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The main result of this paper asserts that a monoid with finitely many left and right ideals is finitely presented if and only if all its Schützenberger groups are finitely presented. The most important part of the proof is a rewriting theorem, giving a presentation for a Schützenberger group, which is similar to the Reidemeister-Schreier rewriting theorem for groups.

1. Introduction.

In [**24**, Theorem 4.1] it was proved that a regular monoid S with finitely many left and right ideals is finitely presented if and only if all its maximal subgroups are finitely presented. Recall that the maximal subgroups of S are precisely the $\mathcal{H}$-classes of S containing idempotents. Schützenberger [**25**, **26**] showed how one can assign to an arbitrary $\mathcal{H}$-class H a group $\Gamma(H)$, called the Schützenberger group of H. Schützenberger groups have many features in common with maximal subgroups; in particular, if the $\mathcal{H}$-class H contains an idempotent (and hence is a maximal subgroup) then H and $\Gamma(H)$ are isomorphic. Since their discovery, they have been used in the structure theory of semigroups (see, for example, [**9**, **10**, **15**, **16**, **20**]), but perhaps, as argued in [**13**], not as much as they deserve.

In this paper we consider connections between presentations for a monoid and for its Schützenberger groups, and we prove the following:

Theorem 1.1. *A monoid with finitely many left and right ideals is finitely presented if and only if all its Schützenberger groups are finitely presented.*

The theorem follows from Corollaries 3.3 and 4.4.

In proving the above theorem we show how one can combine presentations of Schützenberger groups to obtain a presentation for the monoid; see Theorem 3.2. More importantly, we prove a rewriting theorem (Theorem 4.2), in many ways similar to the Reidemeister-Schreier theorem for subgroups of groups, which gives a presentation for a Schützenberger group from a presentation for the monoid. In fact, this presentation is effectively computable, provided that the monoid has finitely many left and right ideals.

This opens the way for our rewriting theorem to be used as a tool in computing with finitely presented monoids, along the similar lines to the use of the Reidemeister-Schreier theorem in computational group theory.

The best known consequence of the Reidemeister-Schreier theorem is that a subgroup of finite index in a finitely presented group is itself finitely presented. Paralleling this are Corollary 2.11 and Proposition 2.16 of [**24**], which combined give that a maximal subgroup H (i.e., an $\mathcal{H}$-class containing an idempotent) of a finitely presented monoid is finitely presented, provided the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes. Given the similarity between maximal subgroups and Schützenberger groups, one could reasonably hope that this last condition would be sufficient to guarantee finite presentability of the Schützenberger group $\Gamma(H)$ of an arbitrary $\mathcal{H}$-class (not necessarily containing an idempotent). However, in Section 6 we use our rewriting theorem to construct a finitely presented monoid which contains an $\mathcal{H}$-class H such that H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class, but the Schützenberger group $\Gamma(H)$ is not finitely presented.

2. Preliminaries.

Green's equivalences. Green's equivalences were introduced in [**8**]. They describe the ideal structure of a monoid (or a semigroup). Since their discovery they have become the principal tool in describing the structure and properties of monoids and semigroups; see [**11**]. We give definitions of the relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$, and some of their basic properties that we need in the sequel. For a more complete treatment we refer the reader to [**11**] or [**12**].

Let S be a monoid. Two elements $s, t \in S$ are said to be $\mathcal{R}$-equivalent (respectively, $\mathcal{L}$-equivalent) if they generate the same right (respectively, left) ideal, i.e., if $sS = tS$ (respectively, $Ss = St$); we write $s\mathcal{R}t$ (respectively, $s\mathcal{L}t$). Two elements are $\mathcal{H}$-equivalent if they are both $\mathcal{R}$-equivalent and $\mathcal{L}$-equivalent.

In the following proposition we list some properties of these relations that we will require later.

Proposition 2.1. *Let S be a monoid.*

(i) *Let $s, t \in S$ be such that $s\mathcal{R}t$, and let $p, q \in S$ be such that $sp = t$ and $tq = s$. Then the mapping $x \mapsto xp$ is a bijection from the $\mathcal{H}$-class of s onto the $\mathcal{H}$-class of t; its inverse is the mapping $x \mapsto xq$. In particular, any two $\mathcal{H}$-classes within the same $\mathcal{R}$-class have the same size.*

(ii) *If $s, p_1, p_2 \in S$ are such that $sp_1p_2\mathcal{R}s$ then $sp_1\mathcal{R}s$.*

(iii) *The relation $\mathcal{R}$ is a left congruence, i.e., for all $s, t_1, t_2 \in S$, if $t_1\mathcal{R}t_2$ then $st_1\mathcal{R}st_2$.*

(iv) *Let $s, t, p \in S$. If $s\mathcal{R}t$ and $ps\mathcal{H}s$ then $pt\mathcal{H}t$.*

(v) *For every $s \in S$ the set sS is a union of $\mathcal{R}$-classes.*

The left-right dual statements hold for $\mathcal{L}$-classes.

Proof. Part (i) is [**11**, Lemma 2.2.1, Lemma 2.2.3]. Parts (ii), (iii) and (v) follow immediately from the definitions. For (iv), if $q \in S$ is such that $sq = t$, then the mapping $x \mapsto xq$ is a bijection from the $\mathcal{H}$-class of s onto the $\mathcal{H}$-class of t by (i), and hence $pt = psq\mathcal{H}t$. ☐

We remark that, unlike $\mathcal{R}$ and $\mathcal{L}$, the relation $\mathcal{H}$ is not, in general, a one sided congruence.

Theorem 1.1 concerns monoids S with finitely many left and right ideals. Since the $\mathcal{R}$-classes of S are in a one-one correspondence with the principal right ideals of S, and since every right ideal of S is a union of principal right ideals of S, we have:

Proposition 2.2. *A monoid has finitely many right ideals if and only if it has finitely many $\mathcal{R}$-classes. Dually, a monoid has finitely many left ideals if and only if it has finitely many $\mathcal{L}$-classes. A monoid has finitely many left and right ideals if and only if it has finitely many $\mathcal{H}$-classes.*

Schützenberger groups. The $\mathcal{H}$-classes in a monoid S exhibit many properties of subgroups of groups. For example, Proposition 2.1 (i) shows that the $\mathcal{H}$-classes within a single $\mathcal{R}$-class behave very much like cosets of a subgroup in a group – a parallel that will be explored in more depth in Sections 4-6. Also it is known that an $\mathcal{H}$-class which contains an idempotent is a maximal subgroup of S, and that all maximal subgroups of S arise in this way; see [**12**, Corollary 2.6].

Schützenberger [**25**, **26**] showed how to assign a group to an arbitrary $\mathcal{H}$-class, so as to reflect the group-like properties of that class. Here we give his construction and some of its basic properties. For more details we refer the reader to [**12**].

Let S be a monoid, and let H be an $\mathcal{H}$-class of S. Denote by $\mathrm{Stab}(H)$ the (right) stabiliser of H in S, i.e., $\mathrm{Stab}(H) = \{s \in S \ : \ Hs = H\}$. On this set define a relation $\sigma(H) = \{(s,t) \in \mathrm{Stab}(H) \times \mathrm{Stab}(H) \ : \ (\forall h \in H)(hs = ht)\}$. It is easy to see that $\sigma(H)$ is a congruence; we call it the *Schützenberger congruence* of H. It is also relatively easy to see that the quotient $\Gamma(H) = \mathrm{Stab}(H)/\sigma(H)$ is a group; it is called the *Schützenberger group* of H. It turns out that $\Gamma(H)$ has the following properties:

- $\Gamma(H)$ acts regularly on H; in particular $|H| = |\Gamma(H)|$;
- if H_1 is an $\mathcal{H}$-class of S belonging to the same $\mathcal{R}$-class, or the same $\mathcal{L}$-class, as H then $\Gamma(H_1) \cong \Gamma(H)$;
- if H contains an idempotent then $\Gamma(H) \cong H$.

For proofs see [**12**, Section 2.3]. Of course, by left-right duality, one may define the *left* Schützenberger group. It turns out, however, that the two are isomorphic.

In the following proposition we list some properties that we will use later. For proofs the reader is again referred to [**12**, Section 2.3].

