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EQUIVARIANT DEFORMATIONS OF MANIFOLDS
AND REAL REPRESENTATIONS

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In the paper we give a partial answer to the following question: Let G be a finite group acting smoothly on a compact (smooth) manifold M , such that for each isotropy subgroup H of G the submanifold M^H fixed by H can be deformed without fixed points; is it true that then M can be deformed without fixed points G -equivariantly? The answer is no, in general. It is yes, for any G -manifold, if and only if G is the direct product of a 2-group and an odd-order group.

1. Introduction.

Let G be a finite group, and M a compact smooth G -manifold. A *deformation* of M is a homotopy of the identity 1_M , i.e., a map $h : M \times I \rightarrow M$, such that $h(-, 0) = 1_M$. We say that the deformation h is *fixed point free*, or *without fixed points* if $h(x, 1) \neq x$ for all $x \in M$. If for all $t \in I$ the self-map $h(-, t)$ of M is equivariant with respect to the G -action, then we say that h is an equivariant deformation of M .

Mainly, the problem is to know when M admits a fixed point free deformation. If M is connected and the action of G is trivial, then the answer is simply given by the Euler characteristic of M : There is a fixed point free deformation if and only if $\chi(M) = 0$. If the action of G is non-trivial, any equivariant deformation of M induces deformations on the subspaces fixed by the subgroups H of G , namely $h^H : M^H \times I \rightarrow M^H$, for $H \subset G$, so we have the necessary condition that M^H must be deformable without fixed points, for all $H \subset G$. With the following theorem, we show that the converse in general is not true, unless G satisfies a hypothesis on its 2-Sylow subgroup.

Theorem 1.1. *Let G a finite group. The following propositions are equivalent:*

- (i) G is the direct product of a 2-group G_2 and an odd-order group G_2' : $G = G_2 \times G_2'$.
- (ii) For any real nontrivial irreducible representation V of G , and any self-normalizing isotropy subgroup H of G , $\dim V^H \neq 1$.

- (iii) Any compact smooth G -manifold M admits a fixed point free equivariant deformation if and only if for all isotropy types (H) the submanifold M^H admits a fixed point free deformation.

The paper is devoted to prove Theorem 1.1. The structure of the paper is closely related to the structure of the Theorem and its proof. First, in Section 2 we start by giving some preliminaries which will be needed later. Secondly, in Section 3 we prove that (i) \implies (iii) (Proposition 3.2). The proof of (iii) \implies (ii) is given in Section 4 by applying Proposition 4.1. The proof of (ii) \implies (i) is given by Proposition 8.1, in Section 8, which requires results from Sections 5, 6 and 7: The Lemmas in Section 5 and 6 are elementary. On the other hand, some results of Section 7 deeply rely on the GAP Character Tables, which are however well-known.

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2. Some preliminaries.

Let G be a finite group, acting on a space X . Then for each $x \in X$, we denote with G_x the *isotropy subgroup* of G at x . If $H \subset G$ is a subgroup of G , then we use the following notation: $N_G(H)$ is the normalizer of H in G , and $W_G(H)$ the Weyl group $N_G(H)/H$; X^H is the H -fixed space of X , i.e., $X^H := \{x \in X \mid Hx = x\}$; X_s^H is the H -singular locus of X , defined by $X_s^H := \{x \in X^H \mid G_x \supsetneq H\}$; X_{s0}^H is the singular set of the action of the group $W_G(H)$ on X^H , i.e., $X_{s0}^H := \{x \in X^H \mid (W_G(H))_x \neq e\}$ where e denotes the identity in $W_G(H)$; it is easy to see that $X_{s0}^H \subset X_s^H$ for all $H \subset G$. If two subgroups H and K of G are conjugate in G , then we use the notation $H \sim K$. We denote with (H) the conjugacy class of H , and with $X_{(H)}$ the (H) -orbit bundle of X , i.e., $X_{(H)} := \{x \in X \mid G_x \sim H\}$. If H is an isotropy subgroup of G , then we say that (H) is an *isotropy type* for X .

Now let X be a compact ENR. Then $X = \sqcup_i X_i$ is the disjoint union of its connected components X_i , with $i = 1 \dots k$. Let

$$\underline{\chi}(X) := (\chi(X_1), \dots, \chi(X_n)) \in \mathbb{Z}^k$$

denote the vector of all the Euler characteristics $\chi(X_i)$.

We follow [Wi84], in order to define an equivariant Euler characteristic $\underline{\chi}_G$. Let X be a compact equivariant Euclidean neighbourhood retract (G -ENR); let $A(X)$ denote the free abelian group generated by the set of all pairs $((H), C)$, where (H) is an isotropy type for X and C is a connected component of $X_{(H)}/G$. Then $\underline{\chi}_G(X)$ is the unique element in $A(X)$ such

that its $((H), C)$ -component is

$$\chi(X^H, X^H \setminus p_H^{-1}C)$$

where $p_H : X_H \rightarrow X_{(H)}/G$ is the projection map. If G is trivial, then $\underline{\chi}_G(X) = \underline{\chi}(X)$.

We will need Theorem B of [Wi84]: A compact smooth G -manifold M admits a fixed point free equivariant deformation if and only if $\underline{\chi}_G(M) = 0$.

We say that a finite group is *2-split* when it is the direct product of a 2-group and a group of odd order (i.e., when G verifies the hypothesis of proposition (i) in Theorem 1.1).

