ISOPERIMETRIC INEQUALITIES FOR SECTORS ON SURFACES

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We discuss sectors on a surface of curvature bounded above by a constant and derive an isoperimetric inequality for a proper sector on such a surface. With this isoperimetric inequality we derive an inequality involving the total length of the cut locus of a point on a closed surface.

1. Introduction.

There have been extensive studies on isoperimetric inequalities on a Riemannian manifold of dimension 2 (shortly, a surface) [Al, BdC, CF, Fi]. C. Bandle derived an isoperimetric inequality for a sector in the Euclidean plane $E^2$ [Ba1, Ba2]: Let $D$ be a sector in $E^2$, which is a simply connected region enclosed by two line segments $\gamma_1, \gamma_2$ starting at a point $p$ and a piecewise smooth simple curve segment $\Gamma$ joining the end points of $\gamma_1, \gamma_2$. Let $\theta_0$ denote the interior angle of $D$ at $p$. C. Bandle showed that for a sector $D$ with $\theta_0 \leq \pi$,

$$L^2(\Gamma) \geq 2 \theta_0 A(D)$$

with equality if and only if $D$ is a circular sector, where $L(\Gamma)$ denotes the length of $\Gamma$ and $A(D)$ the area of $D$.

In this paper, we consider an isoperimetric inequality for a sector on a surface $M$ with curvature $K$ bounded above by a constant $C$. By a sector on a surface $M$ we mean a region of $M$ enclosed by two geodesic segments $\gamma_1, \gamma_2$ and a piecewise smooth curve segment $\Gamma$, which together constitute a simple closed curve $\Gamma^* = \gamma_1 \cup \Gamma \cup \gamma_2$. On a general surface $M$, a sector needs not be simply connected nor bounded (e.g., the cylinder $\mathbb{R} \times S^1$), or could be the whole surface (e.g., the torus $T^2$). On the other hand, such a simple closed curve $\Gamma^*$ may enclose two bounded sectors (e.g., the sphere $S^2$). For our purpose, we will restrict our attention to sectors on a surface $M$ that are closed, simply connected and bounded ones. We will call such a sector by a proper sector, denoted by $D(\Gamma)$ or just $D$. We take parametrizations of two geodesic segments $\gamma_i$ and a piecewise smooth curve segment $\Gamma$ as $\gamma_i : [0, r_i] \to M \ (i = 1, 2)$ and $\Gamma : [a, b] \to M$ such that $\gamma_1(0) = \gamma_2(0)$ and $\Gamma(a) = \gamma_1(r_1)$, $\Gamma(b) = \gamma_2(r_2)$ so that $\Gamma^* = \gamma_1 \cup \Gamma \cup \gamma_2$ is a simple closed curve with a suitable orientation. The vertex of $D$ is the point $\gamma_1(0) = \gamma_2(0)$ where $\gamma_1$ and $\gamma_2$ cross. Our main result is the following:
Isoperimetric Inequality for a Sector. Let $M$ be a surface with curvature $K$ bounded above by a constant $C$. Let $D$ be a proper sector on $M$ with interior angle $\theta_0 \leq \pi$ at the vertex. Then

$$L^2(\Gamma) \geq 2\theta_0 A(D) - C A^2(D).$$

Equality holds only when $D$ is isometric to a geodesic sector on a surface of constant curvature $C$.

Generally, the cut locus $\text{Cut}(p)$ of $p$ on a closed surface $M$ is a local tree which may have infinitely many edges $[M1, M2, GS]$. So, the Hausdorff 1-measure is used to measure a subset of $\text{Cut}(p)$. It is known that every compact subset of $\text{Cut}(p)$ on a complete surface $M$ has finite total length (Hausdorff 1-measure) $[He2, I]$. As an application of our isoperimetric inequality, we derive an inequality involving the total length of the cut locus of a point on a closed surface.

2. Sectors on a Surface.

Definition 2.1. Let $A$ be a subset of $M$ and $p, q \in A$. The distance between $p$ and $q$ in $A$ is defined by

$$d_A(p, q) = \inf_{c \in \Omega^A_{p,q}} \int_c ds,$$

where $\Omega^A_{p,q}$ is the set of all piecewise smooth curve segments contained in $A$ joining $p$ and $q$. Let $B \subset M$ such that $A \cap B \neq \emptyset$. Then $d_A(p, B) = \inf_{q \in A \cap B} d_A(p, q)$ denotes the distance from $p$ to $B$ in $A$.

