When an algebra is graded by a group, any additive character of the group induces a diagonalizable derivation of the ring. This construction is studied in detail for the case of a path algebra modulo relations and its fundamental group. We describe an injection of the character group into the first cohomology group following Assem-de la Peña. Rather general conditions are determined, in this context, which guarantee that a diagonalizable derivation is induced from the fundamental group.

This paper is the second installment in a series devoted to diagonalizable derivations. Suppose that $R$ is a finite-dimensional algebra over the field $k$. We will denote by $\text{Der}(R)$ the space of $k$-algebra derivations from $R$ to itself. Diagonalizable derivations arise naturally whenever $R$ is graded by a group $H$. Indeed, every additive character $\chi \in \text{Hom}(H, k^+)$ can be assigned a derivation $D_\chi \in \text{Der}(R)$ according to the rule

$$D_\chi(r) = \chi(g)r$$

for every $r \in R$ in the homogeneous component of “degree” $g \in H$. Obviously, $D_\chi$ is diagonalizable. Conversely, if $D$ is a diagonalizable derivation of the $k$-algebra $S$ and $H$ is the additive subgroup of $k$ generated by the eigenvalues of $D$ then, for the inclusion map $\iota : H \to k^+$, we have $D = D_\iota$.

In our first paper [FGGM], we proved that the span of all diagonalizable derivations of $R$ comprise a Lie ideal of $\text{Der}(R)$ whenever $k$ has characteristic zero or is algebraically closed of positive characteristic. This result turned out to be a powerful tool in describing what we called spanned-by-split derivations (i.e., those which are sums of diagonalizable derivations) in several classes of algebras. In what follows, we shall use the notation

$$\text{SPDer}(R)$$

for the subspace of $\text{Der}(R)$ consisting of spanned-by-split derivations.

This paper describes some of the examples which motivated our original paper. Consider $R$ presented as $k\Gamma/I$, a path algebra modulo relations, graded by the fundamental group $\pi_1(\Gamma, I)$. In section one, we review and
clarify notions of fundamental group. The second section is devoted to tightening two results of Assem and de la Peña:

- The map which assigns to each character in $\text{Hom}(\pi_1(\Gamma, I), k^+) \text{ a derivation in } \text{SPDer}(k\Gamma/I)$ is injective.
- The induced map of $\text{Hom}(\pi_1(\Gamma, I), k^+) \text{ to } H^1(k\Gamma/I)$ is injective.

Next we give a partial characterization of those diagonalizable derivations which arise in the form $D_{\chi}$. Essentially, we require that the underlying algebra have its radical stabilized by the derivation and that the algebra of constants for the derivation be indecomposable.

We close with a short section placing our constructions in the context of Hopf algebras.

1. Fundamental Groups.

Let $\Gamma$ be a finite connected directed graph. Temporarily forget about the orientation of arrows, obtaining the undirected graph $\Gamma^{\text{un}}$. If $\alpha$ is a walk from vertex $x$ to vertex $y$ and $\beta$ is a walk from vertex $y$ to vertex $z$ then the concatenation $\alpha \beta$ is a walk from $x$ to $z$ and the reverse $\alpha^{-1}$ is a walk from $y$ to $x$. Consider the “homotopy relation”, the smallest equivalence relation on walks which is compatible with right and left concatenation (whenever they make sense) and for which $\sigma \sigma^{-1}$ is equivalent to the trivial walk at $w$ for any walk $\sigma$ beginning at $w$. For a fixed vertex $x$, the classes of closed walks from $x$ to itself comprise the fundamental group, $\pi_1(\Gamma)$. (Different choices of $x$ yield isomorphic groups.)

