(T)|$. □

Theorem 3.10. *Let (X, T) be a minimal Cantor systems such that $|\mathcal{E}(T)| = d$ is finite. The unital ordered vector space $\mathcal{G}_{\mathbb{Z}}(T)/\text{Inf }(T)$ is isomorphic to $\mathbb{R}^d$ where $(\mathbb{R}^d)_+$ are the elements with strictly positive entries along with the zero vector and $(1, 1, \cdots, 1)$ is the order unit.*

Proof. Since the dimension of $G_{\mathbb{R}}(T)/\text{Inf}_{\mathbb{R}}(T)$ and $\mathbb{R}^d$ are the same, we know that they are isomorphic as vector spaces. It remains to show that we can choose an isomorphism which preserves the order structure and unit.

Let $\{[f_1], [f_2], \ldots, [f_d]\}$ be a basis for $G_{\mathbb{R}}(T)/\text{Inf}_{\mathbb{R}}(T)$, and let $\mathcal{E}(T)$ denote the ergodic measures $\mathcal{E}(T) = \{\mu_1, \mu_2, \ldots, \mu_d\}$. We define a map from $\mathbb{R}^d$ to $G_{\mathbb{R}}(T)/\text{Inf}_{\mathbb{R}}(T)$ where a vector $\vec{v} \in \mathbb{R}^d$ gets sent to the equivalence class of a linear combination of the basis functions $g = \sum c_i f_i$ which has $\int g d\mu_j = v_j$ for $j = 1, 2, \ldots, d$.

Such a map preserves positive cones and order units. □

Corollary 3.11. *Let (X, S) and (X, T) be minimal Cantor systems, such that $|\mathcal{E}(S)|$ and $|\mathcal{E}(T)|$ are finite. The unital ordered vector spaces $\mathcal{G}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)$ are isomorphic if and only if $|\mathcal{E}(S)| = |\mathcal{E}(T)|$.*

Proof. We may write the vector space $G_{\mathbb{R}}(T)$ as $G_{\mathbb{R}}(T)/\text{Inf}_{\mathbb{R}}(T) \oplus \text{Inf}_{\mathbb{R}}(T)$. By Theorem 3.10, $\mathcal{G}_{\mathbb{R}}(S)/\text{Inf}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)/\text{Inf}_{\mathbb{R}}(T)$ are isomorphic as unital ordered vector spaces. By Proposition 3.8 the dimension of the infinitesimal subspaces are the same and therefore there is a vector space isomorphism between them. Since there is no order structure on the infinitesimal subspace, the result follows. □

The work of Dougherty, Jackson, and Kechris gives us a dynamical interpretation for equal cardinality of ergodic invariant Borel probabilities.

Definition 3.12 (Borel orbit equivalence)**.** Let $S : X \to X$ and $T : Y \to Y$ be Borel transformations of compact metric spaces X and Y. A Borel orbit equivalence is a Borel bijection $h : X \to Y$ and functions $m : X \to \mathbb{Z}$ and $n : X \to \mathbb{Z}$ such that

$$\forall x \in X,\ hS(x) = T^{m(x)}h(x) \text{ and } hS^{n(x)}(x) = Th(x).$$

Theorem 3.13 (Dougherty, Jackson, Kechris)**.** *Let $S : X \to X$ and $T : Y \to Y$ be Borel transformations of compact metric spaces X and Y. Then S and T are Borel orbit equivalent if and only if $|\mathcal{E}(S)| = |\mathcal{E}(T)|$.*

Therefore, we get the following dynamical interpretation of isomorphism of the unital ordered vector space in the case where the spaces of invariant measures are finite dimensional.

Theorem 3.14. *Let (X,S) and (X,T) be minimal Cantor systems such that $|\mathcal{E}(S)|, |\mathcal{E}(T)|$ are finite. Then the following are equivalent:*

1) $\mathcal{G}_{\mathbb{R}}(S) \cong \mathcal{G}_{\mathbb{R}}(T)$ *as unital ordered vector spaces,*
2) S *and* T *are Borel orbit equivalent.*

The result of Dougherty, Jackson and Kechris is true in the case where the cardinality of the ergodic invariant Borel measures is infinite. However, at present the author does not see how to extend the above theorem to include that case.

4. $\mathcal{G}_{\mathbb{R}}(T)$ as an extension of $\mathcal{G}_{\mathbb{Z}}(T)$.