Note that $d_A(\cdot, \cdot) \geq d_M(\cdot, \cdot)$ for any set $A \subset M$.

Definition 2.2. Let $A$ be a compact subset of $M$ and $G$ a subset of the boundary $\partial A$ of $A$. For $t \geq 0$, the parallel $G_t$ of $G$ in the distance $t$ in $A$ is defined by

$$G_t = \{ q \in A : d_A(q, G) = t \}.$$

For a proper sector $D(\Gamma)$ in $M$, the parallel $\Gamma_t$ of $\Gamma$ in $D(\Gamma)$ is a piecewise smooth simple curve for small $t > 0$. As $t$ gets larger, $\Gamma_t$ can have several components.

Relative Cut Locus. Let $D(\Gamma)$ be the proper sector with two geodesics $\gamma_1, \gamma_2$ on $M$ and a piecewise smooth simple curve segment $\Gamma : [a, b] \to M$ with corners $\Gamma(s_i) = x_i$ at $a = s_1 < s_2 < \cdots < s_{n+1} = b$. Let $N(s)$ be the inward unit normal vector field along $\Gamma$ with the right/left limits $N(s^-)$ at the corners $x_i$ of $\Gamma$, $i = 1, \ldots, n+1$. With notational conventions, $N(s_1^-) = -\gamma_1'(r_1)$ and $N(s_{n+1}^+) = -\gamma_2'(r_2)$, let $N_i$ be the set of all inward
unit tangent vectors in $T_x M$ between $N(s^-_i)$ and $N(s^+_i)$. Let

$$\mathcal{N} = \bigcup_{i=0}^{n+1} \mathcal{N}_i,$$

where $\mathcal{N}_0 = \{N(s) : s \in [a, b]\}$. For each $v \in \mathcal{N} \cap T_q M$, $q \in \Gamma$, let $\gamma_v : [0, r] \to M$ be a geodesic such that $\gamma_v(0) = q$ and $\gamma_v'(0) = v$. The point $z = \gamma_v(t)$ where $\gamma_v$ stops minimizing the distance $d_{D(\Gamma)}(\gamma_v(t), \Gamma)$ is called the relative cut point of $v \in \mathcal{N}$ in $D(\Gamma)$. The set of all such relative cut points of $v \in \mathcal{N}$ in $D(\Gamma)$ is called the relative cut locus of $\Gamma$ in $D(\Gamma)$, denoted by $C_{\text{rel}}(\Gamma)$. If the exterior angle of $\Gamma$ at a corner $x_i$ is positive, then for all $v \in N_i$, the relative cut point of $v$ in $D(\Gamma) = x_i$ itself. Note also that $C_{\text{rel}}(\Gamma)$ need not be a connected set.

**Geodesic Sectors.** For a point $p \in M$, let $U$ be a (simply connected) normal neighborhood of $p$, and take polar coordinates $(r, \theta)$ on $U \setminus \{p\}$ such that the metric can be written as

$$ds^2 = dr^2 + f^2 d\theta^2,$$

where $f = f(r, \theta)$ is the positive-valued function satisfying the initial conditions

$$\lim_{r \to 0} f(r, \theta) = 0, \quad \lim_{r \to 0} \frac{\partial f}{\partial r}(r, \theta) = 1.$$

Let $\gamma_1, \gamma_2 : [0, r] \to M, i = 1, 2$, be two geodesics starting at $p$ with the angle $0 < \theta \leq \pi$ and $\beta : [0, \theta] \to U \subset M$ a geodesic circular arc given by $\beta(s) = (r, s)$ in $U$. The proper sector enclosed by $\gamma_1, \gamma_2$ and $\beta$ is called a geodesic sector denoted by $S_{r,\theta}$. We will call $\beta$ the circular boundary of $S_{r,\theta}$. The area $A_{r,\theta}$ and the arc length $L_{r,\theta}$ of the circular boundary of $S_{r,\theta}$ are respectively given by

$$(2.1) \quad A_{r,\theta} = \int_0^\theta \int_0^r f(t, s) dt ds, \quad L_{r,\theta} = \int_0^\theta f(r, s) ds.$$

**Remark 2.3.** A geodesic sector with vertex $p$ may cross the usual cut locus $\text{Cut}(p)$ of $p$. One can easily construct a geodesic sector $S_{r,\theta}$ crossing the cut locus $\text{Cut}(p)$ of $p$ on the cylinder $\mathbb{R} \times S^1$.

