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LOCAL RESTRICTIONS ON NONPOSITIVELY CURVED n-MANIFOLDS IN $\mathbf{R}^{n+p}$

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In pointwise differential geometry, i.e., linear algebra, we prove two theorems about the curvature operator of isometrically immersed submanifolds. We restrict our attention to Euclidean immersions because here the results are most easily stated and the curvature operator can be simply expressed as the sum of wedges of second fundamental form matrices. First, we reprove and extend a 1970 result of Weinstein to show that for n-manifolds in $\mathbf{R}^{n+2}$ the conditions of positive, nonnegative, nonpositive, and negative sectional curvature imply that the curvature operator is positive definite, positive semidefinite, negative semidefinite, and negative definite, respectively. We provide a simple example to show that this equivalence is no longer true even in codimension 3. Second, we introduce the concept of measuring the amount of curvature at a point x by the rank of the curvature operator at x and prove that surprisingly the rank of a negative semidefinite curvature operator is bounded as a function of only the codimension. Specifically, for an n-manifold in $\mathbf{R}^{n+p}$ this rank is at most $\binom{p+1}{2}$, and this bound is sharp. Under the weaker assumption of nonpositive sectional curvature we prove the rank is at most p^3+p^2-p, and by the proof of the previous theorem this bound can be sharpened to $\binom{p+1}{2}$ for $p=1$ and 2.

1. An expression for the curvature tensor.

Let $\mathcal{N}$ be an n-dimensional Riemannian manifold isometrically immersed in $\mathbf{R}^{n+p}$. We may give $\mathcal{N}$ in nonparametric form near the origin of $\mathbf{R}^{n+p} = \mathbf{R}^n \times \mathbf{R}^p$ by the graph of

$$\psi = (\psi^1, \dots, \psi^p) : V \to \mathbf{R}^p$$

in $\mathbf{R}^{n+p}$ where V is some neighborhood of the origin in $\mathbf{R}^n$. We take as natural coordinates for $\mathcal{N}$ the standard orthonormal coordinates $x_1, \dots, x_n$ in V. We may further assume that our point under consideration is at the origin in $\mathbf{R}^{n+p}$ and that the tangent space at the origin is $\mathbf{R}^n \times \{0\}$. Thus ψ and all its first derivatives vanish at the origin, and consequently the

Christoffel symbols and the first partial derivatives of the metric components vanish at the origin in the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. We take $e_1, \dots, e_{n+p}$ to be the standard orthonormal basis for $\mathbf{R}^{n+p}$ with the identification $e_i = \frac{\partial}{\partial x_i}(0)$ for $1 \le i \le n$. The second fundamental form of $\mathcal{N}$ in $\mathbf{R}^{n+p}$ at the origin takes its values in the the normal space to $\mathcal{N}$ at the origin, i.e., in $\{0\} \times \mathbf{R}^p$. Letting subscripts denote differentiation with respect to the coordinates $x_1, \dots, x_n$, the second fundamental form at the origin in the basis $\frac{\partial}{\partial x_i}(0)$ is the matrix of vectors whose i-j component is $\psi_{ij}(0) \in \mathbf{R}^p$. In this same basis the components of the curvature tensor are

$$R_{ijkl}(0) = \sum_{m=1}^{p} (\psi_{ik}^m(0)\psi_{jl}^m(0) - \psi_{il}^m(0)\psi_{jk}^m(0)).$$

We shall think of the curvature tensor as a symmetric bilinear form on $\wedge_2 \mathbf{R}^n$, in which case $R_{ijkl}(0)$ $(i < j, k < l)$ is just the ij-kl component of the matrix $R(0)$ representing the curvature tensor in the basis $e_i \wedge e_j$ $(1 \le i < j \le n)$ of $\wedge_2 \mathbf{R}^n$.

Assuming the definition of the wedge of a matrix in the next section and letting Ψ^m be the scalar second fundamental form of $\mathcal{N}$ in $\mathbf{R}^{n+p}$ in the normal direction e_{n+m}, i.e., the $n \times n$ matrix whose i-j component is $\psi_{ij}^m(0)$, we see that

$$R(0) = \sum_{m=1}^{p} \Psi^m \wedge \Psi^m$$

with respect to the $e_i \wedge e_j$ basis. $R(0)$ and $\Psi^1, \dots, \Psi^p$ are manifestly real symmetric matrices.

