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For a polynomial ring, R, in $4n$ variables over a field, we consider the submodule of R^4 corresponding to the $4 \times 4n$ matrix made up of n groupings of the linear representation of quarternions with variable entries (which corresponds to the Cauchy-Fueter operator in partial differential equations) and let $\mathcal{M}_n$ be the corresponding quotient module. We compute many homological properties of $\mathcal{M}_n$ including the degrees of all of its syzygies, as well as its Betti numbers, Hilbert function, and dimension. We give similar results for its leading term module with respect to the degree reverse lexicographical ordering. The basic tool in the paper is the theory of Gröbner bases.

1. Introduction.

In several recent papers, [**1**], [**2**] and [**3**], the authors and their colleagues have applied to certain analytic questions some algebraic properties of the following module. Let $x_{i0}, x_{i1}, x_{i2}, x_{i3}$ $(1 \leq i \leq n)$ denote $4n$ variables $(n = 1, 2, \dots)$, let k be any field, and let $R = k[x_{i0}, x_{i1}, x_{i2}, x_{i3} | 1 \leq i \leq n]$ denote the corresponding polynomial ring in the given $4n$ variables (in the analytic applications $k = \mathbb{C}$, but the specific field plays no role in the current paper). We consider the $4 \times 4n$ matrix

$$A_n = \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix},$$

where

$$U_i = \begin{bmatrix} x_{i0} & x_{i1} & x_{i2} & x_{i3} \\ -x_{i1} & x_{i0} & x_{i3} & -x_{i2} \\ -x_{i2} & -x_{i3} & x_{i0} & x_{i1} \\ -x_{i3} & x_{i2} & -x_{i1} & x_{i0} \end{bmatrix},$$

for $i = 1, \dots, n$. It is easy to see that U_i is a linear representation of the quarternion[1] $\xi = x_{i0} + x_{i1}\boldsymbol{i} + x_{i2}\boldsymbol{j} + x_{i3}\boldsymbol{k}$. We denote by $\langle A_n \rangle$ the submodule

[1]Note added in proof: It should be noted that this representation differs slightly from the usual one which may be obtained from this one either by changing the sign of x_{i2} or by interchanging $\boldsymbol{i}$ and $\boldsymbol{j}$. This does not effect any of the computations in this paper or in [**3**].

of R^4 generated by the columns of A_n. The module of study in this paper is

$$\mathcal{M}_n = R^4/\langle A_n\rangle. \tag{1}$$

Very briefly, the connection with analysis is the following. We consider the real quarternions $\mathbb{H}$ and a function $\boldsymbol{f} : \mathbb{H}^n \longrightarrow \mathbb{H}$ with $\boldsymbol{f} = (f_0, f_1, f_2, f_3)$, where the f_i's are the real valued functions which make up the components of $\boldsymbol{f}$. We assume that f_0, f_1, f_2, f_3 are $\mathcal{C}^\infty$-functions in the $4n$ real variables $x_{i0}, x_{i1}, x_{i2}, x_{i3}$ $(1 \leq i \leq n)$. In analogy with the Cauchy-Riemann equations, we say $\boldsymbol{f}$ is left regular if it satisfies the Cauchy-Fueter system

$$\begin{cases} \dfrac{\partial f_0}{\partial \xi_{i0}} - \dfrac{\partial f_1}{\partial \xi_{i1}} - \dfrac{\partial f_2}{\partial \xi_{i2}} - \dfrac{\partial f_3}{\partial \xi_{i3}} = 0 \\[2ex] \dfrac{\partial f_0}{\partial \xi_{i1}} + \dfrac{\partial f_1}{\partial \xi_{i0}} - \dfrac{\partial f_2}{\partial \xi_{i3}} + \dfrac{\partial f_3}{\partial \xi_{i2}} = 0 \\[2ex] \dfrac{\partial f_0}{\partial \xi_{i2}} + \dfrac{\partial f_1}{\partial \xi_{i3}} + \dfrac{\partial f_2}{\partial \xi_{i0}} - \dfrac{\partial f_3}{\partial \xi_{i1}} = 0 \\[2ex] \dfrac{\partial f_0}{\partial \xi_{i3}} - \dfrac{\partial f_1}{\partial \xi_{i2}} + \dfrac{\partial f_2}{\partial \xi_{i1}} + \dfrac{\partial f_3}{\partial \xi_{i0}} = 0 \end{cases} \tag{2}$$

for $i = 1, \ldots, n$. If $\mathcal{S}_n$ denotes a sheaf of generalized functions (e.g., $\mathcal{S}_n$ is the sheaf of hyperfunctions or the sheaf of infinitely differentiable functions) and $\mathcal{S}_n^P$ denotes the left regular functions in $\mathcal{S}_n$, then analytic information concerning left regular functions is deduced from algebraic information concerning $\mathcal{M}_n$ because of the isomorphism

$$\mathrm{Hom}_R(\mathcal{M}_n, \mathcal{S}_n) \cong \mathcal{S}_n^P$$

(see [**1**], [**2**], [**3**], [**13**], and [**15**]).

In this paper we will concentrate just on the algebraic properties of $\mathcal{M}_n$ in (1) and we will continue the developements started in the earlier paper [**3**]. Our main motivation for this paper is to answer some of the questions posed in [**13**] and [**15**] concerning the syzygies of $\mathcal{M}_n$.