We say that a self-normalizing subgroup H of G is a *cutting* subgroup whenever one of the following (equivalent) conditions holds:

- (a) $\exists V$ real representation of G such that $\dim V^H = \dim V^G + 1$.
- (b) $\exists V$ real representation of G such that $\dim V^H = \dim V^G + 1 = 1$.
- (c) $\exists V$ real irreducible non-trivial repr. of G such that $\dim V^H = 1$.

Whenever a cutting subgroup H of G is also an isotropy subgroup with respect to the action on V , we call it an isotropy cutting subgroup.

If V is a real representation of G , by tensoring V with the field of complex numbers \mathbb{C} , we obtain a complex representation $V' = V \otimes \mathbb{C}$ of G . In this case we say that V' is realizable over \mathbb{R} . We also have $\dim V^H = \dim V'^H$ for all $H \subset G$. On the other hand, if W is a complex irreducible representation of G of degree n and character χ , then one of the following sentences is true, according to the value of the second Schur Frobenius indicator

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

- (0) By restriction of scalars we obtain a real irreducible representation of degree $2n$ and character $\chi + \bar{\chi}$.
- (+1) $W = V \otimes \mathbb{C}$ for a real irreducible representation V of degree n and character χ .
- (-1) W is not realizable over \mathbb{R} , and by restriction of scalars we obtain a real irreducible representation of degree $2n$ and character 2χ .

Moreover, every real representation can be obtained by one of the above procedures. Therefore it is readily seen that H is a cutting subgroup of G if and only if there exists a complex non-trivial irreducible representation W of G with character χ such that both the following equations hold

$$\frac{1}{|H|} \sum_{h \in H} \chi(h) = (1_H, \text{Res}_H^G \chi)_H = 1$$

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = 1.$$

Furthermore, H is an isotropy subgroup for the complex representation $W = V \otimes \mathbb{C}$, if and only if it is an isotropy subgroup of G for V .

3. Properties of 2-split groups.

In this section we prove that (i) \implies (ii) and that (i) \implies (iii).

Proposition 3.1. *Let G be a 2-split group. Then G has no cutting subgroups, and in particular no cutting isotropy subgroups.*

Proof. Let $H \subset G$ be a self-normalizing subgroup. Because G is the direct product $G_2 \times G_{2'}$ of a 2-group G_2 and an odd-order group $G_{2'}$, the subgroup H itself is 2-split, being the direct product of $H_2 := G_2 \cap H$ and $H_{2'} := G_{2'} \cap H$ (cf. [Su82], Corollary p. 141). Moreover, the normalizer $N_G(H)$ of H is the direct product of the normalizers $N_{G_2}H_2$ and $N_{G_{2'}}H_{2'}$. Hence H is self-normalizing if and only if H_2 and $H_{2'}$ are self-normalizing in G_2 and $G_{2'}$ respectively. But G_2 is a 2-group, and so $H_2 = G_2$.

Assume now that H is a cutting subgroup, and so that W is a complex irreducible nontrivial representation with second Frobenius Schur indicator $= +1$ and $\dim W^H = 1$. Let χ denote its character. Because $G = G_2 \times G_{2'}$, we deduce that $W = W_2 \otimes W_{2'}$ is the tensor product of two irreducible representations of G_2 and $G_{2'}$, with characters χ_2 and $\chi_{2'}$, and that $\dim W^H = \dim W_2^{G_2} \cdot \dim W_{2'}^{H_{2'}}$. By $\dim(W^H) = 1$, we obtain that $\dim W_2^{G_2} = \dim W_{2'}^{H_{2'}} = 1$; hence W_2 is the trivial representation, and so $W_{2'}$ is nontrivial.

Now, the indicator of W is 1, and so

$$\frac{1}{|G|} \sum_{g_2 \in G_2} \sum_{g_{2'} \in G_{2'}} \chi_2(g_2^2) \cdot \chi_{2'}(g_{2'}^2) = 1$$

which implies

$$\frac{|G_2|}{|G|} \sum_{g_{2'} \in G_{2'}} \chi_{2'}(g_{2'}^2) = 1$$

because W_2 is trivial, and so $W_{2'}$ has indicator 1. But $G_{2'}$ has odd order, and so $W_{2'}$ must be trivial (it is a result of Burnside: cf. [Se77], Exercise 13.9, p. 109), which is a contradiction. \square

It is a classical result that a connected compact smooth manifold admits a deformation if and only if its Euler characteristic is zero, so the following proposition is equivalent to (i) \implies (iii).

Proposition 3.2. *Let G be a 2-split group and M a smooth compact G -manifold. Then M admits a fixed point free equivariant deformation if and only if for all the isotropy types (H) in G the Euler characteristic $\underline{\chi}(M^H) = 0$ vanishes.*

Proof. It is trivial to see that if M admits a fixed point free equivariant deformation, then for all the isotropy types (H) the connected components of M^H admit a fixed a fixed point free deformation, and so $\underline{\chi}(M^H) = 0$. So let us prove the converse, i.e., that if $(\forall(H)) \underline{\chi}(M^H) = 0$ then M has an equivariant f.p.f. deformation, or equivalently $\underline{\chi}_G(M) = 0$, because of Theorem B of [Wi84].