Proposition 2.3. *Let S be a monoid, let H be an $\mathcal{H}$-class of S, and let $h_0 \in H$ be an arbitrary element. Then:*

(i) $\mathrm{Stab}(H) = \{s \in S \;:\; h_0 s \mathcal{H} h_0\}$;

(ii) $\sigma(H) = \{(s,t) \in \mathrm{Stab}(H) \times \mathrm{Stab}(H) \;:\; h_0 s = h_0 t\}$;

(iii) $H = h_0 \mathrm{Stab}(H)$.

Presentations. Along with transformations, presentations are the most general means of constructing monoids. Throughout the development of the theory of monoid presentations, one of the leitmotivs has been the connection with group presentations; see, for example, [**1**, **4**, **18**, **19**, **21**, **24**]. The results of this paper continue and deepen this theme.

A (monoid) *presentation* is a pair $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$, where A is an alphabet, and $\mathfrak{R} \subseteq A^* \times A^*$ is a set of pairs of words over A. A typical pair $(u,v) \in \mathfrak{R}$ is usually written as $u = v$ and is called a *defining relation*. A monoid S is said to be *defined* by $\mathfrak{P}$ if $S \cong A^*/\rho$, where ρ is the smallest congruence on the free monoid A^* containing $\mathfrak{R}$. Thus every word $w \in A^*$ represents an element of S. As is customary, we identify a word and the element of S it represents. To lessen the likelihood of confusion in doing so, for two words $w_1, w_2 \in A^*$ we write $w_1 \equiv w_2$ if they are identical, and $w_1 = w_2$ if they represent the same element of S, i.e., if $w_1/\rho = w_2/\rho$.

For two words $w_1, w_2 \in A^*$ we say that w_2 is *obtained from w_1 by one application of a relation* from $\mathfrak{R}$ if $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$, where $\alpha, \beta \in A^*$ and $(u = v) \in \mathfrak{R}$ or $(v = u) \in \mathfrak{R}$. We shall often use the following standard fact without explicit mention:

Proposition 2.4. *Let $\langle A \mid \mathfrak{R} \rangle$ be a presentation, let S be the monoid defined by it, and let $w_1, w_2 \in A^*$ be two arbitrary words. Then the relation $w_1 = w_2$ holds in S if and only if $w_1 \equiv w_2$ or there exists a sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_m \equiv w_2$ of words in which each α_{i+1} $(1 \le i \le m-1)$ is obtained from α_i by one application of a relation from $\mathfrak{R}$.*

3. From the Schützenberger groups to the monoid.

In this section we show how one can combine presentations of the Schützenberger groups of a monoid to obtain a presentation for the whole monoid. An immediate corollary of this result is the converse part of Theorem 1.1.

Let S be an arbitrary monoid, and let $S/\mathcal{H} = \{H_i \;:\; i \in I\}$ be the collection of all $\mathcal{H}$-classes of S. For each $i \in I$ fix an element

$$h_i \in H_i. \tag{1}$$

Without loss of generality assume that I contains a distinguished element 1 and that

$$1_S = h_1 \in H_1. \tag{2}$$

(In other words, H_1 is the group of units of S.) For $i \in I$ let $\Gamma_i = \Gamma(H_i)$ be the Schützenberger group of H_i. Recall that $\Gamma_i = T_i/\sigma_i$, where $T_i = \mathrm{Stab}(H_i)$ is the stabiliser of H_i, and $\sigma_i = \sigma(H_i)$ is the corresponding Schützenberger congruence.

Proposition 3.1. *With the above notation, if each H_i is generated by a set A_i/σ_i ($i \in I$, $A_i \subseteq S$) then S is generated by the set $B = \{h_i : i \in I\} \cup (\bigcup_{i\in I} A_i)$.*

Proof. Let $s \in S$ be arbitrary. Then there is a unique $i \in I$ such that $s \in H_i$, i.e., $s = h_i t$ for some $t \in T_i$ by Proposition 2.3 (iii). Hence we have $t/\sigma_i \in \Gamma_i = \langle A_i/\sigma_i\rangle = \langle A_i\rangle/\sigma_i$, so that there exists $w \in \langle A_i\rangle$ such that $(t, w) \in \sigma_i$. Now we have $s = h_i t = h_i w \in \langle B\rangle$, completing the proof. □

Our aim now is to find a presentation for S in terms of the generating set B given above. The idea is to note from the above proof that every element of S can be written in the form $h_i w$ ($i \in I$, $w \in A_i^*$), and to find defining relations which allow one to transform any word from B^* into this form. To do this we consider the results of multiplying representatives h_i ($i \in I$) of $\mathcal{H}$-classes by arbitrary generators from B both from left and right.

First for each $i \in I$ and each $x \in B$ we let $\zeta(i, x) \in I$ be the unique element such that

$$h_i x \in H_{\zeta(i,x)}. \tag{3a}$$

By Proposition 2.3 (iii) it follows that $h_i x = h_{\zeta(i,x)} s$ for some $s \in T_{\zeta(i,x)}$. From $s/\sigma_{\zeta(i,x)} \in \Gamma_{\zeta(i,x)} = \langle A_{\zeta(i,x)}\rangle/\sigma_{\zeta(i,x)}$ it follows that there exists $w \in A^*_{\zeta(i,x)}$ such that $s/\sigma_{\zeta(i,x)} = w/\sigma_{\zeta(i,x)}$. For each choice of i and x we choose (arbitrarily) and fix one such word $w = \mu(i, x)$. Thus we have

$$\mu(i, x) \in A^*_{\zeta(i,x)} \tag{3b}$$

and the relation

$$h_i x = h_{\zeta(i,x)}\mu(i, x) \quad (i \in I,\ x \in B) \tag{3}$$

holds in S. In a similar way, for any $i \in I$, $x \in B$ we let

$$\eta(i, x) \in I,\ \nu(i, x) \in A^*_{\eta(i,x)} \tag{4a}$$

be such that the relation

$$x h_i = h_{\eta(i,x)}\nu(i, x) \quad (i \in I,\ x \in B) \tag{4}$$

holds in S, and we also let

$$\theta(i, x) \in I,\ \pi(i, x) \in B^* \tag{5a}$$

be such that

$$h_i x \in H_{\theta(i,x)} \tag{5b}$$

and the relation

$$h_i x = \pi(i,x) h_{\theta(i,x)} \;\; (i \in I,\ x \in B) \tag{5}$$

holds in S.

Theorem 3.2. *If, with the above notation, each Schützenberger group Γ_i $(i \in I)$ is defined by a presentation $\langle A_i \mid \mathfrak{R}_i \rangle$ in terms of generators A_i/σ_i, then S is defined by the presentation with generators $B = \{h_i : i \in I\} \cup (\bigcup_{i \in I} A_i)$ and relations (3), (4), (5) and*

$$h_i u = h_i v \;\; (i \in I,\ (u = v) \in \mathfrak{R}_i), \tag{6}$$

$$h_1 = 1. \tag{7}$$

Proof. First we note that all the relations obviously hold in S. So to prove the theorem it is sufficient to show that every relation holding in S is a consequence of the relations (3)-(7).

To this end we let $w \in B^*$ be arbitrary. Via a series of claims we show that w can be transformed to a particular form using (3)-(7).

Claim 1. There exist $i_1 \in I$ and $w_1 \in B^*$ such that the following two conditions are satisfied:

(i) the relation $w = w_1 h_{i_1}$ is a consequence of the relations (3)-(7); and

(ii) for every suffix w_1' of w we have $w_1' h_{i_1} \mathcal{L} h_{i_1}$ in S.