The first question in [**13**] concerns the first syzygies of $\mathcal{M}_n$. It was conjectured there that they were all quadratic. We will establish this result as our Theorem 3.1. We do not, however, answer the analytic question asked in [**13**], that is, to give an analytic explanation for the quadratic syzygies we construct. The issue is that the third class of quadratic syzygies given in Theorem 3.1 gives some unexpected (from the analytic viewpoint) compatibility conditions for solving the inhomogeneous version of (2).

The second question posed in [**13**] is based upon the observation, using CoCoA(see [**9**]), that for $n = 2, 3, 4$ all the syzygies beyond the first are linear (see Equation (5) in Section 3). We will prove this is a general fact in Section

3 as Theorem 3.2 and this is the main result of this paper.[2] (The analytic consequences of this result are not understood.) There are two main tools used in the proof of this result. The first is that, as seen in [**3**], we can write down an explicit Gröbner basis for $\langle A_n \rangle$ for general n (with respect to the degree reverse lexicographical ordering on R). The second is that we can show that the (Castelnuovo) regularity of $\langle A_n \rangle$ equals 2. This latter, combined with the first syzygies being quadratic, allows us to conclude that all the higher syzygies are linear. It should be pointed out that some of the results of this paper could also be obtained using the theory of shellable complexes (see [**15**] or [**8**]).

In Section 2 we will describe the explicit Gröbner basis for $\langle A_n \rangle$. From this we easily write down the leading term module corresponding to $\langle A_n \rangle$. We then deduce the Hilbert-Poincaré series and consequently the Hilbert polynomial, dimension and degree of $\mathcal{M}_n$.

Also in Section 2, we briefly describe the main algebraic results of [**3**]: Using the explicit Gröbner basis we are able to write down an explicit maximal regular sequence for $\mathcal{M}_n$, showing that $\mathcal{M}_n$ has depth equal $2n+1$ (as a graded R-module) and then from the Auslander-Buchsbaum formula we get the projective dimension of $\mathcal{M}_n$, $\mathrm{pd}_R(\mathcal{M}_n) = 2n-1$. We conclude that $\mathcal{M}_n$ is Cohen-Macaulay. Moreover, using these results and those of Section 3 we give an explicit formula for the Betti numbers of $\mathcal{M}_n$.

2. Dimensions for $\mathcal{M}_n$.

We begin by describing a Gröbner basis for $\langle A_n \rangle$ and as a result a generating set for $\mathrm{Lt}(\langle A_n \rangle)$, the leading term module of $\langle A_n \rangle$ (see [**4**] for the relevant definitions and basic results on Gröbner bases). We use the degree reverse lexicographic (degrevlex) term ordering on R with

$$x_{10} > x_{20} > \cdots > x_{n0} > x_{11} > \cdots > x_{n1} > x_{12} > \cdots > x_{n3}, \tag{3}$$

and the TOP (TOP stands for term over position) ordering on R^4 with $\boldsymbol{e}_1 > \boldsymbol{e}_2 > \boldsymbol{e}_3 > \boldsymbol{e}_4$, where $\boldsymbol{e}_i$ is the ith column of the 4×4 identity matrix. That is, for monomials $X = x_{10}^{\alpha_{10}} \cdots x_{n3}^{\alpha_{n3}}$ and $Y = x_{10}^{\beta_{10}} \cdots x_{n3}^{\beta_{n3}}$ and for $r, s = 1, 2, 3, 4$, we have

$$X\boldsymbol{e}_r > Y\boldsymbol{e}_s \iff \begin{cases} \deg(X) = \sum\limits_{\substack{i=1,\dots,n\\ j=0,1,2,3}} \alpha_{ij} > \deg(Y) = \sum\limits_{\substack{i=1,\dots,n\\ j=0,1,2,3}} \beta_{ij} & \text{or} \\ \deg(X) = \deg(Y) \text{ and } \alpha_{ij} < \beta_{ij} \text{ for the index } ij, & \\ \qquad \text{last with respect to (3), such that } \alpha_{ij} \neq \beta_{ij} & \text{or} \\ X = Y \text{ and } r < s. & \end{cases}$$

[2] After the preparation of this paper, the paper of R. Baston [**5**] was pointed out to the authors which appears to have results very similar to our Theorems 3.1 and 3.2. The paper of Baston is concerned with the possible physical applications of this theory.

Lemma 2.1. *The reduced Gröbner basis for the R-module $\langle A_n\rangle$ is given by the columns of A_n together with the columns of the $\binom{n}{2}$ matrices $U_rU_s - U_sU_r$ $(1 \le r < s \le n)$. Moreover the module generated by the leading terms of all the elements of $\langle A_n\rangle$, denoted* $\mathrm{Lt}(A_n)$, *is*

$$\mathrm{Lt}(A_n) = \langle x_{i0}\boldsymbol{e}_\ell, x_{r2}x_{s1}\boldsymbol{e}_\ell\rangle_{\substack{i=1,\dots,n\\ 1\le r<s\le n\\ \ell=1,2,3,4}}.$$

The proof for this result simply lies in observing that the columns of $U_rU_s - U_sU_r$ clearly lie in $\langle A_n\rangle$, and that all of the S-polynomials of the listed vectors go to zero. See also [**3**].