First we prove the following two Lemmas:

Lemma 3.3. *If G is 2-split and M is a compact smooth G -manifold, then for each isotropy subgroup H , the codimension of $M_s^H \setminus M_{s_0}^H$ in M^H is greater than 2.*

Proof. Let use point out that, because M^H may have components of different dimension, we mean that the codimension must be evaluated component-by-component. So, let $x \in M_s^H \setminus M_{s_0}^H$, and G_x the isotropy subgroup of G at x . Assume that the codimension of $M_s^H \setminus M_{s_0}^H$ in the component of x is 1. By the Slice Theorem (cf. [tD87] - Thm. I.5.6 p. 40) there is a real representation V of G_x such that $G \times_{G_x} V$ is G -diffeomorphic to a G -neighbourhood of x in M . Moreover, for all $K \subset G$ the fixed subspaces $(G \times_{G_x} V)^K \cong V^K$ coincide (cf. [Br72], Thm. II.3.4 p. 82), hence the codimension of V^{G_x} in V^H is 1.

Because x is not in $M_{s_0}^H$, its isotropy $(W_G(H))_x$ with respect to the Weyl group $W_G(H)$ action on M^H is trivial, and therefore $N_G(H) \cap G_x = H$. In other words, $N_{G_x}(H) = H$, and so H is self-normalizing in G_x .

Therefore H is a cutting subgroup of G_x . But G_x is a subgroup of G , which is 2-split, and so G_x itself is 2-split; this implies that G_x cannot have cutting subgroups, because of Proposition 3.1, a contradiction. Hence the codimension must be greater than 1. □

Lemma 3.4. *If for all isotropy types (H) the Euler characteristic vanishes, $\underline{\chi}(M^H) = 0$, then for all H , $\underline{\chi}(M_s^H) = 0$.*

Proof. We can prove it by a kind of induction on isotropy subgroups: let us order the isotropy subgroups (they are finite) by inclusion, i.e., $H < K$ if and only if $H \subset K$. We will denote with the symbols of \mathcal{F} or \mathcal{F}_i , with i integer, the subsets of the (partially ordered) set of isotropy groups with the property that if $H \in \mathcal{F}$ and $K \supset H$ then $K \in \mathcal{F}$. For such an \mathcal{F} , let $M^{\mathcal{F}}$ denote the union $\cup_{H \in \mathcal{F}} M^H$, and $\#\mathcal{F}$ the number of its elements.

We make induction on $\#\mathcal{F}$ to prove that $\underline{\chi}(M^{\mathcal{F}}) = 0$ for all \mathcal{F} . If $\#\mathcal{F} = 1$, then it has just one isotropy subgroup, and by assumption $\underline{\chi}(M^{\mathcal{F}}) = 0$. Assume that $\underline{\chi}(M^{\mathcal{F}}) = 0$ for all \mathcal{F} with $\#\mathcal{F} \leq k$, and let \mathcal{F}_1 be a family such that $\#\mathcal{F}_1 = k + 1$. In \mathcal{F}_1 there are a finite number of minimal elements, say H_0, \dots, H_m . If $m = 0$, i.e., the minimal element is unique, then $M^{\mathcal{F}_1} = M^{H_0}$ and so by hypothesis $\underline{\chi}(M^{\mathcal{F}_1}) = 0$. If $m \geq 1$ then let $\mathcal{F}_2 := \mathcal{F}_1 \setminus \{H_0\}$. By induction hypothesis, $\underline{\chi}(M^{\mathcal{F}_2}) = 0$; moreover, $M^{\mathcal{F}_1} = M^{\mathcal{F}_2} \cup M^{H_0}$ and the

intersection $M^{\mathcal{F}_2} \cap M^{H_0} = M^{\mathcal{F}_3}$, where \mathcal{F}_3 is the family of isotropy groups $K \in \mathcal{F}_2$ such that $K \supset H_0$. Because H_0 is not the only minimal element, $\#\mathcal{F}_3 \leq k$, and so $\underline{\chi}(M^{\mathcal{F}_3}) = 0$. Moreover, each $M^{\mathcal{F}_i}$ is a union of smooth transversal submanifolds in M , and therefore

$$\underline{\chi}(M^{\mathcal{F}_1}) = \underline{\chi}(M^{\mathcal{F}_2}) + \underline{\chi}(M^{H_0}) - \underline{\chi}(M^{\mathcal{F}_3}) = 0$$

which is the thesis. We can now apply this result to the Lemma, because $M_s^H = M^{\mathcal{F}}$ where \mathcal{F} is the family of all the isotropy subgroups strictly greater than H , and this completes the proof. \square

Remark 3.5. With almost the same proof as Lemma 3.4, it is possible to prove that if $\underline{\chi}(M^H) = 0$ for all the isotropy subgroups H in G , then $\underline{\chi}(M^K) = 0$ for any subgroup K of G .

Now we are able to prove Proposition 3.2. Recalling the definition of $\underline{\chi}_G(M)$, we must prove that for each isotropy subgroup H of G and each connected component C of $M_{(H)}/G$, the relative Euler characteristic

$$\chi(M^H, M^H \setminus p_H^{-1}C)$$

is zero. First, let us notice that $M_{(H)}/G \cong M_H/W$, where $W := W_G(H)$, and so we may take $p_H : M_H \rightarrow M_H/W$, and C is a connected component of M_H/W . Now let us consider the projection $q_H : M^H \rightarrow M^H/W$. If D is a connected component of M^H/W , then it contains a connected component C of M_H/W , because M_H is open and dense in M^H . Furthermore, the inclusion $M_H/W \rightarrow (M^H \setminus M_{s_0}^H)/W$ induces a bijection on connected components, because of Lemma 3.3 (let us note that $M_H = M^H \setminus M_s^H$); the same happens to the inclusion $(M^H \setminus M_{s_0}^H)/W \rightarrow M^H/W$ because the action of W is smooth, and $M_{s_0}^H$ is exactly the singular locus for its action. Hence, the connected components of M_H/W are the intersection with M_H/W of the connected components of M^H/W .

