FUNDAMENTAL SOLUTIONS OF INVARIANT DIFFERENTIAL OPERATORS ON A SEMISIMPLE LIE GROUP

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Let $G$ be a connected semisimple Lie group of real rank one. We denote by $\mathcal{U}(\mathfrak{g})^K$ the algebra of left invariant differential operators on $G$ right invariant by $K$, and let $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ be its center.

In this paper we give a sufficient condition for a differential operator $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ to have a fundamental solution on $G$. We verify that this condition implies $PC^\infty(G) = C^\infty(G)$. If $G$ has a compact Cartan subgroup, we also give a sufficient condition for a differential operator $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ to have a parametrix on $G$. Finally we prove a necessary condition for the existence of parametrix of $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ for a connected semisimple Lie group.

1. Introduction.

Let $G$ be a connected semisimple Lie group. The algebra of left invariant differential operators on $G$ identifies canonically with the universal algebra $\mathcal{U}(\mathfrak{g})$. The operators of the center $\mathcal{Z}(\mathfrak{g})$ are the bi-invariant differential operators on $G$, i.e., left and right invariant. More generally, we consider the algebra $\mathcal{U}(\mathfrak{g})^K$ of right $K$-invariant differential operators of $\mathcal{U}(\mathfrak{g})$, where $K$ is a maximally compact subgroup of $G$, and $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ will denote its center.

We denote by $\mathcal{D}(G)$ the space of $C^\infty$ functions with compact support. The dual $\mathcal{D}'(G)$ of continuous linear functionals in $\mathcal{D}(G)$ is the space of distributions of $G$.

An operator $P$ in $\mathcal{U}(\mathfrak{g})$ acts on $\mathcal{D}'(G)$ in the following way:

$$PT(f) = T(P^t f),$$

where $P^t \in \mathcal{U}(\mathfrak{g})$ is such that, if $dx$ is a Haar measure on $G$,

$$\int_G Pf(x)g(x)dx = \int_G f(x)P^t g(x)dx.$$

If $X \in \mathfrak{g}$, $X^t = -X$, so the $P \mapsto P^t$ is the anti automorphism of $\mathcal{U}(\mathfrak{g})$ extending $-Id$ of $\mathfrak{g}$. In addition, this map preserves the subalgebras $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})^K$. 

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Definition 1. A distribution $T \in \mathcal{D}'(G)$ is a fundamental solution of a differential operator $P \in \mathcal{U}(g)$ if $PT = \delta$, where $\delta(f) = f(1)$; and $T$ is a parametrix of $P$ if $PT - \delta \in C^\infty(G)$.

In [5], Cerezo and Rouvière study the construction of parametrix and fundamental solutions in the group $G = SL(2, \mathbb{R})$ for operators in $\mathcal{U}(g)^K$ (note that $\mathcal{U}(g)^K$ is abelian in this case).

When $G$ has only one conjugacy class of Cartan subgroups ($G$ complex semisimple, for example), in [13], Rouvière gives a sufficient condition for the existence of fundamental solution of bi-invariant operators (Theorem 4.2 of [13]). He also proves a necessary condition for $P \in \mathcal{Z}(g)$ to have a parametrix in an any connected semisimple group (Proposition 4.1 of [13]).

In this paper we extend the results of [5] and [13] for operators in $\mathcal{Z}(\mathcal{U}(g)^K)$ to rank one groups and groups with one conjugacy class of Cartan subgroups.

If $G$ is Lie group with Lie algebra $so(n, 1)$ or $su(n, 1)$, it is a well known result that $\mathcal{U}(g)^K$ is abelian, and isomorphic to $\mathcal{Z}(g) \otimes \mathcal{Z}(t)$. This isn’t true for other groups, but in general $\mathcal{Z}(\mathcal{U}(g)^K) \simeq \mathcal{Z}(g) \otimes \mathcal{Z}(t)$ (Knop’s theorem [11]).

Given $\mathfrak{h}_0$ and $\mathfrak{t}_0$ Cartan subalgebras of $g_0$ and $\mathfrak{t}_0$ respectively, we will denote $\gamma^G_0$ and $\gamma^K_0$ the Harish-Chandra homomorphisms of $\mathcal{Z}(g)$ and $\mathcal{Z}(t)$ with respect to the subalgebras $\mathfrak{h}$ and $\mathfrak{t}$; then we have

$$
\mathcal{Z}(g) \otimes \mathcal{Z}(t) \xrightarrow{\gamma^G_0 \otimes \gamma^K_0} \mathcal{U}(\mathfrak{h})^W \otimes \mathcal{U}(\mathfrak{t})^{W_K} \xrightarrow{\delta} \mathcal{Z}(\mathcal{U}(g)^K)
$$

Therefore, by the way of the homomorphisms described above, we can associate to $P \in \mathcal{Z}(\mathcal{U}(g)^K)$ a differential operator $\left(\gamma^G_0 \otimes \gamma^K_0\right)(P)$ in the group $H \times T$, where $H$ and $T$ are the respective Cartan subgroups of $G$ and $K$ with Lie algebras $\mathfrak{h}_0$ and $\mathfrak{t}_0$.

We say that $H^f$ is a fundamental Cartan subgroup of $G$ if $H^f$ has maximal compact factor between $\theta$-stable Cartan subgroups of $G$. All fundamental Cartan subgroups of $G$ are conjugate (c.f. [15], Chapter I).

We will denote $\gamma^G = \gamma^G_{\mathfrak{h}_0}$, where $\mathfrak{h}_0$ is the Lie subalgebra of a fundamental Cartan subgroup $H^f$. Because in $K$ all Cartan subgroups are conjugate, we put $\gamma^K = \gamma^K_{\mathfrak{t}_0}$.

We can now state our main result:

Theorem 1.1. Let $G$ be a connected semisimple Lie group of real rank one or with one conjugacy class of Cartan subgroups. Let $H^f$ be a fundamental Cartan subgroup of $G$ and $P \in \mathcal{Z}(\mathcal{U}(g)^K)$. If $\left(\gamma^G \otimes \gamma^K\right)(P)$ has a fundamental solution in $H^f \times T$, then $P$ has a fundamental solution in $G$. 

When $P$ is a bi-invariant operator, we obtain a complete proof for these groups of the theorem announced in \[3\]:

**Corollary 1.2** (Benabdallah-Rouvière). Let $P \in Z(g)$. If $\gamma^G(P)$ has a fundamental solution in $H^f$, then $P$ has a fundamental solution in $G$.

The proof will consist in the explicit construction of the fundamental solution of $P$, using the Plancherel formula as the main tool. When $G$ is a rank one group having a compact Cartan subgroup, using Zuckerman characters identities combined with the proof of 1.1, we obtain a sufficient condition for the existence of a parametrix of $P$:

**Theorem 1.3.** Let $G$ be a linear connected semisimple Lie group of rank one such that $T$ is a compact Cartan subgroup of $G$, and $P \in Z(U(g)^K)$. If $(\gamma^G \otimes \gamma^K)(P)$ has a parametrix on $T \times T$, then $P$ has a parametrix on $G$.

Finally we extend Proposition 4.1 of \[13\] for $P \in Z(U(g)K)$.

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Finally, I want to thank Professor Michel Duflo who gave a sketch for the proof of Proposition 6.3, which although I suppose well known, I couldn’t find in the references.

## 2. Preliminaries.

In this section we fix notation and summarize the basics known facts about representation theory that will be needed through this paper.

### 2.1. Notation.

Let $G$ be a connected reductive Lie group. $\theta$ will denote a Cartan involution in both $g_0$ and $G$. Let $g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $g_0$ with respect to $\theta$; that is, $\mathfrak{k}_0 = \{ X \in g_0 : \theta X = X \}$ and $\mathfrak{p}_0 = \{ X \in g_0 : \theta X = -X \}$.

Let $K$ be analytic subgroup of $G$ with Lie algebra $\mathfrak{k}_0$, $K$ is a maximally compact subgroup of $G$. We fix $\mathfrak{t}_0$ a Cartan subalgebra of $\mathfrak{k}_0$ coming from a maximal torus $T$ of $K$. We will denote $\tau \in \hat{K}$ an irreducible unitary representation of $K$.

Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{p}_0$. The dimension of $\mathfrak{a}_0$ is the real rank of $G$. We put $m_0 = \{ X \in \mathfrak{t}_0 : [X, \mathfrak{a}_0] = 0 \}$. If $M = Z_K(\mathfrak{a}_0) = \{ x \in K : \text{Ad}(x)|_{\mathfrak{a}_0} = Id \}$, then $M$ is a compact subgroup of $G$ with Lie algebra $m_0$. If $\mathfrak{t}_0^-$ is a Cartan subalgebra of $m_0$, then $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0^-$ is a Cartan subalgebra of $g_0$.

If $B$ is the Killing form of $g_0$, $B|_{\mathfrak{t}_0 \times \mathfrak{t}_0}$ is negative definite and $B|_{\mathfrak{p}_0 \times \mathfrak{p}_0}$ is positive definite. Then $(X, Y) = -B(X, \theta Y)$ defines an inner product on $g_0$. 
The complexification of any real Lie algebra will be denoted without the subscript. If \( \mathfrak{g}_0 = \mathfrak{t}_0 \oplus i\mathfrak{p}_0 \subseteq \mathfrak{g} \), then \( \mathfrak{g}_0 \) is a compact form of \( \mathfrak{g} \). The Killing form of \( \mathfrak{g} \) is the complex bilinear extension of that of \( \mathfrak{g}_0 \), and making an abuse of notation, we will still call it \( B \). On \( \mathfrak{g} \) we define the inner product, \( (X, Y) = -B(X, JY) \), where \( J \) is conjugation with respect to \( \mathfrak{g}_0 \). Let’s note that \( J|_{\mathfrak{g}_0} = \theta \) and so this inner product extends that of \( \mathfrak{g}_0 \).

Given \( \sigma \in \hat{M} \) with infinitesimal character \( \mu_\sigma \in i(t^-_0)' \) and \( \tau \in \hat{K} \) with infinitesimal character \( \mu_\tau \in it^0_0 \), we define

\[
|\sigma| = |\mu_\sigma|, \quad |\tau| = |\mu_\tau|.
\]

We note that this definition makes sense because the norm is both \( W_M \) and \( W_K \) invariant, since the elements of the Weyl groups of \( M \) and \( K \) are inner automorphisms of \( K \).

Every semisimple Lie group of rank one has at most two conjugacy classes of \( \theta \)-stable Cartan subalgebras (or subalgebras). But in the case \( G \) has only one, the hypothesis of rank one is not needed. So from now on we will restrict our attention to the following two cases:

I. \( G \) has only one conjugacy class of \( \theta \)-stable Cartan subalgebras, which will be represented by \( \mathfrak{h}_0 = a_0 \oplus t^-_0 \).

II. \( G \) is a rank one group having a compact Cartan subgroup. In this case \( \mathfrak{t}_0 \) is a Cartan subalgebra of \( \mathfrak{g}_0 \) and we can choose \( \{t_0, h_0\} \) as a representative set of conjugacy classes of \( \theta \)-stable Cartan subalgebras.

We will also suppose that \( G \) has a simply connected complexification \( G^C \). Finally we can assume, with no loss of generality, that \( M \) is connected, because in case I, \( M \) always is ([14, 7.12.7]), and in case II, if \( M \) is disconnected, then \( G \) must be a direct product of \( SL(2, \mathbb{R}) \) with a compact group, and the results of this paper are easily deduced from [5] and [4].