We will need a more traditional description of the fundamental group. Fix a vertex $v$, the “base point”. For each vertex $w \in \Gamma$, choose a walk $\gamma_{v,w}$ from $v$ to $w$ in the underlying undirected graph $\Gamma^{\text{un}}$; we require that $\gamma_{v,v}$ be the empty walk from $v$ to itself. The set

$$\gamma = \{\gamma_{v,w} \mid w \text{ is a vertex}\}$$

will be referred to as a choice of parade data. If $f$ is any walk in $\Gamma^{\text{un}}$ from $x$ to $y$ then we define

$$c_{\gamma}(f) = \gamma_{v,x} f \gamma_{v,y}^{-1},$$

the walk which begins at $v$, takes the parade route to $x$, follows $f$ from $x$ to $y$, and then reverses the parade route to return to $v$ from $y$. Observe that if $g$ is already a closed walk from $v$ to $v$ then $c_{\gamma}(g) = g$. Also, every $c_{\gamma}(f)$ lies in the subgroup generated by

$$\{c_{\gamma}(a) \mid a \text{ is an arrow in } \Gamma\}.$$

Suppose the parade walk $\gamma_{v,w}$ is a sequence

$$a_{\varepsilon(1)}, a_{\varepsilon(2)}, \ldots, a_{\varepsilon(t)}$$
of edges where each $a_j$ is an arrow in $\Gamma$ and $\varepsilon(j) = \pm 1$ according to whether the original orientation is preserved or reversed in the walk. Then

$$c_\gamma(a_1)^{\varepsilon(1)} c_\gamma(a_2)^{\varepsilon(2)} \cdots c_\gamma(a_t)^{\varepsilon(t)} = 1.$$  

We call the word on the left-hand side of the last equation a parade walk relator.

Given fixed parade data $\gamma$, the earlier remark about closed walks through $v$ implies that $\pi_1(\Gamma)$ is generated by $\{c_\gamma(a) \mid a$ is an arrow$\}$. It is less obvious that $\pi_1(\Gamma)$ is the free group on the formal symbols $c_\gamma(a)$ modulo the parade walk relations. (Let $F_\gamma$ be the free group on arrows modulo the parade walk relations for $\gamma$. The obvious map from walks in $\Gamma_{un}$ to $F_\gamma$ respects the homotopy equivalence relation. If we regard $\pi_1(\Gamma)$ as the group of equivalence classes of closed walks through $v$ then the restriction of the factored map is a group homomorphism $\phi : \pi_1(\Gamma) \to F_\gamma$. For any arrow $a$ from $x$ to $y$,

$$\phi(c_\gamma(a)) = \phi(\gamma_{v,x} a \gamma_{v,y}^{-1})$$

$$= \phi(\gamma_{v,x}) \phi(a) \phi(\gamma_{v,y})^{-1}$$

$$= \phi(a)$$

$$= \bar{a}$$

where $\bar{a}$ is the image of the symbol $a$ in $F_\gamma$. Thus $\phi$ is surjective. But $\pi_1(\Gamma)$ is generated by the collection of all such $c_\gamma(a)$ and they are subject to the parade walk relations. It follows that $\phi$ is an isomorphism.) When the context is clear, we drop the subscript $\gamma$.

Suppose that $I$ is an ideal of the path algebra $k\Gamma$ and that $I$ is generated as an ideal by a set of relations $\rho$. The fundamental group $\pi_1(\Gamma, \rho)$ will turn out to be a certain image of $\pi_1(\Gamma)$. While it is possible to describe $\pi_1(\Gamma, \rho)$ abstractly ([S]), we will assume that $\pi_1(\Gamma)$ is already described using parade data $\gamma$. Let $N(\rho)$ be the normal subgroup of $\pi_1(\Gamma)$ generated by $c_\gamma(p)c_\gamma(q^{-1})$ as $p$ and $q$ range over all paths in the support of the same member of $\rho$. Then

$$\pi_1(\Gamma, \rho) = \pi_1(\Gamma)/N(\rho).$$

We denote the canonical homomorphism from $\pi_1(\Gamma)$ to $\pi_1(\Gamma, \rho)$ (which depends on $\gamma$) by $\xi$.

