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Let k be an infinite field of arbitrary characteristic, (A, M, K) a k-algebra of essentially finite type, with K/k separable and P a local property. We say that $LB_k(\mathrm{P})$ holds if: For the generic $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \Rightarrow \mathrm{P}(Ax_\alpha A) \subseteq \mathrm{P}(A) \cap V(x_\alpha) \cap U_P$ ($x_\alpha = \sum \alpha_i x_i$, $\langle x_1, \dots, x_n \rightarrow= M$, U_P non-empty open subset of Spec A and $\mathrm{P}(A) = \{P \in \mathrm{Spec}\, A | A_p$ is P$\}$). We show that: $LB_K(\mathrm{P})$ holds $\Rightarrow LB_K(\mathrm{GP})$ holds for the corresponding geometric property (in particular, for P = regular, normal, reduced, R_s, $LB_K(\mathrm{GP})$ holds). As an appliance we obtain a Bertini Theorem for hypersurgace setions of a variety $X \subseteq P^n_k$ concerning the geometric properties.

1. Introduction.

Bertini showed that, given a smooth projective variety x contained in P^n_k with $k = \mathbf{C}$, the generic hypersurface section of x is also smooth (see [**B**, Chap. 10, n. 25]; for a modern approach, see [**H**, Th. 8.18] or [**J**, Th. 6.3]).

There have been many generalizations of this Theorem: We recall the recent algebraic studies on trasversality made by Kleiman in [**K**] and Speiser in [**S**] where they introduced a fully modern point of view of schemes over an algebraically closed field of arbitrary characteristic.

Another approach to this problem has been proposed by Flenner in [**F**] (following Grothendieck, see [**G**]).

He shows that, given a field k of arbitrary characteristic and given a local k-algebra $(A < M < K)$ with K/k separable, then, for the generic $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \Rightarrow$

$$\mathbf{P}(A/x_\alpha A) \subseteq \mathbf{P}(A) \cap U_P \tag{1}$$

where $x_\alpha = \sum \alpha_i x_i$, $\{x_1, \dots, x_n\}$ is a generator system of M and U_P is a non-empty open subset of Spec A depending on $\mathbf{P}$, being $\mathbf{P}$ one of the following local properties: Regular, normal, reduced, R_s and S_r.

These results applied to the local ring of the vertex of the affine cone corresponding to a projective variety X, imply, by standard techniques, the corresponding global Bertini Theorem for the variety X.

In this work we want to show that every time we have a result like (1) for a property $\mathbf{P}$ we have the same result for the corresponding geometric property **GP** and that the coresponding global results hold (these are known only for geometrically regular, see [**J**, Chap. 1, §6]).

In Section 3 we introduce some topological remarks that we use in next section: We show that if k is a subfield of K, (k infinite), every non-empty open set of K^n can be constructed to a non-empty open set of K^n.

In Section 4, that is main section, we give a local Bertini Theorem for the properties **GP** in an axiomatic form and we show that there are properties **GP** (for example **GP** $= S_r$, geom. R_s, geom. regular, geom. normal, geom. reduced) to which we can apply the Theorem. In these cases we show that the **GP**-locus is open.

In Section 5 we deduce a global Bertini's Theorem for the hypersurface sections of a variety X in a projective space over a field of arbitrary characteristic and for the above cited **GP** (we extend for many geometrically properties Th. 6.3 in [**J**] concerning the only geometrical regular property).

2. Preliminaries and notation.

In this section we fix the standard notation to be used in the following.

The rings considered are always commutative with an identity element.

If A is a ring, $\Omega(A)$ is the set of maximal ideals of A.

We recall here the definition of essentially finite type algebra and some properties of this algebra that we shall have to use in Section 4.

Definition 2.1 ([**EGA**, Chap. IV, 1.3.8])**.** Let T be a ring. A T-algebra S is of essentially finite type (e.f.t. for short) if S is T-isomorphic to $S^{-1}C$ where C is a T-algebra of finite type and S is a multiplicatively closed subset of C.