We use this result to compute the Hilbert-Poincaré series for $\mathcal{M}_n$. Recall that if we write

$$\mathcal{M}_n = \coprod_{\nu\ge 0} M_{\nu n},$$

where $M_{\nu n}$ is the k-space of elements of $\mathcal{M}_n$ of degree ν, then the Hilbert function of $\mathcal{M}_n$ is defined by

$$H_n(\nu) = \dim_k M_{\nu n}$$

and the Hilbert-Poincaré series for $\mathcal{M}_n$ by

$$\mathcal{P}_n(t) = \sum_{\nu=0}^{\infty} H_n(\nu)t^\nu.$$

We obtain

Theorem 2.2.

$$\mathcal{P}_n(t) = \frac{4+4(n-1)t}{(1-t)^{2n+1}}.$$

Proof. We first recall Macaulay's result [**12**] (see also [**6**]) that the Hilbert function and hence the Hilbert-Poincaré series are the same for $\mathcal{M}_n = R^4/\langle A_n\rangle$ and $\mathcal{L}_n = R^4/\mathrm{Lt}(A_n)$. Then from the symmetry of $\mathrm{Lt}(A_n)$ in each component, we see that

$$\mathcal{P}_n(t) = 4\mathcal{Q}_n(t),$$

where $\mathcal{Q}_n(t)$ is the Hilbert-Poincaré series for R/I_n and I_n is the ideal of R defined by

$$I_n = \langle x_{i0}, x_{r2}x_{s1}\rangle_{\substack{i=1,\dots,n\\ 1\le r<s\le n}}.$$

To simplify the notation, for $T_1,\dots,T_r$ monomials in R we denote by $\mathcal{Q}_{T_1,\dots,T_r}(t)$ the Hilbert-Poincaré series for $R/\langle T_1,\dots,T_r\rangle$. The following two computational rules allow us to compute $\mathcal{Q}_n(t)$ (see [**6**]):

Rule 1: If V is a (possibly empty) subset of the variables, then

$$\mathcal{Q}_V(t) = \frac{1}{(1-t)^{4n-|V|}}$$

where $|V|$ denotes the cardinality of V.

Rule 2: If $T_1, \dots, T_r, T$ are monomials with degree $T = d$, then
$$\mathcal{Q}_{T_1,\dots,T_r,T}(t) = \mathcal{Q}_{T_1,\dots,T_r}(t) - t^d \mathcal{Q}_{\frac{T_1}{(T_1,T)},\dots,\frac{T_r}{(T_r,T)}}(t)$$
where (T_i, T) $(1 \le i \le r)$ denotes the greatest common divisor of the two monomials T_i and T.

We first consider the $n+2$ variables that do not appear in the generating set for I_n, namely x_{i3} for $1 \le i \le n$, and x_{11}, x_{n2}. Applying Rule 2 above successively to these $n+2$ variables we obtain
$$\mathcal{Q}_A(t) = (1-t)^{n+2}\mathcal{Q}_n(t),$$
where $A = \{x_{i0}, x_{i3}, x_{11}, x_{n2}, x_{r2}x_{s1}\}_{\substack{i=1,\dots,n \\ 1 \le r < s \le n}}$. It is then clear that the variables x_{i0}, x_{i3} for $1 \le i \le n$, and x_{11}, x_{n2} are no longer relevant to the computation. Thus we may replace R by
$$S = k[x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}]$$
(letting $x_r = x_{r2}$ for $1 \le r \le n-1$ and $y_{s-1} = x_{s1}$ for $2 \le s \le n$) and A by $B = \{x_r y_s | 1 \le r \le s \le n-1\}$. We have, in the obvious notation,
$$\mathcal{Q}_B(t) = \mathcal{Q}_A(t),$$
and to prove the theorem we must prove
$$\mathcal{Q}_B(t) = \frac{1+(n-1)t}{(1-t)^{n-1}}.$$
This will be proved by induction on $n-1$. We note that the number of variables is now $2(n-1)$ and so Rule 1 now reads
$$\mathcal{Q}_V(t) = \frac{1}{(1-t)^{2(n-1)-|V|}}.$$
(Rule 2 is unchanged.)

If $n-1 = 1$, then using first Rule 1 and then Rule 2 with $B = \emptyset$ we get
$$\mathcal{Q}_{x_1y_1}(t) = \mathcal{Q}_\emptyset(t) - t^2\mathcal{Q}_\emptyset(t) = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t},$$
as desired.

For the induction we let $C = \{x_r y_s | 1 \le r \le s \le n-2\}$ so that
$$\begin{aligned}
\mathcal{Q}_B(t) &= \mathcal{Q}_{C,x_1y_{n-1},\dots,x_{n-1}y_{n-1}}(t) \\
&= \mathcal{Q}_{C,x_1y_{n-1},\dots,x_{n-2}y_{n-1}}(t) - t^2\mathcal{Q}_{C,x_1,\dots,x_{n-2}}(t) \\
&= \mathcal{Q}_{C,x_1y_{n-1},\dots,x_{n-2}y_{n-1}}(t) - t^2\mathcal{Q}_{x_1,\dots,x_{n-2}}(t) \\
&= \mathcal{Q}_{C,x_1y_{n-1},\dots,x_{n-2}y_{n-1}}(t) - \frac{t^2}{(1-t)^{2(n-1)-(n-2)}} \\
&= \mathcal{Q}_{C,x_1y_{n-1},\dots,x_{n-2}y_{n-1}}(t) - \frac{t^2}{(1-t)^n};
\end{aligned}$$

here we have used Rule 2 and the equality

$$\langle C, x_1, \ldots, x_{n-2}\rangle = \langle x_1, \ldots, x_{n-2}\rangle,$$

and then Rule 1. We repeat this reasoning on the first term, $\mathcal{Q}_{C,x_1y_{n-1},\ldots,x_{n-2}y_{n-1}}(t)$, with $T = x_{n-2}y_{n-1}$ in Rule 2, only this time there is a non-trivial greatest common denominator with a term in C. We easily obtain

$$\begin{aligned}\mathcal{Q}_{C,x_1y_{n-1},\ldots,x_{n-2}y_{n-1}}(t) &= \mathcal{Q}_{C,x_1y_{n-1},\ldots,x_{n-3}y_{n-1}}(t) - t^2\mathcal{Q}_{x_1,\ldots,x_{n-3},y_{n-2}}(t)\\ &= \mathcal{Q}_{C,x_1y_{n-1},\ldots,x_{n-3}y_{n-1}}(t) - \frac{t^2}{(1-t)^n},\end{aligned}$$

again using Rule 1. Thus

$$\mathcal{Q}_B(t) = \mathcal{Q}_{C,x_1y_{n-1},\ldots,x_{n-3}y_{n-1}}(t) - 2\frac{t^2}{(1-t)^n}.$$