Proof. Write $w \equiv x_1 x_2 \dots x_m$ $(x_i \in B)$. From (7) we have

$$w = h_1 x_1 x_2 \dots x_m.$$

By successively applying relations (5) we obtain

$$h_1 x_1 x_2 \dots x_m = \pi(j_1, x_1)\pi(j_2, x_2) \dots \pi(j_m, x_m) h_{j_{m+1}},$$

where

$$j_1 = 1,\ j_{k+1} = \theta(j_k, x_k) \;\; (k = 1, \dots, m).$$

So if we let $w_1 \equiv \pi(j_1, x_1) \dots \pi(j_m, x_m)$ and $i_1 = j_{m+1}$, the condition (i) is satisfied. From (5b) we have

$$x_1 = h_{j_1} x_1 \mathcal{L} h_{j_2},\ h_{j_2} x_2 \mathcal{L} h_{j_3}, \dots, h_{j_m} x_m \mathcal{L} h_{j_{m+1}} = h_{i_1}.$$

Since $\mathcal{L}$ is a right congruence (the dual of Proposition 2.1 (iii)) it follows that

$$w_1 h_{i_1} = w \equiv x_1 x_2 \dots x_m \mathcal{L} h_{j_2} x_2 \dots x_m \mathcal{L} \dots \mathcal{L} h_{j_m} x_m \mathcal{L} h_{j_{m+1}} = h_{i_1}.$$

Therefore, by the dual of Proposition 2.1 (ii), it follows that the condition (ii) is satisfied as well. □

Claim 2. There exist $i_2 \in I$ and $w_2 \in B^*$ such that the following two conditions are satisfied:

(i) the relation $w_1 h_{i_1} = h_{i_2} w_2$ is a consequence of the relations (3)-(7); and

(ii) for every letter x of w_2 we have $h_{i_2} x \in H_{i_2}$.

Proof. Write $w_1 \equiv x_1 \dots x_m$ and apply (4) successively to obtain

$$w_1 h_{i_1} = h_{j_0} \nu(j_1, x_1) \dots \nu(j_m, x_m),$$

where

$$j_m = i_1,\ j_{k-1} = \eta(j_k, x_k)\ \ (k = m, \dots , 1).$$

So, if we let $i_2 = j_0$ and $w_2 \equiv \nu(j_1, x_1) \dots \nu(j_m, x_m)$, the condition (i) is satisfied.

We next claim that for every k $(1 \leq k \leq m)$ we have

$$x_k x_{k+1} \dots x_m h_{i_1} \in H_{j_{k-1}}. \tag{8}$$

For $k = m$ this follows from (4a) and (4). Assume inductively that (8) holds for some k. Since, by Claim 1 (ii), we have

$$x_{k-1} x_k \dots x_m h_{i_1} \mathcal{L} h_{i_1} \mathcal{L} x_k \dots x_m h_{i_1},$$

it follows by (the dual of) Proposition 2.1 (i) that the mapping $t \mapsto x_{k-1} t$ is a bijection from the $\mathcal{H}$-class of $x_k \dots x_m h_{i_1}$ onto that of $x_{k-1} x_k \dots x_m h_{i_1}$. In particular, we have

$$x_{k-1} x_k \dots x_m h_{i_1} \mathcal{H} x_{k-1} h_{j_{k-1}} \mathcal{H} h_{j_{k-2}}$$

by (4a) and (4), thus completing the inductive proof of (8).

By Claim 1 (ii) we now conclude that

$$h_{i_2} = h_{j_0} \mathcal{L} h_{j_1} \mathcal{L} \dots \mathcal{L} h_{j_m} = h_{i_1}. \tag{9}$$

By (4a) we have $\nu(j_k, x_k) \in A^*_{j_{k-1}}$ $(k = 1, \dots , m)$. By the choice of $A_{j_{k-1}}$ every letter from it stabilises $H_{j_{k-1}}$. Therefore by the dual of Proposition 2.1 (iv), Proposition 2.3 (i) and (9) it follows that every letter x of $\nu(j_k, x_k)$ stabilises H_{i_2}; in particular, $h_{i_2} x \in H_{i_2}$, as required. □

Claim 3. There exist $i_3 \in I$ and $w_3 \in A^*_{i_3}$ such that the relation $h_{i_2} w_2 = h_{i_3} w_3$ is a consequence of the relations (3)-(7).

Proof. Write $w_2 \equiv x_1 \dots x_m$, and note that

$$\zeta(i_2, x_j) = \theta(i_2, x_j) = i_2\ \ (j = 1, \dots , m)$$

by (3a), (5b) and Claim 2 (ii). Therefore we have

$$\begin{array}{rcll} h_{i_2}w_2 & \equiv & h_{i_2}x_1\dots x_m & \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-1})h_{i_2}x_m & \text{(by (5))} \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-1})h_{i_2}\mu(i_2,x_m) & \text{(by (3))} \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-2})h_{i_2}x_{m-1}\mu(i_2,x_m) & \text{(by (5))} \\ & = & \pi(i_2,x_1)\dots\pi(i_2,x_{m-2})h_{i_2}\mu(i_2,x_{m-1})\mu(i_2,x_m) & \text{(by (3))} \\ & = & \dots & \\ & = & h_{i_2}\mu(i_2,x_1)\mu(i_2,x_2)\dots\mu(i_2,x_m). & \end{array}$$

Therefore it is sufficient to let $i_3 = i_2$ and $w_3 \equiv \mu(i_2,x_1)\dots\mu(i_2,x_m)$. □

Now let $w' \in B^*$ be any word, and assume that the relation $w = w'$ holds in S. Write w' as $w' = h_{i'_3}w'_3$ as above. Then, since $h_{i_3}\mathcal{H}w = w'\mathcal{H}h_{i'_3}$, we must have $i_3 = i'_3 = i$ and also $w_3/\sigma_i = w'_3/\sigma_i$ in Γ_i by Proposition 2.3 (ii). Since $\langle A_i \mid \mathfrak{R}_i\rangle$ is a presentation for Γ_i, it follows that w'_3 can be obtained from w_3 by a sequence of applications of relations from $\mathfrak{R}_i$. We are now going to show that this implies that $h_iw'_3$ can be obtained from h_iw_3 by a sequence of applications of relations (3)-(7), which will complete the proof that we indeed have a presentation for S.

Without loss of generality we may assume that w'_3 is obtained from w_3 by one application of a relation from $\mathfrak{R}_i$:

$$w_3 \equiv \alpha u\beta,\ w'_3 \equiv \alpha v\beta\ \ (\alpha,\beta\in A_i^*,\ (u=v)\in\mathfrak{R}_i).$$

Writing $\alpha \equiv x_1\dots x_m$, we have

$$\begin{array}{rcll} h_iw_3 & \equiv & h_ix_1\dots x_mu\beta & \\ & = & \pi(i,x_1)\dots\pi(i,x_m)h_iu\beta & \text{(by (5))} \\ & = & \pi(i,x_1)\dots\pi(i,x_m)h_iv\beta & \text{(by (6))} \\ & = & h_ix_1\dots x_mv\beta \equiv h_iw'_3, & \text{(by (5))} \end{array}$$

as required. □

If the set I is finite (which, by Proposition 2.2 is the case precisely when S has finitely many left and right ideals), and if all the presentations $\langle A_i \mid \mathfrak{R}_i\rangle$ $(i\in I)$ are finite, then so is the above presentation for S. Therefore we have the converse part of Theorem 1.1:

Corollary 3.3. *Let S be a monoid with finitely many left and right ideals. If all the Schützenberger groups of S are finitely presented then S is finitely presented as well.*

4. A rewriting theorem for the Schützenberger group.

The aim of this section is to state a theorem (Theorem 4.2) giving a presentation for the Schützenberger group of an $\mathcal{H}$-class in a monoid defined by a presentation, and to deduce some immediate corollaries, including the direct part of Theorem 1.1. The theorem is proved in the next section.

Let S be a monoid, and let $\langle A \mid \mathfrak{R}\rangle$ be a presentation for S. Let H be an arbitrary $\mathcal{H}$-class of S, and fix a word $h \in A^*$ representing an element of H. Denote by $\Gamma = \Gamma(H)$ the Schützenberger group of H; so $\Gamma = T/\sigma$, where $T = \mathrm{Stab}(H)$ and $\sigma = \sigma(H)$ is the Schützenberger congruence on T.

Let R be the $\mathcal{R}$-class of h, and let $\{H_\lambda \,:\, \lambda \in \Lambda\}$ be the collection of all $\mathcal{H}$-classes of S contained in R. For each $\lambda \in \Lambda$ choose words $p_\lambda, p'_\lambda \in A^*$ such that

$$Hp_\lambda = H_\lambda,\ h_1 p_\lambda p'_\lambda = h_1,\ h_2 p'_\lambda p_\lambda = h_2\ \ (\lambda \in \Lambda,\ h_1 \in H,\ h_2 \in H_\lambda); \tag{10}$$

such words exist by Proposition 2.1 (i). Without loss of generality assume that Λ contains a distinguished element 1, and that

$$H_1 = H,\ p_1 \equiv p'_1 \equiv \epsilon, \tag{11}$$

where ϵ denotes the empty word.

