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Let G be a connected semisimple Lie group of real rank one. We denote by $\mathcal{U}(\mathfrak{g})^K$ the algebra of left invariant differential operators on G right invariant by K, and let $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ be its center.

In this paper we give a sufficient condition for a differential operator $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ to have a fundamental solution on G. We verify that this condition implies $P\,C^\infty(G) = C^\infty(G)$. If G has a compact Cartan subgroup, we also give a sufficient condition for a differential operator $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ to have a parametrix on G. Finally we prove a necessary condition for the existence of parametrix of $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ for a connected semisimple Lie group.

1. Introduction.

Let G be a connected semisimple Lie group. The algebra of left invariant differential operators on G identifies canonically with the universal algebra $\mathcal{U}(\mathfrak{g})$. The operators of the center $\mathcal{Z}(\mathfrak{g})$ are the bi-invariant differential operators on G, i.e., left and right invariant. More generally, we consider the algebra $\mathcal{U}(\mathfrak{g})^K$ of right K-invariant differential operators of $\mathcal{U}(\mathfrak{g})$, where K is a maximally compact subgroup of G, and $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ will denote its center.

We denote by $\mathcal{D}(G)$ the space of C^∞ functions with compact support. The dual $\mathcal{D}'(G)$ of continuous linear functionals in $\mathcal{D}(G)$ is the space of distributions of G.

An operator P in $\mathcal{U}(\mathfrak{g})$ acts on $\mathcal{D}'(G)$ in the following way:

$$PT(f) = T(P^t f),$$

where $P^t \in \mathcal{U}(\mathfrak{g})$ is such that, if dx is a Haar measure on G,

$$\int_G Pf(x)g(x)dx = \int_G f(x)P^t g(x)dx.$$

If $X \in \mathfrak{g}$, $X^t = -X$, so the $P \mapsto P^t$ is the anti automorphism of $\mathcal{U}(\mathfrak{g})$ extending $-Id$ of $\mathfrak{g}$. In addition, this map preserves the subalgebras $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})^K$.

Definition 1. A distribution $T \in \mathcal{D}'(G)$ is a fundamental solution of a differential operator $P \in \mathcal{U}(\mathfrak{g})$ if $PT = \delta$, where $\delta(f) = f(1)$; and T is a parametrix of P if $PT - \delta \in C^\infty(G)$.

In [**5**], Cerezo and Rouvière study the construction of parametrix and fundamental solutions in the group $G = SL(2, \mathbb{R})$ for operators in $\mathcal{U}(\mathfrak{g})^K$ (note that $\mathcal{U}(\mathfrak{g})^K$ is abelian in this case).

When G has only one conjugacy class of Cartan subgroups (G complex semisimple, for example), in [**13**], Rouvière gives a sufficient condition for the existence of fundamental solution of bi-invariant operators (Theorem 4.2 of [**13**]). He also proves a necessary condition for $P \in \mathcal{Z}(\mathfrak{g})$ to have a parametrix in an any connected semisimple group (Proposition 4.1 of [**13**]).

In this paper we extend the results of [**5**] and [**13**] for operators in $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ to rank one groups and groups with one conjugacy class of Cartan subgroups.

If G is Lie group with Lie algebra $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$, it is a well know result that $\mathcal{U}(\mathfrak{g})^K$ is abelian, and isomorphic to $\mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k})$. This isn't true for other groups, but in general $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K) \simeq \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k})$ (Knop's theorem [**11**]).

Given $\mathfrak{h}_0$ and $\mathfrak{t}_0$ Cartan subalgebras of $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively, we will denote $\gamma_{\mathfrak{h}}^G$ and $\gamma_{\mathfrak{t}}^K$ the Harish-Chandra homomorphisms of $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{Z}(\mathfrak{k})$ with respect to the subalgebras $\mathfrak{h}$ and $\mathfrak{t}$; then we have

$$\begin{array}{ccccccc} & & \mathcal{Z}(\mathfrak{g}) \times \mathcal{Z}(\mathfrak{k}) & \xrightarrow{\gamma_{\mathfrak{h}}^G \times \gamma_{\mathfrak{t}}^K} & \mathcal{U}(\mathfrak{h})^W \times \mathcal{U}(\mathfrak{t})^{W_K} & & \\ & & \downarrow \otimes & & \downarrow \otimes & & \\ \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K) & \xrightarrow{\simeq} & \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k}) & \xrightarrow{\gamma_{\mathfrak{h}}^G \otimes \gamma_{\mathfrak{t}}^K} & \mathcal{U}(\mathfrak{h})^W \otimes \mathcal{U}(\mathfrak{t})^{W_K} & \xrightarrow{i \otimes i} & \mathcal{U}(\mathfrak{h} \oplus \mathfrak{t}). \end{array}$$

Therefore, by the way of the homomorphisms described above, we can associate to $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ a differential operator $\left(\gamma_{\mathfrak{h}}^G \otimes \gamma_{\mathfrak{t}}^K\right)(P)$ in the group $H \times T$, where H and T are the respective Cartan subgroups of G and K with Lie algebras $\mathfrak{h}_0$ and $\mathfrak{t}_0$.

We say that H^f is a fundamental Cartan subgroup of G if H^f has maximal compact factor between θ-stable Cartan subgroups of G. All fundamental Cartan subgroups of G are conjugate (c.f. [**15**], Chapter I).

We will denote $\gamma^G = \gamma_{\mathfrak{h}^f}^G$, where $\mathfrak{h}_0^f$ is the Lie subalgebra of a fundamental Cartan subgroup H^f. Because in K all Cartan subgroups are conjugate, we put $\gamma^K = \gamma_{\mathfrak{t}}^K$.

We can now state our main result:

Theorem 1.1. *Let G be a connected semisimple Lie group of real rank one or with one conjugacy class of Cartan subgroups. Let H^f be a fundamental Cartan subgroup of G and $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. If $\left(\gamma^G \otimes \gamma^K\right)(P)$ has a fundamental solution in $H^f \times T$, then P has a fundamental solution in G.*

When P is a bi-invariant operator, we obtain a complete proof for these groups of the theorem announced in [**3**]:

Corollary 1.2 (Benabdallah-Rouvière)**.** *Let $P \in \mathcal{Z}(\mathfrak{g})$. If $\gamma^G(P)$ has a fundamental solution in H^f, then P has a fundamental solution in G.*

The proof will consist in the explicit construction of the fundamental solution of P, using the Plancherel formula as the main tool. When G is a rank one group having a compact Cartan subgroup, using Zuckerman characters identities combined with the proof of 1.1, we obtain a sufficient condition for the existence of a parametrix of P:

Theorem 1.3. *Let G be a linear connected semisimple Lie group of rank one such that T is a compact Cartan subgroup of G, and $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. If $\big(\gamma^G \otimes \gamma^K\big)(P)$ has a parametrix on $T \times T$, then P has a parametrix on G.*

Finally we extend Proposition 4.1 of [**13**] for $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$.

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Finally, I want to thank Professor Michel Duflo who gave a sketch for the proof of Proposition 6.3, which although I suppose well known, I couldn't find in the references.

2. Preliminaries.

In this section we fix notation and summarize the basics known facts about representation theory that will be needed through this paper.

2.1. Notation. Let G be a connected reductive Lie group. θ will denote a Cartan involution in both $\mathfrak{g}_0$ and G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$ with respect to θ; that is, $\mathfrak{k}_0 = \{X \in \mathfrak{g}_0 : \theta X = X\}$ and $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 : \theta X = -X\}$.

Let K be analytic subgroup of G with Lie algebra $\mathfrak{k}_0$, K is a maximally compact subgroup of G. We fix $\mathfrak{t}_0$ a Cartan subalgebra of $\mathfrak{k}_0$ coming from a maximal torus T of K. We will denote $\tau \in \hat{K}$ an irreducible unitary representation of K.

Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{p}_0$. The dimension of $\mathfrak{a}_0$ is the real rank of G. We put $\mathfrak{m}_0 = \{X \in \mathfrak{k}_0 : [X, \mathfrak{a}_0] = 0\}$. If $M = Z_K(\mathfrak{a}_0) = \{x \in K;\ \mathrm{Ad}\ (x)|_{\mathfrak{a}_0} = Id\}$, then M is a compact subgroup of G with Lie algebra $\mathfrak{m}_0$. If $\mathfrak{t}_0^-$ is a Cartan subalgebra of $\mathfrak{m}_0$, then $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0^-$ is a Cartan subalgebra of $\mathfrak{g}_0$.

If B is the Killing form of $\mathfrak{g}_0$, $B|_{\mathfrak{k}_0 \times \mathfrak{k}_0}$ is negative definite and $B|_{\mathfrak{p}_0 \times \mathfrak{p}_0}$ is positive definite. Then $(X, Y) = -B(X, \theta Y)$ defines an inner product on $\mathfrak{g}_0$.

The complexification of any real Lie algebra will be denoted without the subscript. If $\mathfrak{g}_u = \mathfrak{k}_0 \oplus i\mathfrak{p}_0 \subseteq \mathfrak{g}$, then $\mathfrak{g}_u$ is a compact form of $\mathfrak{g}$. The Killing form of $\mathfrak{g}$ is the complex bilinear extension of that of $\mathfrak{g}_0$, and making an abuse of notation, we will still call it B. On $\mathfrak{g}$ we define the inner product, $(X, Y) = -B(X, JY)$, where J is conjugation with respect to $\mathfrak{g}_u$. Let's note that $J|_{\mathfrak{g}_0} = \theta$ and so this inner product extends that of $\mathfrak{g}_0$.

Given $\sigma \in \hat{M}$ with infinitesimal character $\mu_\sigma \in i(\mathfrak{t}_0^-)'$ and $\tau \in \hat{K}$ with infinitesimal character $\mu_\tau \in i\mathfrak{t}_0'$, we define

$$|\sigma| = |\mu_\sigma|, \qquad |\tau| = |\mu_\tau|. \tag{1}$$

We note that this definition makes sense because the norm is both W_M and W_K invariant, since the elements of the Weyl groups of M and K are inner automorphisms of K.

Every semisimple Lie group of rank one has at most two conjugacy classes of θ-stable Cartan subgroups (or subalgebras). But in the case G has only one, the hypothesis of rank one is not needed. So from now on we will restrict our attention to the following two cases:

I. G has only one conjugacy class of θ-stable Cartan subalgebras, which will be represented by $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0^-$.

II. G is a rank one group having a compact Cartan subgroup. In this case $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and we can choose $\{\mathfrak{t}_0, \mathfrak{h}_0\}$ as a representative set of conjugacy classes of θ-stable Cartan subalgebras.

We will also suppose that G has a simply connected complexification $G^{\mathbb{C}}$. Finally we can assume, with no loss of generality, that M is connected, because in case I, M always is ([**14**, 7.12.7]), and in case II, if M is disconnected, then G must be a direct product of $SL(2, \mathbb{R})$ with a compact group, and the results of this paper are easily deduced from [**5**] and [**4**].

2.2. Principal series. Let $\Lambda(\mathfrak{g}_0, \mathfrak{a}_0)$ be the restricted root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$; we choose $\Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0)$ a positive system and we put $\mathfrak{n}_0 = \sum_{\lambda \in \Lambda^+} \mathfrak{g}_0^\lambda$. If A and N are the analytic subgroups with respective Lie algebras $\mathfrak{a}_0$ and $\mathfrak{n}_0$, then MAN is a minimal parabolic subgroup of G.

Given $\sigma \in \hat{M}$ an irreducible representation of M, $\nu \in \mathfrak{a}'$ a complex linear functional on $\mathfrak{a}$. The principal series representation $\pi_{\sigma,\nu}$ is defined inducing the representation $\sigma \otimes e^\nu \otimes 1$ from MAN to G. The representation $\pi_{\sigma,\nu}$ is admissible; moreover the multiplicity of τ in $\pi_{\sigma,\nu}|_K$ is independent of ν ([**10**, Prop. 8.4]) and will be denoted n_τ^σ.