Let $S^C_{r,\theta}$ denote a geodesic sector of radius $r$ and angle $\theta$ on a surface $M^C$ of constant curvature $K \equiv C$. Let $A^C_{r,\theta}$ and $L^C_{r,\theta}$ denote its area and the arc length of the circular boundary, respectively. The explicit expressions are:

$$(2.2) \quad A^C_{r,\theta} = 2\theta \sin^2 \frac{ar}{a^2} \quad \frac{1}{2} \theta r^2 \quad 2\theta \sinh^2 \frac{ar}{a^2},$$

$$L^C_{r,\theta} = \theta \frac{\sin ar}{a} \quad \theta r \quad \theta \frac{\sinh ar}{a}.$$
One can easily verify the following formula:

\[(L_{r,\theta}^C)^2 = 2\theta A_{r,\theta}^C - C(A_{r,\theta}^C)^2.\]  

(2.3)

For the geodesic sectors on a surface with curvature \(K \leq C\), we may have the following lemmas as immediate consequences of the formulas (2.1) and Lemma 7 in [Os]:

**Lemma 2.4.** Let \(M\) be a surface with curvature \(K \leq C\). Let \(S_{r,\theta}\) be a geodesic sector on \(M\) of radius \(r\) and angle \(\theta\). Then

\[A_{r,\theta} \geq A_{r,\theta}^C\]

with equality if and only if \(S_{r,\theta}\) is isometric to \(S_{r,\theta}^C\) on a surface \(M_C\) of constant curvature \(C\).

**Lemma 2.5.** Under the same assumptions as in Lemma 2.4, we have

\[L_{r,\theta} \geq L_{r,\theta}^C.\]

If \(L_{\xi,\theta} = L_{\xi,\theta}^C\) for all \(\xi \in (0,r]\), then \(S_{r,\theta}\) is isometric to \(S_{r,\theta}^C\) on a surface \(M_C\) of constant curvature \(C\).

3. Isoperimetric Inequalities for Sectors on a Surface.

Let \(\beta : [a, b] \rightarrow M\) be a unit speed simple curve, and let \(n\) be a unit normal vector field along \(\beta\). Then one can find a variation \(X : [a, b] \times (-\delta, \delta) \rightarrow M\) of \(\beta\) given by

\[X(s, \xi) = \exp_{\beta(s)} \xi n(s)\]

for some \(\delta > 0\).

For the computation of the length of the parallel \(\Gamma_t\) of \(\Gamma\), we write \(\Gamma = \sum_{i=1}^n \beta^i\) where \(\beta^i (i = 1, \ldots , n)\) are smooth curves with inward unit normal \(N\). For \(t > 0\) small, the parallel \(\Gamma_t\) of \(\Gamma\) in \(D(\Gamma)\) consists of parts of the geodesic parallels \(\beta^i_t\) of \(\beta^i\) in \(D(\Gamma)\) together with the geodesic circular arcs of radius \(t\). Let

\[t_* = \sup_{p \in \Gamma, \, q \in C_{rel}(\Gamma)} d_D(\Gamma)(p, q),\]
and for $0 \leq t \leq t_*$, let

$$D(\Gamma_t) = \{ q \in D(\Gamma) : d_{D(\Gamma)}(q, \Gamma) \geq t \}.$$

Notice that $D(\Gamma_t)$ may not be a proper sector since it is not a connected set in general. We will denote the arc length of $\Gamma_t$ by $L(t)$ and the area of $D(\Gamma_t)$ by $A(t)$ as functions of $t$.