By Proposition 2.1 (i), (ii), for any $s \in S$ and any $\lambda \in \Lambda$, either $H_\lambda s = H_\mu$ for some $\mu \in \Lambda$, or $H_\lambda s s_1 \cap R = \emptyset$ for all $s_1 \in S$. Therefore we can define an action $(\lambda, s) \mapsto \lambda \cdot s$ of S on the set $\Lambda \cup \{0\}$ (assuming $0 \notin \Lambda$) by

$$\lambda \cdot s = \begin{cases} \mu & \text{if } \lambda, \mu \in \Lambda \text{ and } H_\lambda s = H_\mu, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

In the following theorem we give a generating set for Γ, resembling the Schreier generating set for a subgroup of a group (see [**17**, Theorem 2.7]). This result is not new – it is an immediate consequence of Schützenberger's original results [**25**, **26**], and can also be found, in a slightly different notation from ours, in [**14**, Corollary 2.3]. Nevertheless, we will give a proof of this result, because it motivates the definition of a rewriting mapping to follow.

Proposition 4.1. *With the above notation the Schützenberger group Γ of H is generated by the set*

$$X = \{(p_\lambda a p'_{\lambda\cdot a})/\sigma \ :\ \lambda \in \Lambda,\ a \in A,\ \lambda \cdot a \neq 0\}.$$

Proof. First we claim that

$$\Gamma = \{(p_\lambda s p'_{\lambda\cdot s})/\sigma \ :\ \lambda \in \Lambda,\ s \in S,\ \lambda \cdot s \neq 0\}.$$

Denote the right hand side by Γ'. By using (10) and (12) we have

$$Hp_\lambda s p'_{\lambda\cdot s} = H_\lambda s p'_{\lambda\cdot s} = H_{\lambda\cdot s} p'_{\lambda\cdot s} = H.$$

Hence $p_\lambda s p'_{\lambda\cdot s} \in T$, so that Γ' is well defined and $\Gamma' \subseteq \Gamma$. Conversely, if $s/\sigma \in \Gamma$, then from $Hs = H$ it follows that $1 \cdot s = 1$, and hence $s/\sigma = (p_1 s p'_1)/\sigma \in \Gamma'$.

To complete the proof of the proposition we show that an arbitrary element $(p_\lambda s p'_{\lambda\cdot s})/\sigma$ of Γ' can be written as a product of elements of X. We write $s \equiv a_1 \dots a_m$ $(a_i \in A)$ and proceed by induction on m. For $m = 0$

there is nothing to prove, and for $m = 1$ we have an element of X. For $m > 1$ write $a = a_1$, $t \equiv a_2 \dots a_m$; we have

$$\begin{array}{lll} & (p_\lambda s p'_{\lambda\cdot s})/\sigma & \\ = & (p_\lambda a t p'_{\lambda\cdot at})/\sigma & \\ = & (p_\lambda a p'_{\lambda\cdot a} p_{\lambda\cdot a} t p'_{(\lambda\cdot a)\cdot t})/\sigma & \text{(by (10))} \\ = & ((p_\lambda a p'_{\lambda\cdot a})/\sigma)((p_{\lambda\cdot a} t p'_{(\lambda\cdot a)\cdot t})/\sigma) & \text{(since } p_\lambda a p'_{\lambda\cdot a}, p_{\lambda\cdot a} t p'_{(\lambda\cdot a)\cdot t} \in T) \\ \in & \langle X\rangle, & \text{(by induction)} \end{array}$$

completing the proof. □

We are going to find a presentation for Γ in terms of the above generating set X. To this end we introduce a new alphabet

$$B = \{b(\lambda, a) \ : \ \lambda \in \Lambda,\ a \in A,\ \lambda \cdot a \neq 0\}. \tag{13}$$

The letter $b(\lambda, a)$ is thought of as representing the generator $(p_\lambda a p'_{\lambda\cdot a})/\sigma$. To make this more formal we introduce a homomorphism

$$\psi \ : \ B^* \longrightarrow A^*,\ b(\lambda, a) \mapsto p_\lambda a p'_{\lambda\cdot a}; \tag{14}$$

we refer to ψ as the *representation mapping*.

Motivated by the proof of Proposition 4.1 we also define a mapping

$$\phi \ : \ \{(\lambda, w) \in \Lambda \times A^* \ : \ \lambda \cdot w \neq 0\} \longrightarrow B^*,$$

called the *rewriting mapping*, inductively by

$$\phi(\lambda, \epsilon) = \epsilon,\ \phi(\lambda, aw) = b(\lambda, a)\phi(\lambda \cdot a, w). \tag{15}$$

The idea behind this definition is that ϕ simulates the process of rewriting an element $p_\lambda s p'_{\lambda\cdot s}$ into a product of generators from X as in the proof of Proposition 4.1.

Since for each $\lambda \in \Lambda$, $a \in A$ satisfying $\lambda \cdot a \neq 0$ we have $h p_\lambda a p'_{\lambda\cdot a} \in H$, it follows that there is a word $\pi(b(\lambda, a)) \in A^*$ such that

$$h p_\lambda a p'_{\lambda\cdot a} = \pi(b(\lambda, a)) h. \tag{16}$$

We extend the mapping $b(\lambda, a) \mapsto \pi(b(\lambda, a))$ to a homomorphism $\pi \ : B^* \longrightarrow A^*$.

Recall that the relation $\mathcal{R}$ is a left congruence on S. Therefore there is a natural left action $(s, R') \mapsto s * R'$ of S on the set $S/\mathcal{R}$ of all $\mathcal{R}$-classes given by

$$s * R' = R'' \Longleftrightarrow sR' \subseteq R'' \ \ (s \in S,\ R', R'' \in S/\mathcal{R}). \tag{17}$$

Let $\{R_i \ : \ i \in I\}$ be the inverse orbit of R under this action, i.e., let it be the set $\{R' \in S/\mathcal{R} \ : \ (\exists s \in S)(s * R' = R)\}$. Then the action of S on $S/\mathcal{R}$

induces a partial action on $\{R_i : i \in I\}$, which, in turn, translates into an action $(s,i) \mapsto s * i$ of S on the set $I \cup \{0\}$ (assuming $0 \notin I$) given by

$$s * i = \begin{cases} j & \text{if } i, j \in I \text{ and } s * R_i = R_j, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

For each $i \in I$ choose a word $r_i \in A^*$ representing an element of R_i. Without loss of generality assume that I contains two distinguished elements 1 and ω, and that

$$1_S \in R_1,\ r_1 \equiv \epsilon,\ R = R_\omega,\ r_\omega \equiv h. \tag{19}$$

For each $a \in A$ and each $i \in I$ such that $a * i \neq 0$ we have $ar_i \in a * R_i = R_{a*i}$ by (17) and (18). Therefore we can choose words $\tau(a,i) \in A^*$ such that the relations

$$ar_i = r_{a*i}\tau(a,i)\ \ (a \in A,\ i \in I) \tag{20}$$

hold in S. We extend the mapping $(a,i) \mapsto \tau(a,i)$ to a mapping

$$\tau\ :\ \{(w,i) \in A^* \times I\ :\ w * i \neq 0\} \longrightarrow A^*$$

inductively by

$$\tau(\epsilon,i) = \epsilon,\ \tau(wa,i) = \tau(w, a * i)\tau(a,i). \tag{21}$$

We can now formulate the result giving a presentation for Γ.

Theorem 4.2. *With the above notation the Schützenberger group Γ of H is defined by the presentation with generators B and relations*

(22) $\phi(\lambda,u) = \phi(\lambda,v)\ \ (\lambda \in \Lambda,\ (u = v) \in \mathfrak{R},\ \lambda \cdot u \neq 0)$,

(23) $\phi(\lambda,\tau(u,i)) = \phi(\lambda,\tau(v,i))\ \ (\lambda \in \Lambda,\ i \in I,\ (u = v) \in \mathfrak{R},\ H_\lambda \subseteq Sr_{u*i})$,

(24) $\phi(1,\tau(\pi(b(\lambda,a)),\omega)) = b(\lambda,a)\ \ (\lambda \in \Lambda,\ a \in A,\ \lambda \cdot a \neq 0)$,

(25) $\phi(1,\tau(h,1)) = 1$.