If $\mu_\sigma \in i\mathfrak{t}_0^-$ is the infinitesimal character of σ relative to $\mathfrak{t}^-$, then $\pi_{\sigma,\nu}$ has infinitesimal character $\mu_\sigma + \nu$ relative to $\mathfrak{a} + \mathfrak{t}^-$ ([**10**, Prop. 8.22]).

2.3. Discrete series (case II). Let (π, V) be an irreducible unitary representation of G. We say that (π, V) is a discrete series representation of G if all its matrix coefficients $g \mapsto (\pi(g)u, v)$ are square integrable. G admits discrete series representations if and only if G has a compact Cartan subgroup; or equivalently, $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{k}_0$. All discrete series representations can be parametrized by the forms $\lambda \in i\mathfrak{t}_0'$ such that λ is non-singular (i.e., $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$) and $\lambda + \rho$ is analytically integral, or equivalently, λ is integral, because we are assuming $G^{\mathbb{C}}$ simply connected. The functional λ is called the Harish-Chandra parameter. Two discrete series representations are equivalent if and only if their parameters are conjugate by an element of the Weyl group W_K of K. We will denote π_λ the discrete series representation of parameter λ and $\mathcal{S}_d$ the set of Harish-Chandra parameters.

The representation π_λ has infinitesimal character λ, and discrete series representations with parameters $w\lambda$, $w \in W_G$, the Weyl group of G, have the same infinitesimal character. Therefore exactly $|W_G|/|W_K|$ of the representations $\pi_{w\lambda}$ are mutually inequivalent.

Finally, there is a positive number d_λ, called the formal degree of π_λ, such that

$$\int_G (\pi_\lambda(x)u_1, v_1)\overline{(\pi_\lambda(x)u_2, v_2)}\, dx = d_\lambda^{-1}(u_1, u_2)\overline{(v_1, v_2)}$$

for all u_1, u_2, v_1, v_2.

2.4. Global characters. We say that an admissible representation π has a global character Θ_π if the operator

$$\pi(f) = \int_G f(g)\pi(g)dg$$

is a trace class operator for all $f \in \mathcal{D}(G)$ and if the map $f \mapsto \operatorname{tr} \pi(f) = \Theta_\pi(f)$ is a distribution on G.

Every admissible representation π whose decomposition $\pi_{|K} = \sum_{\tau \in \hat{K}} n_\tau \tau$ satisfies $n_\tau \leq C \dim \tau$ has a global character ([**10**, Thm. 10.2]). As a consequence, every irreducible unitary representation has a character ([**10**, Thm. 8.1]). In the same way, induced representations from irreducible unitary representations also have characters. So discrete series as well as principal series representations have characters.

Given $\lambda \in \mathcal{S}_d$, we will denote Θ_λ the character of π_λ; and given $\sigma \in \hat{M}$, $\nu \in \mathfrak{a}'$, $\Theta_{\sigma,\nu}$ the character of $\pi_{\sigma,\nu}$.

2.5. Plancherel formula. We are now in a position to write down the Plancherel formula for the groups we are considering (cf. [**1**, Lemma 5] and [**14**, Thm. 8.15.4]).

Theorem 2.1. *There is a non-negative function $m_\sigma(\nu)$ defined in $\hat{M} \times i\mathfrak{a}_0'$ such that for all $f \in \mathcal{D}(G)$ we have in case* I,

$$f(1) = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathfrak{a}_0'} \Theta_{\sigma,\nu}(f) m_\sigma(\nu)\, d\nu, \tag{2}$$

and in case II,

$$f(1) = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathfrak{a}_0'} \Theta_{\sigma,\nu}(f) m_\sigma(\nu)\, d\nu + \sum_{\lambda \in \mathcal{S}_d} d_\lambda \Theta_\lambda(f). \tag{3}$$

The function $m_\sigma(\nu)$ has the following properties:

(i) *For each $\sigma \in \hat{M}$, $m_\sigma(\nu)$ is the restriction to $i\mathfrak{a}_0'$ of a meromorphic function on $\mathfrak{a}'$ without poles on $i\mathfrak{a}_0'$.*

(ii) *Exist a positive constant C and a positive integer l such that for all $\sigma \in \hat{M}$, $\nu \in i\mathfrak{a}_0'$, we have*

$$|m_\sigma(\nu)| \le C(1 + |\sigma|^2)^l (1 + |\nu|^2)^l.$$

3. Action of P on characters.

If π is an admissible representation with global character Θ_π and infinitesimal character χ_π, and $P \in \mathcal{Z}(\mathfrak{g})$ is a bi-invariant differential operator, then $P\,\Theta_\pi = \chi_\pi(P)\,\Theta_\pi$ ([**10**, Prop. 10.24]).

We want to prove a similar result for an operator $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. First we need to decompose a distribution as a sum of its K-isotypical components.

If we consider the right regular representation R acting on $L^2(G)$ then $\mathcal{D}(G)$ is a dense subspace of C^∞ vectors of $L^2(G)$, and decomposing R in sum of its K-isotypical components we obtain

$$f = \sum_{\tau \in \hat{K}} f_\tau,$$

where

$$f_\tau(x) = d_\tau (f * \Theta_\tau)(x) = d_\tau \int_K f(xk)\Theta_\tau(k^{-1})\, dk$$

with $\Theta_\tau(k) = \operatorname{tr}\,\tau(k)$. Convergence holds not only in $L^2(G)$ but also in $\mathcal{D}(G)$. This decomposition induces a similar decomposition for a distribution $T \in \mathcal{D}'(G)$:

$$T = \sum_{\tau \in \hat{K}} T^\tau$$

with $T^\tau(f) = T(f_{\bar{\tau}})$, where $\bar{\tau}$ is the conjugate representation of τ. Remind that $\bar{\tau}$ is isomorphic to the contragredient representation τ^* of K.

Lemma 3.1. *Let (π, V) be an admissible representation. Suppose that its K-isotypical decomposition*

$$V = \sum_{\tau \in \hat{K}} n_\tau V_\tau \qquad (L^2 \text{ sum})$$

satisfies $n_\tau \le C d_\tau$, where d_τ is the dimension of τ, and $(\pi|_K, V_\tau)$ is equivalent to τ. We choose an orthonormal base of V by joining orthonormal bases $\{e_i^\tau : i = 1 \dots d_\tau\}$ of V_τ.

If Θ_π is the character of π, then:

$$\Theta_\pi^\tau(f) = n_\tau \sum_{i=1}^{d_\tau} (\pi(f)e_i^\tau, e_i^\tau).$$

Proof.

$$\Theta_\pi^\tau(f) = \Theta_\pi(f_{\bar\tau}) = \sum_{\tau\in\hat K} n_\tau \left[\sum_{i=1}^{d_\tau} (\pi(f_{\bar\tau})e_i^\tau, e_i^\tau)\right]$$

with

$$f_{\bar\tau} = d_\tau(f * \Theta_{\bar\tau}) = d_\tau(f * \overline{\Theta}_\tau),$$

where Θ_τ is a C^∞ function on K being τ of finite dimension.

Let $p_\tau : V \longrightarrow V_\tau$ be the orthogonal projection. Now let $\tau' \in \hat K$ be another representation; then, using Fubini's theorem and the bi-invariance of the Haar measures of G and K,

$$\begin{aligned}
&(\pi(f_{\bar\tau})e_i^{\tau'}, e_i^{\tau'})\\
&= \int_G f_{\bar\tau}(x)(\pi(x)e_i^{\tau'}, e_i^{\tau'})\,dx\\
&= \int_G \left[d_\tau \int_K f(xk)\overline{\Theta}_\tau(k^{-1})\,dk\right](\pi(x)e_i^{\tau'}, e_i^{\tau'})\,dx\\
&= \int_G f(x)\left[d_\tau \int_K (\pi(k^{-1})e_i^{\tau'}, \pi(x)^* e_i^{\tau'})\overline{\Theta}_\tau(k^{-1})\,dk\right]dx\\
&= \int_G f(x)\left[\sum_{j=1}^{d_\tau} d_\tau \int_K (\pi(k)e_i^{\tau'}, p_{\tau'}(\pi(x)^* e_i^{\tau'}))\overline{(\tau(k)e_j^\tau, e_j^\tau)}\,dk\right]dx,
\end{aligned}$$

and by Schur orthogonality relations (Corollary 1.10 of [**10**]),

$$\begin{aligned}
(\pi(f_{\bar\tau})e_i^{\tau'}, e_i^{\tau'}) &= \int_G f(x)\left[\sum_{j=1}^{d_\tau} \delta_{\tau,\tau'}(e_i^{\tau'}, e_j^\tau)\overline{(p_{\tau'}(\pi(x)^* e_i^{\tau'}), e_j^\tau)}\,dk\right]dx\\
&= \delta_{\tau,\tau'}\int_G f(x)\overline{(\pi(x)^* e_i^{\tau'}, e_i^{\tau'})}\,dx\\
&= \delta_{\tau,\tau'}\int_G f(x)(\pi(x)e_i^\tau, e_i^\tau)\,dx.
\end{aligned}$$

□

Given π a representation of G, $\tau \in \hat K$, the map $\chi_\pi \otimes \chi_\tau$ is a linear functional on $\mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k})$, and induces a linear functional on $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$, which we will still denote $\chi_\pi \otimes \chi_\tau$.

We can now state the result we need.

Proposition 3.2. *If π is an admissible representation with infinitesimal character χ_π and global character Θ_π, and P is a differential operator in $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$, then*

$$P\,\Theta_\pi^\tau = (\chi_\pi \otimes \chi_\tau)(P)\,\Theta_\pi^\tau.$$

Proof. We preserve the notation of Lemma 3.1. If $P \in \mathcal{Z}(\mathfrak{g})$, $P\,\Theta_\pi^\tau = \chi_\pi(P)\,\Theta_\pi^\tau$. If $P \in \mathcal{Z}(\mathfrak{k})$,

$$P\,\Theta_\pi^\tau(f) = \Theta_\pi^\tau(P^t f) = n_\tau \sum_{i=1}^{d_\tau} (\pi(P^t f) e_i^\tau, e_i^\tau)$$

and

$$\begin{aligned}(\pi(P^t f)e_i^\tau, e_i^\tau) &= \int_G (\pi(x)\pi(P)e_i^\tau, e_i^\tau) f(x)\,dx \\ &= \chi_\tau(P)(\pi(f)e_i^\tau, e_i^\tau) \quad \text{by [\textbf{10}, (8.10)]},\end{aligned}$$

because $\pi(P)|_{V_\tau} = \tau(P)$. Then $P\,\Theta_\pi^\tau = \chi_\tau(P)\,\Theta_\pi^\tau$ if $P \in \mathcal{Z}(\mathfrak{k})$. □

4. Fundamental solutions on abelian connected groups.

In [**4**], Cerezo and Rouvière obtain necessary and sufficient conditions for $P \in \mathcal{U}(\mathfrak{g})$ to have a fundamental solution when G is a connected compact group or a product of a connected compact group with $\mathbb{R}^n$.

In this section we state these results for a connected abelian group, which is the product of a torus with $\mathbb{R}^n$. Theorem 4.1 follows directly from Theorem III of [**4**].