Therefore we need to prove that $\chi(M^H, M^H \setminus p_H^{-1}(D \cap M_H/W)) = 0$ for all connected components D of M^H/W . Let us define $C := D \cap M_H/W$. Because $p_H^{-1}C = q_H^{-1}(D) \cap M_H$, the complement $M^H \setminus p_H^{-1}C = (M^H \setminus q_H^{-1}D) \cup M_s^H$. So, by the exact sequence of the triple $(M^H, M^H \setminus q_H^{-1}D \cup M_s^H, M^H \setminus q_H^{-1}D)$, we obtain that $\chi(M^H, M^H \setminus q_H^{-1}D) = \chi(M^H, M^H \setminus p_H^{-1}C) + \chi(M^H \setminus q_H^{-1}D \cup M_s^H, M^H \setminus q_H^{-1}D)$. By a simple excision argument, the last term is equal to $\chi(M_s^H)$, which is zero by Lemma 3.4, and so for each connected component C the equality

$$\chi(M^H, M^H \setminus q_H^{-1}D) = \chi(M^H, M^H \setminus p_H^{-1}C)$$

holds true. But again applying excision, we see that $\chi(M^H, M^H \setminus q_H^{-1}D) = \chi(q_H^{-1}D)$; but $q_H^{-1}D$ is a disjoint union of connected components of M^H , and

by the assumption $\underline{\chi}(M^H) = 0$ each of them has Euler characteristic zero. Therefore

$$\chi(M^H, M^H \setminus p_H^{-1}C) = \chi(q_H^{-1}D) = 0$$

which completes the proof. □

4. Cutting subgroups and equivariant deformations.

In this section we prove that (iii) \implies (ii), which follows from the following proposition.

Proposition 4.1. *If G has a cutting isotropy subgroup H , then there is a compact connected smooth G -manifold M , without boundary, such that*

$$\underline{\chi}_G(M) \neq 0$$

and

$$\underline{\chi}(M^H) = 0$$

for every isotropy subgroup H of G .

Proof. Let V_0 be the real irreducible non-trivial representation of G , such that $x \in V_0 \setminus 0$ is a generator of $V_0^H \cong \mathbb{R}$, H is self-normalizing and $G_x = H$ is the cutting isotropy subgroup. Let $V := V_0 + \mathbb{R}^3$, i.e., V is equal to V_0 plus three times the trivial representation on \mathbb{R} . Let T denote the 2-dimensional torus and C a closed simple smooth curve in T bounding an (open) disc B . We can embed the pair (T, B) in $(V^H, V^G) \cong (\mathbb{R}^4, \mathbb{R}^3)$ in a way such that $T \setminus V^G$ is the disjoint union of B and the complement of the closure of B . Moreover, we can assume that any small enough smooth regular neighbourhood of T in V^H is a trivial D^2 -bundle on T , and $T \cap V^G \approx C \times D^2$.

Now let GT be the union of all the images gT in V , where g ranges in G . It is a G -space, and let X be a G -regular neighbourhood of GT in V , G -deformable to GT . Without loss of generality, we can assume that X is a smooth compact connected G -manifold with boundary, of dimension $\dim V + 3$. Let $M = 2X$ be G -space obtained by joining two copies of X along the boundary ∂X , with the identity identification map on ∂X . Then M is a compact connected smooth G -manifold without boundary.

For every isotropy subgroup K , the Euler characteristic satisfies $\underline{\chi}(M^K) = 2\underline{\chi}(X^K) - \underline{\chi}(\partial X^K)$. Because $\underline{\chi}(\partial X^K) = (1 + (-1)^n)\underline{\chi}(X^K)$, where $n := \dim X^K - 1$ (cf. [Do72], Corollary VIII.8.8, p. 299), we have that $\underline{\chi}(M^K) = (1 - (-1)^n)\underline{\chi}(X^K)$. But $\underline{\chi}(X^K) = \chi(GT^K)$, and $\chi(GT^K) = 0$ for all K . Therefore $\underline{\chi}(M^K) = 0$ for all isotropy subgroup K .

On the other hand, H is self-normalizing, so the Weyl group $W := W_G H$ is the trivial group. Hence the projection $p_H : M_H \rightarrow M_H/W = M_H$ is just the identity map. The connected components of M_H are now two: $C_+ \approx B \times S^2$, where S^2 denotes the 2-sphere, and $C_- \approx (T \setminus \bar{B}) \times S^2$. Let us compute

$\chi(M^H, M^H \setminus p_H^{-1}C_+)$: it is equal to $\chi(T \times S^2, (T \setminus \bar{B}) \times S^2)$ and therefore to $2\chi(T, T \setminus \bar{B}) = 2\chi(T) - 2\chi(T \setminus \bar{B}) = 2$. Again, $\chi(M^H, M^H \setminus p_H^{-1}C_-) = -2$. These equalities imply that $\underline{\chi}_G(M) \neq 0$ which is the thesis. \square

5. Further preliminaries.

Lemma 5.1. *Let G be a finite group, $N \triangleleft G$ a normal subgroup and $q : G \rightarrow G/N$ the quotient homomorphism. If $K \subset G/N$ is a cutting (isotropy) subgroup of G/N , then $q^{-1}K$ is a cutting (isotropy) subgroup of G .*

Proof. If G/N acts on the vector space V , then this action induces an action of G on V , and if $x \in V$, then its isotropy subgroup G_x of G is exactly the counter-image $q^{-1}(G/N)_x$ of the isotropy subgroup at x of G/N . Moreover, if $K \subset G/N$ is self-normalizing, then $q^{-1}K$ is self-normalizing in G . \square

Lemma 5.2. *If $N \triangleleft G$ is a normal subgroup of G , and $H \subset G$ is a subgroup such that*

- (i) $N \cap H$ is a cutting (isotropy) subgroup of N ;
- (ii) $HN = G$

then H is a cutting (isotropy) subgroup of G .