### 2.2. Principal series.

Let \( \Lambda(\mathfrak{g}_0, a_0) \) be the restricted root system of \( \mathfrak{g}_0 \) with respect to \( a_0 \); we choose \( \Lambda^+(\mathfrak{g}_0, a_0) \) a positive system and we put \( \mathfrak{n}_0 = \sum_{\lambda \in \Lambda^+} \mathfrak{g}_0^\lambda \). If \( A \) and \( N \) are the analytic subgroups with respective Lie algebras \( a_0 \) and \( \mathfrak{n}_0 \), then \( MAN \) is a minimal parabolic subgroup of \( G \).

Given \( \sigma \in \hat{M} \) an irreducible representation of \( M \), \( \nu \in a' \) a complex linear functional on \( a \). The principal series representation \( \pi_{\sigma, \nu} \) is defined inducing the representation \( \sigma \otimes e^\nu \otimes 1 \) from \( MAN \) to \( G \). The representation \( \pi_{\sigma, \nu} \) is admissible; moreover the multiplicity of \( \tau \) in \( \pi_{\sigma, \nu}|_K \) is independent of \( \nu \) ([10, Prop. 8.4]) and will be denoted \( n^\sigma_\tau \).

If \( \mu_\sigma \in it^-_0 \) is the infinitesimal character of \( \sigma \) relative to \( t^- \), then \( \pi_{\sigma, \nu} \) has infinitesimal character \( \mu_\sigma + \nu \) relative to \( a + t^- \) ([10, Prop. 8.22]).

### 2.3. Discrete series (case II).

Let \( (\pi, V) \) be an irreducible unitary representation of \( G \). We say that \( (\pi, V) \) is a discrete series representation of \( G \) if all its matrix coefficients \( g \mapsto (\pi(g)u, v) \) are square integrable. \( G \) admits
discrete series representations if and only if \( G \) has a compact Cartan subgroup; or equivalently, \( t_0 \) is a Cartan subalgebra of \( g_0 \) and \( f_0 \). All discrete series representations can be parametrized by the forms \( \lambda \in i t'_0 \) such that \( \lambda \) is non-singular (i.e., \( (\lambda, \alpha) \neq 0 \) for all \( \alpha \in \Delta(g, t) \)) and \( \lambda + \rho \) is analytically integral, or equivalently, \( \lambda \) is integral, because we are assuming \( G^G \) simply connected. The functional \( \lambda \) is called the Harish-Chandra parameter. Two discrete series representations are equivalent if and only if their parameters are conjugate by an element of the Weyl group \( W_K \) of \( K \). We will denote \( \pi_\lambda \) the discrete series representation of parameter \( \lambda \) and \( S_d \) the set of Harish-Chandra parameters.

The representation \( \pi_\lambda \) has infinitesimal character \( \lambda \), and discrete series representations with parameters \( w\lambda, w \in W_G \), the Weyl group of \( G \), have the same infinitesimal character. Therefore exactly \( |W_G|/|W_K| \) of the representations \( \pi_{w\lambda} \) are mutually inequivalent.

Finally, there is a positive number \( d_\lambda \), called the formal degree of \( \pi_\lambda \), such that
\[
\int_G (\pi_\lambda(x)u_1, v_1)(\pi_\lambda(x)u_2, v_2) \, dx = d_\lambda^{-1}(u_1, u_2)(v_1, v_2)
\]
for all \( u_1, u_2, v_1, v_2 \).

**2.4. Global characters.** We say that an admissible representation \( \pi \) has a global character \( \Theta_\pi \) if the operator
\[
\pi(f) = \int_G f(g)\pi(g) \, dg
\]
is a trace class operator for all \( f \in \mathcal{D}(G) \) and if the map \( f \mapsto \text{tr} \pi(f) = \Theta_\pi(f) \) is a distribution on \( G \).

Every admissible representation \( \pi \) whose decomposition \( \pi|_K = \sum_{\tau \in \hat{K}} n_{\tau} \tau \) satisfies \( n_\tau \leq C \dim \tau \) has a global character ([10, Thm. 10.2]). As a consequence, every irreducible unitary representation has a character ([10, Thm. 8.1]). In the same way, induced representations from irreducible unitary representations also have characters. So discrete series as well as principal series representations have characters.

Given \( \lambda \in S_d \), we will denote \( \Theta_\lambda \) the character of \( \pi_\lambda \); and given \( \sigma \in \hat{M}, \nu \in \mathcal{a}' \), \( \Theta_{\sigma,\nu} \) the character of \( \pi_{\sigma,\nu} \).

**2.5. Plancherel formula.** We are now in a position to write down the Plancherel formula for the groups we are considering (cf. [1, Lemma 5] and [14, Thm. 8.15.4]).

**Theorem 2.1.** There is a non-negative function \( m_\sigma(\nu) \) defined in \( \hat{M} \times i\mathcal{a}'_0 \) such that for all \( f \in \mathcal{D}(G) \) we have in case I,
\[
f(1) = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathcal{a}'_0} \Theta_{\sigma,\nu}(f) m_\sigma(\nu) \, d\nu,
\]
and in case II,

\[(3) \quad f(1) = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathfrak{a}_0'_{\sigma}} \Theta_{\sigma,\nu}(f)m_{\sigma}(\nu) \, d\nu + \sum_{\lambda \in \mathcal{S}_d} d_\lambda \Theta_\lambda(f).\]

The function \(m_{\sigma}(\nu)\) has the following properties:

(i) For each \(\sigma \in \hat{M}\), \(m_{\sigma}(\nu)\) is the restriction to \(i\mathfrak{a}_0'\) of a meromorphic function on \(a'\) without poles on \(i\mathfrak{a}_0'\).

(ii) Exist a positive constant \(C\) and a positive integer \(l\) such that for all \(\sigma \in \hat{M}, \nu \in i\mathfrak{a}_0'\), we have

\[|m_{\sigma}(\nu)| \leq C(1 + |\sigma|^2)^l(1 + |\nu|^2)^l.\]

3. **Action of \(P\) on characters.**

If \(\pi\) is an admissible representation with global character \(\Theta_\pi\) and infinitesimal character \(\chi_\pi\), and \(P \in Z(\mathfrak{g})\) is a bi-invariant differential operator, then \(P \Theta_\pi = \chi_\pi(P) \Theta_\pi\) ([10, Prop. 10.24]).

We want to prove a similar result for an operator \(P \in Z(\mathcal{U}(\mathfrak{g})^K)\). First we need to decompose a distribution as a sum of its \(K\)-isotypical components.

If we consider the right regular representation \(R\) acting on \(L^2(G)\) then \(\mathcal{D}(G)\) is a dense subspace of \(C^\infty\) vectors of \(L^2(G)\), and decomposing \(R\) in sum of its \(K\)-isotypical components we obtain

\[f = \sum_{\tau \in K} f_\tau,\]

where

\[f_\tau(x) = d_\tau(f \ast \Theta_\tau)(x) = d_\tau \int_K f(xk)\Theta_\tau(k^{-1}) \, dk\]

with \(\Theta_\tau(k) = \text{tr} \, \tau(k)\). Convergence holds not only in \(L^2(G)\) but also in \(\mathcal{D}(G)\). This decomposition induces a similar decomposition for a distribution \(T \in \mathcal{D}'(G)\):

\[T = \sum_{\tau \in K} T^\tau\]

with \(T^\tau(f) = T(f_\tau)\), where \(\bar{\tau}\) is the conjugate representation of \(\tau\). Remind that \(\bar{\tau}\) is isomorphic to the contragredient representation \(\tau^*\) of \(K\).

**Lemma 3.1.** Let \((\pi, V)\) be an admissible representation. Suppose that its \(K\)-isotypical decomposition

\[V = \sum_{\tau \in K} n_\tau V_{\tau}\]

satisfies \(n_\tau \leq C d_\tau\), where \(d_\tau\) is the dimension of \(\tau\), and \((\pi|_K, V_{\tau})\) is equivalent to \(\tau\). We choose an orthonormal base of \(V\) by joining orthonormal bases \(\{e_i^\tau : i = 1 \ldots d_\tau\}\) of \(V_{\tau}\).
If $\Theta_\pi$ is the character of $\pi$, then:

$$\Theta_\pi^\tau(f) = n_\tau \sum_{i=1}^{d_\tau} (\pi(f)e_i^\tau, e_i^\tau).$$

Proof.

$$\Theta_\pi^\tau(f) = \Theta_\pi(f_\tau) = \sum_{\tau \in \hat{K}} n_\tau \left[ \sum_{i=1}^{d_\tau} (\pi(f_\tau)e_i^\tau, e_i^\tau) \right]$$

with

$$f_\tau = d_\tau(f \ast \Theta_\tau) = d_\tau(f \ast \overline{\Theta_\tau}),$$

where $\Theta_\tau$ is a $C^\infty$ function on $K$ being $\tau$ of finite dimension.

Let $p_\tau : V \rightarrow V_\tau$ be the orthogonal projection. Now let $\tau' \in \hat{K}$ be another representation; then, using Fubini's theorem and the bi-invariance of the Haar measures of $G$ and $K$,

$$\left( \pi(f_\tau)e_i^{\tau'}, e_i^{\tau'} \right) = \int_G f_\tau(x)(\pi(x)e_i^{\tau'}, e_i^{\tau'}) \, dx$$

$$= \int_G \left[ d_\tau \int_K f(xk)\overline{\Theta_\tau(k^{-1})} \, dk \right] (\pi(x)e_i^{\tau'}, e_i^{\tau'}) \, dx$$

$$= \int_G f(x) \left[ d_\tau \int_K (\pi(k^{-1})e_i^{\tau'}, \pi(x)e_i^{\tau'})\overline{\Theta_\tau(k^{-1})} \, dk \right] \, dx$$

$$= \int_G f(x) \left[ \sum_{j=1}^{d_\tau} d_\tau \int_K (\pi(k)e_j^{\tau'}, p_{\tau'}(\pi(x)e_j^{\tau'}))(\pi(k)e_j^{\tau'}, e_j^\tau) \, dk \right] \, dx,$$

and by Schur orthogonality relations (Corollary 1.10 of [10]),

$$\left( \pi(f_\tau)e_i^{\tau'}, e_i^{\tau'} \right) = \int_G f(x) \left[ \sum_{j=1}^{d_\tau} \delta_{\tau,\tau'}(e_j^{\tau'}, e_j^\tau)(p_{\tau'}(\pi(x)e_j^{\tau'}), e_j^\tau) \, dk \right] \, dx$$

$$= \delta_{\tau,\tau'} \int_G f(x)(\pi(x)e_i^{\tau'}, e_i^\tau) \, dx$$

$$= \delta_{\tau,\tau'} \int_G f(x)(\pi(x)e_i^\tau, e_i^\tau) \, dx.$$

Given $\pi$ a representation of $G$, $\tau \in \hat{K}$, the map $\chi_\pi \otimes \chi_\tau$ is a linear functional on $\mathcal{Z}(g) \otimes \mathcal{Z}(t)$, and induces a linear functional on $\mathcal{Z}(U(g)^K)$, which we will still denote $\chi_\pi \otimes \chi_\tau$.

We can now state the result we need.
Proposition 3.2. If \( \pi \) is an admissible representation with infinitesimal character \( \chi_\pi \) and global character \( \Theta_\pi \), and \( P \) is a differential operator in \( \mathcal{Z}(\mathcal{U}(g)^K) \), then
\[
P \Theta_\pi^\tau = (\chi_\pi \otimes \chi_\tau)(P) \Theta_\pi^\tau.
\]

Proof. We preserve the notation of Lemma 3.1. If \( P \in \mathcal{Z}(g) \),
\[
P \Theta_\pi^\tau = \chi_\tau(P) \Theta_\pi^\tau.
\]
If \( P \in \mathcal{Z}(k) \),
\[
P \Theta_\pi^\tau(f) = \Theta_\pi^\tau(P^t f) = \sum_{i=1}^{d_\tau} (\pi(P^t f)e_i^\tau, e_i^\tau)
\]
and
\[
(\pi(P^t f)e_i^\tau, e_i^\tau) = \int_G (\pi(x)\pi(P)e_i^\tau, e_i^\tau)f(x) \, dx
\]
\[
= \chi_\tau(P)(\pi(f)e_i^\tau, e_i^\tau) \quad \text{by [10, (8.10)]},
\]
because \( \pi(P)|_{V_\tau} = \tau(P) \). Then \( P \Theta_\pi^\tau = \chi_\tau(P) \Theta_\pi^\tau \) if \( P \in \mathcal{Z}(t) \). \( \square \)

4. Fundamental solutions on abelian connected groups.
In [4], Cerezo and Rouvière obtain necessary and sufficient conditions for \( P \in \mathcal{U}(g) \) to have a fundamental solution when \( G \) is a connected compact group or a product of a connected compact group with \( \mathbb{R}^n \).

