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We study the problem of the topological classification of planar polynomial foliations of degree n by giving new lower and upper bounds for the maximum number of inseparable leaves. Moreover, we characterize the planar polynomial foliations that are structural stable under polynomial perturbations and study the exact number of inseparable leaves for this family.

1. Introduction.

In 1940 Kaplan [**14**, **15**] published two large papers on regular families of curves filling the plane, following previous ideas of Whitney [**26**]. A family of curves is called *regular* if it is locally homeomorphic with parallel lines. He proved that each curve of a regular family filling the plane is a homeomorphic line tending to infinity in both directions.

A natural example of generating (orientated) regular families of curves on the plane is given by the solutions of non-singular planar differential systems. Indeed, one major problem from the qualitative theory of differential equations point of view is the topological classification of those differential systems. We say that two planar differential systems are *topologically equivalent* if there exists a homeomorphism on the plane which maps the solution curves of one to the solution curves of the other.

In the second paper, Kaplan characterized the topological classes of regular families based on a certain algebraic structure of the orbits which he called *chordal system*. Ten years later, Markus [**16**] considered the topological classification problem for general (with or without singular points) differential systems on the plane by using different ideas and tools. He pointed out the existence of some key orbits he called *separatrices*. The connected components of the complement of the union of all separatrices are called *canonical regions* where the orbit behavior is tame and all the orbits have the same alpha and omega limit structure. Finally, he defined the *separatrix configuration* of a planar differential systems as the union of all separatrices plus one representative orbit of each canonical region. It follows from Markus and Newmann [**19**] works that two planar differential systems having isolated singular points are topologically equivalent if there exists a homeomorphism sending the separatrix configuration of one system to the separatrix configuration of the other system.

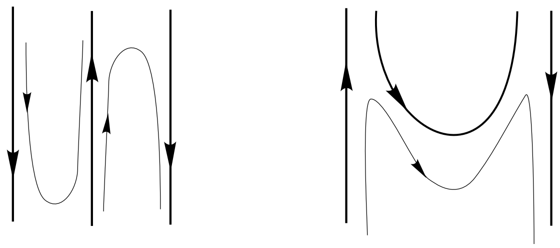
Both, Kaplan and Markus-Newmann theorems, gave a good starting point to obtain all topologically equivalent classes for a given family of planar differential systems (see, for instance, [**12**] and [**4**]). However, it is also clear that a reasonable topologically classification of equivalent classes for general either planar analytic or planar n degree polynomial differential systems with singular points cannot be done. Precisely, when polynomial differential systems of degree n are considered the main difficulties come from the number and distribution of their limit cycles, see the famous Hilbert's sixteenth problem [**10**].

Hence, a natural framework where can be possible to obtain a topological classification is given by the class of polynomial differential systems of degree n without singular points. We will refer to this class as *planar polynomial foliations of degree* n. Here every orbit, which we call a *leave* of the foliation, escapes to infinity in both time directions and divides the plane in two connected unbounded components. The separatrices, in the sense introduced by Markus are now called *inseparable leaves*, and the canonical regions are filled by *parallel* orbits; i.e., they are topologically equivalent to $\mathbb{R}^2$ with the flow defined by $\dot{x} = 0, \dot{y} = 1$.

A different approach for studying planar foliations is due to Haefliger and Reeb [**9**]. They deal with the space of leaves more than the leaves themselves.

Two leaves L_1 and L_2 are said to be *inseparable* if for any arcs T_1 and T_2 transverses to L_1 and L_2 respectively there are leaves which intersects both T_1 and T_2. Since the orbit behavior inside each canonical region is parallel it follows from Markus-Newmann results that the number of topological classes of polynomial foliations of degree n depends on the number of inseparable leaves and the way they are distributed on the plane (Figure 1 show two non-topologically equivalent polynomial foliations with three inseparable leaves). Clearly the maximum number of inseparable leaves gives a measure of the possible different topological classes of polynomial foliations of degree n.