We consider in first place an abelian connected compact Lie group T with Lie algebra $\mathfrak{t}_0$, i.e., T a torus. In this case $\exp : \mathfrak{t}_0 \longrightarrow T$ is a group epimorphism, and $\Gamma = \ker(\exp)$ is a closed discrete subgroup of $\mathfrak{t}_0$. Being T abelian, its irreducible unitary representations are one dimensional and can be parametrized by the linear functionals $\lambda \in i\mathfrak{t}_0'$ such that $\lambda(\Gamma) \subseteq 2\pi i\mathbb{Z}$. So we set

$$\hat{T} = \{\lambda \in i\mathfrak{t}_0' \text{ such that } \lambda(\Gamma) \subseteq 2\pi i\mathbb{Z}\}. \tag{4}$$

A general abelian connected Lie group is of the form $A \times T$, where $A = \mathbb{R}^n$ and T is a m-dimensional torus. The universal algebra $\mathcal{U}(\mathfrak{a} \oplus \mathfrak{t})$ coincides with the symmetric algebra $\mathcal{S}(\mathfrak{a} \oplus \mathfrak{t})$, and so an element P in $\mathcal{U}(\mathfrak{a} \oplus \mathfrak{t})$ can be thought of both as a constant coefficient differential operator on $A \times T$ and as a polynomial function on $\mathfrak{a}' \oplus \mathfrak{t}'$. Given $\lambda \in \hat{T}$, if we put

$$P_\lambda(x) = P(x, \lambda), \; x \in \mathfrak{a}', \tag{5}$$

then P_λ is a polynomial function in $\mathbb{R}^n$, and its norm can be defined

$$||P_\lambda|| = \left(\sum_\alpha \frac{1}{(\alpha!)^2} |P_\lambda^{(\alpha)}(0)|^2 \right)^{\frac{1}{2}}.$$

Theorem 4.1. *Let $P \in \mathcal{U}(\mathfrak{a} \oplus \mathfrak{t})$ be a constant coefficients differential operator on $A \times T$. P has a fundamental solution if and only if exists a positive constant C and a positive integer k such that in any norm of $i\mathfrak{t}_0'$*

$$||P_\lambda|| \geq \frac{C}{(1+|\lambda|^2)^k} \qquad \forall\, \lambda \in \hat{T}. \tag{6}$$

Remark. Let's go back to G a connected semisimple Lie group. Remember that given $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ and $\mathfrak{h}_0$ and $\mathfrak{t}_0$ Cartan subalgebras of $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively we have defined a differential operator $\left(\gamma_{\mathfrak{h}}^G \otimes \gamma_{\mathfrak{t}}^K\right)(P)$ on the group $H \times T$. This operator depends on the choice of the Cartan subgroups of G and K. However, the existence of fundamental solution only depends on the conjugacy class (in G or K) of Cartan subgroups, as we see below.

Let $\tilde{H}$ and $\tilde{T}$ Cartan subgroups of G and K respectively and suppose there exist $g \in G$, $k \in K$ such that $\tilde{H} = g^{-1}Hg$ and $\tilde{T} = k^{-1}Tk$. Then Ad $(g)\mathfrak{h}_0 = \tilde{\mathfrak{h}}_0$ and Ad $(k)\mathfrak{t}_0 = \tilde{\mathfrak{t}}_0$. If $H = A \times T^-$, $\mathfrak{h}_0 = \mathfrak{a}_0 + \mathfrak{t}_0$, then $\tilde{H} = \tilde{A} \times \widetilde{T^-}$, $\tilde{A} = \text{Ad }(g)A$, $\widetilde{T^-} = \text{Ad }(g)T^-$, and is clear from (4) that

(7)

$$\hat{\tilde{T}} = \{\tilde{\lambda} = \lambda \circ \text{Ad }(g)^{-1} : \lambda \in \hat{T}\}, \qquad \widehat{\widetilde{T^-}} = \{\tilde{\mu} = \mu \circ \text{Ad }(k)^{-1} : \mu \in \widehat{T^-}\};$$

besides we have

$$\gamma_{\tilde{\mathfrak{h}}}^G = \text{Ad }(g) \circ \gamma_{\mathfrak{h}}^G, \qquad \gamma_{\tilde{\mathfrak{t}}}^K = \text{Ad }(k) \circ \gamma_{\mathfrak{h}}^K,$$

so if $x \in \mathfrak{a}'$, $\tilde{x} = x \circ \text{Ad }(g)^{-1} \in \tilde{\mathfrak{a}}'$, $(\tilde{\lambda}, \tilde{\mu}) \in \hat{\tilde{T}} \times \widehat{\widetilde{T^-}}$ and if $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$,

$$\begin{aligned} P_{(\tilde{\lambda},\tilde{\mu})}(\tilde{x}) &= \left(\left(\gamma_{\tilde{\mathfrak{h}}}^G \otimes \gamma_{\tilde{\mathfrak{t}}}^K\right)(P)\right)(\tilde{x}, \tilde{\lambda}, \tilde{\mu}) \\ &= \left(\left(\gamma_{\mathfrak{h}}^G \otimes \gamma_{\mathfrak{t}}^K\right)(P)\right)(x, \lambda, \mu) = P_{(\lambda,\mu)}(x), \end{aligned} \tag{8}$$

and so Theorem 4.1 implies that $\left(\gamma_{\mathfrak{h}}^G \otimes \gamma_{\mathfrak{t}}^K\right)(P)$ has a fundamental solution on $H \times T$ if and only if $\left(\gamma_{\tilde{\mathfrak{h}}}^G \otimes \gamma_{\tilde{\mathfrak{t}}}^K\right)(P)$ has one on $\tilde{H} \times \tilde{T}$.

5. Inversion of infinitesimal characters.

5.1. Case I. Let $\mathfrak{h}_0 = \mathfrak{a}_0 + \mathfrak{t}_0^-$ be a Cartan subalgebra of $\mathfrak{g}_0$, where $\mathfrak{t}_0^-$ is a Cartan subalgebra of $\mathfrak{m}_0$, let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. Let $\sigma \in \hat{M}$ with infinitesimal character $\mu_\sigma \in i(\mathfrak{t}_0^-)'$, $\nu \in \mathfrak{a}'$, $\tau \in \hat{K}$ with infinitesimal character $\mu_\tau \in i\mathfrak{t}_0'$.

Recall that to an operator $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ we associate an operator $\left(\gamma_{\mathfrak{h}}^G \otimes \gamma^K\right)(P)$ on the abelian connected group $A \times T^- \times T$. We then have

$$(\chi_{\sigma,\nu} \otimes \chi_\tau)(P) = \left(\left(\gamma_{\mathfrak{h}}^G \otimes \gamma^K\right)(P)\right)(\nu + \mu_\sigma, \mu_\tau), \tag{9}$$

and fixing σ, τ we obtain a polynomial function in $\mathfrak{a}'$, and we put

$$P_{\sigma,\tau}(\nu) = (\chi_{\sigma,\nu} \otimes \chi_\tau)(P). \tag{10}$$

Proposition 5.1. *Given $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$, if $\left(\gamma_{\mathfrak{h}}^G \otimes \gamma^K\right)(P)$ has a fundamental solution on $A \times T^- \times T$ there exist a constant C and a positive integer k such that*

$$||P_{\sigma,\tau}|| \geq \frac{C}{(1+|\sigma|^2)^k\,(1+|\tau|^2)^k} \qquad \forall\,(\sigma,\tau) \in \hat{M} \times \hat{K}.$$

Proof. According to (6) and (9), $P_{\sigma,\tau}(\nu) = \left(\left(\gamma_{\mathfrak{h}}^G \otimes \gamma^K\right)(P)\right)_{(\mu_\sigma,\mu_\tau)}(\nu)$; μ_σ as well as μ_τ are analytically integral forms, so $(\mu_\sigma, \mu_\tau) \in \widehat{T^-} \times \hat{T}$, and the proposition follows directly from Theorem 4.1. □

5.2. Case II: Discrete series. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{k}_0$. Let π_λ be the discrete series representation with parameter $\lambda \in i\mathfrak{t}_0'$, and $\tau \in \hat{K}$ with infinitesimal character $\mu_\tau \in i\mathfrak{t}_0'$.

In this case, given $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$, $\left(\gamma_{\mathfrak{t}}^G \otimes \gamma^K\right)(P)$ is a differential operator on $T \times T$, and we put

$$P_{\lambda,\tau} = (\chi_\lambda \otimes \chi_\tau)(P) = \left(\gamma_{\mathfrak{t}}^G \otimes \gamma^K(P)\right)(\lambda, \mu_\tau). \tag{11}$$

Proposition 5.2. *Given $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$, if $\left(\gamma_{\mathfrak{t}}^G \otimes \gamma^K\right)(P)$ has a fundamental solution on $T \times T$, there exist a constant C and a positive integer k such that*

$$|P_{\lambda,\tau}| \geq \frac{C}{(1+|\lambda|^2)^k\,(1+|\tau|^2)^k} \qquad \forall\,(\lambda,\tau) \in \mathcal{S}_d \times \hat{K}.$$

As in Proposition 5.1, the proof is a direct consequence of Theorem 4.1 with $A = 0$, observing that $(\lambda, \mu_\tau) \in \hat{T} \times \hat{T}$.

5.3. Case II: Principal series. In this case we need to invert simultaneously the infinitesimal characters of principal and discrete series representations. We will prove the following:

Proposition 5.3. *Given $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$, suppose that for a finite set F exist a constant C and a positive integer k such that*

$$|P_{\lambda,\tau}| \geq \frac{C}{(1+|\lambda|^2)^k\,(1+|\tau|^2)^k} \qquad \forall\,(\lambda,\tau) \in (\mathcal{S}_d - F) \times \hat{K}.$$

Then there exist a constant $\tilde{C}$ and a positive integer $\tilde{k}$ such that

$$||P_{\sigma,\tau}|| \geq \frac{\tilde{C}}{(1+|\sigma|^2)^{\tilde{k}}\,(1+|\tau|^2)^{\tilde{k}}} \qquad \forall\,(\sigma,\tau) \in \hat{M} \times \hat{K}.$$

For this purpose we will introduce the Cayley transform which is an inner automorphism of the complex group $G^{\mathbb{C}}$ that conjugates the Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{h}$.

5.4. Cayley transform. Given a compact Cartan subgroup T with Lie algebra $\mathfrak{t}_0$, and given a non-compact root $\beta \in \Delta(\mathfrak{g}, \mathfrak{t})$, we can construct a Cayley transform $\mathbf{c}_\beta$ and a non-compact Cartan subgroup H in the following way:

Let $H_\beta \in \mathfrak{t}$ such that $\beta(H) = B(H, H_\beta)$ for all $H \in \mathfrak{t}$ and we put $H'_\beta = 2|\beta|^{-2} H_\beta$. We choose root vectors $X'_\beta \in \mathfrak{g}_\beta$ and $X'_{-\beta} = -\theta X'_\beta \in \mathfrak{g}_{-\beta}$ such that $B(X'_\beta, X'_{-\beta}) = 2|\beta|^{-2}$ and such that $X'_\beta + X'_{-\beta}$ and $i(X'_\beta - X'_{-\beta})$ are in $\mathfrak{g}_0$. We then define

$$\mathbf{c}_\beta = \mathrm{Ad}\,\left(\exp \frac{\pi}{4}(X'_{-\beta} - X'_\beta)\right) \tag{12}$$

and

$$\mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathbf{c}_\beta(\mathfrak{t}). \tag{13}$$

If we put $\ker \beta = \{H \in \mathfrak{t}_0 : \beta(H) = 0\}$, then $\mathfrak{t}_0 = \ker \beta \oplus \mathbb{R} i H_\beta$. Now $\mathbf{c}_\beta|_{\ker \beta} = Id$ and $\mathbf{c}_\beta(iH_\beta) = i(X'_\beta + X'_{-\beta})$, so

$$\mathfrak{h}_0 = \ker \beta \oplus \mathbb{R}(X'_\beta + X'_{-\beta}), \qquad \mathbf{c}_\beta(\mathfrak{t}_0) = \ker \beta \oplus \mathbb{R} i(X'_\beta + X'_{-\beta}). \tag{14}$$

Let $\mathfrak{a}_0 = \mathbb{R}(X'_\beta + X'_{-\beta})$, $\mathfrak{a}_0$ is a maximal abelian subalgebra in $\mathfrak{p}_0$ because G is of rank one. Besides $\mathfrak{m}_0 = \{X \in \mathfrak{g}_0 : [X, \mathfrak{a}_0] = 0\}$, and it's clear by the choice of $\mathfrak{a}_0$ that $\ker \beta \subset \mathfrak{m}_0$, and $\mathfrak{t}_0^- = \ker \beta$ is maximal abelian subalgebra of $\mathfrak{m}_0$ by dimension. So $\mathbf{c}_\beta$ carries $\mathfrak{t}$ on $\mathfrak{h}$ fixing $\mathfrak{t}^-$.