**Lemma 3.1.** Let $M$ be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on $M$ with interior angle $\theta_0 \leq \pi$ at the vertex. Then

$$L'(0) \leq CA(0) - \theta_0.$$

**Proof.** Let $x_i$ ($i = 1, 2, \ldots, n + 1$) be the corners of $\Gamma$ including end points and let $\alpha_i$ denote the exterior angle of $\Gamma$* at the corner $x_i$ (See Figure 1). We may assume that $\alpha_i \neq \pi$. Let $S = \{2, \ldots, n\}$, and let $A = \{ i \in S : \alpha_i \leq 0 \}$ and $B = \{ j \in S : \alpha_j > 0 \}$. For sufficiently small $\xi > 0$, using the linear approximations for dotted parts of $\beta^*_1$ (say, at $x_2$ in Figure 1), we have, with the help of (3.2),

$$L(\xi) = L(0) - \xi \int_{\Gamma} \kappa ds - \sum_{i \in A} \xi \alpha_i - \sum_{j \in B} 2 \xi \tan(\alpha_j/2)$$

$$- \xi \tilde{\alpha}_1 - \xi \tilde{\alpha}_{n+1} + o(\xi),$$

where $\kappa$ is the geodesic curvature of $\Gamma$ and

$$\tilde{\alpha}_k = \begin{cases} \alpha_k - \pi/2 & \text{if } \alpha_k \leq \pi/2, \\ \tan(\alpha_k - \pi/2) & \text{if } \alpha_k > \pi/2. \end{cases}$$

Using that $\tan \alpha \geq \alpha$ for $\alpha \geq 0$ and $\tilde{\alpha}_k \geq \alpha_k - \pi/2$, we have

$$L'(0) = - \int_{\Gamma} \kappa ds - \sum_{i \in A} \alpha_i - \sum_{j \in B} 2 \tan(\alpha_j/2) - \tilde{\alpha}_1 - \tilde{\alpha}_{n+1}$$

$$\leq - \int_{\Gamma} \kappa ds - \sum_{i=1}^{n+1} \alpha_i + \pi.$$
Since $\gamma_1, \gamma_2$ are geodesics, by the Gauss-Bonnet formula,

$$L'(0) \leq \int_{D(\Gamma)} K \, dA - \theta_0.$$  

Finally, we have

$$L'(0) \leq CA(0) - \theta_0$$

from the curvature condition that $K \leq C$. \hfill \Box

Note that, for sufficiently small $t > 0$, $\Gamma_t$ has one component of a piecewise smooth non-closed simple curve segment. Computation as in Lemma 3.1 thus gives that

$$L'(t) \leq CA(t) - \theta_0$$

for sufficiently small $t > 0$.

Note also that $\Gamma_t$ could be the union of at most finite number of piecewise smooth non-closed curves, piecewise smooth closed curves and points in general. For a piecewise smooth closed curve, by the same way as in Lemma 3.1 we have the following:

**Lemma 3.2.** Let $M$ be a surface with curvature $K \leq C$ and $D$ a nondegenerate compact subset of $M$ with the boundary $\partial D = G$ which is a piecewise smooth simple curve. Let $G_\xi$ denote the parallel of $G = \partial D$ in $D$ and $\ell(t) = L(G_t)$. Then

$$\ell'(0) \leq CA(D) - 2\pi.$$

The following facts come from the results of [Fi, pp. 303-332] by a slight modification (also see [CF, p. 86]): $L(t)$ is continuous for all but at most a finite number of $t$ in $[0, t_\ast]$ at which $L(t)$ has a jump discontinuity, however $A(t)$ is continuous; $A'(t) = -L(t)$ for almost all $t \in [0, t_\ast]$ (cf. [Ha, p. 706]).

**Theorem 3.3.** Let $M$ be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on $M$ with interior angle $\theta_0 \leq \pi$ at the vertex. Then

$$L'(t) \leq CA(t) - \theta_0$$

for almost all $t \in [0, t_\ast]$.

**Proof.** Let $n_t$ and $m_t$ be the numbers of components of piecewise smooth non-closed curves $\Gamma^i_t$ and piecewise smooth closed curves $\Omega^j_t$ of $\Gamma_t$, respectively. For almost all $t \in [0, t_\ast]$, we may write

$$\Gamma_t = \sum_{i=1}^{n_t} \Gamma^i_t + \sum_{j=1}^{m_t} \Omega^j_t.$$

Note that each end point of $\Gamma^i_t$ is either on $\gamma_1$ or on $\gamma_2$ so that $\Gamma^i_t$ and $\gamma_1$ and/or $\gamma_2$ bound a simply connected compact set, denoted by $D(\Gamma^i_t)$. Each
$\Omega^i_t$ itself also bounds simply connected compact set, denoted by $D(\Omega^i_t)$. Thus we may write

$$D(\Gamma_t) = \left( \bigcup_{i=1}^{n_t} D(\Gamma^i_t) \right) \cup \left( \bigcup_{j=1}^{m_t} D(\Omega^j_t) \right).$$