Properties 2.2 ([**EGA**, Chap. IV, 1.3.9 (ii)]; [**M**, 34.A])**.**

(i) If S is a T-algebra of e.f.t. and T' is a T-algebra then $S' = S_{\oplus_T} T'$ is a T'-algebra of e.f.t..

(ii) If S is a T-algebra of e.f.t. and T is an excellent ring then S is an excellent ring.

In the following, all topological spaces are considered with their Zariski topology. If A is a ring we put $V(x_1, \dots, x_n)$ to closed subset of $\operatorname{Spec} A$ corresponding to the ideal generated to the elements $x_1, \dots, x_n$ of A.

Let $F[\underline{T}] = F[T_1, \dots, T_n]$ be the polynomial ring with coefficients in the field F. We identify $F^n\{(\alpha_1, \alpha_n)|\alpha_1 \in F\}$ with the topological subspace $S = \{(T_1 - \alpha_1, \dots, T_n - \alpha_n)|\alpha_1 \in F\}$ $\operatorname{MaxSpec} F[\underline{T}]$. (We observe that $\overline{F}^n = \operatorname{MaxSpec} \overline{F}[\underline{T}]$ where $\overline{F}$ denotes the algebraic closure of the field F.)

The expression "x generic in X", where X is a topological space, means that x is in a dense open subset of X.

We recall here the definition of geometric property.

Definition 2.3. Let $\mathbf{P}$ be a local property and A a local ring containing a field k. We say that A is geometrically $\mathbf{P}$ if $A_{\oplus_k} k$ is $\mathbf{P}$.

(See also [**EGA**, Chap. IV, 6.7.7] for equivalent definitions.)

Finally we put $\mathbf{P}(A) = \{P \in \operatorname{Spec} A | A_{\mathcal{P}}$ verifies the local property $\mathbf{P}\}$.

3. Some topological remarks.

For our aim we have to prove that, given an infinite field k, if K/k is a field extension and $\Im$ is an open dense subset K^n then $\Im \cap k^n$ is an open dense subset of k^n (Prop. 3.3). We prove this fact in two steps (the first one for the 'open' property, the second one for the 'dense' property).

We consider the following commutative diagram:

$$\begin{array}{ccc} K^n & \xrightarrow[i]{} & \operatorname{Spec} K[T_1, \dots, T_n] \\ {\scriptstyle j}\uparrow & & {\scriptstyle f}\downarrow \\ k^n & \xrightarrow[h]{} & \operatorname{Spec} k[T_1, \dots, T_n] \end{array}$$

where i, h are the inclusions of canonical maps and, as well known, K^n (resp. k^n) is a topological subspace of $\operatorname{Spec} K[\underline{T}]$ (resp. $\operatorname{Spec}[\underline{T}]$).

Lemma 3.1. *Let K/k be a field extension, then k^n is a subspace of K^n.*

Proof.
Case 1. *K/k algebraic extension.*

One can suppose that $\mathfrak{I} = V(g)$ is a fundamental closed set.

Consider a representation $g = \sum x_i g_i$ with $x_i \in K$ linearly independent over k and $g_i \in k[X_1, \dots, X_n]$. Then $\mathfrak{I} \cap k^n = V(g_1, \dots, g_r)$. The inclusion $\mathfrak{I} \cap k^n \subseteq (C) \cap k^n$ is trivial. The other one is easy if we remark that $f^{-1}(k^n) = k^n$.

Case 2. *K/k purely transcendental extension.*

Let $\Im = \{(x_1, \dots, x_n) \in K^n | g(x_1, \dots, x_n) = 0$ with $g \in K[\underline{T}]\}$ be a fundamental closed set of K^n. Among the coefficients of g there are only a finite number t of elements of K transcendental over k and so we can reduce to the transcendental extension of finite type. Using induction on t we can consider that there is only one transcendental element Z (i.e., $t = 1$).