Let (X,S) and (X,T) be a minimal Cantor systems. As we saw in the last section, isomorphism of $\mathcal{G}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)$ induces a weaker relation than strong orbit equivalence. We now consider isomorphisms of the unital ordered vector space $\mathcal{G}_{\mathbb{R}}(S)$ to $\mathcal{G}_{\mathbb{R}}(T)$ which when restricted to $\mathcal{G}_{\mathbb{Z}}(S)$ gives an isomorphism of the integer unital ordered groups. We first notice that $\mathcal{G}_{\mathbb{Z}}(T)$ embeds in $\mathcal{G}_{\mathbb{R}}(T)$.

Proposition 4.1. *The natural inclusion map $i : G_{\mathbb{Z}}(T) \to G_{\mathbb{R}}(T)$ is one-to-one and order-preserving.*

Proof. To show that the map is injective, it suffices to show that if an integer-valued function is a real coboundary, then it is an integer coboundary. Assume we have functions $f \in C(X,\mathbb{Z})$ and $g \in C(X,\mathbb{R})$ with $f = g - gT$.

Let x_0 be any point in X and let $\alpha = g(x_0)$. Then for all $n \in \mathbb{Z}$, $g(T^n x_0)$ is an integer plus α. For example if $n > 0$ then

$$g(x_0) - g(T^n x_0) = \sum_{i=0}^{n-1} g(T^i x_0) - g(T^{i+1} x_0) = \sum_{i=0}^{n-1} f(T^i x_0) \in \mathbb{Z}.$$

Since all T-orbits are dense in X, all values of the function g are an integer plus α. Letting $k = g - \alpha$, we obtain an integer-valued function $k \in C(X,\mathbb{Z})$ where $f = k - kT$.

Clearly, if $[f] \in G_{\mathbb{Z}}(T)_+$ then $i([f]) \in G_{\mathbb{R}}(T)_+$. Now suppose that $f \in C(X,\mathbb{Z})$ and $i([f]) \in G_{\mathbb{R}}(T)_+$. That is, suppose there exists a function $h \in C(X,\mathbb{R})$ such that $f(x) + h(x) - hT(x) \geq 0$ for all x. Then either $f \equiv 0$ or for all invariant probability measures μ, $\int f d\mu > 0$. In either case, $[f] \in G_{\mathbb{Z}}(T)_+$. □

Definition 4.2 (pair isomorphism)**.** Suppose that S and T are minimal homeomorphisms of the Cantor set. We will call H a pair isomorphism of $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ if $H : \mathcal{G}_{\mathbb{R}}(S) \to \mathcal{G}_{\mathbb{R}}(T)$ is a real ordered vector space isomorphism such that $H(G_{\mathbb{Z}}(S)) = G_{\mathbb{Z}}(T)$.

In particular, we are interested in the following questions. Does a pair isomorphism $H : (\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S)) \to (\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ induce a stronger relation

than strong orbit equivalence? Does a pair isomorphism imply that S and T are flip conjugate? We will show that the answer to the first question is yes, and the answer to the second is no. To answer these questions, we begin by showing that the isomorphism class of the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ classifies the (topological) *discrete spectrum* of the system (X, T).

Definition 4.3 (discrete spectrum)**.** Let (X, T) be a minimal Cantor system. The discrete spectrum of T is the set of λ such that $FT = \lambda F$ for some continuous function F from X to $\{z \in \mathbb{C}, |z| = 1\}$.

We will call a function F as above an eigenfunction for T and λ an eigenvalue for T.

The strong orbit equivalence class already determines the *rational part* of the discrete spectrum (eigenvalues $\exp(2\pi i\alpha)$ where $\alpha \in \mathbb{Q}$), but strongly orbit equivalent systems may have different irrational spectrum (see [**Orm97**]). The following shows that the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ does indeed carry some additional information beyond strong orbit equivalence.

Theorem 4.4. *Let (X, S) and (X, T) be minimal Cantor systems. Suppose that there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. Then S and T have the same discrete spectrum.*

Proof. We first show that a complex number $\exp(2\pi i\alpha)$ is an eigenvalue for T if and only if there exist $f \in C(X, \mathbb{Z})$ and $k \in C(X, \mathbb{R})$ such that $f = \alpha + k - kT$.