The above presentation is finite, provided that A, $\mathfrak{R}$, Λ and I are all finite. Therefore we have:

Corollary 4.3. *Let S be a finitely presented monoid, and let H be an $\mathcal{H}$-class of S such that the following two conditions are satisfied:*

(i) *the $\mathcal{R}$-class R of H has only finitely many $\mathcal{H}$-classes; and*

(ii) *the inverse orbit of R under the left action of S on its $\mathcal{R}$-classes has only finitely many elements.*

Then the Schützenberger group of H is finitely presented.

Bearing in mind Proposition 2.2, we obtain the direct part of Theorem 1.1 as a special case of Corollary 4.3:

Corollary 4.4. *If S is a finitely presented monoid with finitely many left and right ideals, then all Schützenberger groups of S are finitely presented.*

In conclusion to this section, we emphasise again the significant role of the rewriting mapping ϕ in the above results. This notion is standard in combinatorial group theory, in the context of Reidemeister-Schreier theory; see [**17**]. It has proved equally important in the theory of monoid and semigroup presentations, in which it occurs in a variety of contexts and takes on a number of different forms; see [**2**, **5**, **6**, **7**, **22**, **23**].

5. Proof of the rewriting theorem.

We prove Theorem 4.2 by showing that all the relations (22)-(25) hold in the Schützenberger group Γ (Lemmas 5.5-5.8) and that any other relation which holds in Γ is a consequence of these relations (Lemma 5.11). Since the argument contains a considerable amount of technical detail, we break it up into a number of lemmas. We use the notation introduced in the previous section.

We begin by giving some properties of the rewriting mapping ϕ.

Lemma 5.1. (i) *For every $w_1, w_2 \in A^*$ and every $\lambda \in \Lambda$ such that $\lambda \cdot w_1w_2 \neq 0$ we have*

$$\phi(\lambda, w_1w_2) \equiv \phi(\lambda, w_1)\phi(\lambda \cdot w_1, w_2).$$

(ii) *For every $w \in A^*$ and every $\lambda \in \Lambda$ such that $\lambda \cdot w \neq 0$ the relation*

$$h\psi\phi(\lambda, w) = hp_\lambda w p'_{\lambda\cdot w}$$

holds in S.

(iii) *If $w_1, w_2 \in A^*$ are such that the relation $w_1 = w_2$ holds in S, and if $\lambda \in \Lambda$ is such that $\lambda \cdot w_1 \neq 0$, then the relation $\phi(\lambda, w_1) = \phi(\lambda, w_2)$ is a consequence of the relations (22)-(25).*

Proof. (i) The assertion is proved by a straightforward induction on the length of w_1, using (15).

(ii) This is proved by induction on the length of w, essentially repeating the proof of Proposition 4.1.

(iii) If $w_1 = w_2$ in S then there is a sequence $w_1 \equiv \alpha_1, \alpha_2, \dots, \alpha_n \equiv w_2$ of words from A^* in which each α_{i+1} is obtained from α_i by one application of a relation from $\mathfrak{R}$. If

$$\alpha_i \equiv \beta_i u_i \gamma_i,\ \alpha_{i+1} \equiv \beta_i v_i \gamma_i\ \ (i \in I,\ \beta_i, \gamma_i \in A^*,\ (u_i = v_i) \in \mathfrak{R})$$

then

$$\begin{array}{rcll} \phi(\lambda, \alpha_i) & \equiv & \phi(\lambda, \beta_i)\phi(\lambda \cdot \beta_i, u_i)\phi(\lambda \cdot \beta_i u_i, \gamma_i) & \text{(part (i))} \\ & = & \phi(\lambda, \beta_i)\phi(\lambda \cdot \beta_i, v_i)\phi(\lambda \cdot \beta_i u_i, \gamma_i) & \text{(by (22))} \\ & \equiv & \phi(\lambda, \beta_i)\phi(\lambda \cdot \beta_i, v_i)\phi(\lambda \cdot \beta_i v_i, \gamma_i) & \text{(since } u_i = v_i \text{ in } S) \\ & \equiv & \phi(\lambda, \alpha_{i+1}), & \text{(part (i))} \end{array}$$

a consequence of (22). Therefore $\phi(\lambda, w_2)$ can be obtained from $\phi(\lambda, w_1)$ by using relations (22). ∎

Next we prove two similar properties of the mapping τ.

Lemma 5.2. (i) *For every $w_1, w_2 \in A^*$ and $i \in I$ such that $w_1 w_2 * i \neq 0$ we have*

$$\tau(w_1 w_2, i) \equiv \tau(w_1, w_2 * i)\tau(w_2, i).$$

(ii) *For every $w \in A^*$ and $i \in I$ such that $w * i \neq 0$ the relation*

$$w r_i = r_{w*i}\tau(w, i)$$

holds in S.

Proof. (i) The assertion can be proved by a straightforward induction on the length of w_2, using (21).

(ii) We prove the statement by induction on the length of w. For $|w| = 0$ there is nothing to prove, and for $|w| = 1$ the statement is (20). Let $|w| > 1$ and write $w \equiv w_1 w_2$ with $|w_1|, |w_2| > 0$. By using (i) and induction we have

$$w r_i \equiv w_1 w_2 r_i = w_1 r_{w_2 * i}\tau(w_2, i) = r_{w_1 w_2 * i}\tau(w_1, w_2 * i)\tau(w_2, i) \equiv r_{w*i}\tau(w, i),$$

as required. □

The following lemma describes the connection between the mappings ψ and π.

Lemma 5.3. *For every $w \in B^*$ the relation*

$$h\psi(w) = \pi(w)h$$

holds in S.

Proof. The assertion follows from (14), (16) and the fact that both ψ and π are homomorphisms. □

Next we give two facts relating the mappings τ and π and the actions of S on the sets $I \cup \{0\}$ and $\Lambda \cup \{0\}$.

Lemma 5.4. *For every $w \in B^*$ we have*

(i) $\pi(w) * \omega = \omega$; *and*

(ii) $1 \cdot \tau(\pi(w), \omega) = 1$.

Proof. By Proposition 4.1, Lemma 5.3, (14) and (19) we have

$$\pi(w) r_\omega \equiv \pi(w)h = h\psi(w) \in H \subseteq R = R_\omega,$$

proving (i). Now, by Lemma 5.2 (ii) and (19), we have

$$h\tau(\pi(w), \omega) \equiv r_{\pi(w)*\omega}\tau(\pi(w), \omega) = \pi(w) r_\omega \equiv \pi(w)h = h\psi(w) \in H = H_1,$$

and (ii) follows. □

We can now proceed to prove that the relations (22)-(25) hold in Γ. Recall that a generator $b(\lambda, a)$ from B represents the element $(p_\lambda a p'_{\lambda\cdot a})/\sigma = \psi(b(\lambda, a))/\sigma$ of Γ. Therefore in order to verify that a relation $\alpha = \beta$ ($\alpha, \beta \in B^*$) holds in Γ one needs to verify that $\psi(\alpha)/\sigma = \psi(\beta)/\sigma$, or, equivalently, that $h\psi(\alpha) = h\psi(\beta)$ holds in S.

Lemma 5.5. *For every relation $(u = v) \in \mathfrak{R}$ and every $\lambda \in \Lambda$ such that $\lambda \cdot u \neq 0$ the relation*

$$\phi(\lambda, u) = \phi(\lambda, v)$$

holds in Γ.

Proof. Using Lemma 5.1 (ii) and the fact that $u = v$ in S we have

$$h\psi\phi(\lambda, u) = hp_\lambda u p'_{\lambda\cdot u} = hp_\lambda v p'_{\lambda\cdot v} \equiv h\psi\phi(\lambda, v),$$

as required. $\square$

Lemma 5.6. *For every relation $(u = v) \in \mathfrak{R}$ and every $\lambda \in \Lambda$ and $i \in I$ such that $H_\lambda \subseteq Sr_{u*i}$ the relation*

$$\phi(\lambda, \tau(u, i)) = \phi(\lambda, \tau(v, i))$$

holds in Γ.