So $g(T_1, \dots, T_n) = a_{i_1 \cdots i_n}(z) T^{i_1} \cdots T^{i_n}$ with $a_{i_1 \cdots i_n}(z) \in k(z)$.

$$(k_1, \dots, k_n) \in k^n \cap \Im \Leftrightarrow g(k_1, \dots, k_n) = 0 \Leftrightarrow b_{i_1 \cdots i_n}(z) k^{i_1} \cdots k^{i_n} = 0$$

with $b_{i_1 \cdots i_n}(Z) \in k[Z]$ (obtained by clearing denominators and simplifying) $\Leftrightarrow$ $g_r(k_1, \dots, k_n) Z^r + \cdots + g_0(k_1, \dots, k_n) = 0$ (obtaining ordering $b_{i_1 \cdots i_n}(z) k^{i_1} \cdots k^{i_n}$ like a polynomial in z) where $g_r(T_1, \dots, T_n) \in k[\underline{T}]$.

But Z is transcendental over k and so $(k_1, \dots, k_n) = k^n \cap \Im \Leftrightarrow g(k_1, \dots, k_n) = 0 \; \forall \; i \;\; 0 \leq i \leq r$. Then we have $\Im \cap k^n = V(g_1, \dots, g_r)$.

General case.

It is well known that every field extension can be written as $k \subseteq K' \subseteq K$ with K'/k purely transcendental and K/K' algebraic. So we can apply subsequently Case 2 and Case 1.

Lemma 3.2. *Let k be an infinite field, then k^n is irreducible.*

Proof. We want to show that the intersection of two non-empty open sets is still non-empty.

For this it is clearly sufficient to show that if $f, g \in k[T_1, \dots, T_n]$ and $V(f) \neq k^n$, $V(g) \neq k^n$ then $V(fg) \neq k^n$. We use induction on n. If $n = 1$ we consider the polynomial: $fg = (f_0 + \dots + f_i T^l)(g_0 + \dots + g_h T^h)$. $Fg = 0$ has at most $i+h$ solutions in $\overline{k}$ (and so in k) and this proves that $V(fg) = k$ because k is infinite.

Suppose now that the conclusion is true for any number of variables smaller than n.

We have $fg = (f_0 + f_1 T_n + \dots + f_i T_n^l)(g_0 + g_1 T_n + \dots + g_n T_n^h) = f_0 g_0 + \dots + f_1 g_h T^{l+h}$.

With $f_j g_l \in k[T_1, \dots, T_{n-1}]$ for $0 \leq j \leq i$ and $0 \leq l \leq h$ observe that $f_1 g_h$ is a polynomial in $n-1$ variables $\Rightarrow$ by the induction hypothesis, there exists an element $w = (k_1, \dots, k_n) \in k^{n-1}$ such that $f_1(k_1, \dots, k_n) g_h(k_1, \dots, k_n) \neq 0$. For this w we can find an element $a \in k$ such that $f(k_1, \dots, k_n, a) g(k_1, ..., k_n, a) \neq 0$ because the polynomial in a single variable $f(k_1, \dots, k_n, T_n) g(k_1, \dots, k_n, T_n)$ has at most $i + h$ solutions in k and k is infinite.

Then there exists k_n such that $y = (k_1, \dots, k_{n-1}, k_n) \in V(fg)$.

From the above lemmas we get:

Proposition 3.3. *Let K be an extension of infinite field k. If $\mathfrak{I}$ is an open dense subset of K^n then $\mathfrak{I} \cap k^n$ is an open dense subset of K^n.*

Proof. By Lemma 3.1 we know that $\mathfrak{I} \cap k^n$ is open in k^n. By Lemma 3.2 it is enough to show that $\mathfrak{I} \cap k^n$ is non-empty. It is sufficient to prove this fact for $\mathfrak{I} = k^n - V(f)$ with $f \in K[T_1, \dots, T_n]$, by induction on n.

If $n = 1$, $f(T) = K_0 + \dots + K_r T^r$ has at most r solutions in K and so in k.