A choice of parade data $\gamma$ induces a $\pi_1(\Gamma, \rho)$-grading on $k\Gamma/I$. Explaining this gives us the opportunity to introduce the useful notion of weight ([G]). Suppose that $\Gamma$ is a finite directed graph and $H$ is a group. A weight function for $\Gamma$ with values in $H$ is an assignment $W$ from the arrows of $\Gamma$ to $H$. If we extend $W$ multiplicatively so that vertices have weight $1 \in H$ then the domain of the extension (also called $W$) consists of all directed paths in $\Gamma$. The weight function now induces an $H$-grading on $k\Gamma$. We say that an ideal $I$ of $k\Gamma$ is homogeneous for $W$ provided it is homogeneous with respect to this grading. For such an ideal, the weight induces a grading on $k\Gamma/I$. In
the case of fundamental groups, we can consider the weight with values in $\pi_1(\Gamma, \rho)$ which sends an arrow $a$ to $\xi(c_\gamma(a))$. The ideal $I$ is homogeneous by the construction of $N(\rho)$.

Unfortunately, $\pi_1(\Gamma, \rho)$ is dependent on the choice of relations for $I$. This can be remedied as follows. We say that a nonzero element $r \in I$ is support minimal if it cannot be written as a sum of two elements of $I$, each of whose supports are proper subsets of the support of $r$. (If $\sum \alpha_p p \in I$ is support minimal where the sum runs over paths $p$ with scalars $\alpha_p \neq 0$ then we cannot erase any summands and maintain the subsum in $I$.) It is an immediate consequence of the next proposition that the fundamental group is the same for any two choices of $\rho$ which consist of support minimal relations; this common group is denoted $\pi_1(\Gamma, I)$.

**Proposition 1.1.** Let $\rho$ be any generating set for $I$ and suppose that $s \in I$ is support-minimal. Then $c(p)c(q)^{-1} \in N(\rho)$ for all $p$ and $q$ in the support of $s$.

**Proof.** By definition, if $r \in \rho$ and $\sigma$ is a path in the support of $r$ then
\[ c(\tau) \equiv c(\sigma) \pmod{N(\rho)} \]
for all $\tau$ in the support of $r$. We will abuse notation and write
\[ \text{supp}(r) \equiv c(\sigma) \pmod{N(\rho)}. \]

For any two paths $\alpha$ and $\beta$, we then have
\[ \text{supp}(\alpha r \beta) \equiv c(\alpha \sigma \beta) \pmod{N(\rho)}. \]

An arbitrary $s \in I$ is a linear combination of expressions $\alpha r \beta$ for paths $\alpha, \beta$ and for $r \in \rho$. Given $d \in \pi_1(\Gamma, \rho)$, set $s(d)$ to be the subcombination of all those $\alpha r \beta$ whose support lies in $d$, regarded as a coset. Then
\[ s = \sum_d s(d) \]
with $s(d) \in I$ and the supports of the $s(d)$ pair-wise disjoint. Hence if $s$ is support-minimal it must be equal to a single $s(d)$. \hfill \Box

We mention one last time that we will only be able to speak about a $\pi_1(\Gamma, I)$-grading of $k\Gamma/I$ in the presence of parade data $\gamma$. Thus if $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$, then the induced derivation defined in the introduction depends on some choice of $\gamma$ and, so, will frequently be written $D_{\Psi, \gamma}$.

2. Injectivity Theorems.

In this section, make the standing assumption that $I$ is an admissible ideal of $k\Gamma$. (That is, we assume that $k\Gamma/I$ is finite-dimensional and $I$ lies inside the square of the ideal generated by all arrows.)
Proposition 2.1. Choose parade data for the connected directed graph $\Gamma$. Then $D_\Psi$ is diagonalizable for every $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+).$ Moreover the map

$$\text{Hom}(\pi_1(\Gamma, I), k^+) \rightarrow \text{SPDer}(k\Gamma/I)$$

is injective.