For the sake of brevity, we use the notations: $L_i(t) = L(\Gamma^i_t)$, $A_i(t) = A(D(\Gamma^i_t))$, $\ell_j(t) = L(\Omega^j_t)$ and $B_j(t) = A(D(\Omega^j_t))$. Then by the same computation as (3.3) we have

$$L'_i(t) \leq CA_i(t) - \theta_i,$$

where $\theta_0 \leq \theta_i \leq \pi$. By Lemma 3.2,

$$\ell'_j(t) \leq CB_j(t) - 2\pi.$$

Thus,

$$L'(t) = \sum_i L'_i(t) + \sum_j \ell'_j(t) \leq \sum_i (CA_i(t) - \theta_i) + \sum_j (CB_j(t) - 2\pi) \leq CA(t) - \theta_0$$

for almost all $t \in [0, t_*]$. □

**Theorem 3.4.** Let $M$ be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on $M$ with interior angle $\theta_0 \leq \pi$ at the vertex. Then

$$L^2(s) - L^2(t) \geq 2\theta_0(A(s) - A(t)) - C(A^2(s) - A^2(t))$$

for $s < t \in [0, t_*]$.

**Proof.** By multiplying $A'(t) = -L(t) \leq 0$ to the inequality (3.4), we get

$$-L(t)L'(t) \geq CA(t) - \theta_0 A'(t)$$

for almost all $t \in [0, t_*]$. Note that $L(t)$ is continuous on $[0, t_*]$ except for a finite number of points $0 < t_1 < t_2 < \cdots < t_m < t_*$. Let $I_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, m+1$, where $t_0 = 0$, $t_{m+1} = t_*$. For $s < t$ in $[0, t_*]$, we may assume that $s \in I_i$ and $t \in I_j$ for some $i \leq j$. By direct computation, we have

$$-\int_s^t L(t)L'(t) \, dt = \frac{1}{2}(L^2(s) - L^2(t)) + \frac{1}{2} \sum_{k=i}^{j-1} (L^2(t^+_k) - L^2(t^-_k)), $$
where $h(r^\pm)$, as usual, stands for the right/left limits of a function $h$ at $r$. Notice that $L(t^+_k) < L(t^-_k)$ and $A(t^+_k) = A(t^-_k)$. Thus we have

\begin{equation}
- \int_s^t L(t)L'(t)dt \leq \frac{1}{2}(L^2(s) - L^2(t)).
\end{equation}

Similarly,

\begin{equation}
\int_s^t A(t)A'(t)dt = -\frac{1}{2}(A^2(s) - A^2(t)),
\end{equation}

\begin{equation}
- \int_s^t A'(t)dt = A(s) - A(t).
\end{equation}

Therefore, from (3.9)–(3.12), we have

\begin{equation}
L^2(s) - L^2(t) \geq -2\int_s^t L(t)L'(t)dt
\end{equation}

\begin{equation}
\geq 2C\int_s^t A(t)A'(t)dt - 2\theta_0 \int_s^t A'(t)dt
\end{equation}

\begin{equation}
= 2\theta_0(A(s) - A(t)) - C(A^2(s) - A^2(t)).
\end{equation}

\[ \square \]

We are now ready to state and prove our main result.

**Theorem 3.5.** Let $M$ be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on $M$ with interior angle $\theta_0 \leq \pi$ at the vertex. Then

\begin{equation}
L^2(\Gamma) \geq 2\theta_0 A(D(\Gamma)) - CA^2(D(\Gamma)),
\end{equation}

where equality holds only when $D(\Gamma)$ is isometric to a geodesic sector on a surface $M_C$ of constant curvature $K \equiv C$.

**Proof.** By setting $t \to t_*$ in (3.8) of Theorem 3.4, we get

\begin{equation}
L^2(s) \geq 2\theta_0 A(s) - CA^2(s).
\end{equation}

Now at $s = 0$, we get the inequality (3.13).

For a geodesic sector on a surface $M_C$ of constant curvature $K \equiv C$, it is quite clear that the equality holds in (3.13) by (2.3).