Continuing in this way, we have

$$\mathcal{Q}_B(t) = \mathcal{Q}_C(t) - (n-1)\frac{t^2}{(1-t)^n}.$$

We now can determine $\mathcal{Q}_C(t)$ from the induction assumption. However we note that the current $\mathcal{Q}_C(t)$ occurs in a ring with two more variables than the one in the induction assumption, and so we have

$$\mathcal{Q}_C(t) = \frac{1+(n-2)t}{(1-t)^{n-2+2}}.$$

Thus

$$\mathcal{Q}_B(t) = \frac{1+(n-2)t}{(1-t)^n} - (n-1)\frac{t^2}{(1-t)^n} = \frac{1+(n-1)t}{(1-t)^{n-1}},$$

which is the desired result. □

It is easy to determine the Hilbert function of $\mathcal{M}_n$ from Theorem 2.2. Moreover, since the degree of the numerator in $\mathcal{P}(t)$ is less than the degree of the denominator, we have that the Hilbert function corresponds to the Hilbert polynomial for all degrees ν. So we have

Corollary 2.3. $H_n(\nu) = 4\binom{\nu+2n}{2n} + 4(n-1)\binom{\nu+2n-1}{2n}$.

As a further Corollary, we read off from Theorem 2.2

Corollary 2.4. $\dim \mathcal{M}_n = 2n+1$.

Of course, here we mean the Krull dimension of $R/\operatorname{ann}\mathcal{M}_n$. Also we read off of Theorem 2.2 that the degree or multiplicity of $\mathcal{M}_n$ is $4n(2n+1)$.

We now turn to the projective dimension of $\mathcal{M}_n$. This was the main algebraic result in [**3**] and so we will only summarize the results here. We do this both for completeness and because some of the ideas will be needed in the next section.

In [**3**] we used the Auslander-Buchsbaum formula (see, for example, [**10**, Theorem 19.9 and Exercise 19.8])

$$\mathrm{pd}_R(\mathcal{M}_n) = \mathrm{depth}(\wp_n, R) - \mathrm{depth}(\wp_n, \mathcal{M}_n).$$

Here pd_R denotes the projective dimension of a module over R and $\mathrm{depth}(\wp_n, M)$ is the length of any maximal M-regular sequence in $\wp_n$ ($\wp_n$ denotes the ideal of the variables in R). So in order to compute $\mathrm{pd}_R(\mathcal{M}_n)$ it suffices to compute $\mathrm{depth}(\wp_n, \mathcal{M}_n)$ ($\mathrm{depth}(\wp_n, R) = 4n$, the number of variables). Now $\mathrm{depth}(\wp_n, \mathcal{M}_n)$ is defined to be the length of the longest sequence of polynomials $f_1, \ldots, f_s \in \wp_n$ such that

1) f_ν is a non-zerodivisor on $\mathcal{M}_n/\langle f_1, \ldots, f_{\nu-1}\rangle\mathcal{M}_n$, for $\nu = 1, \ldots, s$;
2) $\mathcal{M}_n \neq \langle f_1, \ldots, f_s\rangle\mathcal{M}_n$.

(Such a sequence is called an $\mathcal{M}_n$-regular sequence in $\wp_n$.)

Theorem 2.5. *We have*

1) $x_{11}, x_{n2}, x_{13}, x_{23}, \ldots, x_{n3}, x_{21} + x_{12}, x_{31} + x_{22}, \ldots, x_{n1} + x_{n-1,2}$ *is a maximal $\mathcal{M}_n$-regular sequence in $\wp_n$.*
2) $\mathrm{depth}(\wp_n, \mathcal{M}_n) = 2n + 1$.
3) $\mathrm{pd}_R(\mathcal{M}_n) = 2n - 1$.

Statements 2 and 3 follow from Statement 1, in light of the comments above. The outline of the proof of Statement 1 is as follows: We first note that $x_{11}, x_{n2}, x_{13}, x_{23}, \ldots, x_{n3}$ are precisely the variables that do not appear in the leading terms of any of the elements of the Gröbner basis of $\langle A_n\rangle$ given in Lemma 2.1. So the verification that they form an $\mathcal{M}_n$-regular sequence is easy. For the verification that the remaining elements in Statement 1 form an appropriate $\mathcal{M}_n$-regular sequence, we consider, for $\nu = 1, \ldots, n$, the submodule of R^4:

$$\begin{aligned} B_{\nu-1} = &\langle A_n, x_{11}\boldsymbol{e}_\ell, x_{n2}\boldsymbol{e}_\ell, x_{13}\boldsymbol{e}_\ell, x_{23}\boldsymbol{e}_\ell, \ldots, x_{n3}\boldsymbol{e}_\ell, \\ &(x_{21} + x_{12})\boldsymbol{e}_\ell, (x_{31} + x_{22})\boldsymbol{e}_\ell, \ldots, (x_{\nu 1} + x_{\nu-1,2})\boldsymbol{e}_\ell\rangle_{\ell=1,2,3,4} \end{aligned} \tag{4}$$