Proof. It is obvious that $D_\Psi$ is diagonalizable, so $\text{Hom}(\pi_1(\Gamma, I), k^+)$ maps into $\text{SPDer}(k\Gamma).$ Thus the issue is injectivity. By the standing assumption, every arrow in $\Gamma$ survives modulo $I$. Since $\pi_1(\Gamma, I)$ is generated by

$$\{\xi(c(a)) \mid a \text{ is an arrow}\},$$

we see that $\pi_1(\Gamma, I)$ is generated by degrees which genuinely occur. (In the literature, the grading is sometimes referred to as full.)

However if $R$ is any $H$-graded algebra and

$$\{h \in H \mid \text{the } h\text{-component of } R \text{ is not 0}\}$$

generates $H$ then the map $\text{Hom}(H, k^+) \rightarrow \text{Der}(R)$ is always injective. 

The next theorem provides a more significant injectivity result, which is based on a similar statement of Assem-de la Peña ([AP]).

Theorem 2.1. Suppose $\chi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ and fix parade data for $\pi_1(\Gamma)$. If the associated derivation $D_\chi$ on $k\Gamma/I$ is inner then $\chi = 0$. Hence the induced map

$$\text{Hom}(\pi_1(\Gamma, I), k^+) \rightarrow H^1(k\Gamma/I)$$

is injective.

Proof. We shall write $\Lambda = k\Gamma/I$. Then $\Lambda = \Lambda_0 \oplus \text{rad}\Lambda$ where we identify $\Lambda_0$ with $(k\Gamma)_0$: A commutative subalgebra with basis consisting of orthogonal idempotents $e(w)$, one for each vertex $w$ of $\Gamma$.

For $s = \sum_w \lambda_w e(w) \in \Lambda_0$ we compute $\text{ad } s$. If $\overline{m} \in \Lambda$ is the image of a path $m$ in $\Gamma$ from vertex $x$ to vertex $y$ then

$$(\text{ad } s)(\overline{m}) = (\lambda_x - \lambda_y)\overline{m}.$$ 

Thus $\text{ad } s$ is always diagonalizable and all images of paths are among its eigenvectors.

Now suppose that $\chi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ and $D_\chi = \text{ad } b$ for some $b \in \Lambda$. Set $b = s + n$ for $s \in \Lambda_0$ and $n \in \text{rad}\Lambda$. The image $\overline{m}$ of every path is an eigenvector for $D_\chi$ corresponding to eigenvalue $(\chi \circ \xi)(c(m))$. Thus $\text{ad } b$ is diagonalizable with a basis of eigenvectors which are images of paths. It follows that $\text{ad } b$ and $\text{ad } s$ must commute. But then $(\text{ad } b) - (\text{ad } s)$ is diagonalizable at the same time that it is equal to $\text{ad } n$, which is nilpotent.

We conclude that $D_\chi = \text{ad } s$ for $s \in \Lambda_0$.

If $s = \sum_w \lambda_w e(w)$ then $(\chi \circ \xi)(c(m)) = \lambda_x - \lambda_y$ for every path $m$ beginning at $x$ and ending at $y$, whose image $\overline{m}$ is nonzero. In particular, if $a$ is an
arrow in $\Gamma$ from $x$ to $y$ and $\varepsilon = \pm 1$ the $(\chi \circ \xi)(c(\pi)) = \varepsilon \cdot (\lambda_x - \lambda_y)$. It follows that if $w$ is an arbitrary vertex in $\Gamma$ and

$$a_1^{\varepsilon(1)}, \ldots, a_t^{\varepsilon(t)}$$

is the parade walk from the base point $v$ to $w$ then

$$(\chi \circ \xi) \left( e \left( a_1^{\varepsilon(1)} \cdots c(a_t)^{\varepsilon(t)} \right) \right) = \lambda_v - \lambda_w.$$ 

On the other hand, $(\chi \circ \xi)(1) = 0$. We conclude that

$$s = \lambda_v \left( \sum_w e(w) \right) = \lambda_v \cdot 1.$$ 

Therefore $D_\chi = \text{ad } s = 0$. □

Corollary 2.1 ([BM]). Let $\Lambda = k\Gamma/I$ be a path-monomial algebra which is finite dimensional. If $H^1(\Lambda, \Lambda) = 0$ then $\Gamma$ is a tree.