2. Some results on wedges of symmetric matrices.

Let $A : V \to W$ be a linear map of vector spaces. By $A \wedge A : \wedge_2 V \to \wedge_2 W$ we mean the map such that $(A \wedge A)(v \wedge w) = Av \wedge Aw$ for every v and w $\in V$ and $A \wedge A$ acts on the rest of $\wedge_2 V$ by linearity. By a *simple vector* in $\wedge_2 V$ we will mean a vector which can be written in the form $v \wedge w$ where $v, w \in V$. If V and W have finite bases $e_1, \dots, e_n$ and $f_1, \dots, f_m$, we can think of A as an $m \times n$ matrix (a_{ij}) and $A \wedge A$ as an $\binom{m}{2} \times \binom{n}{2}$ matrix where $(A \wedge A)_{ij,kl} = a_{ik}a_{jl} - a_{il}a_{jk}$ in the simple bases $e_k \wedge e_l$ $(1 \le k < l \le n)$ and $f_i \wedge f_j$ $(1 \le i < j \le m)$.

Now we fix notation so that A and B are $n \times n$ real symmetric matrices, $n > 1$, and M is the $\binom{n}{2} \times \binom{n}{2}$ real symmetric matrix defined by $M = A \wedge A + B \wedge B$. A, B, and M are to be regarded as symmetric bilinear (or quadratic) forms. We establish some terminology and some facts concerning A, B, and M which will be used implicitly in the following proofs. By *transforming* A and B we mean changing their basis as quadratic forms. Thus if P is a real nonsingular matrix representing a change of basis, $A \to {}^tPAP$ and

$B \to {}^tPBP$. Transforming A and B by P transforms M by $P \wedge P$, i.e., $M \to {}^t(P\wedge P)M(P\wedge P)$. Therefore $P\wedge P$ represents a change of simple basis for M. Transforming A and B by P will change eigenvalues and eigenvectors but not signatures of A, B, and M. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A in the orthonormal basis $v_1, \dots, v_n$, then $\lambda_i\lambda_j$ are the eigenvalues for $A \wedge A$ in the orthonormal simple basis $v_i \wedge v_j$ $(1 \le i < j \le n)$. We say that a matrix is *definite* (*semidefinite*) if it is positive or negative definite (semidefinite). By a *null-cone vector* we mean a nonzero vector whose value under the given quadratic form is 0. We will need the following four elementary facts.

Fact 1. If A is definite, then there exists a real nonsingular matrix P such that ${}^tPAP = I$ or $-I$ and ${}^tPBP$ is diagonal.

Fact 2. If A and B are both semidefinite, then there exists a real nonsingular matrix P such that ${}^tPAP$ and ${}^tPBP$ are both diagonal.

Fact 3 (Identity of Lagrange)**.** For $v, w, y, z \in \mathbf{R}^n$, we have the identity of inner products

$$(v \wedge w) \cdot (y \wedge z) = (v \cdot y)(w \cdot z) - (v \cdot z)(w \cdot y),$$

where $\cdot$ denotes the usual inner products in $\mathbf{R}^n$ and $\wedge_2\mathbf{R}^n$.

Fact 4. For $n = 3$, there exists a real orthogonal 3×3 matrix P such that ${}^t(P \wedge P)M(P \wedge P)$ is diagonal.

From Fact 3,

$$\text{(1)} \quad {}^t(v \wedge w)M(v \wedge w) = ({}^tvAv)({}^twAw) - ({}^twAv)^2 + ({}^tvBv)({}^twBw) - ({}^twBv)^2.$$

We say that M is *positive* (*nonnegative, nonpositive, negative*) *on simple vectors* if the quantity (1) is positive (nonnegative, nonpositive, negative) whenever $v \wedge w$ is nonzero. Letting S be a real symmetric matrix, we define the *positive* (*negative*) *space* of S to be the direct sum of its eigenspaces having positive (negative) eigenvalues. By $\ker(S)$ we will mean the ordinary kernel where S is regarded as a linear mapping. If S represents a fixed quadratic form on a vector space V in a particular basis, then the positive and negative spaces are basis dependent but the kernel is invariant.

If $A_\theta = \cos\theta A + \sin\theta B$ and $B_\theta = -\sin\theta A + \cos\theta B$, it is easy to check that $M = A_\theta \wedge A_\theta + B_\theta \wedge B_\theta$ for $\theta \in \mathbf{R}$. (This is a special case of Fact 5, which appears later.) The following extends Weinstein's result [**8**] on the positive case to the nonnegative case.