5.5. Extension of infinitesimal characters of representations of M. If we fix a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$, the set of infinitesimal characters of $\mathcal{S}_d$ can be parametrized by the set of strongly dominant integral forms, that is,

$$\{\lambda \in i\mathfrak{t}'_0 : \lambda(\Gamma_T) \subseteq 2\pi i\mathbb{Z} \text{ and } (\lambda, \alpha) > 0\, \forall\, \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t})\}, \tag{15}$$

where $\Gamma_T = \ker(\exp|_{\mathfrak{t}_0})$. Now, being $\mathfrak{t}_0$ a Cartan subalgebra of $\mathfrak{g}_u$, this set coincides with the set of infinitesimal characters of irreducible unitary representations of finite dimension of the compact form G_u of G.

Note that $\mathfrak{h}_u = \mathbf{c}_\beta(\mathfrak{t}_0) = i\mathfrak{a}_0 \oplus \mathfrak{t}_0^-$ is another Cartan subalgebra of $\mathfrak{g}_u$, and we can also parametrize the set of infinitesimal characters of $\widehat{G}_u$, and then that of $\mathcal{S}_d$, with the set

$$\{\tilde\lambda \in (\mathfrak{a}_0 \oplus i\mathfrak{t}_0^-)' : \tilde\lambda(\Gamma_{H_u}) \subseteq 2\pi i\mathbb{Z} \text{ and } (\tilde\lambda, \tilde\alpha) > 0\, \forall\, \tilde\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})\}, \tag{16}$$

where $\Gamma_{H_u} = \ker(\exp|_{\mathfrak{h}_u}) = \mathbf{c}_\beta(\Gamma_T)$; putting $\tilde\lambda = \mathbf{c}_\beta(\lambda) = \lambda \circ \mathbf{c}_\beta^{-1}$, it's clear that $\mathbf{c}_\beta$ provides a bijection between both sets.

Besides, if $\Gamma_{T^-} = \ker(\exp|_{\mathfrak{t}_0^-})$, the set of infinitesimal characters of irreducible unitary representations of M is given by

$$\{\mu \in i(\mathfrak{t}_0^-)' : \mu(\Gamma_{T^-}) \subseteq 2\pi i\mathbb{Z} \text{ and } (\mu, \alpha) > 0\, \forall\, \alpha \in \Delta^+(\mathfrak{m}, \mathfrak{t}^-)\}. \tag{17}$$

For any $\lambda \in \mathcal{S}_d$, if we put $\mu = \mathbf{c}_\beta(\lambda)|_{\mathfrak{t}_0^-}$, then

$$\mu(\Gamma_{T^-}) = \lambda(\mathbf{c}_\beta^{-1}\Gamma_{T^-}) \subseteq \lambda(\Gamma_T) \subseteq 2\pi i\mathbb{Z},$$

and if $\alpha \in \Delta(\mathfrak{m}, \mathfrak{t}^-)$, $H_\alpha \in i\mathfrak{t}_0^- \subseteq i\mathfrak{t}_0$, by suitable choice of the respective positive systems,

$$(\mu, \alpha) = \mu(H_\alpha) = \lambda(\mathbf{c}_\beta^{-1}H_\alpha) = \lambda(H_\alpha) = (\lambda, \alpha) > 0,$$

and there exists $\sigma \in \hat{M}$ such that $\mu_\sigma = \mathbf{c}_\beta(\lambda)|_{\mathfrak{t}_0^-}$. We want to see that every infinitesimal character μ_σ can be obtained in this way.

Proposition 5.4. *Given F a finite subset of $\mathcal{S}_d$, for all $\sigma \in \hat{M}$ exists a discrete series parameter $\lambda \in \mathcal{S}_d - F$ such that $\mu_\sigma = \mathbf{c}_\beta(\lambda)|_{\mathfrak{t}_0^-}$. Moreover, λ can be chosen such that $|\lambda| \leq C|\mu|$ for some constant C (independent of μ).*

For the proof we will need some previous lemmas.

We choose positive root systems in $\mathfrak{g}$ and $\mathfrak{m}$ in the following way: Let $\{H_1, \ldots, H_n\}$ be an ordered basis of $\mathfrak{a}_0 + i\mathfrak{t}_0^-$ such that H_1 is a basis of $\mathfrak{a}_0$ and $\{H_2, \ldots, H_n\}$ is basis of $i\mathfrak{t}_0^-$ and let $\Delta^+(\mathfrak{g}, \mathfrak{h})$ be the respective positive root system. Then

$$\Delta^+(\mathfrak{m}, \mathfrak{t}^-) = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) : \alpha|_{\mathfrak{a}_0} = 0\}. \tag{18}$$

We also choose $\Delta^+(\mathfrak{g}, \mathfrak{t})$ as the image of $\Delta^+(\mathfrak{g}, \mathfrak{h})$ by $\mathbf{c}_\beta^{-1}$.

Let $\Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\alpha|_{\mathfrak{a}_0} : \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})\}$ be the respective positive restricted root system. Because of the rank one condition, there exists $\beta_0 \in \mathfrak{a}_0'$ such that $\Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\beta_0\}$ or $\Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\frac{1}{2}\beta_0, \beta_0\}$; and we will still denote β_0 its extension to $(\mathfrak{a}_0 + i\mathfrak{t}_0^-)'$ by 0 in $i(\mathfrak{t}_0^-)'$.

Lemma 5.5. *β_0 belongs to the positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$.*

Proof. Follows directly from Lemmas 1 and 2, p. 33 of [**15**]. □

Lemma 5.6. *If $\{\alpha_1, \ldots, \alpha_n\}$ is a simple root system of $\mathfrak{g}$ with respect to the positive system fixed previously, there exist at most two simple roots such that their restriction to $\mathfrak{a}_0$ are not identically 0.*

Proof. Let $\alpha = m_1\alpha_1 + \cdots + m_n\alpha_n$ be the maximal positive root of $\Delta^+(\mathfrak{g}, \mathfrak{h})$; that is, $m_i \in \mathbb{N}$ for all i. Now $\alpha_i|_{\mathfrak{a}_0} = s_i\beta_0$, con $s_i = 0, 1/2$ or 1 for all i, therefore $m_1s_1 + \cdots + m_ns_n \leq 1$ and so at most two of the s_i can be 0. □

Lemma 5.7. *Let G be a semisimple rank one Lie group. For every $\mu \in (i\mathfrak{t}_0^-)'$ strongly dominant analytically integral form, exists $k \in \mathbb{R}$ such that $k\beta_0 + \mu$ is strongly dominant integral in $(\mathfrak{a}_0 + i\mathfrak{t}_0^-)'$. Moreover, we can choose k with the following properties:*

(i) *$k\beta_0 + \mu \notin F$, where F is a fixed finite subset of strongly dominant integral forms in $(\mathfrak{a}_0 + i\mathfrak{t}_0^-)'$.*
(ii) *$|k| \leq C|\mu|$ for some constant C (independent of μ).*

Proof. With no loss of generality, we can suppose that $\mathfrak{g}$ is simple.

Let $\{\alpha_1, \ldots, \alpha_n\}$ be a simple root system of $\mathfrak{g}$ with respect to the positive system previously chosen. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the fundamental weights of $\mathfrak{g}$ with respect to to this simple root system.

Then $\{\alpha_i : \alpha_i|_{\mathfrak{a}_0} = 0\}$ is a simple root system of the semisimple part of $\mathfrak{m}$. If λ_i^M is the respective fundamental weight in $\mathfrak{m}$ and we extend it by 0 to $(\mathfrak{a}_0 + i\mathfrak{t}_0^-)'$,

$$\lambda_i = \lambda_i^M + k_i\beta_0 \tag{19}$$

because $\lambda_i - \lambda_i^M|_{\mathfrak{t}_0^-} = 0$, and $k_i = \frac{(\lambda_i,\beta_0)}{(\beta_0,\beta_0)} \in \frac{1}{2}\mathbb{Z}$ being β_0 a root. We will analyze two cases:

I) The simple root α_1 is the only one with non-vanishing restriction to $\mathfrak{a}_0$. In this case, M is semisimple. Now, because $(\lambda_1, \alpha_i) = 0$ for all $i \geq 2$, $\lambda_1|_{\mathfrak{t}_0^-} = 0$ then $\lambda_1 = k_1\beta_0$, with $k_1 \in \frac{1}{2}\mathbb{Z}$.

So, if μ is strongly dominant integral, $\mu = m_2\lambda_2^M + \cdots + m_n\lambda_n^M$ with $m_i \in \mathbb{N}$ for all $i \geq 2$ and

$$k\beta_0 + \mu = \left(k - \sum_{i=2}^n m_i k_i\right)\beta_0 + \sum_{i=2}^n m_i\lambda_i = \left(k - \sum_{i=2}^n m_i k_i\right)\frac{1}{k_1}\lambda_1 + \sum_{i=2}^n m_i\lambda_i,$$

so it's enough to choose $k = k_0k_1 + m_2k_2 + \cdots + m_nk_n \in \frac{1}{2}\mathbb{Z}$ with $k_0 \in \mathbb{Z}^+$ minimum subject to the condition $k\beta_0 + \mu \notin F$.

II) There exist two simple roots with non-vanishing restriction to $\mathfrak{a}_0$. In this case, looking at the Satake diagrams ([**15**, Chap. 1], it's clear that the only possibility is $\mathfrak{g}_0 \simeq \mathfrak{su}(n,1)$. We choose $\mathfrak{h}_0$ as the 0 trace diagonal matrices. If $H = \sum ih_iE_{ii}$, let $e_i \in i\mathfrak{h}_0'$ be defined by $e_i(H) = h_i$. We fix $\Delta^+(\mathfrak{g},\mathfrak{h}) = \{e_i - e_j : i < j\}$ and $\{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n\}$ becomes a simple root system. Taking $\mathfrak{a}_0 = i\mathbb{R}(E_{11} - E_{n+1\,n+1})$, then α_1 and α_n have non-vanishing restriction to $\mathfrak{a}_0$ and $\beta_0 = e_1 - e_{n+1}$. In this case $\mathfrak{m} \simeq \mathbb{C} \oplus \mathfrak{sl}(n-1,\mathbb{C})$. The center of $\mathfrak{m}$ is $\mathbb{C}H_0$, where

$$H_0 = \begin{pmatrix} -(n-1)i & & \\ & 2iI & \\ & & -(n-1)i \end{pmatrix}.$$

The fundamental weights of $\mathfrak{g}$ with respect to this simple root system are $\lambda_i = e_1 + \cdots + e_i$, so it follows that $\lambda_1 + \lambda_n = \beta_0$ and $\lambda_1 - \lambda_n$ is the coordinate function of the center of $\mathfrak{m}$.