Suppose that it is true for any integer $m < n$. Put $f(T_1, \dots, T_n) = f_0 + f_1 T_n + \dots + f_i T_n^l$ where $f_j \in K[T_1, \dots, T_{n-1}]$ for $0 \leq j \leq i$. By induction hypothesis there exists $(k_1, \dots, k_{n-1}) \in k^{n-1}$ such that $f_i(k_1, \dots, k_{n-1}) \neq 0$. Considering $f_i(k_1, \dots, k_{n-1}, T^n)$ we observe that f has at most j solutions in k. Let $a \in k$ be a non-solution for $f_i(k_1, \dots, k_{n-1}, T^n)$, then $(k_1, \dots, k_{n-1}, a) \in \mathfrak{I} \cap k^n$.

4. Main result.

The main purpose of this paragraph is to give a local Bertini thorem for the geometric properties. We need some definitions.

Definition 4.1. A local ring (A, M, K) is a Flenner k-algebra if A is a northerian k-algebra, k is an infinite field and K is separable over k.

Definition 4.2. Let $\mathbf{P}$ be a local property of commutative rings. We say that $\mathbf{P}$ is a local Bertini property if, for every local Flenner k-algebra (A, M, K) e.f.t. and every set of generators $\langle x_1, \dots, x_n\rangle$ of M, the following condition holds:

$LB_k(\mathbf{P})$ for generic $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \Rightarrow$

$$\mathbf{P}(Ax_\alpha A) \subseteq (A) \cap V(x_\alpha) \cap U_{\mathbf{P}}$$

where $x_\alpha = \Sigma\alpha_1 x_1$, and $U_{\mathbf{P}}$ is either $\operatorname{Spec} A$ or $\operatorname{Spec} A - \{M\}$, depending on $\mathbf{P}$.

We say briefly that $LB_k(\mathbf{P})$ holds.

Remark 4.3. We observe that $LB_k(\mathbf{P})$ holds for $\mathbf{P}$ = regular, normal, reduced. Serre's properties R_s and S_r (in fact more general statements holds: See [**F**] Theorem 4.1 and Corollaries 4.2 and 4.3).

We want to prove that if A is a Flenner K-algebra of e.f.t. and $LB_k(\mathbf{P})$ holds for some property $\mathbf{P}$ then $LB_k(\mathbf{GP})$ holds too for the corresponding geometric property.

We need some lemmas.

Lemma 4.4. *Let (A, M, K) be a Flenner k-algebra of e.f.t. and $B = A \otimes_k \overline{k}$. Then, for every $M \in \Omega(B)$, (B_M, NB_M, K_M) is a Flenner $\overline{k}$-algebra of e.f.t.*

Proof. Recall that $\varphi : A \longrightarrow B$ is a flat homomorphism.

Case 1. B is a semilocal $\overline{k}$-algebra and $\mathcal{M}B_M = MB_M$ $\forall M \in \Omega(B)$. Clearly B is a $\overline{k}$-algebra of e.f.t. and, being integral over A, we have $\mathcal{M}B \subseteq \underline{\operatorname{Rad}(B)}$.

$B/\mathcal{M}B = K \otimes_A (A \otimes_k \overline{k}) = K \otimes_k \overline{k}$ and $\dim K \otimes_k \overline{k} = 0$. In fact $K \otimes_k \overline{k}$ is noetherian (because B is a $\overline{K}$-algebra of e.f.t. by Prop. 2.2 (i) and so it is noetherian) and integral over K and we can apply Theorem 20 in [**M**]. So $K \otimes_k \overline{k}$ is an artinian ring (Theorem 8.5 in [**A-M**]) and this proves that B is semilocal.

$K \otimes_k \overline{k}$ is also reduced (because K/k is separable and we can apply (27.1) Lemma 1 in [**M**]) and $\dim(K \otimes_k \overline{k})_M = \dim(B/\mathcal{M}B)_M = 0$. This proves that $(B/\mathcal{M}B)_M = B_M/\mathcal{M}B_M$ is a field, that is $\mathcal{M}B_M = MB_M$.

Case 2. K_M is separable over $\overline{k}$ for every $M \in \Omega(B)$ because every extension of an algebraically closed field is separable.