Theorem 1. *If M is positive* (*nonnegative*) *on simple vectors, then*

(i) *either A or B is definite* (*semidefinite*),
(ii) *if A is semidefinite and B is not, then* $\ker(A) \subset \ker(B)$,

(iii) *there exists a basis* $v_1, \dots, v_n$ *for* A *and* B *such that* M *is diagonal in the simple basis* $v_i \wedge v_j$ $(1 \le i < j \le n)$, *and*

(iv) *there exists* $\theta \in \mathbf{R}$ *such that* A_θ *and* B_θ *are both positive definite* (*semidefinite*); *hence* M *is in fact positive definite* (*semidefinite*), (*Weinstein* [**8**, p. 2], *Chen* [**1**, p. 977]).

Proof. (iv): Let

$$T = \{z_v = ({}^t vAv, {}^t vBv) : v \in \mathbf{R}^n \sim \{0\}\} \subset \mathbf{R}^2.$$

Then for $z_v, z_w \in T$ we have $z_v \cdot z_w > 0$ (≥ 0) by (1). (If $v \wedge w = 0$ in the positive case, $z_v \cdot z_w = 0$ would contradict the positivity of M.) This means that T lies in an open (closed) quadrant in $\mathbf{R}^2$. We let θ be the angle through which this quadrant must be rotated (clockwise) to bring it to the first quadrant. Then the quadratic forms A_θ and B_θ each become positive definite (semidefinite).

(iii) is a consequence of (iv) by Fact 2.

(i): That one of A or B is definite (semidefinite) follows from (iv) since the set T, which lies in some open (closed) quadrant, must lie in either the upper, lower, right, or left open (closed) half plane in $\mathbf{R}^2$.

(ii): If there exists $v \in \ker(A)$ such that ${}^t vBv \neq 0$, choose w such that ${}^t wBw$ has the opposite sign of ${}^t vBv$. If there exists $v \in \ker(A)$ such that ${}^t vBv = 0$, but $Bv \neq 0$, choose $w = Bv$. In either case, M is negative on $v \wedge w$ by (1), a contradiction. Hence $Av = 0$ implies $Bv = 0$. $\square$

We give here an example of a sum $M = A \wedge A + B \wedge B + C \wedge C$ that is positive on simple vectors but not positive definite: Let A be diagonal with entries $2\epsilon, \epsilon, 1, 1$ in that order, let B be diagonal with entries $1, 1, 2\epsilon, \epsilon$, in that order, and let C have only four nonzero entries, those being $c_{12} = c_{21} = c_{34} = c_{43} = \sqrt{\sqrt{8}\epsilon}$. Then for $\epsilon > 0$, $\epsilon \neq \frac{1}{\sqrt{2}}$, M is positive semidefinite with exactly one 0 eigenvalue, and the 0 eigenvector is not simple.

Now we treat the case of M negative or nonpositive on simple vectors.

Proposition 1. *The following conditions each imply that* M *is positive on some simple vector:*

(i) *three eigenvalues of* A (*or* B) *have the same sign,*

(ii) $\max\{\operatorname{rank}(A), \operatorname{rank}(B)\} \ge 5$,

(iii) $\min\{\operatorname{rank}(A), \operatorname{rank}(B)\} < \max\{\operatorname{rank}(A), \operatorname{rank}(B)\} = 4$.

Proof. (i): Switching the sign of A if necessary, we may assume that A has three positive eigenvalues. Let V be the vector space spanned by these three eigenvectors. Then $\wedge_2 V$ consists only of simple vectors. In order for M to be nonpositive on simple vectors $B \wedge B$ would actually have to be negative definite on $\wedge_2 V$. But if $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the restriction of B to V, the three pairwise products of the λ_i's (which are the eigenvalues of the restriction of $B \wedge B$ to $\wedge_2 V$) cannot all be negative.

(ii): This condition implies (i).

(iii): Assume $\operatorname{rank}(B) < \operatorname{rank}(A) = 4$. From (i) we may assume that A has two positive and two negative eigenvalues. We let V be the direct sum of the positive and negative spaces of A, and consider the 4×4 matrices A' and B' representing the quadratic forms A and B on V. The hypothesis $\operatorname{rank}(B) < 4$ implies that B' has at least one 0 eigenvalue. Transforming A' and B' so that B' is in diagonal form, we now transform A' and B' by nonsingular real matrices P as follows: We let P differ from the identity matrix only in a fixed diagonal entry which corresponds to one of the 0 entries of B'. In this entry in P we place the real number k. Transforming by P thus preserves B' but multiplies the determinant of A' by k^2. Taking k large enough, we may assume that the product of either the two positive or the two negative eigenvalues of A' strictly exceeds all magnitudes of products of eigenvalues of B'. Consequently, if v and w are the eigenvectors of these chosen eigenvalues for A', $A' \wedge A' + B' \wedge B'$ is strictly positive on $v \wedge w$. $\square$

We need the following two technical lemmas.