Then, if μ is strongly dominant integral, $\mu = m(\lambda_1 - \lambda_n) + m_2\lambda_2^M + \cdots + m_{n-1}\lambda_{n-1}^M$ with $m_i \in \mathbb{N}$ for all $2 \leq i \leq n-1$, $m \in \mathbb{R}$. It's easy to see that μ

analytically integral implies $m \in \mathbb{Z}$. Then

$$\begin{aligned} k\beta_0 + \mu &= k\beta_0 + m(\lambda_1 - \lambda_n) + \sum_{i=2}^{n-1} m_i(\lambda_i - k_i\beta_0) \\ &= \left(k + m - \sum_{i=2}^{n-1} m_i k_i\right)\lambda_1 + \left(k - m - \sum_{i=2}^{n-1} m_i k_i\right)\lambda_n + \sum_{i=2}^{n-1} m_i\lambda_i, \end{aligned}$$

and we choose $k = k_0 + |m| + \sum_{i=2}^{n-1} m_i k_i$ with $k_0 \in \mathbb{Z}^+$ minimum subject to the condition $k\beta_0 + \mu \notin F$. □

Proof of Proposition 5.4. It suffices to take $\lambda = \mathbf{c}_\beta^{-1}(k\beta_0 + \mu)$, with k given by Lemma 5.7, reminding that we are supposing $G^{\mathbb{C}}$ simply connected. □

Proof of Proposition 5.3. By hypothesis there exists a constant C and a positive integer k such that

$$|P_{\lambda,\tau}| \geq \frac{C}{(1+|\lambda|^2)^k\,(1+|\tau|^2)^k} \qquad \forall\,(\lambda,\tau) \in (\mathcal{S}_d - F) \times \hat{K}.$$

Besides, according to the choice of $\mathfrak{t}_0$ and $\mathfrak{h}_0$, $\gamma^G = \mathbf{c}_\beta^{-1} \circ \gamma_{\mathfrak{h}}^G$, therefore

$$\begin{aligned} P_{\lambda,\tau} &= \big((\gamma^G \otimes \gamma^K)(P)\big)(\lambda, \mu_\tau) \\ &= \big((\gamma_{\mathfrak{h}}^G \otimes \gamma^K)(P)\big)(\lambda \circ \mathbf{c}_\beta^{-1}, \mu_\tau) \\ &= \big((\gamma_{\mathfrak{h}}^G \otimes \gamma^K)(P)\big)(\mathbf{c}_\beta(\lambda), \mu_\tau); \end{aligned}$$

given $\sigma \in \hat{M}$, let $\lambda \in \mathcal{S}_d - F$ be given by Proposition 5.4. If we set $x_\sigma = \mathbf{c}_\beta(\lambda)|_{\mathfrak{a}_0} \in i\mathfrak{a}_0'$, then

$$\begin{aligned} P_{\sigma,\tau}(x_\sigma) &= \left(\left(\gamma_{\mathfrak{h}}^G \otimes \gamma^K\right)(P)\right)(x_\sigma + \mu_\sigma, \mu_\tau) \\ &= \left(\left(\gamma_{\mathfrak{h}}^G \otimes \gamma^K\right)(P)\right)(\mathbf{c}_\beta(\lambda), \mu_\tau) = P_{\lambda,\tau}. \end{aligned}$$

If P has order m in $\mathcal{U}(\mathfrak{g})$, $P_{\sigma,\tau}$ is a polynomial function on $\mathfrak{a}'$ of order $\leq m$, and $||P_{\sigma,\tau}||$ is the norm of the vector in $\mathbb{C}^{m+1}$ formed with its coefficients, and by Schwarz inequality

$$|P_{\sigma,\tau}(x_\sigma)| \leq ||P_{\sigma,\tau}|| \left(\sum_{j=0}^{m} |x_\sigma|^{2m}\right)^{1/2} \leq ||P_{\sigma,\tau}||(1+|x_\sigma|^2)^{\frac{m}{2}};$$

besides,

$$1 + |x_\sigma|^2 \leq C_2(1 + |\mu_\sigma|^2),$$

so

$$||P_{\sigma,\tau}|| \geq \frac{\tilde{C}}{(1+|\sigma|^2)^{k+\frac{m}{2}}\,(1+|\tau|^2)^k} \qquad \forall\,(\sigma,\tau) \in \hat{M} \times \hat{K},$$

and it's enough to choose $\tilde{k} = k + \frac{m}{2}$. □

6. Inversion of global characters.

One important step in building the fundamental solution of P is the construction of distributions R_π such that $PR_\pi = \Theta_\pi$ for each representation π that appears in the Plancherel formula. In this section we will define these distributions R_π. We will begin with the principal series representations. Before this we state some necessary results in order to bound the R_π. The next two lemmas follow directly from [**14**, Lemma 5.6.4].

Lemma 6.1. *Let K be a connected compact Lie group. There exist $\Omega \in \mathcal{Z}(\mathfrak{k})$ and a positive constant C such that*

$$|\chi_\tau(\Omega)| \geq C(1+|\tau|^2) \qquad \forall\, \tau \in \hat{K}.$$

Lemma 6.2. *Let G a connected semisimple Lie group with a compact Cartan subgroup. There exist $Z \in \mathcal{Z}(\mathfrak{g})$ and a positive constant C such that*

$$|\chi_\lambda(Z)| \geq C(1+|\lambda|^2) \qquad \forall\, \lambda \in \mathcal{S}_d.$$

Proposition 6.3. *Let G a connected semisimple Lie group. There exist $Z \in \mathcal{Z}(\mathfrak{g})$, a positive constant C, an $\varepsilon > 0$ and a positive integer k such that*

$$|\chi_{\sigma,\nu+z}(Z)| \geq C(1+|\sigma|^2)^k(1+|\nu|^2)^k \qquad \forall\, \sigma \in \hat{M},\ \nu \in i\mathfrak{a}_0',\ z \in \mathfrak{a}',\ |z| < \varepsilon.$$

To prove this proposition, we begin with a lemma.

Lemma 6.4. *Let $\mathfrak{g}_0$ be the Lie algebra of a semisimple Lie group G, $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$, W the Weyl group of $\mathfrak{g}$. There exists $P \in \mathcal{S}(\mathfrak{h})^W$ an homogeneous polynomial function on $\mathfrak{h}'$ such that*

$$P(\lambda) > 0 \quad \forall\, \lambda \in i\mathfrak{h}_0',\ \lambda \neq 0.$$

Proof. We will begin with the construction of a G-invariant polynomial function on $\mathfrak{g}_0$. The symmetric algebra $\mathcal{S}(\mathfrak{g}_0)$ (resp. $\mathcal{S}(\mathfrak{g}_0')$) is identified with the set of polynomial functions on $\mathfrak{g}_0'$ (resp. $\mathfrak{g}_0$). We denote $I(\mathfrak{g}_0)$ (resp. $I(\mathfrak{g}_0')$) the G-invariant elements of $\mathcal{S}(\mathfrak{g}_0)$ (resp. $\mathcal{S}(\mathfrak{g}_0')$).

The Killing form B of $\mathfrak{g}_0$, being non-degenerate and G-invariant, induces canonical isomorphisms between $\mathcal{S}(\mathfrak{g}_0)$ and $\mathcal{S}(\mathfrak{g}_0')$ and between $I(\mathfrak{g}_0)$ and $I(\mathfrak{g}_0')$.

Given $X \in \mathfrak{g}_0$, let

$$p_X(x) = \det(xI - \operatorname{ad} X) = x^n + a_{n-1}(X)x^{n-1} + \cdots + a_l(X)x^l$$

be the characteristic polynomial of ad X, $n = \dim \mathfrak{g}_0$, $l = \dim \mathfrak{h}_0$. The coefficients $a_i(X)$ are G-invariant homogeneous polynomial functions of degree $n-i$ on $\mathfrak{g}_0$. Let d_i and p be positive integers such that $n - i + d_i = 4p$; if we define

$$\tilde{Q} = a_l^{d_l} + \cdots + a_{n-1}^{d_{n-1}},$$

then $\tilde{Q} \in I(\mathfrak{g}_0')$ and is homogeneous of degree $4p$.

B is non-degenerate on $\mathfrak{h}_0$, so $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{h}_0^\perp$, $\mathfrak{h}_0^\perp$ the orthogonal of $\mathfrak{h}_0$ with respect to B. Besides $\mathfrak{h}_0^\perp = (\mathfrak{g}^+ \oplus \mathfrak{g}^-) \cap \mathfrak{g}_0$ so $\tilde{Q}$ vanishes on $\mathfrak{h}_0^\perp$ and is positive on $\mathfrak{h}_0$. Let $Q \in I(\mathfrak{g}_0)$ be the image of $\tilde{Q}$ by the isomorphism mentioned above.

We also note that we can include $\mathfrak{h}_0'$ in $\mathfrak{g}_0'$ by extending a functional from $\mathfrak{h}_0$ to $\mathfrak{g}_0$ by 0 on $\mathfrak{h}_0^\perp$. This inclusion allow us to restrict elements from $\mathcal{S}(\mathfrak{g}_0)$ to $\mathcal{S}(\mathfrak{h}_0)$.

Let $P \in \mathcal{S}(\mathfrak{h}_0)$ be the restriction of Q. P has the following properties:

(i) $P(\lambda) > 0$ for all $\lambda \in \mathfrak{h}_0'$, $\lambda \neq 0$.
(ii) $P \in \mathcal{S}(\mathfrak{h})^W$.

It only remains to see (ii): It's clear that $I(\mathfrak{g}_0)$ is included in $I(\mathfrak{g})$. Besides, $Q = P + \tilde{P}$, with $\tilde{P} \in \mathcal{S}(\mathfrak{g}_0)\mathfrak{h}_0^\perp \subset \mathcal{S}(\mathfrak{g})(\mathfrak{g}^+ \oplus \mathfrak{g}^-)$. Now, $W_G = W(G_u, H_u)$ ([**10**, Thm. 4.41]), so, if $w \in W_G$, $w = \mathrm{Ad}\,(x)$, with $x \in N_{G_u}(H_u)$, and the action of w preserves $\mathfrak{h}$ and $\mathfrak{g}^+ \oplus \mathfrak{g}^-$, then $Q = wQ = wP + w\tilde{P}$, therefore $wP = P$, $w\tilde{P} = \tilde{P}$.

Finally, if $\lambda \in i\mathfrak{h}_0'$, $P(\lambda) = (-i)^{4p}P(i\lambda) > 0$. □

Proof of Proposition 6.3. Let $P \in \mathcal{S}(\mathfrak{h})^W$ be given by the lemma above. If

$$2c = \inf\{P(\lambda) : \lambda \in i\mathfrak{h}_0',\ |\lambda| = 1\} > 0,$$

there exists $0 < \varepsilon < 1/2$ such that $\mathrm{Re}\ P(\lambda + \tilde{\lambda}) \geq c$ for all $\lambda \in i\mathfrak{h}_0'$, $|\lambda| = 1$, $\tilde{\lambda} \in \mathfrak{h}'$, $|\tilde{\lambda}| < 2\varepsilon$. Suppose now that $\lambda \in i\mathfrak{h}_0'$, $|\lambda| \geq 1$, $\tilde{\lambda} \in \mathfrak{h}'$, $|\tilde{\lambda}| < \varepsilon$, then, as $\mathrm{Re}\ P$ is also an homogeneous polynomial of degree $4p$, we have

$$\mathrm{Re}\ P(\lambda + \tilde{\lambda}) = |\lambda|^{4p}\mathrm{Re}\ P\left(\frac{\lambda}{|\lambda|} + \frac{\tilde{\lambda}}{|\lambda|}\right) \geq |\lambda|^{4p}c.$$

On the other hand, if $|\lambda| \leq 1$, $|\tilde{\lambda}| < \varepsilon$, $|P(\lambda + \tilde{\lambda})|$ is bounded by a positive constant A. Now let $Z \in \mathcal{Z}(\mathfrak{g})$ be such that $\gamma_{\mathfrak{h}}(Z) = P + 2A$. If $\sigma \in \hat{M}$, $\nu \in i\mathfrak{a}_0'$, $z \in \mathfrak{a}'$, $|z| < \varepsilon$, then, if $|\mu_\sigma + \nu| \leq 1$,

$$|\chi_{\sigma,\nu+z}(P)| \geq A \geq C_1(1 + |\mu_\sigma|^2)^{2p}(1 + |\nu|^2)^{2p},$$

and, if $|\mu_\sigma + \nu| \geq 1$,

$$|\chi_{\sigma,\nu+z}(P)| \geq 2A + \mathrm{Re}\ P(\mu_\sigma + \nu + z) \geq C_2(1 + |\mu_\sigma|^2)^{2p}(1 + |\nu|^2)^{2p}.$$

□

Now we state a lemma we will use to obtain uniform bounds of the distributions $\Theta_{\sigma,\nu}$ and Θ_λ. The proof follows easily from the proof of Theorem 10.2 in [**10**].