In this section we state these results for a connected abelian group, which is the product of a torus with \( \mathbb{R}^n \). Theorem 4.1 follows directly from Theorem III of [4].

We consider in first place an abelian connected compact Lie group \( T \) with Lie algebra \( t_0 \), i.e., \( T \) a torus. In this case \( \exp: t_0 \rightarrow T \) is a group epimorphism, and \( \Gamma = \ker(\exp) \) is a closed discrete subgroup of \( t_0 \). Being \( T \) abelian, its irreducible unitary representations are one dimensional and can be parametrized by the linear functionals \( \lambda \in i t_0' \) such that \( \lambda(\Gamma) \subseteq 2\pi i \mathbb{Z} \). So we set
\[
\hat{T} = \{ \lambda \in i t_0' \text{ such that } \lambda(\Gamma) \subseteq 2\pi i \mathbb{Z} \}.
\]
A general abelian connected Lie group is of the form \( A \times T \), where \( A = \mathbb{R}^n \) and \( T \) is a \( m \)-dimensional torus. The universal algebra \( \mathcal{U}(a \oplus t) \) coincides with the symmetric algebra \( S(a \oplus t) \), and so an element \( P \) in \( \mathcal{U}(a \oplus t) \) can be thought of both as a constant coefficient differential operator on \( A \times T \) and as a polynomial function on \( a' \oplus t' \). Given \( \lambda \in \hat{T} \), if we put
\[
P_{\lambda}(x) = P(x, \lambda), \quad x \in a',
\]
then \( P_{\lambda} \) is a polynomial function in \( \mathbb{R}^n \), and its norm can be defined
\[
||P_{\lambda}|| = \left( \sum_{\alpha} \frac{1}{|\alpha|!^2} |P_{\lambda}(\alpha)(0)|^2 \right)^{\frac{1}{2}}.
\]
Theorem 4.1. Let \( P \in \mathcal{U}(\mathfrak{a} \oplus \mathfrak{t}) \) be a constant coefficients differential operator on \( A \times T \). \( P \) has a fundamental solution if and only if exists a positive constant \( C \) and a positive integer \( k \) such that in any norm of \( it_0' \)

\begin{equation}
||P_\lambda|| \geq \frac{C}{(1 + |\lambda|^2)^k} \quad \forall \lambda \in \hat{T}.
\end{equation}

Remark. Let's go back to \( G \) a connected semisimple Lie group. Remember that given \( P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K) \) and \( \mathfrak{h}_0 \) and \( \mathfrak{t}_0 \) Cartan subalgebras of \( \mathfrak{g}_0 \) and \( \mathfrak{t}_0 \) respectively we have defined a differential operator \( (\gamma^G_0 \otimes \gamma^K_\mathfrak{t})P \) on the group \( H_0 \times T \). This operator depends on the choice of the Cartan subgroups of \( G \) and \( K \). However, the existence of fundamental solution only depends on the conjugacy class (in \( G \) or \( K \)) of Cartan subgroups, as we see below.

Let \( \tilde{H} \) and \( \tilde{T} \) Cartan subgroups of \( G \) and \( K \) respectively and suppose there exist \( g \in G, k \in K \) such that \( \tilde{H} = \tilde{g}^{-1}Hg \) and \( \tilde{T} = \tilde{k}^{-1}Tk \). Then \( \text{Ad} \,(g)_{\mathfrak{h}_0} = \mathfrak{h}_0 \) and \( \text{Ad} \,(k)_{\mathfrak{t}_0} = \mathfrak{t}_0 \). If \( H = A \times T, \mathfrak{t}_0 = \mathfrak{a}_0 + \mathfrak{t}_0, \) then \( \tilde{H} = \tilde{A} \times \tilde{T}, \tilde{A} = \text{Ad} \,(g)A, \tilde{T} = \text{Ad} \,(g)T \), and is clear from (4) that

\begin{equation}
\tilde{T} = \{ \tilde{\lambda} = \lambda \circ \text{Ad} \,(g)^{-1} : \lambda \in \hat{T} \}, \quad \tilde{\tilde{T}} = \{ \tilde{\mu} = \mu \circ \text{Ad} \,(k)^{-1} : \mu \in \hat{\hat{T}} \};
\end{equation}

besides we have

\begin{equation}
\gamma^G_\mathfrak{h} = \text{Ad} \,(g) \circ \gamma^G_{\mathfrak{h}_0}, \quad \gamma^K_{\mathfrak{t}} = \text{Ad} \,(k) \circ \gamma^K_{\mathfrak{t}_0},
\end{equation}

so if \( x \in \mathfrak{a}', \tilde{x} = x \circ \text{Ad} \,(g)^{-1} \in \tilde{\mathfrak{a}}', (\tilde{\lambda}, \tilde{\mu}) \in \tilde{T} \times \tilde{\tilde{T}} \) and if \( P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K) \),

\begin{equation}
P_{(\tilde{\lambda}, \tilde{\mu})}(\tilde{x}) = \left( \left( \gamma^G_\mathfrak{h} \otimes \gamma^K_{\mathfrak{t}_0} \right)P \right)(\tilde{x}, \tilde{\lambda}, \tilde{\mu})
\end{equation}

and so Theorem 4.1 implies that \( (\gamma^G_\mathfrak{h} \otimes \gamma^K_{\mathfrak{t}})P \) has a fundamental solution on \( H \times T \) if and only if \( (\gamma^G_\mathfrak{h} \otimes \gamma^K_{\mathfrak{t}})P \) has one on \( \tilde{H} \times \tilde{T} \).

5. Inversion of infinitesimal characters.

5.1. Case I. Let \( \mathfrak{h}_0 = \mathfrak{a}_0 + \mathfrak{t}_0' \) be a Cartan subalgebra of \( \mathfrak{g}_0 \), where \( \mathfrak{t}_0' \) is a Cartan subalgebra of \( \mathfrak{m}_0 \), let \( \mathfrak{t}_0 \) be a Cartan subalgebra of \( \mathfrak{t}_0' \). Let \( \sigma \in \hat{M} \) with infinitesimal character \( \mu_\sigma \in i(\mathfrak{t}_0')', \nu \in \mathfrak{a}', \tau \in \hat{K} \) with infinitesimal character \( \mu_\tau \in i\mathfrak{t}_0' \).

Recall that to an operator \( P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K) \) we associate an operator \( (\gamma^G_\mathfrak{h} \otimes \gamma^K_{\mathfrak{t}})P \) on the abelian connected group \( A \times T \times T \). We then have

\begin{equation}
(\chi_{\sigma, \nu} \otimes \chi_\tau)(P) = \left( (\gamma^G_{\mathfrak{h}_0} \otimes \gamma^K_{\mathfrak{t}_0})(P) \right)(\nu + \mu_\sigma, \mu_\tau),
\end{equation}
and fixing $\sigma, \tau$ we obtain a polynomial function in $a'$, and we put
\[
P_{\sigma, \tau}(\nu) = (\chi_{\sigma, \nu} \otimes \chi_{\tau})(P).
\]

**Proposition 5.1.** Given $P \in \mathcal{Z}(\mathcal{U}(g)^K)$, if $(\gamma^G_{\theta} \otimes \gamma^K)(P)$ has a fundamental solution on $A \times T^* \times T$ there exist a constant $C$ and a positive integer $k$ such that
\[
||P_{\sigma, \tau}|| \geq \frac{C}{(1 + |\sigma|^2)^k (1 + |\tau|^2)^k} \quad \forall (\sigma, \tau) \in \hat{M} \times \hat{K}.
\]

**Proof.** According to (6) and (9), $P_{\sigma, \tau}(\nu) = \left((\gamma^G_{\theta} \otimes \gamma^K)(P)\right)_{(\mu_{\sigma}, \mu_{\tau})}(\nu)$; $\mu_{\sigma}$ as well as $\mu_{\tau}$ are analytically integral forms, so $(\mu_{\sigma}, \mu_{\tau}) \in \hat{T} \times \hat{T}$, and the proposition follows directly from Theorem 4.1. \hfill $\blacksquare$

**5.2. Case II: Discrete series.** Let $t_0$ be a Cartan subalgebra of $g_0$ and $\mathfrak{k}_0$. Let $\pi_{\lambda}$ be the discrete series representation with parameter $\lambda \in it'_0$, and $\tau \in \hat{K}$ with infinitesimal character $\mu_{\tau} \in it'_0$.

In this case, given $P \in \mathcal{Z}(\mathcal{U}(g)^K)$, $(\gamma^G_{\theta} \otimes \gamma^K)(P)$ is a differential operator on $T \times T$, and we put
\[
P_{\lambda, \tau} = (\chi_{\lambda} \otimes \chi_{\tau})(P) = (\gamma^G_{\theta} \otimes \gamma^K(P))(\lambda, \mu_{\tau}).
\]

**Proposition 5.2.** Given $P \in \mathcal{Z}(\mathcal{U}(g)^K)$, if $(\gamma^G_{\theta} \otimes \gamma^K)(P)$ has a fundamental solution on $T \times T$, there exist a constant $C$ and a positive integer $k$ such that
\[
|P_{\lambda, \tau}| \geq \frac{C}{(1 + |\lambda|^2)^k (1 + |\tau|^2)^k} \quad \forall (\lambda, \tau) \in S_d \times \hat{K}.
\]

As in Proposition 5.1, the proof is a direct consequence of Theorem 4.1 with $A = 0$, observing that $(\lambda, \mu_{\tau}) \in \hat{T} \times \hat{T}$.