Proof. The normalizer of H in G is $N_G(H) \subset HN_N(H \cap N)$ because $G = HN$, and the last is $H(H \cap N) = H$, hence H is self-normalizing. Let V be a real nontrivial representation of N such that $\dim V^{N \cap H} = \dim V^N + 1 = 1$, i.e., if χ denotes its character,

$$(1_H, \text{Res}_{N \cap H}^N \chi)_{H \cap N} = 1.$$

Then, if W denotes the induced representation $W := \text{Ind}_N^G V$, with character $\text{Ind}_N^G \chi$, we know that W is real and non-trivial; using Frobenius reciprocity, we obtain

$$\begin{aligned} \dim W^G &= (1_G, \text{Ind}_N^G \chi)_G \\ &= (1_N, \chi)_N = \dim V^N = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \dim W^H &= (1_H, \text{Res}_H^G \text{Ind}_N^G \chi)_H \\ &= (\text{Ind}_H^G 1_H, \text{Ind}_N^G \chi)_G = (\text{Res}_N^G \text{Ind}_H^G 1_H, \chi)_N; \end{aligned}$$

because $\text{Res}_N^G \text{Ind}_H^G 1_H = \text{Ind}_{H \cap N}^N 1_{H \cap N}$ (cf. [Se77], Proposition 22, p. 58), the latter is equal to

$$\begin{aligned} (\text{Ind}_{H \cap N}^N 1_{H \cap N}, \chi)_N &= (1_{H \cap N}, \text{Res}_{H \cap N}^N \chi)_{H \cap N} \\ &= \dim V^{H \cap N} = 1 \end{aligned}$$

and so H is a cutting subgroup of G .

If $H \cap N$ is an isotropy subgroup, say of $x \in V^{H \cap N}$, then the element $y := \sum_{r \in [G:N]} rx \in W$, where r ranges over a system of coset representatives of $G/N = H/H \cap N$ in G such that $r \in H$, has isotropy exactly H , and hence the thesis. \square

Lemma 5.3. *Let $N \triangleleft G$ a normal subgroup of G . If the normalizer in N of a p -Sylow subgroup of N is a cutting subgroup, then G has an isotropy cutting subgroup.*

Proof. Let P be a p -Sylow subgroup of N , and let us assume that $N_N(P)$ is a cutting subgroup of N . Then the normalizer $N_G(P)$ in G of P has the following properties:

- $N_G(P)N = G$ (cf. [Su82], Theorem 2.7, p. 99).
- $N_G(P) \cap N = N_N(P)$.
- If $H \supset N_G(P)$ then H is self-normalizing (because $N_G(P) \supset N_G(S) -$ [Su82], Theorem 2.6, p. 98 – and hence apply [Su82], Th.2.8, p. 99).

Hence we can apply Lemma 5.2, to prove that $H := N_G(P)$ is a cutting subgroup of G . Let V denote the irreducible non-trivial real representation of G , such that $\dim V^H = 1$. If x is a generator of V^H , its isotropy subgroup $G_x \supset H$, and hence it is self-normalizing. This is the wanted cutting isotropy subgroup. \square

Lemma 5.4. *Let $N \triangleleft G$ a normal subgroup of G . If M is a maximal self-normalizing subgroup of G such that $M \cap N$ is a maximal subgroup of N , and $M \cap N$ is a cutting subgroup of N , then M is a cutting isotropy subgroup of G .*

Proof. Because $M \cap N$ is maximal, and cutting, it must be a cutting isotropy subgroup of N . Moreover, $MN \supsetneq M$ and hence $MN = G$ by maximality of M . Therefore we can apply Lemma 5.2. \square

6. Solvable groups.

Lemma 6.1. *Let $G = E \rtimes \mathbb{Z}_2$ be the semidirect product of an elementary abelian p -group E and the 2-cyclic group \mathbb{Z}_2 . If G is not 2-split, then the normalizer $N_G(\mathbb{Z}_2)$ is a cutting subgroup of G .*

Proof. Let $\theta : E \rightarrow E$ be the automorphism of E given by conjugation with the generator of \mathbb{Z}_2 . Because G is not 2-split, the prime p is odd and $\text{Fix}(\theta) \neq E$. Also, $\text{Fix}(\theta)$ is a normal subgroup of G , and $G/\text{Fix}(\theta)$ is equal to $(E/\text{Fix}(\theta)) \rtimes \mathbb{Z}_2$, where θ acts as an automorphism of $E' := E/\text{Fix}(\theta)$ without fixed points (different from $0 \in E'$ – we use additive notation for E). It is easy to see that by applying Lemma 5.1 to the projection $G \rightarrow E' \rtimes \mathbb{Z}_2$, it suffices to prove only the case $\text{Fix}(\theta) = 0$. This implies $\theta(e) = -e$ for all $e \in E$. Therefore, G contains a normal subgroup D_{2p} isomorphic to the

dihedral group of order $2p$. For D_{2p} the proposition is true, hence we can apply Lemma 5.3 to obtain that $N_G(\mathbb{Z}_2)$ is a cutting subgroup of G . \square