Lemma 4.5. *Let $(A, \mathcal{M}, K)$ be a Flenner k-algebra of e.f.t., $\{x_1, \dots, x_n\}$ a generator system of $\mathcal{M}$ and $B = A \otimes_k \overline{k}$. If $LB_k(\mathbf{P})$ holds then:*

a) *for the generic $\alpha = (\alpha_1, \dots, \alpha_n) \in \overline{k}^n$: $\mathbf{P}(B/x_\alpha B) \subseteq \mathbf{P}(B) \cap V(x_\alpha B) \cap U_{\mathbf{P}}$,*

b) *for the generic* $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n : \mathbf{P}(B/x_\alpha B) \subseteq \mathbf{P}(B) \cap V(x_\alpha B) \cap U_{\mathbf{P}}$,

where $X_\alpha = \Sigma \alpha_i X_i$ *and* $U_P = \operatorname{Spec} B - \Omega(B)$.

Proof. a) In fact the condition $LB_k(\mathbf{P})$ holds for $(B_M, MB_M, K_M) \, \forall M \in \Omega(B)$ by Lemma 4.4. So we can find an open dense subset $\mathfrak{I}_M$ of $\overline{k}^n$ such that $\forall \alpha \in \mathfrak{I}_M$. $\mathbf{P}(B_M/X_\alpha B_M) \subseteq \mathbf{P}(B_M) \cap V(X_\alpha B_M) \cap U_{\mathbf{P}}$. But B is semilocal by 4.4 so it has a finite number of maximal ideals: $M_1, \dots, M_d$. Putting $\mathfrak{I} = \mathfrak{I}_{M_1} \cap \dots \cap \mathfrak{I}_{M_d}$. This is an open dense subset of $\overline{k}^n$ (by Lemma 3.2), independent from M_1 and so $\forall \alpha \in \mathfrak{I}$ we have $\mathbf{P}(B/X_\alpha B) \subseteq \mathbf{P}(B) \cap V(X_\alpha B) \cap U_{\mathbf{P}}$.

b) Use a) and Proposition 3.3.

Theorem 4.6. *If* $LB_k(\mathbf{P})$ *holds for some local property* $\mathbf{P}$ *then* $LB_k(\mathbf{GP})$ *holds for the corresponding geometric property* $\mathbf{GP}$.

Proof. If (A, M, K) is a Flenner k-algebra of e.f.t. and $\mathcal{P} \in \mathbf{GP}(A) \cap V(X_\alpha) \cap U_{\mathbf{GP}}$ we have to prove that $\mathcal{P} \in \mathbf{GP}(A/X_\alpha A)$.

Clearly we have: $\mathcal{P} \in \mathbf{GP}(A/X_\alpha A) \Leftrightarrow (A_{\mathcal{P}}/X_\alpha A_{\mathcal{P}}) \otimes_k \overline{k}$ is $\mathbf{P} \Leftrightarrow (A/X_\alpha A) \otimes_A (A_{\mathcal{P}} \otimes_k \overline{k})$ is $\mathbf{P}$.

Considering $\varphi : A \longrightarrow B = A \otimes_k \overline{k}$ and $S = A - \mathcal{P} \Rightarrow A_{\mathcal{P}} \otimes_k \overline{k} \cong S^{-1}B$ by Prop. 3.5 in [**A-M**]. If $\mathcal{Q} \in \operatorname{Spec}(A_{\mathcal{P}} \otimes_k \overline{k})$, let Q be its image in $S^{-1}B$. Then, $\forall\ \mathcal{Q} \in \operatorname{Spec}(A\mathfrak{p} \otimes_k \overline{k})$, $(A\mathfrak{p} \otimes_k \overline{k})_{\mathcal{Q}} \cong B_Q$ is $\mathbf{P}$, i.e., $Q \in \mathbf{P}(B)$ It is also $Q \subset (X_\alpha)^e$ and $Q \in U_{\mathbf{P}}$ (because $\mathfrak{p} \neq \mathcal{M} \Rightarrow Q \notin \Omega(B)$). Applying Lemma 4.5 to B we have: $(B_Q)/(X_\alpha)B_Q \cong (A\mathfrak{p} \otimes_k \overline{k})_{\mathcal{Q}}/(X_\alpha)(A\mathfrak{p} \otimes_k \overline{k})_{\mathcal{Q}}$ is $\mathbf{P} \, \forall \, \mathcal{Q} \in \operatorname{Spec}(A\mathfrak{p} \otimes_k \overline{k}) \Rightarrow (A/x_\alpha A\mathfrak{p}) \otimes_k \overline{k}$ is $\mathbf{P} \Rightarrow \mathfrak{p} \in \mathbf{GP}(A/x_\alpha A)$.