Proof. Since $u = v$ holds in S it follows that $ur_i = vr_i$ also holds in S and that $u * i = v * i$. Therefore, by Lemma 5.2 (ii), the relation

$$r_{u*i}\tau(u, i) = r_{v*i}\tau(v, i) \tag{26}$$

holds in S. Let $q \in A^*$ be such that $hp_\lambda = qr_{u*i}$. Premultiplying (26) by q yields

$$hp_\lambda\tau(u, i) = hp_\lambda\tau(v, i); \tag{27}$$

in particular $\lambda \cdot \tau(u, i) = \lambda \cdot \tau(v, i)$. Now, using Lemma 5.1 and (27), we have

$$h\psi\phi(\lambda, \tau(u, i)) = hp_\lambda\tau(u, i)p'_{\lambda\cdot\tau(u,i)} = hp_\lambda\tau(v, i)p'_{\lambda\cdot\tau(v,i)} = h\psi\phi(\lambda, \tau(v, i)),$$

as required. $\square$

Lemma 5.7. *For every $a \in A$ and every $\lambda \in \Lambda$ such that $\lambda \cdot a \neq 0$ the relation*

$$\phi(1, \tau(\pi(b(\lambda, a)), \omega)) = b(\lambda, a)$$

holds in Γ.

Proof. This time we have

$$\begin{array}{lll} & h\psi\phi(1, \tau(\pi(b(\lambda, a)), \omega)) & \\ = & h\tau(\pi(b(\lambda, a)), \omega) & \text{(Lemmas 5.4 and 5.1 (ii) and (11))} \\ = & \pi(b(\lambda, a))r_\omega & \text{(Lemma 5.2 (ii) and (19))} \\ = & h\psi(b(\lambda, a)), & \text{(Lemma 5.3 and (19))} \end{array}$$

as required. $\square$

Lemma 5.8. *The relation*

$$\phi(1, \tau(h, 1)) = 1$$

holds in Γ.

Proof. From Lemma 5.2 (ii) and (19) we have

$$h \equiv hr_1 = r_{h*1}\tau(h, 1) \equiv r_\omega \tau(h, 1) \equiv h\tau(h, 1),$$

and hence $1 \cdot \tau(h, 1) = 1$. By Lemma 5.1 (ii) and (11) we now have

$$h\psi\phi(1, \tau(h, 1)) = h\tau(h, 1) = h \equiv h\psi(\epsilon),$$

completing the proof of the lemma. □

We now turn to the second part of the proof of Theorem 4.2, that is to show that every relation which holds in Γ is a consequence of the relations (22)-(25). The technical part of the argument is contained in the following two lemmas.

Lemma 5.9. *For any word* $w \in B^*$ *the relation*

$$\phi(1, \tau(\pi(w)h, 1)) = w$$

is a consequence of the relations (22)-(25).

Proof. We prove the lemma by induction on the length of w. If $|w| = 0$ then this is the relation (25), and if $|w| = 1$ this is one of the relations (24). Assume that $|w| > 1$ and write $w \equiv w_1 w_2$ with $|w_1|, |w_2| > 0$. Recall that π is a homomorphism, so that $\pi(w) \equiv \pi(w_1)\pi(w_2)$. Now we have

$$\begin{array}{cll}
 & \phi(1, \tau(\pi(w)h, 1)) & \\
\equiv & \phi(1, \tau(\pi(w_1), \omega)\tau(\pi(w_2), \omega)\tau(h, 1)) & \text{(Lemmas 5.2 (i) and 5.4)} \\
\equiv & \phi(1, \tau(\pi(w_1), \omega))\phi(1, \tau(\pi(w_2), \omega))\phi(1, \tau(h, 1)) & \\
 & & \text{(Lemmas 5.1 (i) and 5.4)} \\
= & \phi(1, \tau(\pi(w_1), \omega))\phi(1, \tau(h, 1))\phi(1, \tau(\pi(w_2), \omega))\phi(1, \tau(h, 1)) & \\
 & & \text{(by (25))} \\
\equiv & \phi(1, \tau(\pi(w_1)h, 1))\phi(1, \tau(\pi(w_2)h, 1)) & \text{(Lemmas 5.1 (i), 5.2 (i), 5.4)} \\
= & w_1 w_2 \equiv w, & \text{(induction)}
\end{array}$$

as required. □

Lemma 5.10. *Let* $\alpha, \beta \in A^*$ *and* $(u = v) \in \mathfrak{R}$ *be such that* $\alpha u \beta$ *represents an element of* R. *Then the relation*

$$\phi(1, \tau(\alpha u \beta, 1)) = \phi(1, \tau(\alpha v \beta, 1))$$

is a consequence of the relations (22)-(25).

Proof. Since $u = v$ in S it follows that $u\beta * 1 = v\beta * 1$, and hence

$$\phi(1, \tau(\alpha, u\beta * 1)) \equiv \phi(1, \tau(\alpha, v\beta * 1)). \tag{28}$$

Next we claim that

$$\phi(1\cdot\tau(\alpha,u\beta*1),\tau(u,\beta*1))=\phi(1\cdot\tau(\alpha,v\beta*1),\tau(v,\beta*1)) \tag{29}$$

is one of the relations (23). Indeed, from $\alpha u\beta\in R$, $u\beta\in R_{u\beta*1}$ and Proposition 2.1 (iii), it follows that $\alpha r_{u\beta*1}\in R$. Next, by Lemma 5.2 (ii) and (19), in S we have

$$\alpha r_{u\beta*1}=r_{\alpha u\beta*1}\tau(\alpha,u\beta*1)\equiv r_\omega\tau(\alpha,u\beta*1)=h\tau(\alpha,u\beta*1).$$

We conclude that $1\cdot\tau(\alpha,u\beta*1)\neq 0$ and $\alpha r_{u\beta*1}\in H_{1\cdot\tau(\alpha,u\beta*1)}\cap Sr_{u\beta*1}$. By the dual of Proposition 2.1 (v) we conclude that $H_{1\cdot\tau(\alpha,u\beta*1)}\subseteq Sr_{u\beta*1}$, which is precisely the condition for (29) to be one of the relations (23).

Finally, by Lemma 5.2 (ii) and (19) in S we have

$$h\tau(\alpha u,\beta*1)\equiv r_{\alpha u\beta*1}\tau(\alpha u,\beta*1)=\alpha ur_{\beta*1}=\alpha vr_{\beta*1}=h\tau(\alpha v,\beta*1).$$

Therefore $1\cdot\tau(\alpha u,\beta*1)=1\cdot\tau(\alpha v,\beta*1)$, and hence

$$\phi(1\cdot\tau(\alpha u,\beta*1),\tau(\beta,1))\equiv\phi(1\cdot\tau(\alpha v,\beta*1),\tau(\beta,1)). \tag{30}$$

Using Lemmas 5.1 (i) and 5.2 (i) and (28), (29), (30) we obtain

$$\begin{aligned}
&\phi(1,\tau(\alpha u\beta,1))\\
\equiv\ &\phi(1,\tau(\alpha,u\beta*1)\tau(u,\beta*1)\tau(\beta,1))\\
\equiv\ &\phi(1,\tau(\alpha,u\beta*1))\phi(1\cdot\tau(\alpha,u\beta*1),\tau(u,\beta*1))\\
&\quad\cdot\phi(1\cdot\tau(\alpha,u\beta*1)\tau(u,\beta*1),\tau(\beta,1))\\
\equiv\ &\phi(1,\tau(\alpha,u\beta*1))\phi(1\cdot\tau(\alpha,u\beta*1),\tau(u,\beta*1))\\
&\quad\cdot\phi(1\cdot\tau(\alpha u,\beta*1),\tau(\beta,1))\\
=\ &\phi(1,\tau(\alpha,v\beta*1))\phi(1\cdot\tau(\alpha,v\beta*1),\tau(v,\beta*1))\\
&\quad\cdot\phi(1\cdot\tau(\alpha v,\beta*1),\tau(\beta,1))\\
\equiv\ &\phi(1,\tau(\alpha v\beta,1)),
\end{aligned}$$

as a consequence of (22)-(25). □

Lemma 5.11. *If $w_1,w_2\in B^*$ are any two words such that $w_1=w_2$ holds in Γ then the relation $w_1=w_2$ is a consequence of the relations (22)-(25).*