Proof. We are assuming that the generating set $\rho$ for $I$ consists of monomials. As a consequence, $\pi_1(\Gamma, I)$ is a free group; it is trivial if and only if $\Gamma$ is a tree. Thus if $\Gamma$ is not a tree then $\text{Hom}(\pi_1(\Gamma, I), k^+) = \text{nonzero}$. By the theorem, $H^1$ is nonzero. □

3. Fundamental Derivations.

The argument presented in the previous theorem rests on the following property of the diagonalizable derivation $D_\chi$. For any pair of vertices $x$ and $y$ there exists an undirected walk $a_1^{\varepsilon(1)}, \ldots, a_t^{\varepsilon(t)}$ such that $\sum_j \varepsilon(j) D_\chi(\pi_j) = 0$. This property turns out to be crucial in trying to characterize those diagonalizable derivations which arise from a fundamental group.

Definition 3.1. Let $R$ be a finite-dimensional $k$-algebra. We say that a derivation $D \in \text{Der}(R)$ is fundamental provided that there exists a finite directed graph $\Gamma$ and an admissible ideal $I$ of $k\Gamma$ such that $R \simeq k\Gamma/I$ and there is parade data $\gamma$ together with some $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ so that

$$D = D_\Psi, \gamma$$

under the identification given by the isomorphism.

There are some obvious things we can say about a fundamental derivation $D$ of $R$. First, $D$ must be diagonalizable. Indeed, the images of paths in $R$ are all eigenvectors. As another consequence, $D(\text{rad } R) \subseteq \text{rad } R$. Notice that the algebra $R$ is $k$-elementary, which means that $R/\text{rad } R$ is a finite product of copies of $k$. More is true: The algebra complement in $k\Gamma$ to the ideal generated by all arrows, which coincides with the span of the vertex
idempotents, survives as an algebra complement to \( \text{rad} R \) in \( R \). Thus \( \text{rad} R \) has an algebra complement which lies inside the “subalgebra of constants”

\[
R^D = \{ r \in R \mid D(r) = 0 \}.
\]

We shall see that these properties come close to characterizing fundamental derivations.

**Lemma 3.1.** Let \( W \) be an \( H \)-valued weight on the arrows of \( \Gamma \) and let \( I \) be a \( W \)-homogeneous ideal of \( k\Gamma \). Suppose that for a fixed vertex \( x \) and every other vertex \( y \) there exists a walk in \( \Gamma^{\text{un}} \),

\[
\gamma_{x,y} : b_1^{\varepsilon(1)}, \ldots, b_t^{\varepsilon(t)}
\]

from \( x \) to \( y \) such that

\[
W(b_1)^{\varepsilon(1)} \cdots W(b_t)^{\varepsilon(t)} = 1
\]

in \( H \). Then there is a homomorphism \( \theta : \pi_1(\Gamma, I) \to H \) such that

\[
(\theta \circ \xi)(c_{\gamma}(a)) = W(a)
\]

for all arrows \( a \).

**Proof.** As we remarked earlier, \( \pi_1(\Gamma) \) is isomorphic to the free group on \( \{ c_{\gamma}(a) \mid a \text{ is an arrow of } \Gamma \} \) modulo the parade walk relators

\[
\{ c_{\gamma}(\gamma_{x,y}) \mid x \neq y \}.
\]

Hence \( W \) induces a group homomorphism \( \theta : \pi_1(\Gamma) \to H \) such that

\[
\theta(c_{\gamma}(a)) = W(a)
\]

for all arrows \( a \).

Suppose that \( \rho \) is a support-minimal set of relations for the homogeneous ideal \( I \). We claim that the elements of \( \rho \) are homogeneous. If \( r \in \rho \) write \( r = \sum_h r_h \) where each \( r_h \) is a nontrivial linear combination of paths with weight \( h \). By homogeneity, each \( r_h \) lies in \( I \). But the support of \( r_h \) is clearly a subset of the support of \( r \). Hence \( r = r_h \) for some choice of \( h \).