Suppose functions k, f exist as above. Multiplying both sides of $f = \alpha + k - kT$ by $2\pi i$ and exponentiating one obtains

$$\exp(2\pi ikT(x)) \exp(2\pi if(x)) = \exp(2\pi i\alpha) \exp(2\pi ik(x)).$$

Since $f(x) \in \mathbb{Z}$ for all x, we see that $\exp(2\pi if(x)) = 1$ and therefore $F(x) = \exp(2\pi ik(x))$ is an eigenfunction for T with eigenvalue $\exp(2\pi i\alpha)$.

Now suppose that $F : X \to S^1$ is an eigenfunction for the eigenvalue $\exp(2\pi i\alpha)$. Let $U_1, U_2, \ldots, U_n$ be clopen sets such that a logarithm function L_j can be continuously defined on each $F(U_j)$. For $x \in U_j$ we define $k(x) = L_j(F(x))$. With this definition, $k : X \to \mathbb{R}$ is a continuous function and $k(x) - k(Tx) + \alpha$ is an integer for all $x \in X$. Therefore, there is a $f : X \to \mathbb{Z}$, $k : X \to \mathbb{R}$ such that $f = \alpha + k - kT$.

This completes the proof since if $f = \alpha + k - kS$ as above and there is a pair isomorphism $H : (\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S)) \to (\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ then $H([f]) = \alpha H([1_S]) = \alpha[1_T]$. Taking a representative function g from $H([f])$, we see that there must be a function $k' \in C(X, \mathbb{R})$ such that $g = \alpha + k' - k'T$. Therefore if $\exp(2\pi i\alpha)$ is an eigenvalue for S then $\exp(2\pi i\alpha)$ is an eigenvalue for T as well. □

The above theorem extends to determine the possible discrete spectrum of an *induced system* (A, T_A) of (X, T). An induced system (A, T_A) of (X, T)

is a minimal Cantor system obtained by taking a clopen subset $A \subseteq X$ and the map $T_A : A \to A$. The map T_A is defined to be $T_A(x) = T^n(x)$ where n is the smallest positive integer such that $T^n(x) \in A$.

Theorem 4.5. *Let (X, S) and (X, T) be minimal Cantor systems. Suppose that there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. Then there is an induced system (A, S_A) of (X, S) which has λ in the discrete spectrum if and only if there is an induced system of (B, T_B) of (X, T) which has λ in the discrete spectrum.*

Proof. Suppose that $\lambda = \exp(2\pi i\alpha)$ is in the topological discrete spectrum of an induced system (A, S_A). This occurs if and only if for some $f \in C(A, \mathbb{Z})$ and $h \in C(A, \mathbb{R})$ we have $f - \alpha = h - hS_A$ on the set A. Extend f to $\hat{f} : X \to \mathbb{Z}$ by defining $\hat{f} \equiv 0$ on the complement of A. Then the function $(\hat{f} - \alpha 1_A)$ is an S-coboundary by Theorem 2.6. (Notice that for $x \in A$, $\sum_{i=0}^{n}(\hat{f} - \alpha 1_A)(S^i x) = \sum_{i=0}^{m}(\hat{f} - \alpha 1_A)((S_A)^i x)$ for some $m \leq n$.)

Now since $\hat{f} - \alpha 1_A$ is an S-coboundary and we have a pair isomorphism $H : (\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S)) \to (\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$, we know that $H([\hat{f}]) - \alpha H([1_A]) = 0$ in $\mathcal{G}_{\mathbb{R}}(T)$.

Claim. There is an indicator function $1_B \in H([1_A])$ for some clopen set $B \subseteq X$.

Proof of Claim. Since $[1_A] \in G_{\mathbb{Z}}(S)_+$ there is a $g_1 \in H([1_A])$ such that $g_1(x) \geq 0$ for all x. Since $[1]_S - [1_A] \in G_{\mathbb{Z}}(S)_+$ there is a $g_2 \in H([1_A])$ such that $g_2(x) \leq 1$ for all x. We know that the function $g_1 - g_2$ is an integer T-coboundary, $g_1 - g_2 = k - kT$ for some $k \in C(X, \mathbb{Z})$. Let C be a clopen set on which k is constant. Let $C_1, C_2, \ldots, C_r$ be the clopen subsets of C such that $x \in C_n$ if and only if n is the smallest positive integer such that $T^n(x) \in C$. By minimality, $C = \bigcup_{n=1}^{r} C_n$ for some r.