and we write down an explicit Gröbner basis $G_{\nu-1}$ for $B_{\nu-1}$. If $x_{\nu+1,1} + x_{\nu 2}$ is a zero-divisor on $R^4/B_{\nu-1}$, then there is a vector $\boldsymbol{g} \in R^4 - B_{\nu-1}$ such that $(x_{\nu+1,1} + x_{\nu 2})\boldsymbol{g} \in B_{\nu-1}$ and so $(x_{\nu+1,1} + x_{\nu 2})\boldsymbol{g}$ reduces to zero with respect to $G_{\nu-1}$. Using all this explicit information we can arrive at a contradiction. Similarly, we show every element in $\wp_n$ is a zero-divisor on R^4/B_{n-1} by showing that for all $f \in \wp_n$ we have $f^2\boldsymbol{e}_1 \in B_{n-1}$ by showing that $f^2\boldsymbol{e}_1$ reduces to zero using G_{n-1}. For details see [**3**].

We obtain the striking

Corollary 2.6. *The module $\mathcal{M}_n$ is Cohen-Macaulay.*

This follows immediately from Corollary 2.4 and Theorem 2.5 since the definition of a Cohen-Macaulay graded module over a graded ring is that

$$\mathrm{depth}(\wp_n, \mathcal{M}_n) = \dim \mathcal{M}_n.$$

We also obtain the standard decomposition of Cohen-Macaulay modules.

Corollary 2.7. *Let*

$$S = k[x_{11}, x_{n2}, x_{13}, x_{23}, \ldots, x_{n3}, x_{21} + x_{12}, x_{31} + x_{22}, \ldots, x_{n1} + x_{n-1,2}].$$

We have the following S-module direct sum decomposition:

$$\mathcal{M}_n = S^4 \oplus x_{12}S^4 \oplus \cdots \oplus x_{n-1,2}S^4.$$

Proof. It is easy to verify that the reduced Gröbner basis of B_{n-1} in (4) consists precisely of the elements of $\mathcal{M}_n$ listed in (4) for $\nu = n$ together with $x_{s2}x_{r2}\boldsymbol{e}_\ell$ for $1 \leq r \leq s \leq n-1$ and $\ell = 1, 2, 3, 4$. This gives rise to the leading term module for B_{n-1} consisting of the vectors $x_{i0}e_\ell, x_{i1}e_\ell, x_{i3}e_\ell$ for $1 \leq i \leq n$, $x_{s2}x_{r2}\boldsymbol{e}_\ell$ for $1 \leq r \leq s \leq n-1$, and $x_{n2}e_\ell$, all for $\ell = 1, 2, 3, 4$. This in turn gives rise to the list of standard monomials $\boldsymbol{e}_\ell, x_{12}\boldsymbol{e}_\ell, \ldots, x_{n-1,2}\boldsymbol{e}_\ell$ ($\ell = 1, 2, 3, 4$), which form a k basis for $\mathcal{M}_n/B_{n-1}$. It is then a standard and easily proven fact that these elements form a basis of $\mathcal{M}_n$ as a free S-module. $\square$

3. Degrees of the Syzygies.

We now turn to more explicit statements concerning the minimal free resolution of $\mathcal{M}_n$, or equivalently of $\langle A_n \rangle$. For $n = 2, 3, 4$ these minimal free resolutions of $\langle A_n \rangle$ were computed using the computer algebra package CoCoA (see [**9**]). We obtained:

(5)

for n=2:

$$0 \longrightarrow R^4(-4) \longrightarrow R^8(-3) \longrightarrow R^8(-1) \longrightarrow \langle A_2 \rangle \longrightarrow 0,$$

for n=3:

$$0 \longrightarrow R^8(-6) \longrightarrow R^{36}(-5) \longrightarrow R^{60}(-4)$$

$$\longrightarrow R^{40}(-3) \longrightarrow R^{12}(-1) \longrightarrow \langle A_3 \rangle \longrightarrow 0,$$

for n=4:

$$0 \longrightarrow R^{12}(-8) \longrightarrow R^{80}(-7) \longrightarrow R^{224}(-6) \longrightarrow R^{336}(-5)$$

$$\longrightarrow R^{280}(-4) \longrightarrow R^{112}(-3) \longrightarrow R^{16}(-1) \longrightarrow \langle A_4 \rangle \longrightarrow 0.$$

Here, as usual, $R^\nu(-j)$ means the graded free R-module of rank ν with the grading translated j places. (For a general graded R-module $N = \coprod_{i \in \mathbb{Z}} N_i$, we denote by $N(j)$ the graded R-module such that $N(j)_i = N_{i+j}$).

In the three examples above, the matrices that define the maps (i.e., the syzygies) have a special form: the first has quadratic entries and all the others have linear entries. That is, the first syzygy module of $\mathcal{M}_n$ has a 2-linear resolution in the sense of Eisenbud and Goto [**11**]. Our main goal in this section is to prove this is true for general n.

Our first result then is that the first syzygies are generated by quadratics.