It follows that if \( p \) and \( q \) are paths in the support of some \( r \) in \( \rho \) then \( W(p) = W(q) \). Therefore \( \theta \) is the identity on \( N(\rho) \), the normal subgroup generated by all possible \( c_{\gamma}(p)c_{\gamma}(q)^{-1} \) of this sort. We conclude that \( \theta \) factors through \( \pi_1(\Gamma, I) \).

**Theorem 3.1.** Assume that \( I \) is an admissible ideal of \( k\Gamma \). Suppose that

(a) \( E \) is a diagonalizable derivation of \( k\Gamma/I \) which vanishes on images of vertices and for which the images of arrows are eigenvectors, i.e., for each arrow \( a \) in \( \Gamma \) there is a scalar \( \omega(\overline{a}) \) such that \( E(\overline{a}) = \omega(\overline{a}) \overline{a} \);

(b) there is a vertex \( x \) such that for every other vertex \( y \) there exists a walk

\[
\gamma_{x,y} : b_1^{\varepsilon(1)}, \ldots, b_t^{\varepsilon(t)}
\]
from $x$ to $y$ such that
\[ \sum_j \varepsilon(j)\omega(b_j) = 0. \]

Then there is some $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ with $E = D_{\Psi, \gamma}$ for $\gamma = \{\gamma_x, y\}$.

Proof. We apply the lemma with $H = k^+$. Since $I$ is admissible, we see that different arrows cannot have the same images in $k\Gamma/I$. Thus it makes sense to define a function $W$ on arrows via $W(a) = \omega(a)$, thereby lifting the $k^+$-grading to $k\Gamma$. We conclude from the lemma that there is an additive character $\Psi$ on $\pi_1(\Gamma, I)$ such that $(\Psi \circ \xi)(c_\gamma(a)) = \omega(a)$.

Thus the two derivations $D_\Psi$ and $E$ agree on the images of arrows and vertices. But these elements generate $k\Gamma/I$ as an algebra. \qed

The next result is a close relative to Theorem 3.4 in [G].

Lemma 3.2. Let $R$ be a finite-dimensional $k$-elementary algebra. Assume that $D$ is a diagonalizable derivation of $R$ such that $D(\text{rad}\, R) \subseteq \text{rad}\, R$ and $R^D$ contains an algebra complement to $\text{rad}\, R$. Then there exists a finite directed graph $\Gamma$, an admissible ideal $I$ of $k\Gamma$, and a derivation $\tilde{D}$ of $k\Gamma$ such that

(a) $\tilde{D}(I) \subseteq I$;
(b) $\tilde{D}$ vanishes on the vertex idempotents of $k\Gamma$;
(c) each arrow is an eigenvector for $\tilde{D}$;
(d) $R \cong k\Gamma/I$ with $\tilde{D}$ inducing $D$.

Proof. Let $e(1), \ldots, e(n)$ be orthogonal idempotents whose sum is 1, which span a complement to $\text{rad}\, R$, and which satisfy $D(e(j)) = 0$ for $j = 1, \ldots, n$. Then
\[ D(e(i)\text{rad}\, Re(j)) \subseteq e(i)\text{rad}\, Re(j) \]
for all $i$ and $j$. An elementary eigenspace argument using the diagonalizability of $D$ implies that the pair of $D$-stable spaces
\[ e(i)\text{rad}\, Re(j) \cap (\text{rad}\, R)^2 \subseteq e(i)\text{rad}\, Re(j) \]
splits with a $D$-stable vector space complement $A(i, j)$.