Lemma 6.5. *Given $\tilde{K} \subset G$ a compact subset, for every admissible representation π whose K-types have the multiplicity property $n_\tau \leq N \dim \tau$ there* *exists a differential operator* $\tilde{\Omega} \in \mathcal{Z}(\mathfrak{k})$ *and a constant* C *which only depends on* $\tilde{K}$ *such that if* $\operatorname{supp} f \subseteq \tilde{K}$

$$|\Theta_\pi(f)| \le CN \left(\int_G |\tilde{\Omega} f(g)|^2 ||\pi(g)||^2 \, dg \right)^{1/2}.$$

In particular, if π *is unitary,* $|\Theta_\pi(f)| \le CN ||\tilde{\Omega} f||_{L^2(G)}$.

6.1. Principal series. Before proceeding to the definition of the distributions $R_{\sigma,\nu}$ we state two lemmas about polynomial functions.

If m is a positive integer we denote $\mathrm{Pol}(m)$ the complex vector space of polynomial functions in $\mathbb{C}^n$ of degree $\le m$, and $\mathrm{Pol}^0(m)$ this vector space with the origin removed.

Given $Q \in \mathrm{Pol}(m)$, $\zeta \in \mathbb{C}^n$, the equation

$$\tilde{Q}(\zeta) = \left(\sum_\alpha \frac{1}{(\alpha!)^2} |Q^{(\alpha)}(\zeta)|^2 \right)^{\frac{1}{2}} \tag{20}$$

defines a norm in $\mathrm{Pol}(m)$ for a fixed $\zeta \in \mathbb{C}^n$. Let

$$||Q|| = \tilde{Q}(0). \tag{21}$$

The following lemma is used to construct a fundamental solution of a constant coefficients differential operator in $\mathbb{R}^n$ (Theorem 7.3.10 of [**8**]). For a proof see Lemmas 7.3.11 and 7.3.12 in [**8**].

Lemma 6.6. *For all* $\varepsilon > 0$ *exists a non-negative function* $\Phi \in C^\infty(\mathrm{Pol}^0(m) \times \mathbb{C}^n)$ *such that*

(i) Φ *is absolutely homogeneous of degree* 0, *i.e.,* $\Phi(zQ, \zeta) = \Phi(Q, \zeta)$ *for all* $z \in \mathbb{C} - \{0\}$.
(ii) $\Phi(Q, \zeta) = 0$ *if* $|\zeta| \ge \varepsilon$.
(iii) *If* F *is an entire function in* $\mathbb{C}^n$ *and* $d\zeta$ *is Lebesgue measure in* $\mathbb{C}^n$,

$$\int F(\zeta) \Phi(Q, \zeta) \, d\zeta = F(0).$$

(iv) *There exists a constant* C_ε *such that* $\tilde{Q}(0) \le C_\varepsilon |Q(\zeta)|$ *if* $\Phi(Q, \zeta) \ne 0$.

Being $\mathrm{Pol}(m)$ a finite dimensional vector space, all norms $\tilde{Q}(\zeta)$ are equivalent. The following lemma estimates the constants which give this equivalence. The proof is an easy consequence of Taylor's formula.

Lemma 6.7. *Exists a constant* $C > 0$ *depending only on* n *and* m *such that*

$$C \left(1 + |\zeta|^2\right)^{-m} \tilde{Q}(0)^2 \le \tilde{Q}(\zeta)^2 \le C \left(1 + |\zeta|^2\right)^m \tilde{Q}(0)^2$$

for all $Q \in \mathrm{Pol}(m)$.

We are now in position to define the distributions $R_{\sigma,\nu}$ for $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfying the hypothesis of Theorem 1.1. Let $\varepsilon > 0$ be given by Proposition 6.3, m the order of P and let $\Phi \in C^\infty(\mathrm{Pol}^0(m) \times \mathbb{C})$ be the non-negative function given by Lemma 6.6.

Given $\sigma \in \hat{M}$, $\tau \in \hat{K}$, and a fixed $\nu \in \mathfrak{a}'$, we put

$$P^\nu_{\sigma,\tau}(z) = P_{\sigma,\tau}(\nu + z) = (\chi_{\sigma,\nu+z} \otimes \chi_\tau)(P). \tag{22}$$

Finally, if dz is Lebesgue measure in $\mathfrak{a}'$, $f \in \mathcal{D}(G)$, we define

$$R_{\sigma,\nu}(f) = \sum_{\tau \in \hat{K}} \int_{|z|<\varepsilon} \frac{1}{P^\nu_{\sigma,\tau}(z)} \Theta^\tau_{\sigma,\nu+z}(f) \Phi(P^\nu_{\sigma,\tau}, z)\, dz. \tag{23}$$

This definition makes sense because $P^\nu_{\sigma,\tau}(z) \neq 0$ if $\Phi(P^\nu_{\sigma,\tau}, z) \neq 0$ (Lemma 6.6 (iv)).

Proposition 6.8. *The map defined by* (23) *is a finite order distribution for all $\sigma \in \hat{M}$, $\nu \in \mathfrak{a}'$. This map has also the following properties:*

(i) $PR_{\sigma,\nu} = \Theta_{\sigma,\nu}$.

(ii) *For every positive integer k and $f \in \mathcal{D}(G)$, exist a constant $C > 0$ which only depends on the support of f and a differential operator $D_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ such that*

$$|R_{\sigma,\nu}(f)| \leq \frac{C}{(1+|\nu|^2)^k(1+|\sigma|^2)^k} ||D_k f||_{L^2(G)} \qquad \forall\, \sigma \in \hat{M},\, \nu \in i\mathfrak{a}'_0.$$

Proof. By Proposition 3.2, $P\Theta^\tau_{\sigma,\nu+z} = P^\nu_{\sigma,\tau}(z)\Theta^\tau_{\sigma,\nu+z}$, and, as $\Theta_{\sigma,\nu+z}(f)$ is an entire function in z ([**6**], Section 21) for fixed f, so it is $\Theta^\tau_{\sigma,\nu+z}(f)$, and by Lemma 6.6 (ii),

$$\int_{|z|<\varepsilon} \Theta^\tau_{\sigma,\nu+z}(f)\Phi(P^\nu_{\sigma,\tau}, z)\, dz = \Theta^\tau_{\sigma,\nu}(f),$$

and this proves (i).

Let's see that (23) defines a distribution: According to Lemma 6.6 (iv) and Lemma 6.7 together with (20) and (21), if $\Phi(P^\nu_{\sigma,\tau}, z) \neq 0$, we have $||P_{\sigma,\tau}|| \leq C_1(1+|\nu|^2)^m |P^\nu_{\sigma,\tau}(z)|$, and by the hypothesis on P (Prop. 5.1 or 5.3), exists a positive integer $\tilde{k}$ such that

$$\frac{1}{|P^\nu_{\sigma,\tau}(z)|} \leq C_2(1+|\nu|^2)^m \left(1+|\sigma|^2\right)^{\tilde{k}} \left(1+|\tau|^2\right)^{\tilde{k}}. \tag{24}$$

On the other hand, if $Z \in \mathcal{Z}(\mathfrak{g})$ and $\Omega \in \mathcal{Z}(\mathfrak{k})$ are given by Proposition 6.3 and Lemma 6.1 respectively and if s_1 and s_2 are positive integers, we have, for some positive integer $\bar{k}$,

$$|\Theta^{\tau}_{\sigma,\nu+z}(f)| = \frac{|\Theta^{\tau}_{\sigma,\nu+z}\left((Z^t)^{s_1}(\Omega^t)^{s_2}f\right)|}{|\chi^{s_1}_{\sigma,\nu+z}(Z)||\chi^{s_2}_{\tau}(\Omega)|} \leq \frac{C_3\,|\Theta_{\sigma,\nu+z}\left((Z^t)^{s_1}(\Omega^t)^{s_2}f_{\bar{\tau}}\right)|}{\left((1+|\sigma|^2)(1+|\nu|^2)\right)^{s_1-\bar{k}}(1+|\tau|^2)^{s_2}},$$

besides, it holds for $D \in \mathcal{U}(\mathfrak{g})^K$ that $Df_{\bar{\tau}} = (Df)_{\bar{\tau}}$; also $|\Theta_\pi(f_{\bar{\tau}})| \leq |\Theta_\pi(f)|$, so

$$|\Theta^{\tau}_{\sigma,\nu+z}(f)| \leq \frac{C_3\,|\Theta_{\sigma,\nu+z}\left((Z^t)^{s_1}(\Omega^t)^{s_2}f\right)|}{\left((1+|\sigma|^2)(1+|\nu|^2)\right)^{s_1-\bar{k}}(1+|\tau|^2)^{s_2}}. \tag{25}$$

Let $\tilde{K} \subseteq G$ be a compact subset. Note that for principal series it holds $n_\tau \leq \dim\tau$ ([**10**, p. 207]); so by Lemma 6.5 exist $\tilde{\Omega} \in \mathcal{Z}(\mathfrak{k})$ and a constant C_1 independent of $\sigma \in \hat{M}$, ν, $z \in \mathfrak{a}'$ such that

$$|\Theta_{\sigma,\nu+z}(f)| \leq C_1\left(\int_G |\tilde{\Omega}f(g)|^2||\pi_{\sigma,\nu+z}(g)||^2\,dg\right)^{1/2};$$

on the other hand, given φ in the space V^σ where the $\pi_{\sigma,\nu+z}$ acts, if $a(g)$ is the A-component of g in the Iwasawa decomposition, then (cf. [**10**, p. 169]),

$$\left(\pi_{\sigma,\nu+z}(g)\varphi\right)(k) = e^{-z\log a(g^{-1}k)}\left(\pi_{\sigma,\nu}(g)\varphi\right)(k),$$

and taking $A = \sup_{g\in\tilde{K},k\in K,|z|<\varepsilon}|e^{-z\log a(g^{-1}k)}|$ and $B_{\sigma,\nu} = \sup_{g\in\tilde{K}}||\pi_{\sigma,\nu}(g)||$, then $||\pi_{\sigma,\nu+z}(g)|| \leq AB_{\sigma,\nu}$ uniformly on $\tilde{K}$, so for all f such that $\operatorname{supp} f \subseteq \tilde{K}$,

$$|\Theta^{\tau}_{\sigma,\nu+z}(f)| \leq AB_{\sigma,\nu}||\tilde{\Omega}f||_{L^2(G)}. \tag{26}$$

Now combining (24), (25) and (26) and the fact that Φ is uniformly bounded we obtain

$$\left|\int_{|z|<\varepsilon}\frac{\Theta^{\tau}_{\sigma,\nu+z}(f)\Phi(P^{\nu}_{\sigma,\tau},z)}{P^{\nu}_{\sigma,\tau}(z)}\,dz\right| \leq \frac{C_4B_{\sigma,\nu}\,||\tilde{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2}f||_{L^2(G)}}{(1+|\nu|^2)^{s_1-m-\bar{k}}(1+|\sigma|^2)^{s_1-\tilde{k}-\bar{k}}(1+|\tau|^2)^{s_2-\tilde{k}}};$$

therefore for all f such that $\operatorname{supp} f \subseteq \tilde{K}$

$$|R_{\sigma,\nu}(f)| \leq \left(\sum_{\tau\in\hat{K}}\frac{1}{(1+|\tau|^2)^{s_2-\tilde{k}}}\right)\frac{C_4B_{\sigma,\nu}||\tilde{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2}f||_{L^2(G)}}{(1+|\nu|^2)^{s_1-m-\bar{k}}(1+|\sigma|^2)^{s_1-\tilde{k}-\bar{k}}}, \tag{27}$$

and $\displaystyle\sum_{\tau\in\hat{K}}\frac{1}{(1+|\tau|^2)^{s_2-\tilde{k}}}$ is finite if we choose $s_2 > \tilde{k}+1/2\dim K$ ([**14**, Lemma 5.6.7]), so $R_{\sigma,\nu}$ is a finite order distribution.