Theorem 3.1. *There is a generating set for the first syzygy module of A_n consisting of vectors with quadratic entries. More specifically a generating set for the first syzygy module of A_n comes from expanding the following 3 formulas:*

1) *For each of the $\binom{n}{2}$ pairs of indices r and s $(1 \leq r, s \leq n)$*

$$\left[U_r, U_s U_s^t\right] = 0 \text{ and } \left[U_r U_r^t, U_s\right] = 0.$$

2) *For each of the $\binom{n}{3}$ triples of indices r, s, and ℓ $(1 \leq r, s, \ell \leq n)$*

$$\left[U_r, U_s U_\ell^t + U_\ell U_s^t\right] = 0 \text{ and } \left[U_s, U_r U_\ell^t + U_\ell U_r^t\right] = 0.$$

3) *For each of the $\binom{n}{3}$ triples of indices r, s, and ℓ $(1 \leq r, s, \ell \leq n)$*

$$[U_r, U_s]\, J\, [U_\ell, I] + [U_s, U_\ell]\, J\, [U_r, I] + [U_\ell, U_r]\, J\, [U_s, I] = 0$$

and

$$[U_r, U_s]\, I\, [U_\ell, J] + [U_s, U_\ell]\, I\, [U_r, J] + [U_\ell, U_r]\, I\, [U_s, J] = 0.$$

Here $[-,-]$ denotes the commutator and I and J denote the matrices for the quaternions $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively. The $4\left[2\binom{n}{2} + 4\binom{n}{3}\right]$ generators above then form a minimal generating set (note that each of the matrix identities above yield 4 syzygies).

That the first two formulas are valid follows since in the first case $U_s U_s^t$ is diagonal (i.e., $q\bar{q}$ is "real" for any quaternion) and in the second case since $U_s U_\ell^t + U_\ell U_s^t$ is diagonal (i.e., $\bar{q}_1 q_2 + \bar{q}_2 q_1$ is "real" for any 2 quaternions). The third formula can be verified, but we know of no "simple" explanation for it as in the previous cases.

Proof. We have in Lemma 2.1 that a Gröbner basis for $\langle A_n \rangle$ consists of vectors involving just one or two of the matrices U_i. Now to compute the syzygy module of the Gröbner basis we use S-polynomials and reduce them (see, for example, [**4**], Theorem 3.7.3). Since the reduction process cannot involve any variables not already in the S-polynomial, we see that the generating set for the syzygy module of the Gröbner basis obtained this way can involve at most four of the matrices U_i as multipliers of the same four matrices U_i. That is, the generating set for the syzygies of the Gröbner basis can be taken as the union of the generating sets of the syzygies of four of the U_i at a time. From these we obtain the syzygies of A_n following the procedure given in [**4**], Theorem 3.7.6. Namely, we consider the $4n \times (4n + 4\binom{n}{2})$ transformation matrix, T, between the vectors in A_n and the Gröbner basis, which is easily seen to consist of a $4n \times 4n$ identity matrix in its first $4n$ columns and in its remaining $4\binom{n}{2}$ columns the matrices $U_1, \ldots, U_n$ as follows:

$$\begin{bmatrix} U_2 & U_3 & \cdots & U_n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ -U_1 & 0 & \cdots & 0 & U_3 & \cdots & U_n & 0 & \cdots & 0 & \cdots \\ 0 & -U_1 & \cdots & 0 & -U_2 & \cdots & 0 & U_4 & \cdots & U_n & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & -U_1 & 0 & \cdots & -U_2 & 0 & \cdots & -U_3 & \cdots \end{bmatrix}.$$

If we multiply the above type of syzygy of the Gröbner basis by this matrix in order to obtain a syzygy of A_n we again see that it can involve no more than four of the matrices U_i, and indeed, again we see that the desired minimal generating set of the first syzygies of A_n can be obtained by taking the union of the minimal generating set of the syzygies of four of the U_i taken at a time.

Now, using CoCoAwe have computed the minimal generating set for the syzygies of A_n for $n = 2, 3, 4$ and have obtained the formulas enunciated in the Theorem. The author's have made this file and the computations giving rise to the formulas available on the web at the URL www.math.umd.edu/˜wwa/syzygies. This then completes the proof of the theorem. □

We now move on to the higher syzygies.

Theorem 3.2. *All of the syzygy modules in the minimal resolution of A_n of order greater than 1 are generated by linear polynomials.*

The proof of this result rests on the concept of (Castelnuovo-Mumford) regularity. We say that the homogeneous submodule M of a free R-module is *m-regular* provided that for the minimal graded free resolution

$$0 \longrightarrow \bigoplus_j R(-e_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_j R(-e_{1j})$$
$$\longrightarrow \bigoplus_j R(-e_{0j}) \longrightarrow M \longrightarrow 0$$

of M, we have $e_{ij} - i \leq m$ for all i, j. We say the *regularity of M* is m ($\mathrm{reg}(M) = m$) provided m is the least integer for which M is m-regular. For a general discussion of this concept see [**10**].

For example we see in Equation (5) that for $n = 2, 3, 4$, we have $\mathrm{reg}(\langle A_n \rangle) = 2$. Also for the leading term module $\mathrm{Lt}(A_n)$ we have (again using CoCoA) for $n = 2$:

$$0 \longrightarrow R^4(-4) \longrightarrow R^4(-2) \oplus R^8(-3) \longrightarrow R^8(-1) \oplus R^4(-2) \longrightarrow \mathrm{Lt}(A_2) \longrightarrow 0$$

and for $n = 3$:

$$0 \longrightarrow R^8(-6) \longrightarrow R^{36}(-5) \longrightarrow R^4(-3) \oplus R^{60}(-4) \longrightarrow R^{12}(-2) \oplus R^{44}(-3)$$

$$\longrightarrow R^{12}(-1) \oplus R^{12}(-2) \longrightarrow \mathrm{Lt}(A_3) \longrightarrow 0.$$

Thus we see that for $n = 2, 3$ we also have $\mathrm{reg}(\mathrm{Lt}(A_n)) = 2$. These results hold in general.