Lemma 6.2. *Let $G = D \rtimes \mathbb{Z}_2$ be the semidirect product of an odd-order finite group D and the 2-cyclic group \mathbb{Z}_2 . If G is not 2-split, then $N_G(\mathbb{Z}_2)$ is a cutting subgroup of G .*

Proof. As in the proof of Lemma 6.1, let $\theta : D \rightarrow D$ the automorphism given by conjugation with the generator of \mathbb{Z}_2 . Because G is not 2-split, $\text{Fix}(\theta) \neq D$. By the Feit-Thompson Theorem, D is solvable. Let us prove the assertion by induction on the composition length $l(D)$ of D , i.e., the length of a composition series in D (cf. [Su82], p. 43). If $l(D) = 1$, then it follows from Lemma 6.1. Assume it holds whenever $G = D' \rtimes \mathbb{Z}_2$ where D' is any odd order group with $l(D') \leq l(D) - 1$. Let D_0 be a characteristic nontrivial subgroup of D . Then $D_0 \rtimes \mathbb{Z}_2$ is a normal subgroup of G . Now, if $G_0 := D_0 \rtimes \mathbb{Z}_2$ is not 2-split, then by induction hypothesis the normalizer $N_{G_0}(\mathbb{Z}_2)$ is a cutting subgroup of G_0 , and so we can apply Lemma 5.3 to finish. On the other hand, if G_0 is 2-split, then $G/D_0 = (D/D_0) \rtimes \mathbb{Z}_2$ cannot be 2-split: Otherwise for all $x \in D$ we should have $x^{-1}\theta(x) \in D_0$ where $D_0 \subset \text{Fix}(\theta)$, which implies $\theta(x) = x$ for all x (i.e., G 2-split). But again we can apply the induction hypothesis, with Lemma 5.1, to see that $N_G(\mathbb{Z}_2)$ is a cutting subgroup of G . \square

Lemma 6.3. *Let G be a 2-nilpotent group (i.e., the 2-Sylow subgroup of G has a normal complement). If G is not 2-split, then it has a cutting isotropy subgroup.*

Proof. By hypothesis $G = D \rtimes S_2$, the semidirect product of the (odd-order) normal complement D and the 2-Sylow subgroup S_2 . Let $\alpha : S_2 \rightarrow \text{Aut}(D)$ be the homomorphism which defines the semi-direct product. Because G is not 2-split, $\ker \alpha \neq S_2$; also, the quotient $G/\ker \alpha$ is still 2-nilpotent and not 2-split. By Lemma 5.1 therefore it suffices to consider the case $\ker \alpha = 1$. Because S_2 is a 2-group, there exists a normal 2-cyclic subgroup $\mathbb{Z}_2 \triangleleft S_2$. Now, $N := D \rtimes \mathbb{Z}_2$ is a normal subgroup of G , and fulfills the assumptions of Lemma 6.2. Therefore $N_N(\mathbb{Z}_2)$ is a cutting subgroup. Hence, by Lemma 5.3, G has a cutting isotropy subgroup. \square

Lemma 6.4. *Let $G = E \rtimes F$ be the semidirect product of an elementary abelian 2-group E and an elementary abelian p -group F for an odd prime p . If G is not 2-split, then $N_G(F)$ is a cutting subgroup of G .*

Proof. Because G is not 2-split, the intersection $\bigcap_{f \in F} \text{Fix}(f) \neq E$, where $\text{Fix}(f)$ denotes the subgroup of E fixed by the automorphism induced by conjugation with $f \in F$ (i.e., the centralizer of f in E). Let $e \in E \setminus \bigcap_f \text{Fix}(f)$. Let $V := \mathbb{R}$ be a the representation of E such that $\chi(e) = -1$, and $\chi(\bigcap_f \text{Fix}(f)) = 1$, where χ is the character of V . If we consider the induced

representation $W := \text{Ind}_E^G V$, then it is easy to see that $\dim W^H = 1$, and $\dim W^G = 0$, where $H := N_G(F)$. Hence the thesis. \square

Lemma 6.5. *Let $G = E \rtimes F$ be the semidirect product of an elementary abelian p -group, and a 2-split group F . If G is not 2-split, then it has a cutting isotropy subgroup.*

Proof. If p is an odd prime, then G is 2-nilpotent, because F is 2-split, and so we can apply Lemma 6.3. If $p = 2$, then $F = F_2 \times F_{2'}$, where F_2 is a 2-group and $F_{2'}$ an odd-order group. Because G is not 2-split, in particular it cannot be a 2-group, and hence $F_{2'}$ is not trivial. Now, if $\alpha : F_{2'} \rightarrow \text{Aut} E$ is the homomorphism given by the semidirect product (conjugation), as before we know that $\ker \alpha \neq F_{2'}$; therefore, up to using Lemma 5.1, we can assume $\ker \alpha = 1$. Let N be a minimal normal subgroup of $F_{2'}$. It is also a minimal normal subgroup of F , and $E \rtimes N$ is a normal subgroup of G . But N is an elementary q -abelian group, with q odd prime, and so we can apply Lemma 6.4 to $E \rtimes F$. It is not 2-split because we assume $\ker \alpha = 1$. Furthermore, the conclusion follows by Lemma 5.3. \square

Proposition 6.6. *Let G be a finite solvable group. Then G has a cutting isotropy subgroup if and only if G is not 2-split.*