Proof. The assumption that $w_1=w_2$ holds in Γ is equivalent to the relation $h\psi(w_1)=h\psi(w_2)$ holding in S, which is, in turn, equivalent to the relation $\pi(w_1)h=\pi(w_2)h$ by Lemma 5.3. Therefore there is a sequence of words $\pi(w_1)h\equiv\gamma_1,\gamma_2,\dots,\gamma_n\equiv\pi(w_2)h$ from A^* such that each γ_{i+1} is obtained from γ_i by one application of a relation from $\mathfrak{R}$. By Lemma 5.10 we have that each relation $\phi(1,\tau(\gamma_i,1))=\phi(1,\tau(\gamma_{i+1},1))$ is a consequence of the relations (22)-(25). Therefore the relation

$$\phi(1,\tau(\pi(w_1)h,1))=\phi(1,\tau(\pi(w_2)h,1)) \tag{31}$$

is a consequence of the relations (22)-(25). By Lemma 5.9 the relations

$$w_j = \phi(1, \tau(\pi(w_j)h, 1)) \ \ (j = 1, 2) \tag{32}$$

are also consequences of the relations (22)-(25). Combining (31) and (32) we conclude that $w_1 = w_2$ is a consequence of the relations (22)-(25). □

The proof of Theorem 4.2 is now complete.

6. Example: A non-finitely presented Schützenberger group.

Recall that if an $\mathcal{H}$-class of a monoid S contains an idempotent then H is a maximal subgroup of S and $H \cong \Gamma(H)$. Therefore, in this case, a presentation for $\Gamma(H)$ can be obtained from [**24**, Theorem 2.9]. In particular, by [**24**, Corollary 2.11], we have that $\Gamma(H)$ is finitely presented provided that S is finitely presented and the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes. Comparing this to Corollary 4.3, we see that, in this case, condition (ii) is not needed. So one may ask whether the same is true in general. In this section we present an example which answers this question in negative. More precisely, we are going to construct a finitely presented monoid which contains an $\mathcal{H}$-class H which is the only $\mathcal{H}$-class in its $\mathcal{R}$-class, but $\Gamma(H)$ is not finitely presented. This difference in behaviour between the group and non-group $\mathcal{H}$-classes is relatively surprising, because Schützenberger groups usually have the same properties as maximal subgroups. For instance, the generation theorems for the two are essentially identical (compare Proposition 4.1 and [**24**, Theorem 2.7]), leading to the same rewriting mapping (compare (15) with [**24**, Equality (2)]), and also they satisfy the same global results with respect to finite presentability (compare Theorem 1.1 and [**24**, Theorem 4.1]). It is also worth pointing out that the presentation for $\Gamma(H)$ we have obtained here essentially contains the presentation for H from [**24**, Theorem 2.9] – the relations (22), (24) and (25) correspond to [**24**, (3), (4), (5)] respectively.

Let A be the alphabet

$$A = \{a_1, a_2, a_3, a_4, a_1', a_2', a_3', a_4', b, c, d\},$$

and consider the presentation

$$\mathfrak{P} = \langle A \mid a_j a_j' = a_j' a_j = \epsilon,\ a_1 a_2 = a_3 a_4,\ a_j b = b a_j^2,\ cb^2 = cb,\ a_j d = d a_j,$$
$$cbda_j = a_j cbd \ \ (j = 1, 2, 3, 4)\rangle.$$

Let S be the monoid defined by $\mathfrak{P}$, and let H be the $\mathcal{H}$-class of $h \equiv cbd$. First we are going to show that H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class, and then we use our rewriting theorem (Theorem 4.2) to find a presentation for $\Gamma(H)$ and show that it is not finitely presented.

We begin by finding some properties of equal words in the semigroup S. We denote by A_0 the alphabet $\{a_j, a_j' \ : \ j = 1, 2, 3, 4\}$.

Lemma 6.1. *Let $w_1, w_2 \in A^*$ be arbitrary two words such that the relation $w_1 = w_2$ holds in S. Then the following statements are true.*

(i) *If w_1 contains a letter $x \in \{b, c, d\}$ then w_2 contains x as well.*
(ii) *The number of occurrences of each of the letters c and d is the same for w_1 and w_2.*
(iii) *If $w_1 \equiv \alpha x \beta$ and $w_2 \equiv \gamma y \delta$ with $\alpha, \gamma \in (A_0 \cup \{b\})^*$, $x, y \in \{c, d\}$ and $\beta, \delta \in A^*$ then the number of occurrences of b in α is equal to the number of occurrences of b in γ.*
(iv) *If w_1 has an occurrence of b preceding an occurrence of c (i.e., if w_1 has the form $w_1 \equiv \alpha b \beta c \gamma$, $\alpha, \beta, \gamma \in A^*$) then so does w_2.*
(v) *If w_1 has an occurrence of d preceding an occurrence of c then so does w_2.*
(vi) *If w_1 has an occurrence of d preceding an occurrence of b then so does w_2.*

Proof. Each part can be proved by noting that the property in question is invariant under the defining relations of S. □

Lemma 6.2. *If $w \in A^*$ is such that $cbdw\mathcal{R}cbd$ in S, then $w \in A_0^*$.*

Proof. Let $w_1 \in A^*$ be such that $cbdww_1 = cbd$ in S. By Lemma 6.1 (ii) and (vi) it follows that the word ww_1 contains no occurrence of either b, c or d. Therefore $w \in A_0^*$, as required. □

Lemma 6.3. *H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class.*

Proof. Let $s \in S$ be an arbitrary element which is $\mathcal{R}$-equivalent to cbd. Then s can be written as $s = cbdw$. By Lemma 6.2, it follows that $w \in A_0^*$. By using relations $a_j cbd = cbda_j$, we see that $s = wcbd$, and, since all a_j are invertible, we conclude that $s\mathcal{H}cbd$. □

In the notation of Sections 4 and 5 we have $\Lambda = \{1\}$ and $R = H$. The action of S on $\Lambda \cup \{0\}$ is given by

$\cdot$	a_j	a_j'	b	c	d
1	1	1	0	0	0
0	0	0	0	0	0

which follows immediately from Lemmas 6.2 and 6.3. The formal generators (13) for $\Gamma(H)$ are $b(1, a_j)$, $b(1, a_j')$ ($j = 1, 2, 3, 4$). If we identify these symbols with a_j and a_j' respectively, we obtain $B = A_0$. The definitions (15) and (16) of the mappings ϕ and π now simplify to

$$\phi(w) \equiv \pi(w) \equiv w \quad (w \in A_0^*). \tag{33}$$

Next we find the set I and the action of S on it.

Lemma 6.4. *The words ϵ, $b^i d$ ($i \geq 0$), cbd, form a system of representatives of $\mathcal{R}$-classes in the inverse orbit of H.*

Proof. Recall that an $\mathcal{R}$-class R' is in the inverse orbit of H if and only if there exists $s \in S$ such that $sR' \subseteq H$. Therefore, from $cbd\epsilon = cb(b^i d) = \epsilon(cbd) = cbd$, it follows that the $\mathcal{R}$-class of each of the given words is indeed in the inverse orbit of H. It remains to be proved that these are all the $\mathcal{R}$-classes in the inverse orbit, and that they are all distinct.

First of all, by Lemma 6.1 (i) there does not exist a word $w \in A^*$ such that $b^i dw = \epsilon$ or $cbdw = \epsilon$ in S. Therefore ϵ is not $\mathcal{R}$-equivalent to either $b^i d$ or cbd. Similarly, $b^i d$ is not $\mathcal{R}$-equivalent to cbd. Finally, if $k \neq i$, then $b^i d$ is not $\mathcal{R}$-equivalent to $b^k d$ by Lemma 6.1 (iii).

Let $w \in A^*$ be any word whose $\mathcal{R}$-class is in the inverse orbit of H. By the dual of Proposition 2.1 (v) this means that there exists a word $w_1 \in A^*$ such that $w_1 w = cbd$ in S. By Lemma 6.1 (ii), w contains at most one occurrence of the letter c, as well as at most one occurrence of the letter d. Thus we can distinguish the following four cases:

Case 1: *w contains no occurrences of c or d.* By Lemma 6.1 (i), w_1 must contain occurrences of both c and d. Hence, by Lemma 6.1 (vi), w does not contain any occurrences of b. In other words, $w \in A_0^*$, and hence $w\mathcal{R}\epsilon$.