**5.3. Case II: Principal series.** In this case we need to invert simultaneously the infinitesimal characters of principal and discrete series representations. We will prove the following:

**Proposition 5.3.** Given $P \in \mathcal{Z}(\mathcal{U}(g)^K)$, suppose that for a finite set $F$

exist a constant $C$ and a positive integer $k$ such that
\[
|P_{\lambda, \tau}| \geq \frac{C}{(1 + |\lambda|^2)^k (1 + |\tau|^2)^k} \quad \forall (\lambda, \tau) \in (S_d - F) \times \hat{K}.
\]

Then there exist a constant $\tilde{C}$ and a positive integer $\tilde{k}$ such that
\[
||P_{\sigma, \tau}|| \geq \frac{\tilde{C}}{(1 + |\sigma|^2)^k (1 + |\tau|^2)^k} \quad \forall (\sigma, \tau) \in \hat{M} \times \hat{K}.
\]

For this purpose we will introduce the Cayley transform which is an inner automorphism of the complex group $G^C$ that conjugates the Cartan subalgebras $t$ and $\mathfrak{h}$. 
5.4. Cayley transform. Given a compact Cartan subgroup $T$ with Lie algebra $t_0$, and given a non-compact root $\beta \in \Delta(g, t)$, we can construct a Cayley transform $c_\beta$ and a non-compact Cartan subgroup $H$ in the following way:

Let $H_\beta \in t$ such that $\beta(H) = B(H, H_\beta)$ for all $H \in t$ and we put $H'_\beta = 2|\beta|^{-2}H_\beta$. We choose root vectors $X'_\beta \in g_\beta$ and $X'_{-\beta} = -\theta X'_\beta \in g_{-\beta}$ such that $B(X'_\beta, X'_{-\beta}) = 2|\beta|^{-2}$ and such that $X'_\beta + X'_{\beta}$ and $i(X'_\beta - X'_{\beta})$ are in $g_0$. We then define

$$c_\beta = \text{Ad} \left( \exp \frac{\pi}{4}(X'_{-\beta} - X'_\beta) \right)$$

and

$$b_0 = g_0 \cap c_\beta(t).$$

If we put $\ker \beta = \{ H \in t_0 : \beta(H) = 0 \}$, then $t_0 = \ker \beta \oplus \mathbb{R}iH_\beta$. Now $c_\beta|_{\ker \beta} = Id$ and $c_\beta(iH_\beta) = i(X'_\beta + X'_{-\beta})$, so

$$b_0 = \ker \beta \oplus \mathbb{R}(X'_\beta + X'_{-\beta}), \quad c_\beta(t_0) = \ker \beta \oplus \mathbb{R}(X'_\beta + X'_{-\beta}).$$

Let $a_0 = \mathbb{R}(X'_\beta + X'_{-\beta})$, $a_0$ is a maximal abelian subalgebra in $p_0$ because $G$ is of rank one. Besides $m_0 = \{ X \in g_0 : [X, a_0] = 0 \}$, and it’s clear by the choice of $a_0$ that $\ker \beta \subset m_0$, and $t_0 = \ker \beta$ is maximal abelian subalgebra of $m_0$ by dimension. So $c_\beta$ carries $t$ on $\mathfrak{h}$ fixing $\mathfrak{t}^\perp$.

5.5. Extension of infinitesimal characters of representations of $M$.

If we fix a positive root system $\Delta^+(g, t)$, the set of infinitesimal characters of $\mathcal{S}_d$ can be parametrized by the set of strongly dominant integral forms, that is,

$$\{ \lambda \in i\mathfrak{t}^\perp_0 : \lambda(\Gamma_T) \subseteq 2\pi i\mathbb{Z} \text{ and } (\lambda, \alpha) > 0 \forall \alpha \in \Delta^+(g, t) \},$$

where $\Gamma_T = \ker(\exp|_{t_0})$. Now, being $t_0$ a Cartan subalgebra of $g_0$, this set coincides with the set of infinitesimal characters of irreducible unitary representations of finite dimension of the compact form $G_u$ of $G$.

Note that $b_u = c_\beta(t_0) = i\mathfrak{a}_0 \oplus i\mathfrak{t}^\perp_0$ is another Cartan subalgebra of $g_u$, and we can also parametrize the set of infinitesimal characters of $\hat{G}_u$, and then that of $\mathcal{S}_d$, with the set

$$\{ \hat{\lambda} \in (a_0 \oplus i\mathfrak{t}^\perp_0)' : \hat{\lambda}(\Gamma_{H_u}) \subseteq 2\pi i\mathbb{Z} \text{ and } (<\hat{\lambda}, \hat{\alpha}) > 0 \forall \hat{\alpha} \in \Delta^+(g, h) \},$$

where $\Gamma_{H_u} = \ker(\exp|_{h_u}) = c_\beta(\Gamma_T)$; putting $\hat{\lambda} = c_\beta(\lambda) = \lambda \circ c_\beta^{-1}$, it’s clear that $c_\beta$ provides a bijection between both sets.

Besides, if $\Gamma_{T^-} = \ker(\exp|_{t_0^-})$, the set of infinitesimal characters of irreducible unitary representations of $M$ is given by

$$\{ \mu \in i(t_0^-)' : \mu(\Gamma_{T^-}) \subseteq 2\pi i\mathbb{Z} \text{ and } (\mu, \alpha) > 0 \forall \alpha \in \Delta^+(m, t^-) \}. $$
For any \( \lambda \in S_d \), if we put \( \mu = c_\beta(\lambda)|_{t_0} \), then
\[
\mu(\Gamma_T^-) = \lambda(c_\beta^{-1}\Gamma_T^-) \subseteq \lambda(\Gamma_T) \subseteq 2\pi i \mathbb{Z},
\]
and if \( \alpha \in \Delta(\mathfrak{m}, t^-) \), \( H_\alpha \in i t_0^- \subseteq i t_0 \), by suitable choice of the respective positive systems,
\[
(\mu, \alpha) = \mu(H_\alpha) = \lambda(c_\beta^{-1}H_\alpha) = \lambda(H_\alpha) = (\lambda, \alpha) > 0,
\]
and there exists \( \sigma \in \hat{M} \) such that \( \mu_\sigma = c_\beta(\lambda)|_{t_0} \). We want to see that every infinitesimal character \( \mu_\sigma \) can be obtained in this way.

**Proposition 5.4.** Given \( F \) a finite subset of \( S_d \), for all \( \sigma \in \hat{M} \) exists a discrete series parameter \( \lambda \in S_d - F \) such that \( \mu_\sigma = c_\beta(\lambda)|_{t_0} \). Moreover, \( \lambda \) can be chosen such that \( |\lambda| \leq C|\mu| \) for some constant \( C \) (independent of \( \mu \)).

For the proof we will need some previous lemmas.

We choose positive root systems in \( \mathfrak{g} \) and \( \mathfrak{m} \) in the following way: Let \( \{H_1, \ldots, H_n\} \) be an ordered basis of \( \mathfrak{a}_0 + i t_0^- \) such that \( H_1 \) is a basis of \( \mathfrak{a}_0 \) and \( \{H_2, \ldots, H_n\} \) is basis of \( i t_0^- \) and let \( \Delta^+(\mathfrak{g}, \mathfrak{h}) \) be the respective positive root system. Then
\[
\Delta^+(\mathfrak{m}, t^-) = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) : \alpha|_{\mathfrak{a}_0} = 0\}.
\]
We also choose \( \Delta^+(\mathfrak{g}, t) \) as the image of \( \Delta^+(\mathfrak{g}, \mathfrak{h}) \) by \( c_\beta^{-1} \).

Let \( \Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\alpha|_{\mathfrak{a}_0} : \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})\} \) be the respective positive restricted root system. Because of the rank one condition, there exists \( \beta_0 \in \mathfrak{a}_0' \) such that \( \Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\beta_0\} \) or \( \Lambda^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\frac{1}{2}\beta_0, \beta_0\} \); and we will still denote \( \beta_0 \) its extension to \( (\mathfrak{a}_0 + it_0^-)' \) by \( 0 \) in \( it_0^- \).

**Lemma 5.5.** \( \beta_0 \) belongs to the positive root system \( \Delta^+(\mathfrak{g}, \mathfrak{h}) \).

**Proof.** Follows directly from Lemmas 1 and 2, p. 33 of [15]. \( \square \)

**Lemma 5.6.** If \( \{\alpha_1, \ldots, \alpha_n\} \) is a simple root system of \( \mathfrak{g} \) with respect to the positive system fixed previously, there exist at most two simple roots such that their restriction to \( \mathfrak{a}_0 \) are not identically 0.

**Proof.** Let \( \alpha = m_1\alpha_1 + \cdots + m_n\alpha_n \) be the maximal positive root of \( \Delta^+(\mathfrak{g}, \mathfrak{h}) \); that is, \( m_i \in \mathbb{N} \) for all \( i \). Now \( \alpha|_{\mathfrak{a}_0} = s_i\beta_0 \), con \( s_i = 0, 1/2 \) or 1 for all \( i \), therefore \( m_1s_1 + \cdots + m_ns_n \leq 1 \) and so at most two of the \( s_i \) can be 0. \( \square \)

**Lemma 5.7.** Let \( G \) be a semisimple rank one Lie group. For every \( \mu \in (it_0^-)' \) strongly dominant analytically integral form, exists \( k \in \mathbb{R} \) such that \( k\beta_0 + \mu \) is strongly dominant integral in \( (\mathfrak{a}_0 + it_0^-)' \). Moreover, we can choose \( k \) with the following properties:

(i) \( k\beta_0 + \mu \notin F \), where \( F \) is a fixed finite subset of strongly dominant integral forms in \( (\mathfrak{a}_0 + it_0^-)' \).

(ii) \( |k| \leq C|\mu| \) for some constant \( C \) (independent of \( \mu \)).
Proof. With no loss of generality, we can suppose that $g$ is simple.

Let $\{\alpha_1, \ldots, \alpha_n\}$ be a simple root system of $g$ with respect to the positive system previously chosen. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the fundamental weights of $g$ with respect to this simple root system.

Then $\{\alpha_i : \alpha_i \mid a_0 = 0\}$ is a simple root system of the semisimple part of $m$. If $\lambda_i^M$ is the respective fundamental weight in $m$ and we extend it by 0 to $(a_0 + it_0)'$,

\begin{equation}
\lambda_i = \lambda_i^M + k_i \beta_0
\end{equation}

because $\lambda_i - \lambda_i^M \mid a_0 = 0$, and $k_i = (\lambda_i, \beta_0)/(\beta_0, \beta_0) \in 1/2\mathbb{Z}$ being $\beta_0$ a root. We will analyze two cases:

I) The simple root $\alpha_1$ is the only one with non-vanishing restriction to $a_0$. In this case, $M$ is semisimple. Now, because $(\lambda_1, \alpha_i) = 0$ for all $i \geq 2$, $\lambda_1 \mid a_0 = 0$ then $\lambda_1 = k_1 \beta_0$, with $k_1 \in 1/2\mathbb{Z}$.

So, if $\mu$ is strongly dominant integral, $\mu = m_2 \lambda_2^M + \cdots + m_n \lambda_n^M$ with $m_i \in \mathbb{N}$ for all $i \geq 2$ and

$k \beta_0 + \mu = \left( k - \sum_{i=2}^n m_i k_i \right) \beta_0 + \sum_{i=2}^n m_i \lambda_i = \left( k - \sum_{i=2}^n m_i k_i \right) \frac{1}{k_1} \lambda_1 + \sum_{i=2}^n m_i \lambda_i,$

so it’s enough to choose $k = k_0 k_1 + m_2 k_2 + \cdots + m_n k_n \in 1/2\mathbb{Z}$ with $k_0 \in \mathbb{Z}^+$ minimum subject to the condition $k \beta_0 + \mu \notin F$.

II) There exist two simple roots with non-vanishing restriction to $a_0$. In this case, looking at the Satake diagrams ([15, Chap. 1], it’s clear that the only possibility is $g_0 \simeq su(n, 1)$. We choose $h_0$ as the 0 trace diagonal matrices. If $H = \sum_i i h_i E_{ii}$, let $e_i \in i h_0'$ be defined by $e_i(H) = h_i$. We fix $\Delta^+(g, h) = \{e_i - e_j : i < j\}$ and $\{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n\}$ becomes a simple root system. Taking $a_0 = i \mathbb{R}(E_{11} - E_{n+1,n+1})$, then $\alpha_1$ and $\alpha_n$ have non-vanishing restriction to $a_0$ and $\beta_0 = e_1 - e_{n+1}$. In this case $m \simeq \mathbb{C} \oplus sl(n - 1, \mathbb{C})$. The center of $m$ is $\mathbb{C} H_0$, where

\begin{equation*}
H_0 = \begin{pmatrix}
-(n-1)i & 2i & \cdots & \cdots \\
 & & 2i & \\
 & & & \ddots \\
 & & & & -i(n-1)
\end{pmatrix}.
\end{equation*}

The fundamental weights of $g$ with respect to this simple root system are $\lambda_i = e_1 + \cdots + e_i$, so it follows that $\lambda_1 + \lambda_n = \beta_0$ and $\lambda_1 - \lambda_n$ is the coordinate function of the center of $m$.