We argue that $\{A(i, j) \mid 1 \leq i, j \leq n\}$ generates $\text{rad}\, R$ as an algebra. Denote by $A$ the algebra generated by these subspaces. Certainly $A \subseteq \text{rad}\, R$. If the algebras do not coincide, the nilpotence of $\text{rad}\, R$ implies that there must be a largest $m$ such that $(\text{rad}\, R)^m$ does not lie in $A$. That is, there exist $r_1, \ldots, r_m \in \text{rad}\, R$ such that
\[ r_1r_2 \cdots r_m \notin A. \]
We may assume that \( r_j \in e_{f(j)} \text{rad} R e_{g(j)} \) for some choice of indices \( f(j) \) and \( g(j) \). Write \( r_j = a_j + s_j \) with \( a_j \in A(f(j), g(j)) \) and \( s_j \in (\text{rad} R)^2 \). Then
\[
hr_1 r_2 \cdots r_m = a_1 a_2 \cdots a_m + z
\]
where \( z \in (\text{rad} R)^{m+1} \). In other words, \( a_1 \cdots a_m + z \in A \). We have reached the contradiction that \( r_1 \cdots r_m \in A \).

We can now describe \( \Gamma \). Its vertices are the idempotents \( e(1), \ldots, e(n) \). Choose a basis for each \( A(i, j) \) which consists of eigenvectors for \( D \). These basis vectors comprise the set of arrows which begin at \( e(i) \) and end at \( e(j) \). There is an obvious algebra map from \( k \Gamma \) onto \( R \). If \( a \) is one of the designated eigenvectors for \( D \) in \( A(i, j) \) and \( D(a) = \lambda a \) then we define the function \( \tilde{D} \) on the arrow \( a \) by \( \tilde{D}(a) = \lambda a \). It is easy to see that \( \tilde{D} \) extends uniquely to a derivation of \( k \Gamma \) which vanishes on vertex idempotents.

The lemma follows with \( I \) the kernel of the map \( k \Gamma \to R \).

\[ \square \]

**Theorem 3.2.** Let \( R \) be a finite-dimensional \( k \)-elementary algebra. Suppose that \( D \) is a diagonalizable derivation of \( R \) with \( D(\text{rad} R) \subseteq \text{rad} R \). Suppose, further, that \( R^D \) contains an algebra complement to \( \text{rad} R \) and that \( R^D \) is indecomposable as an algebra. Then \( D \) is fundamental.

**Proof.** We carry over all of the notation in the previous lemma. Define a new graph \( G \) whose vertices are \( e(1), \ldots, e(n) \) (the vertices of \( \Gamma \)) and construct an arrow from \( e(i) \) to \( e(j) \) provided \( e(i) R^D e(j) \neq 0 \). The sum of the \( e(h) \) over all those vertex idempotents in a connected component of \( G \) is a central idempotent of \( R^D \). Since \( R^D \) is indecomposable, \( G \) is connected.

We claim that if \( e(i) R^D e(j) \neq 0 \) then there is a path \( m \) in \( \Gamma \) such that \( m \) begins at \( e(i) \), ends at \( e(j) \), and \( D(\overline{m}) = 0 \). To see this, suppose that \( e(i) x e(j) \neq 0 \) for \( x \in R^D \). Write \( x = \sum \alpha_p \overline{p} \) with \( \alpha_p \) a nonzero scalar and \( \overline{p} \) the image in \( R \) of a path \( p \) in \( k \Gamma \) which begins at \( e(i) \) and ends at \( e(j) \). We may assume that the \( \overline{p} \) which appear in the sum are linearly independent in \( R \). Each such \( \overline{p} \) is an eigenvector for \( D \). Since eigenvectors for distinct eigenvalues are linearly independent, we conclude that \( D(\overline{p}) = 0 \) for every \( \overline{p} \) which appears.