Lemma 1. *If $n = 3$, A has one zero and two positive* (*or two negative*) *eigenvalues, and M is nonpositive on simple vectors, then B has one zero and two nonzero eigenvalues, and* $\ker(A) = \ker(B)$.

Proof. Note that since $n = 3$, all vectors in $\wedge_2 \mathbf{R}^3$ are simple. First, $\operatorname{rank}(B) > 1$ for otherwise $M = A \wedge A$ which is positive on some simple vector. B cannot have two eigenvalues of the same sign, or M would be positive on some simple vector. Consequently B has one positive, one zero, and one negative eigenvalue. Assume A is diagonal in the basis e_1, e_2, e_3, and that e_1 is the 0 eigenvector. Let f_1, f_2, f_3 be orthonormal eigenvectors for B, such that f_1 is the 0 eigenvector for B. Then $B \wedge B$ is zero on the plane spanned by $f_1 \wedge f_2, f_1 \wedge f_3$. In order that M be nonpositive on this plane consisting of simple vectors, $A \wedge A$ must be equal to 0 on it. But this means that the $f_1 \wedge f_2$-$f_1 \wedge f_3$ plane must exactly coincide with the $e_1 \wedge e_2$-$e_1 \wedge e_3$ plane, which is the *unique* plane on which $A \wedge A$ is 0. Since the $f_1 \wedge f_2$-$f_1 \wedge f_3$ plane uniquely determines the vector f_1 up to a sign (and similarly for e_1), we have $f_1 = e_1$ (up to a sign). $\square$

Lemma 2. *Let $n = 4$, let B have eigenvalues $1, 1, -1, -1$, and let B_1 and B_2 be the upper left and lower right 2×2 matrices within B. If $\det(B_1)$, $\det(B_2) \leq -1$, then $B = B_1 \oplus B_2$ and $\det(B_1) = \det(B_2) = -1$.*

Proof. Let V_1 and V_2 be the vector spaces on which B_1 and B_2 act (as quadratic forms), and consider B_i as mapping V_i into V_i. (Note that V_1 and V_2 are orthogonal.) $\det(B_1) \leq -1$ implies that $|\lambda| \geq 1$ for one eigenvalue λ of B_1. Consequently, a unit eigenvector $v \in V_1$ for λ satisfies $|{}^t vBv| \geq 1$. Since 1 and -1 are the extremal values for B on unit vectors, $|\lambda|$ must be exactly 1 and v must be an eigenvector for B. $\det(B_1) \leq -1$ now implies that $|\lambda'| \geq 1$ for the remaining eigenvalue λ' of B_1, and the same argument as for λ, v shows $|\lambda'| = 1$, and that its eigenvector v' is also an eigenvector for B. The condition $\det(B_1) \leq -1$ now shows $\{\lambda, \lambda'\} = \{1, -1\}$. Since a similar argument applies to B_2, we obtain two more eigenvectors for B. The four eigenvectors thus obtained are orthogonal (since V_1 and V_2 are orthogonal). Since each eigenvector was contained in either V_1 or V_2, the entries in B outside of B_1 and B_2 vanish. □

Proposition 2. *Suppose* $\operatorname{rank}(A) = \operatorname{rank}(B) = 4$ *and* M *is nonpositive on simple vectors. Then* M *is diagonal in some simple basis* $v_i \wedge v_j$ $(1 \leq i < j \leq n)$, $\operatorname{rank}(M) = 2$, *and therefore* M *is zero on some nonzero simple vector.*