Suppose now that the equality $L^2(0) = 2\theta_0 A(0) - CA^2(0)$ holds. Then from (3.8) with $s = 0$ together with (3.14) we get

\begin{equation}
L^2(t) = 2\theta_0 A(t) - CA^2(t),
\end{equation}

for all $0 \leq t \leq t_*$. Since $\Gamma_{t_*}$ is in the relative cut locus $C_{rel}(\Gamma)$ of $\Gamma$ in $D(\Gamma)$, $D(\Gamma_{t_*})$ is contained in $C_{rel}(\Gamma)$. That is, $A(t_*) = 0$ and so by (3.15) $L(t_*) = 0$.

By differentiation,

\begin{equation}
L'(t) = -\theta_0 + CA(t)
\end{equation}
for all $0 \leq t \leq t_*$. Therefore, equalities hold for all $0 \leq t \leq t_*$ in inequalities in the proof of Theorem 3.3. This implies that, for $t < t_*$, the exterior angles at the end points of $\Gamma_t$ are less than or equal to $\pi/2$ and there are no corners (which are not end points) on $\Gamma_t$ at which the exterior angle of $\Gamma_t^*$ is positive. In addition, for each $v \in T_q M \cap \mathcal{N}$, which is the set defined as in Section 2, the geodesic $\gamma_v : [0, t_*) \to M$ such that $\gamma_v(0) = q \in \Gamma$, $\gamma_v'(0) = v$ satisfies $d_{D(\Gamma)}(\gamma_v(t), q) = t$ and $\gamma_v(t) \in \Gamma_t$ for each $t \in [0, t_*]$. That is, $C_{rel}(\Gamma) \subseteq \Gamma_*$. Therefore, $C_{rel}(\Gamma) = \Gamma_*$ is the set of a single point, say $\{p\}$. Moreover, no geodesics starting at $p$ intersect before the distance $t_*$ in $D(\Gamma)$, so $D(\Gamma)$ is a geodesic sector of radius $t_*$ and angle $\theta_0$ centered at $p$. By (3.15) and (3.16), $L : [0, t_*) \to \mathbb{R}$ satisfies the following ODE

\[(3.17) \quad L''(t) = -CL(t), \quad L(t_*) = 0, \quad L'(t_*) = -\theta_0.\]

By comparing the solution of (3.17) with $L_{C, \theta}$ in (2.2) for a geodesic sector on $M_C$, one can see that $D(\Gamma)$ is isometric to $S_{t_*, \theta_0}$ by Lemma 2.5.

If $M = \mathbb{E}^2$ and we set $C = 0$, then Theorem 3.5 implies the result of C. Bandle [Ba1, Ba2]. Similar isoperimetric inequalities on Lorentzian surfaces were obtained by the authors [BH, B].

**Remark 3.6.** The condition that $\theta_0 \leq \pi$ in Theorem 3.5 is essential. One can construct a proper sector for which the isoperimetric inequality (3.13) does not hold in the following way: In $\mathbb{E}^2$, consider a proper sector with $\Gamma = \Gamma^1 \cup \Gamma^2$, where $\Gamma^1$ is a semi-circle of radius $r$ centered at $q$ and $\Gamma^2$ is a circular arc of angle $0 < \varphi < \pi$ and radius $\alpha r$ ($0 < \alpha < 1$) centered at $p$ (see Figure 2). Take $C = 0$, then

$$L^2(\Gamma) = (\pi + \varphi\alpha)^2r^2 < (\pi + \varphi)(\pi + \varphi\alpha^2)r^2 = 2\theta_0 A(D(\Gamma)).$$

![Figure 2.](image-url)
Remark 3.7. The isoperimetric inequality (3.13) holds for a closed, simply connected, bounded set $D$ having the boundary $\Gamma^* = \gamma_1 \cup \Gamma \cup \gamma_2$, where $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(r_1) = \gamma_2(r_2) = \Gamma(a) = \Gamma(b)$ by approximating $\Gamma^*$ by $\Gamma^*_\varepsilon$, where $\Gamma^*_\varepsilon$ is a closed curve obtained from $\Gamma^*$ by changing the parts of $\Gamma^*$ contained in a geodesic ball of radius $\varepsilon$ at $\gamma_1(r_1) = \gamma_2(r_2)$ so that $\Gamma^*_\varepsilon$ is piecewise smooth and simple.