Then, if $\mu$ is strongly dominant integral, $\mu = m(\lambda_1 - \lambda_n) + m_2 \lambda_2^M + \cdots + m_{n-1} \lambda_{n-1}^M$ with $m_i \in \mathbb{N}$ for all $2 \leq i \leq n - 1$, $m \in \mathbb{R}$. It’s easy to see that $\mu$
analytically integral implies \( m \in \mathbb{Z} \). Then

\[
k\beta_0 + \mu = k\beta_0 + m(\lambda_1 - \lambda_n) + \sum_{i=2}^{n-1} m_i(\lambda_i - k_i\beta_0)
= \left( k + m - \sum_{i=2}^{n-1} m_i k_i \right) \lambda_1 + \left( k - m - \sum_{i=2}^{n-1} m_i k_i \right) \lambda_n + \sum_{i=2}^{n-1} m_i \lambda_i,
\]

and we choose \( k = k_0 + |m| + \sum_{i=2}^{n-1} m_i k_i \) with \( k_0 \in \mathbb{Z}^+ \) minimum subject to the condition \( k\beta_0 + \mu \notin F \). \( \square \)

**Proof of Proposition 5.4.** It suffices to take \( \lambda = \mathbf{c}_\beta^{-1}(k\beta_0 + \mu) \), with \( k \) given by Lemma 5.7, reminding that we are supposing \( G^C \) simply connected. \( \square \)

**Proof of Proposition 5.3.** By hypothesis there exists a constant \( C \) and a positive integer \( k \) such that

\[
|P_{\lambda, \tau}| \geq \frac{C}{(1 + |\lambda|^2)^k (1 + |\tau|^2)^k} \quad \forall (\lambda, \tau) \in (S_d - F) \times \hat{K}.
\]

Besides, according to the choice of \( t_0 \) and \( \beta_0 \), \( \gamma^G = \mathbf{c}_\beta^{-1} \circ \gamma^G \), therefore

\[
P_{\lambda, \tau} = \left( (\gamma^G \circ \gamma^K)(P) \right)(\lambda, \mu_\tau).
= \left( (\gamma^G_b \circ \gamma^K)(P) \right)(\lambda \circ \mathbf{c}_\beta^{-1}, \mu_\tau)
= \left( (\gamma^G_b \circ \gamma^K)(P) \right)(\mathbf{c}_\beta(\lambda), \mu_\tau);
\]

given \( \sigma \in \hat{M} \), let \( \lambda \in S_d - F \) be given by Proposition 5.4. If we set \( x_\sigma = \mathbf{c}_\beta(\lambda)|_{a_0} \in i\mathfrak{a}_0' \), then

\[
P_{\sigma, \tau}(x_\sigma) = \left( (\gamma^G_b \circ \gamma^K)(P) \right)(x_\sigma + \mu_\sigma, \mu_\tau)
= \left( (\gamma^G_b \circ \gamma^K)(P) \right)(\mathbf{c}_\beta(\lambda), \mu_\tau) = P_{\lambda, \tau}.
\]

If \( P \) has order \( m \) in \( \mathcal{U}(\mathfrak{g}) \), \( P_{\sigma, \tau} \) is a polynomial function on \( \mathfrak{a}' \) of order \( \leq m \), and \( ||P_{\sigma, \tau}|| \) is the norm of the vector in \( \mathbb{C}^{m+1} \) formed with its coefficients, and by Schwarz inequality

\[
||P_{\sigma, \tau}(x_\sigma)|| \leq ||P_{\sigma, \tau}|| \left( \sum_{j=0}^{m} |x_\sigma|^{2j} \right)^{1/2} \leq ||P_{\sigma, \tau}||(1 + |x_\sigma|^2)^{\frac{m}{2}};
\]

besides,

\[
1 + |x_\sigma|^2 \leq C_2 (1 + |\mu_\sigma|^2),
\]

so

\[
||P_{\sigma, \tau}|| \geq \frac{\tilde{C}}{(1 + |\sigma|^2)^{k+\frac{m}{2}} (1 + |\tau|^2)^k} \quad \forall (\sigma, \tau) \in \hat{M} \times \hat{K},
\]

and it’s enough to choose \( \tilde{k} = k + \frac{m}{2} \). \( \square \)
6. Inversion of global characters.

One important step in building the fundamental solution of $P$ is the construction of distributions $R_\pi$ such that $PR_\pi = \Theta_\pi$ for each representation $\pi$ that appears in the Plancherel formula. In this section we will define these distributions $R_\pi$. We will begin with the principal series representations. Before this we state some necessary results in order to bound the $R_\pi$.

**Lemma 6.1.** Let $K$ be a connected compact Lie group. There exist $\Omega \in Z(t)$ and a positive constant $C$ such that

$$|\chi_\tau(\Omega)| \geq C(1 + |\tau|^2) \quad \forall \tau \in \hat{K}.$$

**Lemma 6.2.** Let $G$ a connected semisimple Lie group with a compact Cartan subgroup. There exist $Z \in Z(g)$ and a positive constant $C$ such that

$$|\chi_\lambda(Z)| \geq C(1 + |\lambda|^2) \quad \forall \lambda \in S_d.$$

**Proposition 6.3.** Let $G$ a connected semisimple Lie group. There exist $Z \in Z(g)$, a positive constant $C$, an $\epsilon > 0$ and a positive integer $k$ such that

$$|\chi_{\sigma,\nu+z}(Z)| \geq C(1 + |\sigma|^2)^k(1 + |\nu|^2)^k \quad \forall \sigma \in \hat{M}, \nu \in i\hat{a}_0', z \in a', |z| < \epsilon.$$

To prove this proposition, we begin with a lemma.

**Lemma 6.4.** Let $g_0$ be the Lie algebra of a semisimple Lie group $G$, $h_0$ a Cartan subalgebra of $g_0$, $W$ the Weyl group of $g$. There exists $P \in S(h)^W$ an homogeneous polynomial function on $h'$ such that

$$P(\lambda) > 0 \quad \forall \lambda \in i\hat{h}_0', \lambda \neq 0.$$

**Proof.** We will begin with the construction of a $G$-invariant polynomial function on $g_0$. The symmetric algebra $S(g_0)$ (resp. $S(g_0')$) is identified with the set of polynomial functions on $g_0$ (resp. $g_0'$). We denote $I(g_0)$ (resp. $I(g_0')$) the $G$-invariant elements of $S(g_0)$ (resp. $S(g_0')$).

The Killing form $B$ of $g_0$, being non-degenerate and $G$-invariant, induces canonical isomorphisms between $S(g_0)$ and $S(g_0')$ and between $I(g_0)$ and $I(g_0')$.

Given $X \in g_0$, let

$$p_X(x) = \det(xI - \text{ad } X) = x^n + a_{n-1}(X)x^{n-1} + \cdots + a_l(X)x^l$$

be the characteristic polynomial of $\text{ad } X$, $n = \dim g_0$, $l = \dim h_0$. The coefficients $a_i(X)$ are $G$-invariant homogeneous polynomial functions of degree $n - i$ on $g_0$. Let $d_i$ and $p$ be positive integers such that $n - i + d_i = 4p$; if we define

$$\tilde{Q} = a_1^{d_1} + \cdots + a_{n-1}^{d_{n-1}},$$

then $\tilde{Q} \in I(g_0')$ and is homogeneous of degree $4p$. 

$B$ is non-degenerate on $\mathfrak{h}_0$, so $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{h}_0^\perp$, $\mathfrak{h}_0^\perp$ the orthogonal of $\mathfrak{h}_0$
with respect to $B$. Besides $\mathfrak{h}_0^\perp = (\mathfrak{g}^+ \oplus \mathfrak{g}^-) \cap \mathfrak{g}_0$ so $\tilde{Q}$ vanishes on $\mathfrak{h}_0^\perp$ and
is positive on $\mathfrak{h}_0$. Let $Q \in I(\mathfrak{g}_0)$ be the image of $\tilde{Q}$ by the isomorphism
mentioned above.

We also note that we can include $\mathfrak{h}_0^\perp$ in $\mathfrak{g}_0^*$ by extending a functional from
$\mathfrak{h}_0$ to $\mathfrak{g}_0$ by $0$ on $\mathfrak{h}_0^\perp$. This inclusion allow us to restrict elements from $S(\mathfrak{g}_0)$
to $S(\mathfrak{h}_0)$.

Let $P \in S(\mathfrak{h}_0)$ be the restriction of $Q$. $P$ has the following properties:

(i) $P(\lambda) > 0$ for all $\lambda \in \mathfrak{h}_0^\perp$, $\lambda \neq 0$.
(ii) $P \in S(\mathfrak{h})^W$.

It only remains to see (ii): It’s clear that $I(\mathfrak{g}_0)$ is included in $I(\mathfrak{g})$. Besides, $Q = P + \tilde{P}$, with $\tilde{P} \in S(\mathfrak{g}_0)\mathfrak{h}_0^\perp \subset S(\mathfrak{g})(\mathfrak{g}^+ \oplus \mathfrak{g}^-)$. Now, $W_G = W(G_a, H_a)$
([10, Thm. 4.41]), so, if $w \in W_G$, $w = \text{Ad} (x)$, with $x \in N_{G_a}(H_a)$, and the
action of $w$ preserves $\mathfrak{h}$ and $\mathfrak{g}^+ \oplus \mathfrak{g}^-$, then $Q = wQ = wP + w\tilde{P}$, therefore
$wP = P$, $w\tilde{P} = \tilde{P}$.

Finally, if $\lambda \in i\mathfrak{h}_0^\perp$, $P(\lambda) = (-i)^{4p}P(i\lambda) > 0$. \hfill \qed

Proof of Proposition 6.3. Let $P \in S(\mathfrak{h})^W$ be given by the lemma above. If

$$2c = \inf \{ |P(\lambda) : \lambda \in i\mathfrak{h}_0^\perp, |\lambda| = 1 \},$$

there exists $0 < \varepsilon < 1/2$ such that $\text{Re} P(\lambda + \tilde{\lambda}) \geq c$ for all $\lambda \in i\mathfrak{h}_0^\perp, |\lambda| = 1$,
$\tilde{\lambda} \in \mathfrak{h}_0^\perp, |\tilde{\lambda}| < 2\varepsilon$. Suppose now that $\lambda \in i\mathfrak{h}_0^\perp, |\lambda| \geq 1$, $\tilde{\lambda} \in \mathfrak{h}_0^\perp, |\tilde{\lambda}| < \varepsilon$, then,
as $\text{Re} P$ is also an homogeneous polynomial of degree $4p$, we have

$$\text{Re} P(\lambda + \tilde{\lambda}) = |\lambda|^{4p} \text{Re} P\left(\frac{\lambda}{|\lambda|} + \frac{\tilde{\lambda}}{|\tilde{\lambda}|}\right) \geq |\lambda|^{4p} c.$$ 

On the other hand, if $|\lambda| \leq 1, |\tilde{\lambda}| < \varepsilon, |P(\lambda + \tilde{\lambda})|$ is bounded by a positive
constant $A$. Now let $Z \in Z(\mathfrak{g})$ be such that $\gamma_{\mathfrak{h}}(Z) = P + 2A$. If $\sigma \in \hat{M}$,
$\nu \in i\mathfrak{a}_0^\perp, z \in \mathfrak{a}', |z| < \varepsilon$, then, if $|\mu_{\sigma} + \nu| \leq 1$,

$$|\chi_{\sigma,\nu+z}(P)| \geq A \geq C_1(1 + |\mu_{\sigma}|^2)^{2p}(1 + |\nu|^2)^{2p},$$

and, if $|\mu_{\sigma} + \nu| \geq 1$,

$$|\chi_{\sigma,\nu+z}(P)| \geq 2A + \text{Re} P(\mu_{\sigma} + \nu + z) \geq C_2(1 + |\mu_{\sigma}|^2)^{2p}(1 + |\nu|^2)^{2p}.$$ \hfill \qed

Now we state a lemma we will use to obtain uniform bounds of the distributions $\Theta_{\sigma,\nu}$ and $\Theta_{\lambda}$. The proof follows easily from the proof of Theorem
10.2 in [10].