Proof. By Proposition 1 A has two positive and two negative eigenvalues. So we may assume that A is diagonalized with eigenvalues $1, 1, -1, -1, 0, \ldots, 0$ in that order. Focusing on the upper left 4×4 matrices A' and B' within A and B, it follows from Proposition 1 that B' has rank 4. Applying Lemma 1 to the 3×3 submatrices within A corresponding to rows and columns $1, 2, i$ and rows and columns $3, 4, i$ for $i > 4$, we conclude that the corresponding 3×3 submatrices within B have zeros in their third rows and third columns. But this information together with the assumption that B' has rank 4 leads to the conclusion that every element in B outside of B' must be 0, for otherwise there would be submatrices within B of rank 6. (E.g., if the i-j element of B $(4 < i < j)$ were not 0, then the 6×6 matrix corresponding to rows and columns $1, 2, 3, 4, i, j$ would have rank 6.) Again by Proposition 1 B' also has two positive and two negative eigenvalues. By relabeling if necessary and transforming A' and B' we may now assume that $\det(A') \geq 1$ and that B' is diagonal with eigenvalues $1, 1, -1, -1$ in that order. However, by wedging eigenvectors together and using the hypothesis we see that no product of a pair of eigenvalues of A' can be greater than 1. Hence, the eigenvalues of A' must be of the form a, $\frac{1}{a}$, $-b$, and $-\frac{1}{b}$ $(a, b > 0)$. By an orthogonal matrix Q transform A' and B' so that A' is diagonal with these eigenvalues in the order given. Let B_1 and B_2 be the upper left and lower right 2×2 submatrices within B'. Our hypothesis on M implies that $\det B_1$, $\det B_2 \leq -1$, so by Lemma 2 we conclude that $B' = B_1 \oplus B_2$ and $\det(B_1) = \det(B_2) = -1$. If we transform A' and B' by the diagonal matrix P having $\frac{1}{\sqrt{a}}$, $\sqrt{a}$, $\frac{1}{\sqrt{b}}$, $\sqrt{b}$ as entries, then A' has diagonal entries $1, 1, -1, -1$ and the determinants of B_1 and B_2 have been preserved. We can now diagonalize B' while preserving A' by using 2×2 orthogonal blocks in P. Thus A' and B' are simultaneously diagonal and M is seen to be diagonal in a some basis $v_i \wedge v_j$ $(1 \leq i < j \leq n)$. B' is now diagonal with entries of the form c, $\frac{-1}{c}$, d, $\frac{-1}{d}$ $(c,\ d > 0)$. Our hypothesis on M implies that $cd \leq 1$ and $\frac{1}{cd} \leq 1$, hence $cd = 1$. Consequently M is diagonal with only 2 nonzero entries, so $\operatorname{rank}(M) = 2$. In fact, if $\tan\theta = c$ then A_θ and B_θ each have rank 2. □

Proposition 3. *Suppose* $\mathrm{rank}(B) \leq \mathrm{rank}(A) = 3$ *and* M *is nonpositive on simple vectors. Then* M *is diagonal in some simple basis* $v_i \wedge v_j$ $(1 \leq i < j \leq n)$, *and* $\mathrm{rank}(M) \leq 3$.

Proof. By Proposition 1 one of A and $-A$ has one negative and two positive eigenvalues, so we may assume that A is diagonalized with eigenvalues $-1, 1, 1, 0, \ldots, 0$ in that order. Applying Lemma 1 to the 3×3 submatrices of A associated to rows and columns $2, 3, i$ for all $i > 3$, we obtain that the corresponding 3×3 submatrices of B have only zeros in their third rows and third columns. We note that the 2×2 submatrix within B corresponding to rows and columns 2 and 3 must be nondegenerate or our hypothesis on M would be violated. By now considering determinants of the 4×4 matrices within B corresponding to rows and columns $1, 2, 3, i$ (for $i > 3$) and rows and columns $2, 3, i, j$ (for $3 < i < j$) we see that all elements in B except for those in the upper left 3×3 block must vanish, for otherwise B would have rank at least 4. Letting $e_1, \ldots, e_n$ be the basis which we have assumed for A and B, we find that M will have nonzero entries only in the 3×3 submatrix corresponding to the basis vectors $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_2 \wedge e_3$. This shows that $\mathrm{rank}(M) \leq 3$. Applying Fact 4 to this 3×3 submatrix yields 3 orthogonal vectors which can be extended to an orthonormal basis $v_1, \ldots, v_n$ which satisfy the remaining conclusion of the proposition. $\square$

Theorem 2. *If* M *is negative* (*resp. nonpositive*) *on simple vectors, then* M *is diagonal in some simple basis* $v_i \wedge v_j$ $(1 \leq i < j \leq n)$. *Furthermore,* $\mathrm{rank}(M) \leq 3$.