To see (ii), just observe that $B_{\sigma,\nu} = 1$ if $\nu \in i\mathfrak{a}_0'$, so given k if we take $D_k = \tilde{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2}$ with the s_2 chosen above and $s_1 \geq k + \bar{k} + \max(\tilde{k}, m)$, (27) becomes

$$|R_{\sigma,\nu}(f)| \leq \frac{C}{(1+|\nu|^2)^k(1+|\sigma|^2)^k}\,||D_k f||_{L^2(G)}$$

with C depending only $\tilde{K}$. □

6.2. Discrete Series. Suppose T is a compact Cartan subgroup of G.

Proposition 6.9. *Let $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ such that exist a constant C and a positive integer k such that*

$$|P_{\lambda,\tau}| \geq \frac{C}{(1+|\lambda|^2)^k\,(1+|\tau|^2)^k} \qquad \forall\,(\lambda,\tau) \in \mathcal{S}_d \times \hat{K},$$

then the map

$$R_\lambda(f) = \sum_{\tau \in \hat{K}} \frac{1}{P_{\lambda,\tau}} \Theta_\lambda^\tau(f)$$

defines a finite order distribution with the following properties:

(i) $PR_\lambda = \Theta_\lambda$.

(ii) *For each positive integer k and $f \in \mathcal{D}(G)$, exist a constant $C > 0$ which only depends on the support of f and a differential operator $E_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ such that*

$$|R_\lambda(f)| \leq \frac{C}{(1+|\lambda|^2)^k}\,||E_k f||_{L^2(G)} \qquad \forall\,\lambda \in \mathcal{S}_d.$$

Proof. First note that R_λ is well defined because by Proposition 5.2 exist a constant C_1 and a positive integer k_1 such that

$$|P_{\lambda,\tau}| \geq \frac{C_1}{(1+|\lambda|^2)^k_1\,(1+|\tau|^2)^k_1} \qquad \forall\,(\lambda,\tau) \in \mathcal{S}_d \times \hat{K},$$

and in particular $P_{\lambda,\tau} \neq 0$; (i) is clear because $P\Theta_\lambda^\tau = P_{\lambda,\tau}\Theta_\lambda^\tau$ (Proposition 3.2).

To see (ii), observe that by Lemma 6.5 exist a constant C_2 depending only on supp f and $\tilde{\Omega} \in \mathcal{Z}(\mathfrak{k})$ such that $|\Theta_\lambda(f)| \leq C_2||\tilde{\Omega}f||_{L^2(G)}$.

Then, if $Z \in \mathcal{Z}(\mathfrak{g})$ and $\Omega \in \mathcal{Z}(\mathfrak{k})$ are given by Lemmas 6.2 and 6.1 respectively, we have

$$\begin{aligned}|R_\lambda(f)| &\le \sum_{\tau\in\hat{K}} \frac{1}{|P_{\lambda,\tau}|}|\Theta_\lambda^\tau(f)| \le \sum_{\tau\in\hat{K}} \frac{1}{C_3}\left(1+|\lambda|^2\right)_1^k\left(1+|\tau|^2\right)_1^k|\Theta_\lambda(f_{\bar\tau})|\\ &\le \sum_{\tau\in\hat{K}} \frac{\tilde{C}}{(1+|\lambda|^2)^{s_1-k_1}(1+|\tau|^2)^{s_2-k_1}}|\Theta_\lambda((Z^t)^{s_1}(\Omega^t)^{s_2}f_{\bar\tau})|\\ &\le \left(\sum_{\tau\in\hat{K}}\frac{1}{(1+|\tau|^2)^{s_2-k_1}}\right)\frac{\tilde{C}C_1}{(1+|\lambda|^2)^{s_1-k_1}}||(\tilde{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2}f)||_{L^2(G)},\end{aligned}$$

and it suffices to take $E_k = \tilde{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2}$ with $s_1 = k + k_1$ and s_2 such that $\displaystyle\sum_{\tau\in\hat{K}}\frac{1}{(1+|\tau|^2)^{s_2-k_1}}$ is finite. □

7. Demonstration of Theorem 1.1.

Now we are ready to complete the proof of Theorem 1.1 with the explicit construction of the fundamental solution of P.

Proposition 7.1. *Let $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ and suppose that exist a constant C and a positive integer k such that in case* I,

$$||P_{\sigma,\tau}|| \ge \frac{C}{(1+|\sigma|^2)^k(1+|\tau|^2)^k} \qquad \forall\,(\sigma,\tau)\in\hat{M}\times\hat{K}, \tag{28}$$

and in case II,

$$|P_{\lambda,\tau}| \ge \frac{C}{(1+|\lambda|^2)^k(1+|\tau|^2)^k} \qquad \forall\,(\lambda,\tau)\in\mathcal{S}_d\times\hat{K}; \tag{29}$$

if $R_{\sigma,\nu}$ and R_λ are the distributions defined respectively by (23) *and* 6.9*, then the map R defined in case* I,

$$R = \sum_{\sigma\in\hat{M}}\int_{\nu\in i\mathfrak{a}_0'} R_{\sigma,\nu}\, m_\sigma(\nu)\,d\nu, \tag{30}$$

and in case II,

$$R = \sum_{\sigma\in\hat{M}}\int_{\nu\in i\mathfrak{a}_0'} R_{\sigma,\nu}\, m_\sigma(\nu)\,d\nu + \sum_{\lambda\in\mathcal{S}_d} d_\lambda\, R_\lambda, \tag{31}$$

is a finite order distribution which is a fundamental solution of P.

Remark. We note that in case II, T is a fundamental Cartan subgroup of G, and in case I, $H = A\times T^-$ is.

So if $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ is such that $\left(\gamma^G \otimes \gamma^K\right)(P)$ has a fundamental solution in $H^f \times T$, Propositions 5.2 and 5.1 imply (28) and (29) respectively in each case, so Theorem 1.1 is a direct consequence of Proposition 7.1.

Proof. First of all, we note that in case II, Proposition 5.3 says that (28) implies (29) (changing, maybe, C and k), so $R_{\sigma,\nu}$ is well defined and Proposition 6.8 applies in this case.

Equality $PR = \delta$ is clear by Plancherel formula (Theorem 2.1) and because $PR_{\sigma,\nu} = \Theta_{\sigma,\nu}$ and $PR_\lambda = \Theta_\lambda$ (Propositions 6.8 and 6.9 respectively); it only remains to prove that R is a finite order distribution in each case. So we will prove that each of the following are finite order distributions:

$$R_{sp} = \sum_{\sigma\in\hat{M}} \int_{\nu\in i\mathfrak{a}_0'} R_{\sigma,\nu}\, m_\sigma(\nu)\, d\nu, \qquad R_{sd} = \sum_{\lambda\in\mathcal{S}_d} d_\lambda\, R_\lambda.$$

Let $\tilde{K}$ be a compact subset and $f \in \mathcal{D}_{\tilde{K}}(G)$; for each positive integer k let $D_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ be given by Proposition 6.8 (ii), then, using 2.1 (ii),

$$|R_{sp}(f)| \le C_1 C_2 \left(\sum_{\sigma\in\hat{M}} \frac{1}{(1+|\sigma|^2)^{k-l_2}}\right)\left(\int_{\nu\in i\mathfrak{a}_0'} \frac{1}{(1+|\nu|^2)^{k-l_1}}\right) ||D_k f||_{L^2(G)},$$

and choosing k large enough so that the sum and the integral are finite, we obtain an operator $D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ and a constant C depending only on $\tilde{K}$ such that

$$|R_{sp}(f)| \le C||Df||_{L^2(G)},$$

and this proves that R_{sp} is a distribution of finite order less or equal that the order of D.

In the same way, for each positive integer k let $E_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ be given by Proposition 6.9 (ii), therefore

$$|R_{sd}(f)| \le C_3 \left(\sum_{\tau\in\hat{K}} \frac{d_\lambda}{(1+|\lambda|^2)^k}\right) ||E_k f||_{L^2(G)};$$

if $k > 1/2 \dim G$, $\displaystyle\sum_{\tau\in\hat{K}} \frac{d_\lambda}{(1+|\lambda|^2)^k}$ is finite ([**14**, Lemma 5.6.7]), there exist $E \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ and a constant $\tilde{C}$ depending only on $\tilde{K}$ such that

$$|R_{sd}(f)| \le \tilde{C}||Ef||_{L^2(G)},$$

so R_{sd} is a distribution of finite order less or equal that the order of E. ☐

8. P-convexity of G.

Suppose that $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfies the conditions of Proposition 7.1. So P has a fundamental solution $R \in \mathcal{D}'(G)$. This implies that the differential equation $Pu = f$ has a solution $u \in C^\infty(G)$ for all $f \in \mathcal{D}(G)$; just taking $u = f * R$ because $Pu = f * PR = f * \delta = f$. Now, in order to guarantee the solvability of $Pu = f$ when $f \in C^\infty(G)$, it is necessary to analyze the P-convexity of G.

Definition 2. Given $D \in \mathcal{U}(\mathfrak{g})$, we say that G is D-convex if for every compact subset $\Omega \subseteq G$ exists another compact subset $\tilde{\Omega} \subseteq G$ such that

$$\operatorname{supp}(Df) \subseteq \Omega \implies \operatorname{supp}(f) \subseteq \tilde{\Omega}.$$

Using Johnson's injectivity criterion ([**9**]), we will verify that G is P-convex.

Let $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfying the conditions of 7.1. Then $P_{\sigma,\tau}$ is a non-zero polynomial on $\mathfrak{a}'$ for all $(\sigma, \tau) \in \hat{M} \times \hat{K}$.

Given $\sigma \in \hat{M}$, $\nu \in \mathfrak{a}'$, we denote V^σ the space where the principal series representation $\pi_{\sigma,\nu}$ acts (remind that we can choose V^σ independent of ν). Let V_F^σ be the subspace of K-finite vectors of $\pi_{\sigma,\nu}$.

Now, if U is a $\pi_{\sigma,\nu}(P)$-invariant finite dimensional subspace of V_F^σ, then

$$U \subseteq W = \sum_{j=i}^{k} n_{\tau_j}^{\sigma} V_{\tau_j};$$

on the other hand, if $v_j \in V_{\tau_j}$, $\pi_{\sigma,\nu}(P)v_j = (\chi_{\sigma,\nu} \otimes \chi_{\tau_j})(P)v_j = P_{\sigma,\tau_j}(\nu)v_j$, that is, $\pi_{\sigma,\nu}(P)$ is diagonalizable on W, so U has a $\pi_{\sigma,\nu}(P)$-invariant complement $\tilde{U}$ in W.