Lemma 6.5. Given $\tilde{K} \subset G$ a compact subset, for every admissible representation $\pi$ whose $K$-types have the multiplicity property $n_\tau \leq N \text{dim} \tau$ there
exists a differential operator $\tilde{\Omega} \in \mathcal{Z}(\mathfrak{k})$ and a constant $C$ which only depends on $\tilde{K}$ such that if $\operatorname{supp} f \subseteq \tilde{K}$

$$|\Theta_{\pi}(f)| \leq CN \left( \int_{G} |\tilde{\Omega} f(g)|^2 \|\pi(g)\|^2 \, dg \right)^{1/2}.$$ 

In particular, if $\pi$ is unitary, $|\Theta_{\pi}(f)| \leq CN \|\tilde{\Omega} f\|_{L^2(G)}$.

### 6.1. Principal series.

Before proceeding to the definition of the distributions $R_{\sigma, \nu}$ we state two lemmas about polynomial functions.

If $m$ is a positive integer we denote $\operatorname{Pol}(m)$ the complex vector space of polynomial functions in $\mathbb{C}^n$ of degree $\leq m$, and $\operatorname{Pol}^0(m)$ this vector space with the origin removed.

Given $Q \in \operatorname{Pol}(m)$, $\zeta \in \mathbb{C}^n$, the equation

$$\tilde{Q}(\zeta) = \left( \sum_{\alpha} \frac{1}{(\alpha!)^2} |Q^{(\alpha)}(\zeta)|^2 \right)^{1/2}$$

defines a norm in $\operatorname{Pol}(m)$ for a fixed $\zeta \in \mathbb{C}^n$. Let

$$||Q|| = \tilde{Q}(0).$$

The following lemma is used to construct a fundamental solution of a constant coefficients differential operator in $\mathbb{R}^n$ (Theorem 7.3.10 of [8]). For a proof see Lemmas 7.3.11 and 7.3.12 in [8].

**Lemma 6.6.** For all $\epsilon > 0$ exists a non-negative function $\Phi \in C^\infty(\operatorname{Pol}^0(m) \times \mathbb{C}^n)$ such that

(i) $\Phi$ is absolutely homogeneous of degree 0, i.e., $\Phi(zQ, \zeta) = \Phi(Q, \zeta)$ for all $z \in \mathbb{C} - \{0\}$.

(ii) $\Phi(Q, \zeta) = 0$ if $|\zeta| \geq \epsilon$.

(iii) If $F$ is an entire function in $\mathbb{C}^n$ and $d\zeta$ is Lebesgue measure in $\mathbb{C}^n$,

$$\int F(\zeta)\Phi(Q, \zeta) \, d\zeta = F(0).$$

(iv) There exists a constant $C_\epsilon$ such that $\tilde{Q}(0) \leq C_\epsilon |Q(\zeta)|$ if $\Phi(Q, \zeta) \neq 0$.

Being $\operatorname{Pol}(m)$ a finite dimensional vector space, all norms $\tilde{Q}(\zeta)$ are equivalent. The following lemma estimates the constants which give this equivalence. The proof is an easy consequence of Taylor’s formula.

**Lemma 6.7.** Exists a constant $C > 0$ depending only on $n$ and $m$ such that

$$C \left( 1 + |\zeta|^2 \right)^{-m} \tilde{Q}(0)^2 \leq \tilde{Q}(\zeta)^2 \leq C \left( 1 + |\zeta|^2 \right)^m \tilde{Q}(0)^2$$

for all $Q \in \operatorname{Pol}(m)$. 
We are now in position to define the distributions $R_{\sigma,\nu}$ for $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfying the hypothesis of Theorem 1.1. Let $\varepsilon > 0$ be given by Proposition 6.3, $m$ the order of $P$ and let $\Phi \in C^\infty(\text{Pol}^0(m) \times \mathbb{C})$ be the non-negative function given by Lemma 6.6.

Given $\sigma \in \hat{M}$, $\tau \in \hat{K}$, and a fixed $\nu \in \mathfrak{a}'$, we put

\[
P_{\sigma,\tau}(z) = P_{\sigma,\tau}(\nu + z) = (\chi_{\sigma,\nu+z} \otimes \chi_{\tau})(P).
\]

Finally, if $dz$ is Lebesgue measure in $\mathfrak{a}'$, $f \in \mathcal{D}(G)$, we define

\[
R_{\sigma,\nu}(f) = \sum_{\tau \in \hat{K}} \int_{|z| < \varepsilon} \frac{1}{P_{\sigma,\tau}(z)} \Theta_{\sigma,\nu+\tau}(f) \Phi(P_{\sigma,\tau}, z) \, dz.
\]

This definition makes sense because $P_{\sigma,\tau}(z) \neq 0$ if $\Phi(P_{\sigma,\tau}, z) \neq 0$ (Lemma 6.6 (iv)).

**Proposition 6.8.** The map defined by (23) is a finite order distribution for all $\sigma \in \hat{M}$, $\nu \in \mathfrak{a}'$. This map has also the following properties:

(i) $PR_{\sigma,\nu} = \Theta_{\sigma,\nu}$.

(ii) For every positive integer $k$ and $f \in \mathcal{D}(G)$, exist a constant $C > 0$ which only depends on the support of $f$ and a differential operator $D_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ such that

\[
|R_{\sigma,\nu}(f)| \leq \frac{C}{(1 + |\nu|^2)(1 + |\sigma|^2)^k} ||D_k f||_{L^2(G)} \quad \forall \sigma \in \hat{M}, \nu \in i\mathfrak{a}'_0.
\]

**Proof.** By Proposition 3.2, $P\Theta_{\sigma,\nu+\tau} = P_{\sigma,\tau}(z)\Theta_{\sigma,\nu+\tau}$, and, as $\Theta_{\sigma,\nu+\tau}(f)$ is an entire function in $z$ ([6], Section 21) for fixed $f$, so it is $\Theta_{\sigma,\nu+\tau}(f)$, and by Lemma 6.6 (ii),

\[
\int_{|z| < \varepsilon} \Theta_{\sigma,\nu+\tau}(f) \Phi(P_{\sigma,\tau}, z) \, dz = \Theta_{\sigma,\nu}(f),
\]

and this proves (i).

Let’s see that (23) defines a distribution: According to Lemma 6.6 (iv) and Lemma 6.7 together with (20) and (21), if $\Phi(P_{\sigma,\tau}, z) \neq 0$, we have $||P_{\sigma,\tau}|| \leq C_1(1 + |\nu|^2)^m ||P_{\sigma,\tau}(z)||$, and by the hypothesis on $P$ (Prop. 5.1 or 5.3), exists a positive integer $\bar{k}$ such that

\[
\frac{1}{|P_{\sigma,\tau}(z)|} \leq C_2(1 + |\nu|^2)^m (1 + |\sigma|^2)^{\bar{k}} (1 + |\tau|^2)^{\bar{k}}.
\]

On the other hand, if $Z \in \mathcal{Z}(\mathfrak{g})$ and $\Omega \in \mathcal{Z}(\mathfrak{t})$ are given by Proposition 6.3 and Lemma 6.1 respectively and if $s_1$ and $s_2$ are positive integers, we have,
for some positive integer \( \bar{k} \),
\[
|\Theta_{\sigma,\nu+\bar{k}}(f)| = \frac{\left| \Theta_{\sigma,\nu+\bar{k}} ^{(\Omega^t)^{s_1}(\Omega^t)^{s_2}} f \right|}{|\chi_{\sigma,\nu+\bar{k}}(Z)||\chi_{\sigma}^t(\Omega)|} \leq \frac{C_3 \left| \Theta_{\sigma,\nu+\bar{k}} ^{(\Omega^t)^{s_1}(\Omega^t)^{s_2}} f \right|}{((1 + |\sigma|^2)(1 + |\nu|^2))^{s_1-k} (1 + |\tau|^2)^{s_2}},
\]
besides, it holds for \( D \in \mathcal{U}(g) \) that \( Df_{\tau} = (Df)_{\tau} \); also \( |\Theta_{\pi}(f)| \leq |\Theta_{\pi}(f)| \), so
\[
(25) \quad |\Theta_{\sigma,\nu+\bar{k}}(f)| \leq \frac{C_3 \left| \Theta_{\sigma,\nu+\bar{k}} ^{(\Omega^t)^{s_1}(\Omega^t)^{s_2}} f \right|}{((1 + |\sigma|^2)(1 + |\nu|^2))^{s_1-k} (1 + |\tau|^2)^{s_2}}.
\]

Let \( \tilde{K} \subseteq G \) be a compact subset. Note that for principal series it holds \( n_{\tau} \leq \dim \tau \) ([10, p. 207]); so by Lemma 6.5 exist \( \hat{\Omega} \in Z(\mathfrak{t}) \) and a constant \( C_1 \) independent of \( \sigma \in \mathcal{M} \), \( \nu \) \( z \in \mathfrak{a}^* \) such that
\[
|\Theta_{\sigma,\nu+z}(f)| \leq C_1 \left( \int_G |\hat{\Omega}(g)|^2 ||\pi_{\sigma,\nu+z}(g)||^2 dg \right)^{1/2};
\]
on the other hand, given \( \varphi \) in the space \( V^\sigma \) where the \( \pi_{\sigma,\nu+z} \) acts, if \( a(g) \) is the A-component of \( g \) in the Iwasawa decomposition, then (cf. [10, p. 169]),
\[
(\pi_{\sigma,\nu+z}(g)\varphi)(k) = e^{-z\log a(g^{-1}k)} (\pi_{\sigma,\nu}(g)\varphi)(k),
\]
and taking \( A = \sup_{g \in K, k \in K, |z| < \varepsilon} |e^{-z\log a(g^{-1}k)}| \) and \( B_{\sigma,\nu} = \sup_{g \in K} ||\pi_{\sigma,\nu}(g)|| \), then \( ||\pi_{\sigma,\nu+z}(g)|| \leq AB_{\sigma,\nu} \) uniformly on \( \tilde{K} \), so for all \( f \) such that \( \text{supp} \ f \subseteq \tilde{K} \),
\[
(26) \quad |\Theta_{\sigma,\nu+z}(f)| \leq AB_{\sigma,\nu} ||\hat{\Omega}(\mathfrak{t})||_{L^2(G)}.
\]
Now combining (24), (25) and (26) and the fact that \( \Phi \) is uniformly bounded we obtain
\[
\left| \int_{|z| < \varepsilon} \frac{\Theta_{\sigma,\nu+z}(f)\Phi(P_{\sigma,\tau}, z)}{P_{\sigma,\tau}(z)} dz \right| \leq \frac{C_4 B_{\sigma,\nu} ||\hat{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2} f||_{L^2(G)}}{(1 + |\nu|^2)^{s_1-m-k} (1 + |\sigma|^2)^{s_1-k} (1 + |\tau|^2)^{s_2}},
\]
therefore for all \( f \) such that \( \text{supp} \ f \subseteq \tilde{K} \)
\[
(27) \quad |R_{\sigma,\nu}(f)| \leq \sum_{\tau \in \mathcal{K}} \frac{1}{(1 + |\tau|^2)^{s_2-k}} \frac{C_4 B_{\sigma,\nu} ||\hat{\Omega}(Z^t)^{s_1}(\Omega^t)^{s_2} f||_{L^2(G)}}{(1 + |\nu|^2)^{s_1-m-k} (1 + |\sigma|^2)^{s_1-k}},
\]
and \( \sum_{\tau \in \mathcal{K}} \frac{1}{(1 + |\tau|^2)^{s_2-k}} \) is finite if we choose \( s_2 > \bar{k} + 1/2 \dim K \) ([14, Lemma 5.6.7]), so \( R_{\sigma,\nu} \) is a finite order distribution.
To see (ii), just observe that $B_{\sigma,\nu} = 1$ if $\nu \in i\mathfrak{a}_0'$, so given $k$ if we take $D_k = \tilde{\Omega}(Z^2)^{s_1}(\Omega^k)^{s_2}$ with the $s_2$ chosen above and $s_1 \geq k + \tilde{k} + \max(k, m)$, (27) becomes

$$|R_{\sigma,\nu}(f)| \leq \frac{C}{(1 + |\nu|^2)^k (1 + |\sigma|^2)^k} \|D_k f\|_{L^2(G)}$$

with $C$ depending only $\tilde{K}$. □

6.2. Discrete Series. Suppose $T$ is a compact Cartan subgroup of $G$.

Proposition 6.9. Let $P \in \mathcal{Z}(U(\mathfrak{g})^K)$ such that exist a constant $C$ and a positive integer $k$ such that

$$|P_{\lambda,\tau}| \geq \frac{C}{(1 + |\lambda|^2)^k (1 + |\tau|^2)^k} \quad \forall (\lambda, \tau) \in S_d \times \hat{K},$$

then the map

$$R_\lambda(f) = \sum_{\tau \in \hat{K}} \frac{1}{P_{\lambda,\tau}} \Theta_\lambda^{\tau}(f)$$

defines a finite order distribution with the following properties:

(i) $PR_\lambda = \Theta_\lambda$.