Proof. First we prove the negative case. This is vacuously true for $n > 3$ by Proposition A (below). It is true for $n = 3$ by Fact 4. The case $n = 2$ is trivial. Propositions 1, 2, and 3 leave only the possibility of $\mathrm{rank}(B) \leq \mathrm{rank}(A) \leq 2$ in the nonpositive case. However, in this case there exist matrices A' and B' such that $A' \wedge A' = -A \wedge A$ and $B' \wedge B' = -B \wedge B$, for we only need negate one of the eigenvalues of each of A and B. Therefore $M' = -M$, and now the argument for the nonnegative case (Theorem 1) applies to show that M is diagonal in some simple basis as claimed. Of course in this case $\mathrm{rank}(M) \leq 2$ since $A \wedge A$ and $B \wedge B$ are each of rank at most 1. $\square$

Now let

$$M_p = \sum_{i=1}^{p} A_i \wedge A_i,$$

where each A_i is a real symmetric $n \times n$ matrix. The following proposition is due to Otsuki.

Proposition A.

(i) *If $n > p$ and M_p is nonpositive on simple vectors, then there is some nonzero vector $v \in \mathbf{R}^n$ such that ${}^t v A_i v = 0$ for $1 \leq i \leq p$* [**4**].

(ii) *If $n > p + 1$, M_p is nonnegative on some nonzero simple vector* [**5**, p. 233].

We now need one final fact.

Fact 5. The curvature tensor is an invariant and does not depend on the choice of orthonormal basis of the normal space. Equivalently, given any real orthogonal $p \times p$ matrix $Q = (q_{ij})$, the transformation $A'_i = \sum_{j=1}^{p} q_{ij} A_j$ $(1 \leq i \leq p)$ leaves M_p invariant; i.e., $\sum_{i=1}^{p} A'_i \wedge A'_i = \sum_{i=1}^{p} A_i \wedge A_i$. (This is a generalization of the previous assertion that $A_\theta \wedge A_\theta + B_\theta \wedge B_\theta = A \wedge A + B \wedge B$.)

Proposition 4. *Let M_p be nonpositive on simple vectors and let $n > p$. Then there is a basis for $\mathbf{R}^n$ such that the upper left $n - p \times n - p$ block of each A_i is identically* 0.

Proof. The proof is by induction on n. Throughout the proof we let $\langle \cdots \rangle$ denote the linear span of a collection of vectors. By Proposition A(i) there is a simultaneous null-cone vector v of the A_i's. If $n = p + 1 = 2$ we are done, so assume $n \geq 3$ and let $r = \dim\langle A_1 v, \ldots, A_p v\rangle$. Since the transformation in Fact 5 does not affect the conclusion of the proposition, we may assume that $A_1 v, \ldots, A_r v$ are linearly independent and that $A_{r+1} v = \cdots = A_p v = 0$. Choose a basis for $\mathbf{R}^n$ of the form $v, v_2, \ldots, v_n$ such that $\langle v, v_2, \ldots, v_{n-r}\rangle \perp \langle A_1 v, \ldots, A_r v\rangle$ and ${}^t v_{n-r+i} A_j v = \delta_{ij}$ for $1 \leq i, j \leq r$. We claim that in this basis the upper left $n - r \times n - r$ block of A_i for $1 \leq i \leq r$ is identically 0. Let $w \in \langle v, v_2, \ldots, v_{n-r}\rangle$ and let $\epsilon \in \mathbf{R}$ be small. To first order in ϵ,

$$\begin{aligned} &{}^t(w \wedge (v + \epsilon v_{n-r+i})) M_p (w \wedge (v + \epsilon v_{n-r+i})) \\ &\approx \sum_{j=1}^{p} 2\epsilon ({}^t w A_j w)({}^t v A_j v_{n-r+i}) \\ &= \sum_{j=1}^{p} 2\epsilon ({}^t w A_j w) \delta_{ij} = 2\epsilon {}^t w A_i w. \end{aligned}$$

Since M_p is nonpositive on simple vectors and since ϵ can be made arbitrarily small with either sign, we must have ${}^t w A_i w = 0$, which proves our claim. If $r \geq 1$ we apply the induction hypothesis to the upper left $n - r \times n - r$ blocks of the $p - r$ matrices $A_{r+1}, \ldots, A_p$ to obtain a new basis for $\langle v, v_2, \ldots, v_{n-r}\rangle$ such that the upper left blocks of order $n - p = (n - r) - (p - r)$ of $A_{r+1}, \ldots, A_p$ are identically 0, and we are done. If $r = 0$ then the first rows and first columns of $A_1, \ldots, A_p$ are identically 0, so we apply the induction hypothesis to the lower right $n - 1 \times n - 1$ blocks of the p matrices to obtain a new basis such that the upper left $n-p-1 \times n-p-1$ blocks are identically 0. Remembering that the first rows and first columns are also zero, we are done. $\square$

Theorem 3. *If M_p is nonpositive on simple vectors, then* $\operatorname{rank}(M_p) \leq p^3 + p^2 - p$; *in case $p = 2$,* $\operatorname{rank}(M_2) \leq 3$.