Suppose that $\det \pi_{\sigma,\nu}(P)|_U = 0$ for all $\nu \in \mathfrak{a}'$. Then

$$0 = \det \pi_{\sigma,\nu}(P)|_U \det \pi_{\sigma,\nu}(P)|_{\tilde{U}} = \prod_{j=1}^{k} \left(P_{\sigma,\tau_j}(\nu)\right)^{d_{\tau_j} n_{\tau_j}^{\sigma}} \qquad \forall\, \nu \in \mathfrak{a}',$$

therefore exist $\sigma \in \hat{M}$, $\tau \in \hat{K}$ such that $P_{\sigma,\tau}(\nu) = 0$ for all $\nu \in \mathfrak{a}'$ which is absurd. So Theorems 5.1 and 5.2 in [**9**] imply that G is P-convex.

9. Parametrix of operators in $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$.

In this section, using Zuckerman characters identities combined with the work done so far, we will prove:

Proposition 9.1. *Let G be a linear connected semisimple Lie group of rank one having a compact Cartan subgroup, and $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. A sufficient condition for the existence of a parametrix of P is:*

Exist a finite set $F \subset \mathcal{S}_d$, a positive constant C and a positive integer k such that

$$|P_{\lambda,\tau}| \geq \frac{C}{(1+|\lambda|^2)^k\,(1+|\tau|^2)^k} \qquad \forall\,(\lambda,\tau) \in (\mathcal{S}_d - F) \times \hat{K}. \tag{32}$$

Remark. The fact that if $\big(\gamma^G \otimes \gamma^K\big)(P)$ has a parametrix on $T \times T$ then (32) holds is an easy consequence of [**4**, Thm. II]. So Proposition 9.1 clearly implies Theorem 1.3.

Before proceeding to the proof of this proposition, we state Zuckerman characters identities in the form we need (cf. Proposition 5.13 from [**16**] for a more precise statement together with the proof).

Proposition 9.2. *Let G be as in Proposition* 9.1. *We put $r = \frac{|W(\mathfrak{g},\mathfrak{h})|}{|W(G,H)|}$. Given $\lambda \in \mathcal{S}_d$ exist $\sigma_1^\lambda, \ldots, \sigma_r^\lambda \in \hat{M}$ and $\nu_1^\lambda, \ldots, \nu_r^\lambda \in \mathfrak{a}_0'$ such that*

$$\frac{1}{|W_K|} \sum_{w \in W_G} \Theta_{w\lambda} = a_0(\lambda)\Theta_\lambda^f + \sum_{j=1}^r a_j(\lambda)\Theta_{\sigma_j^\lambda,\nu_j^\lambda},$$

where Θ_λ^f is the character of the finite dimensional representation with infinitesimal character λ, and $a_j(\lambda) = \pm 1$ for all $0 \leq j \leq r,\ \lambda \in \mathcal{S}_d$.

Proof of Proposition 9.1. We will construct the parametrix R of P as the sum of three distributions $R = R_{sp} + R_{sd} + R_F$.

As in the proof of Proposition 7.1, applying Proposition 5.3 together with 6.8, we see that

$$R_{sp} = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathfrak{a}_0'} R_{\sigma,\nu}\, m_\sigma(\nu)\, d\nu$$

defines a finite order distribution such that

$$PR_{sp} = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathfrak{a}_0'} \Theta_{\sigma,\nu}\, m_\sigma(\nu)\, d\nu.$$

In the same way, if we define

$$R_{sd} = \sum_{\lambda \in \mathcal{S}_d - F} d_\lambda\, R_\lambda,$$

then, applying Proposition 6.9,

$$PR_{sd} = \sum_{\lambda \in \mathcal{S}_d - F} d_\lambda\, \Theta_\lambda.$$

For the definition of R_F, we note in first place that having all the representations $\pi_{w\lambda}$ infinitesimal character λ, F is closed by the action of W_G.

Besides $\pi_{w\lambda}$ is equivalent to $\pi_{w'\lambda}$ if and only if $w^{-1}w' \in W_K$, and we can write

$$\sum_{\lambda\in F}\Theta_\lambda = \frac{|W_K|}{|W_G|}\sum_{\lambda\in F}\frac{1}{|W_K|}\sum_{w\in W_G}\Theta_{w\lambda}. \tag{33}$$

Now, for each $\lambda \in F$, let $\sigma_1^\lambda, \dots, \sigma_r^\lambda \in \hat{M}$ and $\nu_1^\lambda, \dots, \nu_r^\lambda \in \mathfrak{a}_0'$ be given by Proposition 9.2; then we can define

$$R_F = \frac{|W_K|}{|W_G|}\sum_{\lambda\in F}\left(\sum_{j=1}^{r} a_j(\lambda) R_{\sigma_j^\lambda,\nu_j^\lambda}\right) \tag{34}$$

which is a finite order distribution.

Finally, Propositions 6.8 and 9.2 imply

$$PR_F = \frac{|W_K|}{|W_G|}\sum_{\lambda\in F}\left(\sum_{j=1}^{r} a_j(\lambda)\Theta_{\sigma_j^\lambda,\nu_j^\lambda}\right) = \sum_{\lambda\in F}\Theta_\lambda - \frac{|W_K|}{|W_G|}\sum_{\lambda\in F} a_o(\lambda)\Theta_\lambda^f;$$

therefore, putting $R = R_{sp} + R_{sd} + R_F$, the above equalities together with Plancherel formula (Theorem 2.1) imply

$$PR - \delta = -\frac{|W_K|}{|W_G|}\sum_{\lambda\in F} a_0(\lambda)\Theta_\lambda^f,$$

and this distribution is given by a C^∞ function because Θ_λ^f are characters of a finite dimensional representation. $\square$

10. Casimir Operator.

The fundamental solution constructed for $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfying the conditions of Theorem 1.1 is invariant by inner automorphisms of K. In the case that P is a bi-invariant operator in the conditions of Corollary 1.2, we obtain a fundamental solution of P invariant by inner automorphisms of all G.

Let's analyze the case of the Casimir operator of G. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and if $\lambda \in \mathfrak{h}'$, then $\chi_\lambda(\Omega) = B(\lambda,\lambda) - B(\rho,\rho)$ ([**14**, 5.6.4]); in particular, if $\sigma \in \hat{M}$, $\nu \in i\mathfrak{a}_0'$,

$$\chi_{\sigma,\nu}(\Omega) = \chi_{\mu_\sigma+\nu}(\Omega) = |\mu_\sigma| - |\nu|^2 - |\rho|^2, \tag{35}$$

and $||\chi_{\mu,-}(\Omega)|| \geq 1$ for all $\mu \in \widehat{T^-}$, so, by Theorem 4.1, $\gamma_{\mathfrak{h}}^G(\Omega)$ has a fundamental solution on $A \times T^-$. Besides if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, in this case if $\lambda \in i\mathfrak{t}_0'$ we have

$$\chi_\lambda(\Omega) = |\lambda|^2 - |\rho|^2; \tag{36}$$

now, being $G^{\mathbb{C}}$ simply connected, $\lambda = \rho$ is a discrete series parameter, and consequently we cannot apply Corollary 1.2.

So Theorem 1.1 only provides a fundamental solution of the Casimir operator when G has one conjugacy class of Cartan subgroups.

We note, on the other hand, that solvability of Ω have been proved by Rauch and Wigner ([**12**]) in a non-constructive way. In [**2**] an explicit fundamental solution of Ω is constructed for $G = SL(2,\mathbb{R})$ and it's also proved that an invariant one doesn't exists for this group.

However, note that $|\chi_\lambda(\omega)| \geq C$ for all $\lambda \in \mathcal{S}_d - F$, where $F = \{\lambda \in \mathcal{S} : |\lambda| = |\rho|\}$ is a finite subset, so by Proposition 9.1, Ω has a parametrix on G.

11. A necessary condition.

Let G be a connected semisimple Lie group. Recall from [**7**, Thm. 5.17] that we can define a Harish-Chandra homomorphism $\gamma_{\mathfrak{a}} : \mathcal{U}(\mathfrak{g})^K \to \mathcal{S}(\mathfrak{a})^{W_0}$, where $W_0 = W(\mathfrak{g}_0, \mathfrak{a}_0)$ is the Weyl group of the restricted root system. The kernel of $\gamma_{\mathfrak{a}}$ is $\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$.

Proposition 4.1 in [**13**] states that if $P \in \mathcal{Z}(\mathfrak{g})$ is in the kernel of $\gamma_{\mathfrak{a}}$, then P doesn't have a parametrix. In this section we will extend this proposition for $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$.

Let $\mathcal{D}_K(G)$, resp. $C_K^\infty(G)$, be the space of left and right K-invariant functions in $\mathcal{D}(G)$, resp. $C^\infty(G)$, $\mathcal{D}'_K(G)$ the dual of $\mathcal{D}_K(G)$, identified with the space of left and right K-invariant distributions in $\mathcal{D}'(G)$, and similarly $\mathcal{D}_{W_0}(A)$, resp. $\mathcal{D}'_{W_0}(A)$, the space of W_0-invariant elements of $\mathcal{D}(A)$, resp. $\mathcal{D}'(A)$. Let

$$F_f(a) = a^\rho \int_N f(an)\, dn,$$

$f \in \mathcal{D}_K(G)$, $a \in A$; then the map $f \mapsto F_f$ is an isomorphism of $\mathcal{D}_K(G)$ onto $\mathcal{D}_{W_0}(A)$ for the Schwartz topologies ([**7**, Cor. 7.9]). If $P \in \mathcal{Z}(\mathfrak{g})$, $F_{Pf} = \gamma_{\mathfrak{a}}(P)F_f$ ([**7**, p. 307]). Transposing F^{-1} we get an isomorphism F^t of $\mathcal{D}'_K(G)$ onto $\mathcal{D}'_{W_0}(A)$, and

$$F^t_{PT} = \gamma_{\mathfrak{a}}(P)F^t_T$$

for all $P \in \mathcal{Z}(\mathfrak{g})$, $T \in \mathcal{D}'_K(G)$.

Now let $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. If $\tau \in \hat{K}$, we can form the homomorphism $\gamma_{\mathfrak{a}} \otimes \chi_\tau : \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K) \longrightarrow \mathcal{S}(\mathfrak{a})^{W_0}$.

Proposition 11.1. *Let G be a connected semisimple Lie group and $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. If P has a parametrix in G, then for all $\tau \in \hat{K}$, $(\gamma_{\mathfrak{a}} \otimes \chi_\tau)(P) \neq 0$.*

Proof. Suppose that P has a parametrix E and that exists $\tau \in \hat{K}$ such that $(\gamma_{\mathfrak{a}} \otimes \chi_\tau)(P) = 0$.

An easy computation shows that if $\Omega \in \mathcal{Z}(\mathfrak{k})$, then $\Omega E^\tau = \chi_\tau(\Omega) E^\tau$, so if $P^\tau = (Id \otimes \chi_\tau)(P) \in \mathcal{Z}(\mathfrak{g})$, then

$$PE^\tau = P^\tau E^\tau.$$

Taking τ-components, $PE - \delta \in C^\infty(G)$ implies $P^\tau E^\tau - \delta^\tau \in C^\infty(G)$, and making everything K-bi-invariant we get $P^\tau E_K^\tau - \delta_K^\tau \in C_K^\infty(G)$.

Applying F^t yields

$$\gamma_{\mathfrak{a}}(P^\tau) F^t_{E_K^\tau} - F^t_{\delta_K^\tau} \in C^\infty_{W_0}(A);$$

now $\gamma_{\mathfrak{a}}(P^\tau) = (\gamma_{\mathfrak{a}} \otimes \chi_\tau)(P) = 0$ would imply $\delta_K^\tau \in C_K^\infty(G)$, which is absurd because it's easy to see that $\delta^\tau = d_\tau \Theta_\tau m_K$, where $m_K(f) = \int_K f$ is the distribution induced by the Haar measure of K. $\square$

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