We now consider a special case of a proper sector: Suppose that $\gamma_i : [0, r_i] \to M$ ($i = 1, 2$) are two geodesic segments such that $\gamma_1(0) = \gamma_2(0) = p$, $\gamma_1(r_1) = \gamma_2(r_2) = q$ and $\gamma_1((0, r_1)) \cap \gamma_2((0, r_2)) = \emptyset$. Such a sector will be called an oval sector. Notice that there are no such oval sectors on a surface with curvature $K \leq 0$. Let $\theta_1, \theta_2$ be the interior angles of $D$ at $p, q$, respectively. From Theorem 3.5, we have:

**Corollary 3.8.** Let $M$ be a surface with curvature $K \leq C$ for a positive constant $C$. Let $D$ be an oval sector enclosed by $\gamma_1, \gamma_2$ with $\theta_1, \theta_2 \leq \pi$ on $M$. Then

$$A(D) \geq \frac{2\theta_*}{C},$$

where $\theta_* = \max\{\theta_0, \theta_1\}$. Equality holds only when $D$ is isometric to a geodesic sector of radius $\pi/\sqrt{C}$ and angle $\theta_1 = \theta_2$ on a surface of constant curvature $K \equiv C$.

The equality case of Corollary 3.8 is based on the following fact: For an oval sector $D$ with $\theta_1, \theta_2 \leq \pi$ on a surface $M$ with curvature $K \leq C$, by the Gauss-Bonnet formula, one can obtain the inequality

$$A(D) \geq \frac{1}{C}(\theta_0 + \theta_1),$$

where equality holds only when $M$ is a surface of constant curvature $C$.

The following is an immediate consequence of Corollary 3.8:

**Corollary 3.9.** Let $M$ be a surface with curvature $K \leq C$ for a positive constant $C$. Suppose that $\gamma_1, \gamma_2$ are two geodesics starting at a point $p \in M$ with angle $\theta$ at $p$ and $D$ is a simply connected domain on $M$ containing $\gamma_1, \gamma_2$ with area less than $\frac{2\theta}{C}$. Then $\gamma_1$ and $\gamma_2$ never meet again in $D$.

4. **Lengths of the cut locus.**

Let $M$ be a closed surface (i.e., a compact surface without boundary) with area $A(M)$. For $v \in T_pM$, $p \in M$, denote by $\gamma_v$ the unique geodesic satisfying $\gamma_v(0) = v$. Define $\rho(v) = \sup \{t \in \mathbb{R} : \gamma_v$ is minimal on $[0, t]\}$. Then $\rho(v)$ is continuous on the set $S_p = \{v \in T_pM : ||v|| = 1\}$ and the values of $\rho$ on $S_p$ are bounded (by the diameter of $M$). Note that, if $w = \lambda v \in T_pM$ ($\lambda \geq 0$), then $\rho(v) = \lambda \rho(w)$. Let

$$U_p = \{v \in T_pM : \rho(v) > 1\}.$$
Then \( U_p \) is a bounded set in \( T_pM \) and the (usual) cut locus of \( p \) is
\[
\text{Cut}(p) = \exp_p(\partial U_p).
\]
It is well known that, for \( p \in M \),
\[
M = U_p \cup \text{Cut}(p),
\]
where \( U_p = \exp_p(U_p) \). Note that \( \text{Cut}(p) \) is a deformation retract of \( M \setminus \{p\} \).

Hence, on any orientable closed surface \( M \) of genus \( g \), \( \text{Cut}(p) \) of any point \( p \in M \) contains \( 2g \) closed curves, which form a set of generators for the fundamental group of \( M \). It is known that any compact subset of \( \text{Cut}(p) \) of \( p \) on a closed surface \( M \) has finite Hausdorff 1-measure (cf. [He2, I]). Thus any path in \( \text{Cut}(p) \) is rectifiable ([He1, Proposition 5.1]) and the Hausdorff 1-measure of a path in \( \text{Cut}(p) \) is its arc length ([Fa, p. 29]). Using our isoperimetric inequality (4.1) in Theorem 4.2, we will derive an inequality involving the Hausdorff 1-measure of the cut locus of a point in a closed orientable surface.

**Lemma 4.1.** Let \( M \) be a closed surface and \( p \in M \). Then there is a geodesic segment through \( p \) such that its end points are in \( \text{Cut}(p) \) and it bisects \( U_p \) in area.