Proof. We already know that if G is 2-split then it has no cutting subgroups (Proposition 3.1). So let us assume that there are not 2-split groups without cutting isotropy subgroups. Let G be the minimal example, with respect to the number of elements. By assumption, G is solvable, and hence it has a nontrivial minimal normal subgroup N , which is elementary p -abelian. The quotient G/N is 2-split, otherwise we could use Lemma 5.1 to find a cutting isotropy subgroup in G , by minimality of G . Moreover, G cannot be 2-nilpotent (by Lemma 6.3), while G/N is 2-nilpotent. By an easy consequence of Proposition 9.5.2 of [Ro82], p. 270, N must have a complement $F \cong G/N$, hence G is the semidirect product of an elementary abelian p -group and a 2-split complement F . But these are the assumptions of Lemma 6.5, and so G has a cutting isotropy subgroup: a contradiction. \square

7. Almost simple groups.

7.1. Alternating groups. The simple alternating groups and their automorphism groups are easy to check, as it can be seen in the following Lemmas. The only exceptional case is $\text{Aut}(A_6)$, which needs an ad-hoc argument.

Lemma 7.1. *Let A_n be the alternating group, with $n \geq 5$. Then $H := A_{n-1} \subset G$ is a cutting isotropy subgroup of A_n .*

Proof. It suffices to consider the $(n - 1)$ -dimensional representation of A_n : let S be the set of n elements on which A_n acts. Then $\mathbb{R}[S]$ minus the trivial

representation gives rise to the $(n - 1)$ -representation V of A_n . The dimensions of the fixed subspaces $\dim V^G$ and $\dim V^H$ are 0 and 1 respectively. The subgroup H is given by the elements in A_n which fix one element in S , and it is easy to see that it is self-normalizing. \square

Lemma 7.2. *If G is a group such that $A_n \subset G \subset \text{Aut}(A_n)$, with $n \geq 5$, then G has a cutting isotropy subgroup.*

Proof. If $n \neq 6$, then $\text{Aut}(A_n) = \Sigma_n$ the n -symmetric group. Because of Lemma 7.1, we only need to prove it for $G := \Sigma_n$, and the proof is identical to the alternating case. For $n = 6$, let us consider A_6 acting on the 6 elements s_1, \dots, s_6 , and let C_5 be the cyclic group generated in A_6 by the 5-cyclic permutation σ of s_1, \dots, s_5 . It is a 5-Sylow group, and its normalizer is contained in $A_5 \subset A_6$ (i.e., its normalizer fixes s_6). Therefore $N := A_6$ is a normal subgroup of G such that the normalizer of its 5-Sylow is a cutting subgroup. By Lemma 5.3 we have the conclusion. \square

7.2. Groups of Lie type. In this section we just quote some well-known results in representation theory of simple groups of Lie type.

Lemma 7.3. *Let G be a simple group of Lie type. Then there exists a p -Sylow subgroup P of G such that its normalizer $N_G(P)$ is a cutting subgroup of G .*

Proof. Let $V := \text{St}_G$ denote the real Steinberg representation of G and B the Borel subgroup of G . Then $\dim V^G = 0$ because St_G is nontrivial and irreducible (cf. [Ca85], Corollary 6.2.4, page 190); moreover, B is the normalizer in G of a p -Sylow subgroup U of G (cf. [Ca85], Section 2.5, [St57], Section 4), and $\dim V^B = 1$ because of Proposition 6.2.3, p. 190, of [Ca85]. All the details on the Steinberg representation may be found in [St57, Cu66, CR87, Ca85]. \square

Lemma 7.4. *Let G be a group such that $L \subset G \subset \text{Aut}(L)$ where L is a simple group of Lie type. Then G has a cutting isotropy subgroup.*

Proof. It follows from Lemma 7.3, applied to L , and then Lemma 5.3. \square

7.3. Sporadic groups. For the simple sporadic groups and their automorphism groups, it is enough to check the character tables, as given in [CC85]. The best and most easy way to do it, is to use GAP [Sc95], and its Character Table Libraries for simple groups. We refer to GAP Version 3 (Release 4.4).¹

¹ Detailed information on GAP can be found in the following URL's:

<http://www.math.rwth-aachen.de/~GAP/WWW> (Germany),

<http://www-gap.dcs.st-and.ac.uk/~gap> (UK),

<http://www.ccs.neu.edu/mirrors/GAP> (USA) and

<http://wwwmaths.anu.edu.au/research.groups/algebra/GAP/www> (Australia).

Lemma 7.5. *If G is an almost simple sporadic group, then it has a cutting isotropy subgroup.*

Proof. We now give the GAP code of a function that can help in make computations on characters of groups, to detect cutting subgroups.

```
IsCutting:=function(G,H,fm)
local tg,th,realchars,res,cut,trivialch;
tg:=CharTable(G);
th:=CharTable(H);
trivialch:=List(th.classes,c -> 1);
realchars:=Filtered(tg.irreducibles,chi ->
                    Indicator(tg,[chi],2)=[1]);
if fm = [] then fm:=GetFusionMap(th,tg); fi;
res:=Restricted(realchars,fm);
cut:=Filtered(res,chi ->
              ScalarProduct(th,trivialch,chi)=1);
return Length(cut)>1;
end;
```

The variables G and H must be the names of the group G and of the subgroup $H \subset G$, as given e.g., in Table 1. The variable fm is the fusion map from H to G , given as a list (cf. [Sc95]). If fm is the empty list, then its value is the fusion map as recovered from the Character Table Libraries, by the function `GetFusionMap(th,tg)` where th is the source table, and tg is the destination table. The function `IsCutting(G,H,fm)` returns true if H is a cutting subgroup of G , and false otherwise. For example, to know whether $M11M1$ is a cutting subgroup of $M11$, we need to digit

```
gap>IsCutting("M11","M11M1",[]);
true
gap>
```