Case 2: *w contains one occurrence of c and no occurrences of d.* By Lemma 6.1 (i) w_1 must contain an occurrence of d, but then we obtain a contradiction with Lemma 6.1 (v). Therefore, this case never occurs.

Case 3: *w contains one occurrence of d and no occurrences of c.* By Lemma 6.1 (vi), w cannot contain any occurrences of b after the only occurrence of d. Hence w can be written as

$$w \equiv \alpha_1 b \alpha_2 b \dots \alpha_m b \alpha_{m+1} d \alpha_{m+2},$$

where $m \geq 0$ and $\alpha_k \in A_0^*$, $k = 1, \dots, m+2$. By using relations $a_j b = b a_j^2$ and $a_j d = d a_j$ we see that w is equal in S to a word of the form $b^m d\alpha$ with $\alpha \in A_0^*$, and hence $w\mathcal{R}b^m d$.

Case 4: *w contains one occurrence of c and one occurrence of d.* By Lemma 6.1 (i), (iv), (v) and (vi) we have that w must have occurrences of b, that all these occurrences must supercede the occurrence of c and also must precede the occurrence of d. In other words, w has the form

$$w \equiv \alpha_1 c \alpha_2 b \alpha_3 b \dots \alpha_m b \alpha_{m+1} d \alpha_{m+2},$$

with $m \geq 2$ and $\alpha_k \in A_0^*$, $k = 1, \dots, m+2$. By applying relations $a_j b = b a_j^2$, $a_j d = d a_j$, $cb^2 = cb$ and $a_j cbd = cbd a_j$, we see that w is equal in S to a word of the form $cbd\alpha$, $\alpha \in A_0^*$, and hence $w\mathcal{R}cbd$.

This completes the proof of the lemma. $\square$

Following the notation from Sections 4 and 5, we let $I = \{1, 2, \dots\} \cup \{\omega\}$, and then we denote by R_1 the $\mathcal{R}$-class of $r_1 \equiv \epsilon$, by R_i ($i = 2, 3, \dots$) the $\mathcal{R}$-class of $r_i \equiv b^{i-2} d$, and by R_ω the $\mathcal{R}$-class of $r_\omega \equiv cbd$ (i.e., H).

Lemma 6.5. *The left action of S on $I \cup \{0\}$ is as given in Table 1.*

a_j	a'_j	b	c	d	$*$
1	1	0	0	2	1
2	2	3	0	0	2
3	3	4	ω	0	3
4	4	5	ω	0	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	i	$i+1$	ω	0	i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ω	ω	0	0	0	ω

Table 1. The left action of S on $I \cup \{0\}$.

Proof. Since a_j is a unit it follows that $a_j\epsilon\mathcal{R}\epsilon$, and hence $a_j * 1 = 1$. A multiple application of the relation $a_jb = ba_j^2$ yields $a_jb^i = b^ia_j^{2^i}$. Since we also have $a_jd = da_j$ it follows that $a_j(b^{i-2}d) = b^{i-2}da_j^{2^{i-2}} \in R_i$, and so $a_j * i = i$. Finally, from the relation $a_jcbd = cbda_j$ it follows that $a_j * \omega = \omega$. This completes the proof for the a_j column of the table. The proof for the a'_j column is analogous.

Assume that $b \in R_i$ for some $i \in I$. Since $\{R_i : i \in I\}$ is the inverse orbit of H, it follows that there exists a word $w \in A^*$ such that $wb = cbd$. Now w must contain an occurrence of the letter d by Lemma 6.1 (i), and this yields a contradiction with Lemma 6.1 (vi). Therefore $b * 1 = 0$. Similarly from Lemma 6.1 (iv) it follows that $b(cbd) \notin R_i$ for all $i \in I$, and hence $b * \omega = 0$. Finally, we have $b(b^{i-2}d) \equiv b^{i-1}d \in R_{i+1}$, so that $b * i = i + 1$, and this completes the proof for the b column of the table.

For the c column we have $c * 1 = c * 2 = c * \omega = 0$ by Lemma 6.1 (iii) and (ii). We also have $c * i = \omega$ $(i \geq 3)$ because $cb^{i-2}d = cbd \in R_\omega$. Finally, for the d column we have $d * 1 = 2$, since $d\epsilon \in R_2$, and $d * i = 0$ for $i \geq 2$ and $i = \omega$, by Lemma 6.1 (i). $\square$

The mapping τ is defined by (21), once the values $\tau(x, i)$ $(x \in A, i \in I, x * i \neq 0)$ are chosen in accord with (20). One possible choice is given in Table 2. All the entries are easily verified by direct computation. For example, the entry $a_j^{2^{i-2}}$ in the position (a_j, i) follows from

$$a_jr_i \equiv a_jb^{i-2}d = b^{i-2}a_j^{2^{i-2}}d = b^{i-2}da_j^{2^{i-2}} \equiv r_ia_j^{2^{i-2}}.$$

a_j	a'_j	b	c	d	τ
a_j	a'_j	$-$	$-$	ϵ	1
a_j	a'_j	ϵ	$-$	$-$	2
a_j^2	$(a'_j)^2$	ϵ	ϵ	$-$	3
a_j^4	$(a'_j)^4$	ϵ	ϵ	$-$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_j^{2^{i-2}}$	$(a'_j)^{2^{i-2}}$	ϵ	ϵ	$-$	i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_j	a'_j	$-$	$-$	$-$	ω

Table 2. Values for $\tau(x, i)$ ($x \in A$, $i \in I$, $x * i \neq 0$).

With (33) in mind, the presentation for $\Gamma(H)$ given in Theorem 4.2 has generators A_0 and the relations

$$u = v \quad ((u = v) \in \mathfrak{R},\ 1 \cdot u \neq 0), \tag{34}$$

$$\tau(u, i) = \tau(v, i) \quad ((u = v) \in \mathfrak{R},\ i \in I,\ u * i \neq 0), \tag{35}$$

$$\tau(x, \omega) = x \quad (x \in A_0), \tag{36}$$

$$\tau(h, 1) = 1, \tag{37}$$

where, as usual, $\mathfrak{R}$ denotes the defining relations of S. The group (34) clearly consists of the relations

$$a_j a'_j = a'_j a_j = 1 \ (j = 1, 2, 3, 4),\ a_1 a_2 = a_3 a_4. \tag{38}$$

Consider now the relations (35). Let $u = v$ be the relation $a_1 a_2 = a_3 a_4$, and let $i \geq 2$ be arbitrary. By using (21) and Table 2 we have

$$\tau(a_1 a_2, i) \equiv \tau(a_1, a_2 * i)\tau(a_2, i) \equiv \tau(a_1, i)\tau(a_2, i) \equiv a_1^{2^{i-2}} a_2^{2^{i-2}}$$

and, similarly, $\tau(a_3 a_4, i) \equiv a_3^{2^{i-2}} a_4^{2^{i-2}}$. Therefore we obtain the relations

$$a_1^{2^{i-2}} a_2^{2^{i-2}} = a_3^{2^{i-2}} a_4^{2^{i-2}} \quad (i \geq 2). \tag{39}$$

In a similar way we may check that all the remaining defining relations are identical. Therefore, as a group, $\Gamma(H)$ is defined by (39). In particular, $\Gamma(H)$ is an amalgamated product of two free groups of rank two with a free group of infinite rank amalgamated (see [**17**, Section 4.2]) and is not finitely presented by [**3**]. To summarise:

Proposition 6.6. *Let S be the semigroup defined by the presentation*

$$\mathfrak{P} = \langle A \mid a_j a'_j = a'_j a_j = \epsilon,\ a_1 a_2 = a_3 a_4,\ a_j b = b a_j^2,\ a_j d = d a_j,\ cb^2 = cb,\\ cbda_j = a_j cbd \ \ (j = 1, 2, 3, 4)\rangle,$$

and let H be the $\mathcal{H}$-class of the element cbd. Then H is the only $\mathcal{H}$-class in its $\mathcal{R}$-class. However, the Schützenberger group $\Gamma(H)$ of H is defined by the presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i} \quad (i = 0, 1, 2, \dots) \rangle,$$

and is not finitely presented.

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