**Proof.** For a unit vector \( v \in T_pM \), denote \( c_v : [-\rho(-v), \rho(v)] \to M \) the unique geodesic segment such that \( c_v(s) = \gamma_{-v}(-s) \) for \( s \in [-\rho(-v), 0] \) and \( c_v(t) = \gamma_v(t) \) for \( t \in [0, \rho(v)] \). Then \( c_v(-\rho(-v)) = \gamma_{-v}(\rho(-v)) \), \( c_v(\rho(v)) = \gamma_v(\rho(v)) \), \( c_v(0) = p \). Clearly, for each \( v \in \mathbb{S}_p \), \( c_v \) splits \( U_p \) into two pieces. We take one piece of these for each \( v \) continuously and name it \( D_v \). Let \( A(v) \) be the area of \( D_v \). Then \( A(v) \) is a continuous function on \( \mathbb{S}_p \) since \( \rho \) is continuous on \( \mathbb{S}_p \). By the mean value theorem, there is a \( v_0 \in \mathbb{S}_p \) such that \( A(v_0) = \frac{1}{2}A(M) \), and \( c_{v_0} \) is a desired one. \( \square \)

**Theorem 4.2.** Let \( M \) be a closed surface with curvature \( K \leq C \) and \( \ell \) the total length (the Hausdorff 1-measure) of the cut locus \( \text{Cut}(p) \) of \( p \in M \). Then
\[
\ell^2 \geq \pi A(M) - \frac{C}{4} A^2(M). \tag{4.1}
\]

**Proof.** Let \( \gamma : [a, b] \to M \) be a geodesic segment bisecting \( U_p \) into \( D_1, D_2 \) with \( A(D_1) = A(D_2) \) as in Lemma 4.1. Then there is a (continuous) path \( \tilde{\Gamma} \) in \( \text{Cut}(p) \) joining \( \gamma(a) \) and \( \gamma(b) \) so that \( \gamma \) and \( \tilde{\Gamma} \) constitute the common boundary of \( D_1 \) and \( D_2 \). As the path \( \tilde{\Gamma} \) may not be piecewise smooth and \( \tilde{\Gamma} \) is compact, for any \( \varepsilon > 0 \) we choose a piecewise simple curve segment \( \Gamma \) joining \( \gamma(a) \) and \( \gamma(b) \) such that \( \Gamma \) is contained in the \( \varepsilon \)-neighborhood of \( \tilde{\Gamma} \) and \( L(\Gamma) \leq L(\tilde{\Gamma}) \). Let \( D'_1, D'_2 \) be two domains with boundary \( \gamma \) and \( \Gamma \) corresponding to \( D_1, D_2 \), respectively. By Theorem 3.5 and Remark 3.7,
\[
L^2(\Gamma) \geq 2\pi A(D'_1) - CA^2(D'_1), \quad L^2(\Gamma) \geq 2\pi A(D'_2) - CA^2(D'_2).
\]
Combining these with the fact that $A(D'_i) = A(D_i) + O(\varepsilon^2)$ for $i = 1, 2$,

$$L^2(\tilde{\Gamma}) \geq 2\pi A(D_1) - CA^2(D_1), \quad L^2(\tilde{\Gamma}) \geq 2\pi A(D_2) - CA^2(D_2).$$

Since $\tilde{\Gamma} \subset \text{Cut}(p)$ and $L(\tilde{\Gamma})$ is equal to the Hausdorff 1-measure of $\tilde{\Gamma}$, which is less than or equal to the total length $\ell$ of $\text{Cut}(p)$, we get

$$\ell^2 \geq \pi A(M) - \frac{C}{4} A^2(M).$$

□

Example 4.3. Let $T^2$ be the flat torus obtained by identifying the opposite sides of quadrilateral ABCD (see Figure 3). Then the cut locus $\text{Cut}(p)$ of middle point $p \in T^2$ is the set formed by the line segments AB and BC, and so $\ell = a + b$, where $a$ and $b$ are the length of the line segments AB and BC, respectively. The area of $T^2$ is $ab$. Take $C = 0$ as usual, then (4.1) gives a well-known inequality

$$(a + b)^2 \geq \pi ab.$$
Corollary 4.5. Let $M$ be a closed orientable surface of genus $g > 1$ with curvature $-1$. Then
\[ \ell^2 \geq 4\pi^2 g(g - 1). \]

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References


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