Corollary 4.7. $LB_k(\mathbf{GP})$ *holds for Flenner* K *algebra of e.f.t.* (A, M, K) *if:*

i) $\mathbf{GP} =$ *geom. Regular and* $U_{\mathbf{GP}}(A) = \operatorname{Spec} A$;

ii) $\mathbf{GP} = S_r$, *geom. Serre's property* R_s, *geom. normal, geom. reduced and* $U_{\mathbf{GP}}(A) = \operatorname{Spec} A - \{M\}$;

(*with the notation given in Def.* 4.2)

Proof. By Remark 4.3 and Theorem 4.6.

In connection with Theorem 4.6 it is important know that the **GP**-locus of an e.f.t. K-algebra is open, at least for the properties $\mathbf{P}$ cited above. This will be shown in Theorem 4.8 below.

Theorem 4.8. *Let* A *be a* K*-algebra of finite type, then* $\mathbf{GP}(A)$ *is an open subset of* $\operatorname{Spec} A$ *for* $\mathbf{GP} = S_r$, *geom. Serre's property* R_s, *geom. Regular, geom. Normal, geom. Reduced.*

Proof. We may assume that A is a K-algebra of finite type. Indeed if A is a K-algebra of e.f.t. then (Def. 2.1) $A = S^{-1}C$ where C is a K-algebra of finite type and S is a multiplicatively closed subset of C. If U is an open subset of $\operatorname{Spec} C$ and if we call φ the continuous map defined from $\operatorname{Spec}(S^{-1}, C)$ to $\operatorname{Spec} C$ induced by the canonical homomorphism $\varphi^* : C \to S^{-1}C$, then $\varphi^{-1}(U)$ is an open subset of $\operatorname{Spec}(S^{-1}C) = \operatorname{Spec} A$. Moreover the properties **GP** are preserved by localization.

(a) Case **GP** = geom. Normal, geom. R_n.

We use a proof that looks like Zariski's Theorem in [**EGA**, Chap. IV, 6.12.5].

We consider $A \otimes_k K'$ where $K' = K^{P^{-\infty}}$. The morphism $\operatorname{Spec}(K') \to \operatorname{Spec}(k)$ is a universal homomorphism and so the morphism $\operatorname{Spec}(A \otimes_k K') \to \operatorname{Spec} A$ is a homomorphism.

Then the projection of $\mathbf{P}(A \otimes_k K')$ in $\operatorname{Spec} A$ is just the set $\mathbf{GP}(A)$ (by [**EGA**, Theorem 6.7.7 Chap. IV].

We have only to show that $\mathbf{P}(A \otimes_k K')$ is open in $\operatorname{Spec}(A \otimes_k K')$. But this is true:

i) for $\mathbf{P}$ = regular by [**EGA**, Chap. IV 6.12.5];
ii) for $\mathbf{P} = R_n$ by i) and [**EGA**, Chap. IV 6.12.9];
iii) for $\mathbf{P}$ = normal by i) and [**EGA**, Chap. IV 6.13.5].

(b) Case $\mathbf{GP} = S_n$ and geom. Reduced.