For $x \in C_n$, we know

$$\begin{aligned}\sum_{i=0}^{n-1} g_1 T^i(x) - g_2 T^i(x) &= \sum_{i=0}^{n-1} kT^i(x) - kT^{i+1}(x) \\ &= k(x) - kT^n(x) \\ &= 0.\end{aligned}$$

Therefore, for all n and $x \in C_n$, we have

$$0 \leq \sum_{i=0}^{n-1} g_1 T^i(x) = \sum_{i=0}^{n-1} g_2 T^i(x) \leq n.$$

Fix n and $x \in C_n$. Let B_n be the union of exactly $\sum_{i=0}^{n-1} g_1 T^i(x)$ of the sets $C_n, TC_n, \ldots, T^{n-1}C_n$. Let $B = \bigcup_{n=1}^{r} B_n$.

The difference between g_1 and 1_B must be a T-coboundary by Theorem 2.6. This follows since for $x \in C_n$, $\sum_{i=0}^{n-1} g_1 T^i(x) = \sum_{i=0}^{n-1} 1_B T^i(x)$.

Thus the difference $g_1 - 1_B$ is bounded along T-orbits. This proves the claim.

Let g be a representative of $H([\hat{f}])$ and let B be a clopen set such that $1_B \in H([1_A])$. Then there exists a function $k \in C(X, \mathbb{R})$ such that $g - \alpha 1_B = k - kT$.

Let $K(x) = \exp(2\pi i k(x))$. The function g is integer-valued so

$$\begin{aligned} K(T(x)) &= \exp(-2\pi i g(x)) \exp(2\pi i \alpha 1_B(x)) K(x) \\ &= \exp(2\pi i \alpha 1_B(x)) K(x). \end{aligned}$$

If $x \in B$, we have $K(T(x)) = \lambda K(x)$. If $x \notin B$, we have $K(T(x)) = K(x)$. Therefore, for $x \in B$, we have $K(T_B(x)) = \lambda K(x)$. Thus, T has an induced system with λ in the spectrum. □

Remark. If $\lambda = \exp(2\pi i \alpha)$ where $\alpha \in \mathbb{Q}$ then the conclusion of the previous theorem is trivially true. For any minimal Cantor system T and any $p \in \mathbb{Z}$ there is an induced system T_A such that a periodic orbit of cardinality p is a factor of T_A. To see this, let $B \subseteq X$ be a clopen set with small enough diameter so that if $x \in B$ and $T^n(x) \in B$ then $n \geq p$. Let $A = \bigcup_{i=0}^{p-1} T^i B$. Then the induced system (A, T_A) has as a factor of a finite orbit of length p.

In the case where $\lambda = \exp(2\pi i \alpha)$, $\alpha \notin \mathbb{Q}$, the statement is nontrivial as we will see in Corollary 4.8.

The following theorem shows some of the limitations on dynamical information that one can get from the pair $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. In particular, it shows that one cannot deduce flip conjugacy from a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. For a unital ordered group $\mathcal{G}$, when we say $\mathcal{G} \subseteq \mathbb{Q}$ we mean that $\mathcal{G}$ is isomorphic to a subgroup of $(\mathbb{Q}, \mathbb{Q}_+, 1)$ with the induced order.

Theorem 4.6. *Let S and T be minimal homeomorphisms of the Cantor set. Suppose $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ if and only if S and T are strongly orbit equivalent.*

Proof. Since $\mathcal{G}_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ every integer-valued function can be written as a constant plus a integer coboundary. The embeddings for $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$ are given by $[f] \mapsto q[1]$ where q is the rational number corresponding to $[f]$. Since the integer unital ordered groups are subsets of $\mathbb{Q}$, the maps S and T are uniquely ergodic. Therefore, the real ordered groups $\mathcal{G}_{\mathbb{R}}(S)$ and $\mathcal{G}_{\mathbb{R}}(T)$ are isomorphic by Theorem 3.10. Since the isomorphism maps the constant function one to the constant function one, it must map the subgroup of integer-valued functions onto one another. □

In particular, this shows that systems (X,T) with $\mathcal{G}_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ cannot be strongly orbit equivalent to systems with any irrational discrete spectrum. Systems with $\mathcal{G}_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$ include all odometer systems (see example from Section 2). For an odometer system with d_i digits in the ith place the group $G_{\mathbb{Z}}(T)$ is isomorphic to the subgroup of the rationals formed by all rationals whose denominators are products of the d_i's [**HPS92**]. For the dyadic adding machine $G_{\mathbb{Z}}(T)$ is the dyadic rationals $\mathbb{Z}[\frac{1}{2}]$.