Theorem 3.3. *We have* $\mathrm{reg}(\langle A_n \rangle) = \mathrm{reg}(\mathrm{Lt}(A_n)) = 2$.

Given this result the proof of Theorem 3.2 is immediate. Namely, from Theorem 3.1 we know that the first syzygies of A_n are quadratic. Hence, in order to have a regularity of 2 we must have all higher order syzygies be linear.

Proof of Theorem 3.3. Since we have the general fact that $\mathrm{reg}(\mathrm{Lt}(A_n)) \geq \mathrm{reg}(\langle A_n \rangle)$ (see [**7**]) and we have from Theorem 3.1 that $\mathrm{reg}(\langle A_n \rangle) \geq 2$, we see that it suffices to show that $reg(\mathrm{Lt}(A_n)) = 2$. Recall that from Lemma 2.1 we have

$$\mathrm{Lt}(A_n) = \langle x_{i0}\boldsymbol{e}_\ell, x_{r2}x_{s1}\boldsymbol{e}_\ell \rangle_{i=1,\dots,n, 1 \leq r < s \leq n, \ell=1,2,3,4}.$$

Since the vectors here are concentrated in one coordinate and are the same for all coordinates we see that $\mathrm{reg}(\mathrm{Lt}(A_n)) = \mathrm{reg}(I_n)$ where

$$I_n = \langle x_{i0}, x_{r2}x_{s1} \rangle_{i=1,\dots,n, 1 \leq r < s \leq n},$$

an ideal in R.

We use the following criterion of Bayer and Stillman ([**7**]):

An ideal I in R is m-regular if and only if there are $h_1, \dots, h_\ell \in R_1$, for some $\ell \geq 0$, such that

$$(\langle I, h_1, \dots, h_{i-1} \rangle : h_i)_m = \langle I, h_1, \dots, h_{i-1} \rangle_m$$

for $i = 1, \dots, \ell$ and

$$\langle I, h_1, \dots, h_\ell \rangle_m = R_m.$$

We will show that the regular sequence we defined in Theorem 2.5 works for $h_1, \dots, h_\ell$ in the present case. Namely,

$$\begin{gathered} h_1 = x_{11}, h_2 = x_{n2}, h_3 = x_{13}, h_4 = x_{23}, \dots, h_{n+2} = x_{n3}, \\ h_{n+3} = x_{21} + x_{12}, h_{n+4} = x_{31} + x_{22}, \dots, h_{2n+1} = x_{n1} + x_{n-1,2} \end{gathered} \tag{6}$$

works with $m = 2$, and $\ell = 2n + 1$, and $I = I_n$. We will also show that this sequence forms a maximal regular sequence for R/I_n and so also for $R^4/\mathrm{Lt}(A_n)$. This will give, as it did for Theorem 2.5:

Theorem 3.4. *We have the following equalities*

1) $\mathrm{depth}(\wp_n, R^4/\mathrm{Lt}(A_n)) = \mathrm{depth}(\wp_n, R/I_n) = 2n + 1$
2) $\mathrm{pd}_R(R^4/\mathrm{Lt}(A_n)) = \mathrm{pd}_R(R/I_n) = 2n - 1$.

The first part of the sequence in (6) are the variables that do not appear in the generators for I_n and so form a regular sequence. To continue the process we add these variables into I_n. In the ideal they have no effect on the regular sequence or on the Bayer-Stillman criterion. So we simplify the notation by letting $x_j = x_{j1}$ for $2 \le j \le n$ and $y_j = x_{j2}$ for $1 \le j \le n-1$ and prove the following lemma.

Lemma 3.5. *We consider the polynomial ring* $S = k[x_2, \dots, x_n, y_1, \dots, y_{n-1}]$ *and the ideal in* S, $I = \langle x_i y_j \,|\, 1 \le j < i \le n \rangle$. *Then*

1) $\langle I, x_2 + y_1, \dots, x_\ell + y_{\ell-1} \rangle : (x_{\ell+1} + y_\ell) = \langle I, x_2 + y_1, \dots, x_\ell + y_{\ell-1} \rangle$ *for* $\ell = 1, \dots, n-1$.
2) $\langle I, x_2 + y_1, \dots, x_n + y_{n-1} \rangle : f \ne \langle I, x_2 + y_1, \dots, x_n + y_{n-1} \rangle$ *for all* $f \in \wp_n$.
3) $\langle I, x_2 + y_1, \dots, x_n + y_{n-1} \rangle_2 = S_2$.

We note that Statements 1 and 2 imply that $x_2 + y_1, \dots, x_n + y_{n-1}$ forms a maximal I-regular sequence in $\wp_n$ and Statements 1 and 3 imply that this same sequence satisfies the Bayer-Stillman criterion for the regularity of I to be equal to 2. So once this lemma is proved all the above unproved assertions are proved.