Proof. By Proposition 4 we may assume that the upper left $n-p \times n-p$ block of each A_h is identically 0. The first $n-p$ columns of each A_h have rank at most p, as do the last p columns. (At this point we could already obtain the bound $p\binom{2p}{2}$ since each A_h has rank at most $2p$.) Therefore the rank of the columns $A_h e_j \wedge A_h e_l$ $(1 \leq j \leq n-p,\ n-p+1 \leq l \leq n)$ of $A_h \wedge A_h$ is at most p^2. The corresponding columns of M_p thus have rank at most p^3. The columns $M_p(e_j \wedge e_l)$ for $n-p+1 \leq j < l \leq n$ obviously have rank at most $\binom{p}{2}$. The columns $M_p(e_j \wedge e_l)$ for $1 \leq j < l \leq n-p$ also have rank at most $\binom{p}{2}$, since the entries $(M_p)_{ik,jl}$ of M_p are only nonzero when $n-p+1 \leq i < k \leq n$. Thus the rank of M_p is at most $p^3 + 2\binom{p}{2} = p^3 + p^2 - p$. Our special assertion for the case $p = 2$ is contained in Theorem 2. $\square$

Remark. The general bound for $\operatorname{rank}(M_p)$ supplied by the proof of Theorem 3 is obtained in a crude way, and for $p = 2$ we have proved a sharper bound, namely 3. We do not know whether the bound of the theorem can in general be sharpened to $\binom{p+1}{2}$.

Theorem 4. *If M_p is negative semidefinite, then* $\operatorname{rank}(M_p) \leq \binom{p+1}{2}$. *Furthermore, this bound is sharp.*

Proof. For brevity M will denote M_p. By Proposition 4 we may assume that the upper left $n-p \times n-p$ block of each A_h is identically 0. Let $M_{ij,kl} = {}^t(e_i \wedge e_j) M (e_k \wedge e_l)$ for all $1 \leq i, j, k, l \leq n$. For the rest of the proof let $1 \leq i, k \leq n-p$ and $n-p+1 \leq j, l \leq n$. Since $M_{ik,ik} = 0$, and since any null-cone vector of a semidefinite matrix must be a 0-eigenvector, the first $\binom{n-p}{2}$ columns of M are 0, and in particular, $M_{ik,jl} = 0$. We have

$$M_{ij,kl} = \sum_{h=1}^{p} (a_{ik}^h a_{jl}^h - a_{il}^h a_{jk}^h) = \sum_{h=1}^{p} -a_{il}^h a_{jk}^h$$

since $a_{ik}^h = 0$. But since each A_h is symmetric

$$\sum_{h=1}^{p} (a_{ij}^h a_{kl}^h - a_{il}^h a_{jk}^h) = \sum_{h=1}^{p} (a_{ij}^h a_{kl}^h - a_{il}^h a_{kj}^h) = M_{ik,jl} = 0,$$

so $M_{ij,kl} = \sum_{h=1}^{p} -a_{ij}^h a_{kl}^h$. We now see that the $p(n-p) \times p(n-p)$ block of M corresponding to its restriction to the span of the $e_i \wedge e_j$'s has rank at most p because the matrix $[a_{ij}^h a_{kl}^h]$ has rank at most one for each h. Again because null-cone vectors must be 0-eigenvectors, this means that the middle $p(n-p)$

columns of M have rank at most p. There remain $\binom{p}{2}$ columns of M so the rank of M is at most $p+\binom{p}{2}=\binom{p+1}{2}$. To achieve the bound, we make $M=-I$ with the following p diagonal matrices of order $p+1$ having the following diagonal entries: $(-1,1,1,\dots,1),(0,-2,1,\dots,1),\dots,(0,0,0,\dots,-p,1)$. □

It is interesting to note that a specialization of this theorem to the case when each A_i is diagonal is equivalent to Coxeter's theorem that a positive semidefinite connected a-form has kernel of dimension at most one [**2**, pp. 173-175].