The meaning of Table 1 is the following: For each sporadic group S in the list, we have two cases: $\text{Out}(S) = 1$ or $\text{Out}(S) = 2$. In the first case, we only need to check for isotropy cutting subgroups of the group itself, and it turns out that among maximal subgroups there is always a cutting subgroup (which is therefore an isotropy subgroup). We give an example of such a maximal subgroup in the third column of the Table. On the other hand, if $\text{Out}(S) = 2$, this argument does not work in general, so we must use Lemma 5.3 and the tables of Sylow normalizers as given in [Os86], readable in GAP with the names listed in the second column. Those subgroups are the nontrivial Sylow normalizers in the simple group S , which already have a fusion map stored in the Character Table Libraries, except for the case of the Suzuki group `Suz`: in this case the (unique) fusion map must be explicitly recovered by the built-in function `SubgroupFusions` as follows:

```
gap>fm:=SubgroupFusions(CharTable("SuzN2","Suz"))[1];;
gap>IsCutting("Suz","SuzN2",fm);
true
gap>
```

(GAP) Name		Sylow Normalizer	Maximal subgroup	Out
M11	Mathieu		A6.2.3 = M11M1	1
M12	Mathieu	M12N2		2
M22	Mathieu	M22N2		2
M23	Mathieu		M22 = M23M1	1
M24	Mathieu		M23 = M24M1	1
J1	Janko		L2(11) = J1M1	1
J2	Janko	J2N5		2
J3	Janko	J3N3		2
J4	Janko		mx1j4 = J4M1	1
Co1	Conway		Co2 = Co1M1	1
Co2	Conway		U6(2).2 = Co2M1	1
Co3	Conway		McL.2 = Co3M1	1
Fi22	Fischer	Fi22N5		2
Fi23	Fischer		2.Fi22 = Fi23M1	1
F3+	Fischer		Fi23 (*)	2
HS	Higman-Sims	HSN2		2
McL	McLaughlin	McLN5		2
He	Held	HeN7		2
Ru	Rudvalis		2F4(2)'.2 = RuM1	1
Suz	Suzuki	SuzN2		2
ON	O'Nan	ONN3		2
HN	Harada-Norton		2.HS.2 = HNM2 (*)	2
Ly	Lyons		GL(5) = LyM1	1
Th	Thompson		3D4(2).3 = ThM1	1
B	Baby Monster		Fi23	1
M	Monster		2.B	1

Table 1. Examples of cutting subgroups of sporadic simple groups.

So, only F3+ and HN, (marked with (*)) are left, because there are no such tables for their Sylow normalizers: In this case we must use Lemma 5.4, and the fact that Fi23 and HNM2 are maximal subgroups of F3+ and HN which are intersections of maximal subgroups in Aut(F3+) and Aut(HN) (cf. [NW86], [LW91]).

Therefore the conclusion of the Lemma follows by applying the function IsCutting to the list of groups and subgroups in Table 1. Let use note that these are only examples of cutting subgroups: As it is readily seen, most of

the maximal subgroups and Sylow normalizers are cutting subgroups. The groups in the list are the ones which are more or less ready to use in the Character Table Libraries. \square

8. Non 2-split groups.

We are now able to prove the last proposition, which is equivalent to (ii) \implies (i).

Proposition 8.1. *If G is a finite group which is not 2-split, then it has a cutting isotropy subgroup.*

Proof. Let us assume on the contrary that there is at least one group not 2-split and without cutting isotropy subgroups. Let G be the smallest group of this type (with respect to the order). Because of the Lemmas in Section 7, G cannot be simple, hence there is a nontrivial minimal normal subgroup $1 \neq N \triangleleft G$; because of minimality and Lemma 5.1, G/N must be 2-split, hence solvable. Therefore N is a direct product of n copies of a finite simple group S , say $N \cong S^n = S \times \times \cdots \times S$. Moreover, $C_G(N) = 1$: In fact, if $C_G(N) \neq 1$ by the same minimality argument we obtain that $G/C_G(N)$ is 2-split and hence solvable, which contradicts the fact that S is simple non-abelian. Therefore G is contained in $\text{Aut}(N) \cong \text{Aut}(S)^n \rtimes \Sigma_n$, where Σ_n denotes the n -th symmetric group.

$$N = S^n \triangleleft G \subset \text{Aut}(S)^n \rtimes \Sigma_n.$$

Because N is minimal in G , $G \cap \text{Aut}(S)^n$ must be of the form $G \cap \text{Aut}(S)^n \cong G_0^n$, for a subgroup $G_0 \subset \text{Aut}(S)$ such that $G_0 \supset S$. In other words, G_0 is almost simple, and hence by applying the results of Section 7 it has a cutting isotropy subgroup H_0 . By taking the n -th power of the tensor product of the representation of G_0 , we can see that H_0^n is a cutting isotropy subgroup of G_0^n . Furthermore, G_0^n is a normal subgroup of G , and there exists a subgroup H of G such that $H \cap G_0^n = H_0^n$ and $HG_0^n = G$. Therefore we can apply Lemma 5.2, to prove that H is a cutting isotropy subgroup of G . But this is a contradiction, because we assumed G to have no cutting isotropy subgroups, hence the thesis. \square

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