(ii) For each positive integer $k$ and $f \in D(G)$, exist a constant $C > 0$ which only depends on the support of $f$ and a differential operator $E_k \in \mathcal{Z}(U(\mathfrak{g})^K)$ such that

$$|R_\lambda(f)| \leq \frac{C}{(1 + |\lambda|^2)^k} \|E_k f\|_{L^2(G)} \quad \forall \lambda \in S_d.$$

Proof. First note that $R_\lambda$ is well defined because by Proposition 5.2 exist a constant $C_1$ and a positive integer $k_1$ such that

$$|P_{\lambda,\tau}| \geq \frac{C_1}{(1 + |\lambda|^2)^k_1 (1 + |\tau|^2)^k_1} \quad \forall (\lambda, \tau) \in S_d \times \hat{K},$$

and in particular $P_{\lambda,\tau} \neq 0$; (i) is clear because $P\Theta_\lambda^{\tau} = P_{\lambda,\tau} \Theta_\lambda^{\tau}$ (Proposition 3.2).

To see (ii), observe that by Lemma 6.5 exist a constant $C_2$ depending only on supp $f$ and $\tilde{\Omega} \in \mathcal{Z}(\mathfrak{t})$ such that $|\Theta_\lambda(f)| \leq C_2 \|\tilde{\Omega} f\|_{L^2(G)}$. 
Then, if \( Z \in \mathcal{Z}(\mathfrak{g}) \) and \( \Omega \in \mathcal{Z}(\mathfrak{t}) \) are given by Lemmas 6.2 and 6.1 respectively, we have

\[
|R_{\lambda}(f)| \leq \sum_{\tau \in \hat{K}} \frac{1}{|P_{\lambda,\tau}|} |\Theta_{\lambda}(f)| \leq \sum_{\tau \in \hat{K}} \frac{1}{C_3 (1 + |\lambda|^2)^{k_1} (1 + |\tau|^2)^{k_1}} |\Theta_{\lambda}(f)|
\]

\[
\leq \sum_{\tau \in \hat{K}} \frac{1}{(1 + |\lambda|^2)^{s_1-k_1} (1 + |\tau|^2)^{s_2-k_1}} |\Theta_{\lambda}((Z^I)^{s_1}(\Omega^I)^{s_2} f)|
\]

\[
\leq \left( \sum_{\tau \in \hat{K}} \frac{1}{(1 + |\tau|^2)^{s_2-k_1}} \right) \frac{C_1}{(1 + |\lambda|^2)^{s_1-k_1}} ||(\tilde{\Omega}(Z^I)^{s_1}(\Omega^I)^{s_2} f)||_{L^2(G)},
\]

and it suffices to take \( E_k = \tilde{\Omega}(Z^I)^{s_1}(\Omega^I)^{s_2} \) with \( s_1 = k + k_1 \) and \( s_2 \) such that

\[
\sum_{\tau \in \hat{K}} \frac{1}{(1 + |\tau|^2)^{s_2-k_1}}
\]

is finite. \( \square \)

7. Demonstration of Theorem 1.1.

Now we are ready to complete the proof of Theorem 1.1 with the explicit construction of the fundamental solution of \( P \).

**Proposition 7.1.** Let \( P \in \mathcal{Z}(U(\mathfrak{g})) \) and suppose that exist a constant \( C \) and a positive integer \( k \) such that in case I,

\[
||P_{\sigma,\tau}|| \geq \frac{C}{(1 + |\sigma|^2)^k (1 + |\tau|^2)^k} \quad \forall (\sigma, \tau) \in \hat{M} \times \hat{K},
\]

and in case II,

\[
|P_{\lambda,\tau}| \geq \frac{C}{(1 + |\lambda|^2)^k (1 + |\tau|^2)^k} \quad \forall (\lambda, \tau) \in S_d \times \hat{K};
\]

if \( R_{\sigma,\nu} \) and \( R_{\lambda} \) are the distributions defined respectively by (23) and 6.9, then the map \( R \) defined in case I,

\[
R = \sum_{\sigma \in \hat{M}} \int_{\nu \in \text{id}_0'} R_{\sigma,\nu} m_\sigma(\nu) \, d\nu,
\]

and in case II,

\[
R = \sum_{\sigma \in \hat{M}} \int_{\nu \in \text{id}_0'} R_{\sigma,\nu} m_\sigma(\nu) \, d\nu + \sum_{\lambda \in S_d} d_{\lambda} R_{\lambda},
\]

is a finite order distribution which is a fundamental solution of \( P \).

**Remark.** We note that in case II, \( T \) is a fundamental Cartan subgroup of \( G \), and in case I, \( H = A \times T^- \) is.
So if $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ is such that $(\gamma^G \otimes \gamma^K)(P)$ has a fundamental solution in $H^I \times T$, Propositions 5.2 and 5.1 imply (28) and (29) respectively in each case, so Theorem 1.1 is a direct consequence of Proposition 7.1.

**Proof.** First of all, we note that in case II, Proposition 5.3 says that (28) implies (29) (changing, maybe, $C$ and $k$), so $R_{\sigma,\nu}$ is well defined and Proposition 6.8 applies in this case.

Equality $PR = \delta$ is clear by Plancherel formula (Theorem 2.1) and because $PR_{\sigma,\nu} = \Theta_{\sigma,\nu}$ and $PR_\lambda = \Theta_\lambda$ (Propositions 6.8 and 6.9 respectively); it only remains to prove that $R$ is a finite order distribution in each case. So we will prove that each of the following are finite order distributions:

$$R_{sp} = \sum_{\sigma \in \hat{M}} \int_{\nu \in i\mathfrak{a}_0'} R_{\sigma,\nu} m_{\sigma}(\nu) \, d\nu, \quad R_{sd} = \sum_{\lambda \in S_d} d_\lambda R_\lambda.$$  

Let $\tilde{K}$ be a compact subset and $f \in \mathcal{D}_{\tilde{K}}(G)$; for each positive integer $k$ let $D_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ be given by Proposition 6.8 (ii), then, using 2.1 (ii),

$$|R_{sp}(f)| \leq C_1 C_2 \left( \sum_{\sigma \in \hat{M}} \frac{1}{(1 + |\sigma|^2)^{k-l_2}} \right) \left( \int_{\nu \in i\mathfrak{a}_0'} \frac{1}{(1 + |\nu|^2)^{k-l_1}} \right) \|D_k f\|_{L^2(G)},$$

and choosing $k$ large enough so that the sum and the integral are finite, we obtain an operator $D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ and a constant $C$ depending only on $\tilde{K}$ such that

$$|R_{sp}(f)| \leq C \|D f\|_{L^2(G)},$$

and this proves that $R_{sp}$ is a distribution of finite order less or equal that the order of $D$.

In the same way, for each positive integer $k$ let $E_k \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ be given by Proposition 6.9 (ii), therefore

$$|R_{sd}(f)| \leq C_3 \left( \sum_{\tau \in \tilde{K}} \frac{d_\lambda}{(1 + |\lambda|^2)^k} \right) \|E_k f\|_{L^2(G)};$$

if $k > 1/2 \dim G$, $\sum_{\tau \in \tilde{K}} \frac{d_\lambda}{(1 + |\lambda|^2)^k}$ is finite ([14, Lemma 5.6.7]), there exist $E \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ and a constant $\tilde{C}$ depending only on $\tilde{K}$ such that

$$|R_{sd}(f)| \leq \tilde{C} \|E f\|_{L^2(G)},$$

so $R_{sd}$ is a distribution of finite order less or equal that the order of $E$. □
8. P-convexity of $G$.

Suppose that $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfies the conditions of Proposition 7.1. So $P$ has a fundamental solution $R \in \mathcal{D}'(G)$. This implies that the differential equation $Pu = f$ has a solution $u \in C^\infty(G)$ for all $f \in \mathcal{D}(G)$; just taking $u = f \ast R$ because $Pu = f \ast P \mathcal{R} = f \ast \delta = f$. Now, in order to guarantee the solvability of $Pu = f$ when $f \in C^\infty(G)$, it is necessary to analyze the $P$-convexity of $G$.

**Definition 2.** Given $D \in \mathcal{U}(\mathfrak{g})$, we say that $G$ is $D$-convex if for every compact subset $\Omega \subseteq G$ exists another compact subset $\tilde{\Omega} \subseteq G$ such that

$$\text{supp } (Df) \subseteq \Omega \implies \text{supp } (f) \subseteq \tilde{\Omega}.$$ 

Using Johnson’s injectivity criterion ([9]), we will verify that $G$ is $P$-convex.

Let $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$ satisfying the conditions of 7.1. Then $P_{\sigma,\tau}$ is a non-zero polynomial on $a'$ for all $(\sigma, \tau) \in \hat{M} \times \hat{K}$.

Given $\sigma \in \hat{M}$, $\nu \in a'$, we denote $V^\sigma$ the space where the principal series representation $\pi_{\sigma,\nu}$ acts (remind that we can choose $V^\sigma$ independent of $\nu$).

Let $V^\sigma_{P}$ be the subspace of $K$-finite vectors of $\pi_{\sigma,\nu}$.

Now, if $U$ is a $\pi_{\sigma,\nu}(P)$-invariant finite dimensional subspace of $V^\sigma_{P}$, then

$$U \subseteq W = \sum_{j=1}^{k} n^\sigma_{\tau_j} V_{\tau_j};$$

on the other hand, if $v_j \in V_{\tau_j}$, $\pi_{\sigma,\nu}(P)v_j = (\chi_{\sigma,\nu} \otimes \chi_{\tau_j})(P)v_j = P_{\sigma,\tau_j}(\nu)v_j$, that is, $\pi_{\sigma,\nu}(P)$ is diagonalizable on $W$, so $U$ has a $\pi_{\sigma,\nu}(P)$-invariant complement $\tilde{U}$ in $W$.