3. Applications to n-manifolds in $\mathbf{R}^{n+p}$.

Using the fact that sectional curvatures are simply the values taken by (unit) simple vectors under the curvature operator R_y regarded as a quadratic form on $\wedge_2 T_y(\mathcal{N})$, we now apply our results from the last section to obtain geometrical theorems. We will always assume $n>1$ and $p\geq 1$.

Theorem 5. *Let $\mathcal{N}$ be an n-dimensional Riemannian manifold isometrically immersed in $\mathbf{R}^{n+2}$ and $y\in\mathcal{N}$. Then if all sectional curvatures are positive (nonnegative, nonpositive, negative) at y, there is a basis $v_1,\dots,v_n$ for $T_y(\mathcal{N})$ such that the curvature operator R_y is diagonal in the basis $v_i\wedge v_j$ $(1\leq i<j\leq n)$ of $\wedge_2 T_y(\mathcal{N})$. Consequently,*

R_y has positive sectional curvature $\iff$ R_y is positive definite (Weinstein).
R_y has nonnegative sectional curvature $\iff$ R_y is positive semidefinite.
R_y has nonpositive sectional curvature $\iff$ R_y is negative semidefinite.
R_y has negative sectional curvature $\iff$ R_y is negative definite.

Proof. This follows from Theorems 1 and 2. □

Corollary 1. *Let $\mathcal{N}$ be a compact n-dimensional Riemannian manifold isometrically immersed in $\mathbf{R}^{n+2}$.*

(i) *If $\mathcal{N}$ has positive sectional curvature, then $H^r(\mathcal{N};\mathbf{R})=0$ for $0<r<n$. If $\mathcal{N}$ is additionally connected and orientable, $\mathcal{N}$ is a rational homology sphere.*
(ii) *If $\mathcal{N}$ has nonnegative sectional curvature, then a differential form on $\mathcal{N}$ is harmonic if and only if it is parallel.*

Proof. Statements (i) and (ii) hold with the hypotheses that the curvature operator is positive definite, positive semidefinite respectively [**6**, Chap. 4]. Hence (i) and (ii) hold as stated by Theorem 5 above. □

Corollary 2 (Weinstein)**.** *No metric on $S^2\times S^2$ with positive sectional curvature can be induced by an immersion in $\mathbf{R}^6$.*

Proof. This is immediate from Corollary 1(i) since $\mathrm{H}^2(\mathrm{S}^2 \times \mathrm{S}^2; \mathbf{R}) = \mathbf{R} \times \mathbf{R}$. □

The following theorem is immediate from Proposition A(ii).

Theorem A (Otsuki)**.** *If $\mathcal{N}$ is an n-dimensional Riemannian manifold and all sectional curvatures are negative at the point $y \in \mathcal{N}$, then $\mathcal{N}$ cannot be isometrically immersed in $\mathbf{R}^{n+p}$ if $n > p+1$.*

Theorem 6. *Suppose $\mathcal{N}$ is an n-dimensional Riemannian manifold isometrically immersed in $\mathbf{R}^{n+p}$, $y \in \mathcal{N}$, and R_y has nonpositive sectional curvature. Then the rank of R_y is at most $p^3 + p^2 - p$, and if $p = 2$, the rank of R_y is at most* 3.

Proof. This is immediate from Theorem 3. □

We leave it as an open question whether the bound in Theorem 6 can in general be sharpened to $\binom{p+1}{2}$, but we have proved this sharper bound for $p = 1$ and $p = 2$, since we have shown in these cases that the rank of the curvature tensor can be at most 1 and 3, respectively.

Theorem 7. *Suppose $\mathcal{N}$ is an n-dimensional Riemannian manifold isometrically immersed in $\mathbf{R}^{n+p}$, $y \in \mathcal{N}$, and R_y is negative semidefinite. Then the rank of R_y is at most $\binom{p+1}{2}$. Furthermore, this bound is sharp.*

Proof. This follows from Theorem 4 and from the fact that there exist (noncomplete) n-manifolds of constant negative curvature in $\mathbf{R}^{2n-1}$ [**7**, p. 196]. □

Note. Since this paper was first submitted for publication, Florit [**3**, Proposition 7] has proved Proposition 4 independently. An easy application of his Theorem 1 is an improvement of the bound in our Theorem 3 to $2p^2 - p$. We wish to acknowledge the late Professor Fred Almgren for his encouragement in the writing of this paper.

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