Corollary 4.7. *Suppose (X,T) is a minimal Cantor system where $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then T cannot have irrational discrete spectrum.*

Proof. To prove this, one simply needs an S with $\mathcal{G}_{\mathbb{Z}}(S) = \mathcal{G}_{\mathbb{Z}}(T)$ such that S has no irrational spectrum. Then by Theorem 4.6, and Theorem 3.3, there is a pair isomorphism between $(\mathcal{G}_{\mathbb{R}}(S), \mathcal{G}_{\mathbb{Z}}(S))$ and $(\mathcal{G}_{\mathbb{R}}(T), \mathcal{G}_{\mathbb{Z}}(T))$. But by Theorem 4.4, S and T must have the same discrete spectrum.

To create such an S, make a list of the denominators $\{d_1 < d_2 < \cdots\}$ which appear in elements of $\mathcal{G}_{\mathbb{Z}}(T)$, then construct an odometer system with d_1 digits in the first place, d_2 digits in the second place, and so on.

The odometer systems have no irrational spectrum. This follows from the fact that for every clopen set A in an odometer system (X,T) there is an integer n such that $T^n A = A$. If there were a map $F : X \to S^1$ and a λ such that $FT = \lambda F$, then there would be a clopen set $A \subseteq X$ whose image under F lies within $\{\exp(2\pi i\theta) \ : \ 0 \leq \theta \leq \pi\}$ such that $FT^n(A) = F(A)$ for some n. If $\lambda = \exp(2\pi i\alpha)$ with α irrational, then $\lambda^n F(A)$ can never equal $F(A)$. □

Corollary 4.8. *Suppose (X,T) is a minimal Cantor system where $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then T cannot have an induced system with irrational discrete spectrum.*

Proof. Suppose that $\mathcal{G}_{\mathbb{Z}}(T)$ is a subgroup of $\mathbb{Q}$. Then any induced system T_A must also have $\mathcal{G}_{\mathbb{Z}}(T_A) \subseteq \mathbb{Q}$. This follows from results of [**GPS95**], or by the following argument. Since $G_{\mathbb{Z}}(T) \subseteq \mathbb{Q}$, T is uniquely ergodic. Moreover, the integral of any integer-valued continuous function with this measure must be rational. The same holds for an induced system (A, T_A), so $G_{\mathbb{Z}}(T_A) \subseteq \mathbb{Q}$. □

Appendix A. A homeomorphism good on measures and bad on orbits.

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Suppose S and T are minimal homeomorphisms of the Cantor set X. Giordano, Putnam and Skau proved that if $h : X \to X$ is a homeomorphism which identifies integer coboundaries for S and T then S and T are orbit equivalent (Theorem 2.7 of this paper). This result is a spinoff of

their beautiful algebraic characterization ("K_0 modulo the infinitesimals") of orbit equivalence of homeomorphisms of the Cantor set. This work developed constructions involving Bratteli diagrams, C^*-algebras and some homological algebra. It is natural to ask whether the theorem above could be proved directly, i.e., without reference to this associated machinery. This seems problematic even at first glance–given h as in the theorem, how could one recover orbit information? Here is an example (circulated informally in 1992) which reinforces this impression. I thank Chris Skau for helpful comments.

Example. Let X be the domain of the dyadic adding machine S. There is a homeomorphism T from X to X such that

- if x and y are any two points in the same S-orbit, then the points $T(x)$ and $T(y)$ are in different S-orbits, and
- for all clopen sets U, there are continuous functions $f, g : X \to \mathbb{Z}$ such that $1_U - 1_{TU} = f - fS$ and $1_U - 1_{SU} = g - gT$.

The dyadic adding machine is defined as an example of an odometer system in Section 2. We recapitulate the definition here. The space X is $\{0,1\}^{\mathbb{N}}$. A point x in X is a one-sided sequence $x_1x_2x_3...$ with each x_i in $\{0,1\}$. The map S sends the sequence $x = 1^\infty$ ($x_i = 1$, for all i) to the sequence 0^∞. Otherwise, x has for some nonnegative k an initial word 1^k0 and Sx is obtained by replacing this word with 0^k1.

Two sequences in X are *cofinal* if they disagree in only finitely many coordinates. Two sequences x, y are in the same S-orbit if and only if either (1) they are cofinal or (2) one is cofinal to 0^∞ and the other is cofinal to 1^∞.