A is a K-algebra of finite type and so it is excellent by Prop. 2.2 (ii). So we can apply consideration [**EGA**, 7.9.7 Chap. IV] for $\mathbf{P} = S_n$ and Prop. 4.6.13 Chap. IV [**EGA**] for $\mathbf{P}$ = reduced.

Using Theorem 4.8 we have:

Corollary 4.9. *If (A, M, K) is a Flenner K-algebra of e.f.t. then $\mathbf{GP}(A)$ is an open subset of $\operatorname{Spec} A$ for $\mathbf{GP} = S_r$, geom. Serre's property R_s, geom. regular, geom. normal, geom. reduced.*

5. Application to Global Bertini Theorems.

We want now to deduce from Theorem 4.6 a global Bertini Theorem for geometric properties of hypersurface sections of a projective variety over an arbitrary field.

For this we use a standard technique involving the vertex of the affine cone (see also [**F**, §5]).

We give some notation: Let k be a field, $X \subseteq \mathbf{P}^n_k$ a projective variety over the field k and $Y \subseteq X$ a closed subset of X. Let $Y^+ \subset X^+ \subseteq \mathbf{A}^{n+1}_k$ be the corresponding affine cones; put $A = 0_{x^+,v}$ (where v is the vertex) and let I be the ideal of Y^+ in A. Let $X(\overline{k})$, $Y(\overline{k})$ be the varieties obtained from X and Y by making the base extension field $\to \overline{k}$.

Proposition 5.1. *Let $\mathbf{P}$ be a local property which is preserved by polynomials and fractions and which descends by faithful flatness. With the notation given above, the following are equivalent:*

(i) $X - Y$ *is* **GP** *over* k;
(ii) $X^+ - Y^+$ *is* **GP** *over* k;
(iii) $\operatorname{Spec} A - V(I)$ *is* **GP** *over* k.

Proof. $X - y$ is **GP** over $k \Leftrightarrow X(\bar{k}) - Y(\bar{k})$ is $\mathbf{P} \underset{(1)}{\Leftrightarrow} X^+(\bar{k}) - Y^+(\bar{k})$ is $\mathbf{P}(\Leftrightarrow X^+ - Y^+$ is **GP** over $k) \underset{(2)}{\Leftrightarrow} \operatorname{Spec} A(\bar{k}) - V(I(\bar{k}))$ is $\mathbf{P} \Leftrightarrow \operatorname{Spec} A - V(I)$ is **GP** over k, where the equivalencies (1) and (2) are due to Proposition 2.1 in [**CGM**].

In the following let $S = \otimes S_d$ be graded k-algebra of finite type so that $S_0 \cong k$ and $S = k[S_1]$.

Theorem 5.2. *$S = k[S_1]$ a graded k-algebra, k a field with infinitely many elements and $\{f_0, \dots, f_{n(q)}\}$ a generator system of S_q as a k-vector space. Let* **P** *be as in* 5.1*.*

If $LB_k(\mathbf{GP})$ holds for some geometrical property **GP** *then, for the generic $\alpha = (\alpha_0, \dots, \alpha_{n(q)}) \in k^{n(q)+1}$ we have that,*

$$\mathbf{GP}(\operatorname{Proj}(S/f_\alpha S)) \subseteq \mathbf{GP}(\operatorname{Proj}(S)) \cap V^+(f_\alpha)$$

where $f_\alpha = \Sigma \alpha_1 f_1$.

Proof. For $q = 1$ we can apply Prop. 5.1 and Th. 4.6. (Observe that K, the residue field of A, coincides with k and so it is separable over k.) For $q > 1$ we can reduce to the hyperplane case using the Veronese map of degree q.

Corollary 5.3. *With the hypothesis and notation as in Theorem* 5.2 *we have $\mathbf{GP}(\operatorname{Proj}(S/x_\alpha S)) \subseteq \mathbf{GP}(\operatorname{Proj}(S)) \cap V^+(X_\alpha)$ for $\mathbf{GP} = S_r$, geom. Serre's property R_s, geom. regular, geom. normal, geom. reduced, regular, etc.*

Proof. Apply Theorem 5.2 and Corollary 4.7.

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