Suppose that $\det \pi_{\sigma,\nu}(P)|_U = 0$ for all $\nu \in a'$. Then

$$0 = \det \pi_{\sigma,\nu}(P)|_U \det \pi_{\sigma,\nu}(P)|_{\tilde{U}} = \prod_{j=1}^{k} (P_{\sigma,\tau_j}(\nu))^d_{\nu} n^\sigma_{\nu} \quad \forall \nu \in a',$$

therefore exist $\sigma \in \hat{M}$, $\tau \in \hat{K}$ such that $P_{\sigma,\tau}(\nu) = 0$ for all $\nu \in a'$ which is absurd. So Theorems 5.1 and 5.2 in [9] imply that $G$ is $P$-convex.

9. Parametrix of operators in $\mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$.

In this section, using Zuckerman characters identities combined with the work done so far, we will prove:

**Proposition 9.1.** Let $G$ be a linear connected semisimple Lie group of rank one having a compact Cartan subgroup, and $P \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})^K)$. A sufficient condition for the existence of a parametrix of $P$ is:
Exist a finite set $F \subset S_d$, a positive constant $C$ and a positive integer $k$ such that

$$\left| P_{\lambda,\tau} \right| \geq \frac{C}{(1 + |\lambda|^2)^k (1 + |\tau|^2)^k} \quad \forall (\lambda, \tau) \in (S_d - F) \times \hat{K}. \tag{32}$$

**Remark.** The fact that if $(\gamma^G \otimes \gamma^K)(P)$ has a parametrix on $T \times T$ then (32) holds is an easy consequence of [4, Thm. II]. So Proposition 9.1 clearly implies Theorem 1.3.

Before proceeding to the proof of this proposition, we state Zuckerman characters identities in the form we need (cf. Proposition 5.13 from [16] for a more precise statement together with the proof).

**Proposition 9.2.** Let $G$ be as in Proposition 9.1. We put $r = \frac{|W(G,H)|}{|W(G)|}$. Given $\lambda \in S_d$ exist $\sigma_{\lambda}^1, \ldots, \sigma_{\lambda}^r \in \hat{M}$ and $\nu_{\lambda}^1, \ldots, \nu_{\lambda}^r \in a'_0$ such that

$$\frac{1}{|W_K|} \sum_{w \in W_G} \Theta_{w,\lambda} = a_0(\lambda)\Theta_{\lambda}^f + \sum_{j=1}^r a_j(\lambda)\Theta_{\sigma_{\lambda}^j,\nu_{\lambda}^j},$$

where $\Theta_{\lambda}^f$ is the character of the finite dimensional representation with infinitesimal character $\lambda$, and $a_j(\lambda) = \pm 1$ for all $0 \leq j \leq r$, $\lambda \in S_d$.

**Proof of Proposition 9.1.** We will construct the parametrix $R$ of $P$ as the sum of three distributions $R = R_{sp} + R_{sd} + R_F$.

As in the proof of Proposition 7.1, applying Proposition 5.3 together with 6.8, we see that

$$R_{sp} = \sum_{\sigma \in \hat{M}} \int_{\nu \in a'_0} R_{\sigma,\nu} m_{\sigma}(\nu) \, d\nu$$

defines a finite order distribution such that

$$PR_{sp} = \sum_{\sigma \in \hat{M}} \int_{\nu \in a'_0} \Theta_{\sigma,\nu} m_{\sigma}(\nu) \, d\nu.$$ 

In the same way, if we define

$$R_{sd} = \sum_{\lambda \in S_d - F} d_{\lambda} R_{\lambda},$$

then, applying Proposition 6.9,

$$PR_{sd} = \sum_{\lambda \in S_d - F} d_{\lambda} \Theta_{\lambda}.$$ 

For the definition of $R_F$, we note in first place that having all the representations $\pi_{w,\lambda}$ infinitesimal character $\lambda$, $F$ is closed by the action of $W_G$. 

Besides $\pi_{w^\lambda}$ is equivalent to $\pi_{w^\lambda'}$ if and only if $w^{-1}w' \in W_K$, and we can write

$$\sum_{\lambda \in F} \Theta_{\lambda} = \frac{|W_K|}{|W_G|} \sum_{\lambda \in F} \sum_{w \in W_G} \Theta_{w^\lambda}. \tag{33}$$

Now, for each $\lambda \in F$, let $\sigma_1^\lambda, \ldots, \sigma_r^\lambda \in \hat{M}$ and $\nu_1^\lambda, \ldots, \nu_r^\lambda \in a'_0$ be given by Proposition 9.2; then we can define

$$R_F = \frac{|W_K|}{|W_G|} \sum_{\lambda \in F} \left( \sum_{j=1}^r a_j(\lambda) R_{\sigma_j^\lambda, \nu_j^\lambda} \right) \tag{34}$$

which is a finite order distribution.

Finally, Propositions 6.8 and 9.2 imply

$$PR = \frac{|W_K|}{|W_G|} \sum_{\lambda \in F} \left( \sum_{j=1}^r a_j(\lambda) \Theta_{\sigma_j^\lambda, \nu_j^\lambda} \right) = \sum_{\lambda \in F} \Theta_{\lambda} - \frac{|W_K|}{|W_G|} \sum_{\lambda \in F} a_0(\lambda) \Theta_f^\lambda;$$

therefore, putting $R = R_{sp} + R_{sd} + R_F$, the above equalities together with Plancherel formula (Theorem 2.1) imply

$$PR - \delta = -\frac{|W_K|}{|W_G|} \sum_{\lambda \in F} a_0(\lambda) \Theta_f^\lambda,$$

and this distribution is given by a $C^\infty$ function because $\Theta_f^\lambda$ are characters of a finite dimensional representation. □


The fundamental solution constructed for $P \in \mathcal{Z}(U(g)^K)$ satisfying the conditions of Theorem 1.1 is invariant by inner automorphisms of $K$. In the case that $P$ is a bi-invariant operator in the conditions of Corollary 1.2, we obtain a fundamental solution of $P$ invariant by inner automorphisms of all $G$.

Let’s analyze the case of the Casimir operator of $G$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and if $\lambda \in \mathfrak{h}'$, then $\chi_{\lambda}(\Omega) = B(\lambda, \lambda) - B(\rho, \rho)$ ([14, 5.6.4]); in particular, if $\sigma \in \mathfrak{h}$, $\nu \in i\mathfrak{a}_0'$,

$$\chi_{\sigma, \nu}(\Omega) = \chi_{\mu_{\sigma} + \nu}(\Omega) = |\mu_{\sigma}| - |\nu|^2 - |\rho|^2, \tag{35}$$

and $||\chi_{\mu, -}(\Omega)|| \geq 1$ for all $\mu \in \mathfrak{t}^\perp$, so, by Theorem 4.1, $\gamma_C(\Omega)$ has a fundamental solution on $A \times T^\perp$. Besides if $t$ is a Cartan subalgebra of $\mathfrak{g}$, in this case if $\lambda \in i\mathfrak{t}_0'$ we have

$$\chi_{\lambda}(\Omega) = |\lambda|^2 - |\rho|^2; \tag{36}$$
now, being $G^C$ simply connected, $\lambda = \rho$ is a discrete series parameter, and consequently we cannot apply Corollary 1.2.

So Theorem 1.1 only provides a fundamental solution of the Casimir operator when $G$ has one conjugacy class of Cartan subgroups.

We note, on the other hand, that solvability of $\Omega$ have been proved by Rauch and Wigner ([12]) in a non-constructive way. In [2] an explicit fundamental solution of $\Omega$ is constructed for $G = SL(2, \mathbb{R})$ and it’s also proved that an invariant one doesn’t exists for this group.

However, note that $|\chi_\lambda(\omega)| \geq C$ for all $\lambda \in S_d - F$, where $F = \{ \lambda \in S : |\lambda| = |\rho| \}$ is a finite subset, so by Proposition 9.1, $\Omega$ has a parametrix on $G$.

11. A necessary condition.

Let $G$ be a connected semisimple Lie group. Recall from [7, Thm. 5.17] that we can define a Harish-Chandra homomorphism $\gamma_a : \mathcal{U}(g)^K \to \mathcal{S}(a)^{W_0}$, where $W_0 = W(g_0, a_0)$ is the Weyl group of the restricted root system. The kernel of $\gamma_a$ is $\mathcal{U}(g)^K \cap \mathcal{U}(g)f$.

Proposition 4.1 in [13] states that if $P \in \mathcal{Z}(g)$ is in the kernel of $\gamma_a$, then $P$ doesn’t have a parametrix. In this section we will extend this proposition for $P \in \mathcal{Z}(\mathcal{U}(g)^K)$.

Let $\mathcal{D}_K(G)$, resp. $C^K_\infty(G)$, be the space of left and right $K$-invariant functions in $\mathcal{D}(G)$, resp. $C^K_\infty(G)$, $\mathcal{D}'_K(G)$ the dual of $\mathcal{D}_K(G)$, identified with the space of left and right $K$-invariant distributions in $\mathcal{D}'(G)$, and similarly $\mathcal{D}_{W_0}(A)$, resp. $\mathcal{D}'_{W_0}(A)$, the space of $W_0$-invariant elements of $\mathcal{D}(A)$, resp. $\mathcal{D}'(A)$. Let

$$F_f(a) = a^\rho \int_N f(an) \, dn,$$

$f \in \mathcal{D}_K(G)$, $a \in A$; then the map $f \mapsto F_f$ is an isomorphism of $\mathcal{D}_K(G)$ onto $\mathcal{D}_{W_0}(A)$ for the Schwartz topologies ([7, Cor. 7.9]). If $P \in \mathcal{Z}(g)$, $F_{Pf} = \gamma_a(P)F_f$ ([7, p. 307]). Transposing $F^{-1}$ we get an isomorphism $F^t$ of $\mathcal{D}'_K(G)$ onto $\mathcal{D}'_{W_0}(A)$, and

$$F^t_{PT} = \gamma_a(P)F^t_T$$

for all $P \in \mathcal{Z}(g)$, $T \in \mathcal{D}'_K(G)$.

Now let $P \in \mathcal{Z}(\mathcal{U}(g)^K)$. If $\tau \in \hat{K}$, we can form the homomorphism $\gamma_a \otimes \chi_\tau : \mathcal{Z}(\mathcal{U}(g)^K) \to \mathcal{S}(a)^{W_0}$.

**Proposition 11.1.** Let $G$ be a connected semisimple Lie group and $P \in \mathcal{Z}(\mathcal{U}(g)^K)$. If $P$ has a parametrix in $G$, then for all $\tau \in \hat{K}$, $(\gamma_a \otimes \chi_\tau)(P) \neq 0$.

**Proof.** Suppose that $P$ has a parametrix $E$ and that exists $\tau \in \hat{K}$ such that $(\gamma_a \otimes \chi_\tau)(P) = 0$. 


An easy computation shows that if $\Omega \in \mathcal{Z}(\mathfrak{t})$, then $\Omega E^\tau = \chi^\tau(\Omega)E^\tau$, so if $P^\tau = (Id \otimes \chi^\tau)(P) \in \mathcal{Z}(\mathfrak{g})$, then

$$P E^\tau = P^\tau E^\tau.$$ 

Taking $\tau$-components, $PE - \delta \in C^\infty(G)$ implies $P^\tau E^\tau - \delta^\tau \in C^\infty(G)$, and making everything $K$-bi-invariant we get $P^\tau E^\tau_K - \delta^\tau_K \in C^\infty_K(G)$.

Applying $F^\tau$ yields

$$\gamma_a(P^\tau)(F^\tau_{E^\tau_K} - F^\tau_{\delta^\tau_K}) \in C^\infty_W(A);$$ 

now $\gamma_a(P^\tau) = (\gamma_a \otimes \chi^\tau)(P) = 0$ would imply $\delta^\tau_K \in C^\infty_K(G)$, which is absurd because it’s easy to see that $\delta^\tau = d^\tau\Theta^\tau m_K$, where $m_K(f) = \int_K f$ is the distribution induced by the Haar measure of $K$. □

References


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