Proof of Lemma 3.5, part 1. The proof will use the theory of Gröbner bases. We will use the degree reverse lexicographical (degrevlex) term ordering with $x_2 > \dots > x_n > y_1 > \dots > y_{n-1}$. We let $I_\ell = \langle I, x_2 + y_1, \dots, x_\ell + y_{\ell-1} \rangle$ (so that $I_1 = I$). Then a Gröbner basis for I_ℓ is

$$G_\ell = \{y_i y_j | 1 \le i \le j \le \ell - 1\} \cup \{x_i y_j | \ell + 1 \le i \le n, 1 \le j < i\} \\ \cup \{x_2 + y_1, \dots, x_\ell + y_{\ell-1}\} \tag{7}$$

as is readily checked. From this we can explicitly list the standard (reduced) power products:

$$\begin{aligned} & y_i y_\ell^{\nu_\ell} \cdots y_{n-1}^{\nu_{n-1}} && (0 \le i \le \ell - 1, y_0 = 1) \\ & x_{\ell+1}^{\mu_{\ell+1}} \cdots x_i^{\mu_i} y_i^{\nu_i} \cdots y_{n-1}^{\nu_{n-1}} && (\ell + 1 \le i \le n) \end{aligned} \tag{8}$$

where the integers $\nu_\ell, \dots, \nu_{n-1}$ and $\mu_{\ell+1}, \dots, \mu_n$ are all nonnegative and, in order to guarantee that all the power products in (8) are distinct, we assume that all the μ_i in the second set of power products in (8) are ≥ 1. This is again readily checked using G_ℓ above. Now, by way of contradiction, let us assume that there is an $f \notin I_\ell$ with

$$(x_{\ell+1} + y_\ell) f \in I_\ell.$$

We may assume that f is reduced, that is, f is a linear combination of the power products in (8). Now

$$x_{\ell+1} y_i y_\ell^{\nu_\ell} \cdots y_{n-1}^{\nu_{n-1}} \in I_\ell \quad (0 \le i \le \ell - 1)$$

unless $i = 0$ and $\nu_\ell = 0$. Also

$$y_\ell x_{\ell+1}^{\mu_{\ell+1}} \cdots x_i^{\mu_i} y_i^{\nu_i} \cdots y_{n-1}^{\nu_{n-1}} \in I_\ell \quad (\ell + 1 \le i \le n),$$

since $\mu_i \geq 1$. This means that $(x_{\ell+1} + y_\ell)f$ is congruent modulo I_ℓ to a linear combination of monomials all of which are standard and different and include all terms in f in the first list multiplied by y_ℓ and in the second list multiplied by $x_{\ell+1}$. Since $(x_{\ell+1} + y_\ell)f \in I_\ell$ all of these terms must be 0 and we immediately conclude that $f = 0$, a contradiction. Thus Statement 1 in Lemma 3.5 is proved.

Proof of Lemma 3.5, part 2. We are to show that $I_n : f \neq I_n$ for all $f \in \wp_n$, and for this it suffices to show that for all $f \in \wp_n$, we have that $f^2 \in I_n$. From Equation (7) we have that the Gröbner basis for I_n is

$$G_n = \{y_i y_j | 1 \leq i \leq j \leq n-1\} \cup \{x_2 + y_1, \dots, x_n + y_{n-1}\}.$$

Since $f \in \wp_n$ every term of f^2 is of degree 2 or higher and using $x_2 + y_1, \dots, x_n + y_{n-1}$ we may replace every x variable with a y variable and have the result congruent modulo I_n and then we see the result is in I_n using the first set of generators in G_n above.

Proof of Lemma 3.5, part 3. This is the same argument as for the last case as every element of S_2 has degree 2 and so lies in I_n by the argument above. This completes the proof of Lemma 3.5 and so of Theorem 3.4 as well.

From Theorems 3.1 and 3.2 we see that the minimal free resolution of $\mathcal{M}_n = R^4/\langle A_n \rangle$ has the following shape:

$$\begin{aligned} &0 \longrightarrow R^{\beta_{2n-1}}(-2n) \longrightarrow R^{\beta_{2n-2}}(-2n+1) \longrightarrow \cdots \\ &\longrightarrow R^{\beta_3}(-4) \longrightarrow R^{\beta_2}(-3) \longrightarrow R^{\beta_1}(-1) \longrightarrow R^{\beta_0} \longrightarrow \mathcal{M}_n \longrightarrow 0. \end{aligned} \tag{9}$$

Here, of course, $\beta_0 = 4$ and $\beta_1 = 4n$. It is not in general possible to read off the Betti numbers, β_ν $(0 \leq \nu \leq 2n-1)$, from the minimal resolution of a module. However, because of the simple nature of the resolution (9) we may combine resolution (9) with the Hilbert-Poincaré series of Theorem 2.2 to get the Betti numbers for $\mathcal{M}_n$.

Corollary 3.6. *The Betti numbers of the module $\mathcal{M}_n = R^4/\langle A_n \rangle$ are given by $\beta_0 = 4$, $\beta_1 = 4n$ and for $2 \leq \nu \leq 2n-1$ by the formula*

$$\beta_\nu = 4\binom{2n-1}{\nu}\frac{n(\nu-1)}{\nu+1}. \tag{10}$$

Proof. As noted, for example, in Stanley [**14**], we can read off the Hilbert-Poincaré series from the minimal free resolution (9) as

$$\mathcal{P}_n(t) = \frac{\beta_0 - \beta_1 t + \beta_2 t^3 - \beta_3 t^4 - \cdots + \beta_{2n-2}t^{2n-1} - \beta_{2n-1}t^{2n}}{(1-t)^{4n}}.$$

From Theorem 2.2 we have that

$$\mathcal{P}_n(t) = \frac{4 + 4(n-1)t}{(1-t)^{2n+1}}.$$

Equating coefficients in the last two expressions gives the result. □

Note: Corollary 3.6 gives $\beta_2 = 4\binom{2n-1}{2}\frac{n}{3}$, and so we recover the number of generators of the first syzygy module given in Theorem 3.1, since

$$4\binom{2n-1}{2}\frac{n}{3} = 4\left[2\binom{n}{2} + 4\binom{n}{3}\right].$$

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