We put the previous two paragraphs together. For each \( i \neq j \) there is a walk from \( e(i) \) to \( e(j) \) in \( G \) with edge sequence
\[
g_{\epsilon(1)}^e, g_{\epsilon(2)}^e, \ldots, g_{\epsilon(v)}^e.
\]
This walk gives rise to an “expanded” walk
\[
m_{\epsilon(1)}^e, m_{\epsilon(2)}^e, \ldots, m_{\epsilon(v)}^e
\]
from \( e(i) \) to \( e(j) \) in \( \Gamma \), where each \( m_d \) is a path with the same endpoints as \( g_d \) and \( D(\overline{m_d}) = 0 \). If we rewrite the second walk as
\[
a_{\eta(1)}^e, a_{\eta(2)}^e, \ldots, a_{\eta(t)}^e
\]
for arrows $a_i$ in $\Gamma$ then
\[ \sum \eta(i) D(a_i) = 0. \]
(The point is that if the sum of the eigenvalues along a path $m_d$ is zero then the same is true for the sum of the negatives of those eigenvalues in reverse order along the path.) For each $2 \leq j \leq n$ pick such a walk $\gamma_{1,j}$ from the base point $e(1)$ to $e(j)$.

According to Theorem 3.1, there exists some $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ such that $D = D\Psi_{\gamma}$. □

**Corollary 3.1.** Assume that $k$ is an algebraically closed field of characteristic zero and $R$ is a finite-dimensional local $k$-algebra. Then every diagonalizable derivation of $R$ is fundamental.

**Proof.** Since $k$ is algebraically closed and $R$ is local, we have $R/\text{rad} R \simeq k$. It follows that $R^D$ is local, and so, indecomposable. Finally, it is well known that $D(\text{rad} R) \subseteq \text{rad} R$ for any $D \in \text{Der}(R)$, by virtue of $\text{char} k = 0$. □

### 4. Hopf Algebras.

We end with a hint that there may be other classes of derivations beside diagonalizable ones for which an interesting theory exists. Every diagonalizable derivation of a $k$-algebra corresponds to a group grading by a subgroup of the additive group $k^+$. Equivalently, every diagonalizable derivation has the form $D\chi$ where $\chi \in \text{Hom}(G, k^+)$ for some group $G$ which grades the algebra. A group grading for an algebra $R$ is an example of an $H$-comodule algebra action on $R$, where $H$ is a Hopf algebra. (In the special case, $H = kG$ with the standard Hopf structure.) In general, if $H$ is any Hopf algebra then $R$ is an $H$-comodule algebra provided that $R$ is a left $H$-comodule, via

\[ \lambda : R \to H \otimes R \]

(so $\lambda(a) = \sum a_0 \otimes a_1$ for $a \in R$) and

\[ \lambda(ab) = \sum a_0 b_0 \otimes a_1 b_1 \text{ for } a,b \in R; \]

\[ \lambda(1) = 1 \otimes 1. \]

See [Mo], Section 4.1 for more details. In the particular case of the group algebra, $R$ is a $kG$-comodule algebra if and only if it is $G$-graded as an algebra.

If $\epsilon$ is the augmentation for $H$ then those functionals which are $\epsilon$-derivations,

\[ \text{Der}_\epsilon^*(H, k) = \{ f \in H^* \ | \ f(ab) = \epsilon(a)f(b) + f(a)\epsilon(b) \}, \]
comprise a Lie algebra under the commutator \([f, g] = f \ast g - g \ast f\). (Here \(*\) is convolution on \(H^*\).) When \(H = kG\) then

\[ \text{Der}_k^\epsilon(kG, k) = \text{Hom}(G, k^+). \]

Back to the general set-up, for each \(f \in \text{Der}_k^\epsilon(H, k)\) define \(D_f \in \text{Hom}_k(R, R)\) by

\[ D_f(a) = \sum f(a_0)a_1 \quad \text{for all } a \in R. \]

We leave it as an exercise that \(D_f \in \text{Der}(R)\) and the map \(\text{Der}_k^\epsilon(H, k) \to \text{Der}(R)\) sending \(f\) to \(D_f\) is a Lie algebra homomorphism. This construction subsumes our earlier \(D^\chi\).

The subspace of derivations spanned by all \(D_f\), as one runs over all co-module algebra actions of one or more Hopf algebras \(H\), deserves future scrutiny.

**References**


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