Choose a collection of infinite pairwise disjoint sets A_n, $1 \leq n < \infty$, such that $\mathbb{N}$ is the union of the A_n. Enumerate the finite words on $\{0,1\}$ as $W(1), W(2), ...$ such that $n > m$ implies the length $|W(n)|$ of $W(n)$ is at least $|W(m)|$. Define $B_n = \{m \in A_n : m > |W(n)|\}$, an infinite subset of $\mathbb{N}$. For each $n > 0$, we define a homeomorphism $\phi_n : X \to X$ by

$$(\phi_n x)_i = \begin{cases} x_i + 1 \pmod 2 & \text{if } i \in B_n \text{ and } x_1...x_{|W(n)|} = W(n) \\ x_i & \text{otherwise.} \end{cases}$$

Now define $\psi_n = \phi_n \circ \phi_{n-1} \circ ... \circ \phi_1$. In other words, $\psi_1 = \phi_1$ and $\psi_n(x) = \phi_n(\psi_{n-1}(x))$. Finally, let $\psi = \lim \psi_n$. Apart from a technical detail, ψ will be the homeomorphism T of the example.

For each n and x,

$$(\psi_n x)_i = (\psi x)_i, \qquad 1 \leq i \leq |W(n)|.$$

Therefore the maps ψ_n are converging uniformly and ψ is a homeomorphism. Also, for every k and x, the word $(\psi x)_1...(\psi x)_k$ is determined by the word $x_1...x_k$. So, for every k, ψ induces a permutation of the initial cylinders of length k. This means that for any cylinder set U, there is a unique integer l

such that $\psi U = S^l U$ and $1_U - 1_{\psi U} = 1_U - 1_{S^l U}$. In particular, we may deduce that any integer ψ-coboundary is an integer S-coboundary and vice-versa.

If x and y are distinct points in X, then let $N = N(x,y)$ denote the largest integer such that $x_i = y_i$ if $i < N$. Notice, if ϕ_i corresponds to a word $W(i)$ such that $|W(i)| < N(x,y)$ (equivalently, $i \leq 2^{N-1}$) then $N(x,y) = N(\phi_i x, \phi_i y)$. On the other hand, if ϕ_i corresponds to a word $W(i)$ of length N, then ϕ_i fixes the initial word of length N in every point, and ϕ_i changes a point x (by flipping symbols in the coordinates indexed by B_i) if and only if $x_1 ... x_N = W(i)$.

Now given distinct points x and y, set $\gamma = \psi_{2^{N-1}}$, where $N = N(x,y)$. Let ϕ_n be the map corresponding to the initial word $W(n)$ of γx of length N. Our discussion above gives the following implications:

$$(\psi x)_i \neq x_i \qquad \text{if } i \in B_n,$$
$$(\psi y)_i = y_i \qquad \text{if } i \in B_n.$$

It follows immediately that if x and y are distinct cofinal points, the ψx and ψy are not cofinal.

Next note that if x and y are distinct points, then for some B_n,

$$x_i = (\psi x)_i \qquad \text{and} \qquad y_i = (\psi y)_i, \qquad \text{for} \qquad i \in B_n.$$

(In fact we can use n such that B_n corresponds to a word $W(n)$ of length 2 which begins neither $\psi_2 x$ nor $\psi_2 y$.) Consequently, if x is cofinal to 0^∞ and y is cofinal to 1^∞, then $\psi x, \psi y$ are not cofinal.

This finishes the proof for the example, except for a technical detail: It might be the case that there are points x, y in the same S-orbit such that ψx is cofinal to 0^∞ and ψy is cofinal to 1^∞. To take care of this, choose points u and v such that the preimage under ψ of the S-orbit of u does not intersect any S-orbit containing a point in the preimage under ψ of the S-orbit of v. Let β be a cofinal homeomorphism of X (x, y are cofinal iff $\beta x, \beta y$ are cofinal) which exchanges 0^∞ with u and which exchanges 1^∞ with v. Let T be the composition, ψ followed by β. Now we have for all distinct points x and y: If x, y are cofinal then Tx, Ty are not cofinal; if x is cofinal to 0^∞ and y is cofinal to 1^∞, then Tx,Ty are not cofinal; if Tx is cofinal to 0^∞ and Ty is cofinal to 1^∞, then x, y are not in the same S-orbit.

This finishes the proof.

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