We denote by $s(n)$ the maximum number of inseparable leaves that a planar polynomial foliation of degree n can have. Although, the problem of getting $s(n)$ has been studied for many authors, it still remains open if $n \geq 4$. In 1972 Markus [**17**] proved that $s(n) \leq 6n$. Four years later, Muller [**18**] proved that $s(n) \leq 2n$ (see also [**23**]). Pluvinage [**20**] gave a family of planar polynomial foliation of degree $n \geq 3$ with at least $n-2$ inseparable leaves and at most $2n-4$ inseparable leaves, but without knowing the exact number of inseparable leaves. Finally, if n is even, Schecter and Singer gave in [**23**] an explicit example with $2n-4$ inseparable leaves. On the other hand, it is easy to show that $s(0) = s(1) = 0$ and from [**7**] it follows that $s(2) = 3$. The cubic case had been studied by Camacho and Palmeira [**5**], however we have found some gaps in their proof. In [**11**] and [**3**] we proved



(a) (b)

Figure 1. Two non-topologically equivalent differential systems with three inseparable leaves.

that $s(3) = 3$. In summary, $s(0) = s(1) = 0$, $s(2) = s(3) = 3$ and $s(n) \leq 2n$ if $n \geq 4$, and $s(n) \geq 2n - 4$ if $n \geq 4$ is even.

The aim of this paper is twofold. On one hand, we give new better lower bounds on the maximum number of inseparable leaves for planar polynomial foliations. More precisely, for all $n \geq 4$ we find an explicit planar polynomial foliation of degree n belonging to the Pluvinage family with $2n - 4$ inseparable leaves. Hence, $s(n) \geq 2n - 4$ for all $n \geq 4$. Moreover we improve the general lower bound for $s(n)$ when $n = 4$ and $n = 6$, in fact we show that $s(4) \geq 6$ and $s(6) \geq 9$.

On the other hand we characterize the structurally stable planar polynomial foliations under polynomial perturbations, and study the number of inseparable leaves in this family of planar polynomial foliations. Finally we give examples which realize all possible number of inseparable leaves of structurally stable planar polynomial foliations. Most of the techniques we use involve the Poincaré compactification and the blow up method for studying the local phase portrait of a degenerate singular point (at infinity).

This work is organized as follows. In Section 2 we give some notation and preliminary definitions. We also show the relation between inseparable leaves and hyperbolic sectors at infinity (in the Poincaré sphere as well as in the Bendixson sphere). In Section 3 we improve the lower bounds on the maximum number of inseparable leaves for $n = 4$, $n = 6$ and for all $n \geq 5$ odd. Finally, in Section 4 we study the number of inseparable leaves of all structurally stable planar polynomial foliations.

2. Preliminaries.

We start by introducing notation and definitions we will use later on the paper. We denote by $\mathcal{P}_n(\mathbb{R}^2)$ the set of all polynomial vector fields on $\mathbb{R}^2$ given by

$$X(x,y) = \big(P(x,y), Q(x,y)\big), \tag{1}$$

where P and Q are polynomials in the variables x and y of degree at most n, and P^2+Q^2 has degree $2n$. We denote by $\mathcal{PF}_n(\mathbb{R}^2) \subset \mathcal{P}_n(\mathbb{R}^2)$ the subset of all planar polynomial foliations.

For $X \in \mathcal{P}_n(\mathbb{R}^2)$ we define the *Bendixson compactified vector field* $b(X)$ *corresponding to* X which is an analytic vector field induced on $\mathcal{S}^2$ as follows (see, for instance [**23**]). The sphere in $\mathbb{R}^3$ given by $x^2+y^2+z^2=\frac{1}{4}$ is called the *Bendixson sphere*. We identify the xy-plane with the tangent plane at this sphere on the point $S=(0,0,-\frac{1}{2})$ given by the equation $z=-\frac{1}{2}$. Let p_N be the stereographic projection from the north pole $N=(0,0,\frac{1}{2})$ to the plane $z=-\frac{1}{2}$, and let p_S be the stereographic projection from the south pole $S=(0,0,-\frac{1}{2})$ to the plane $z=\frac{1}{2}$.

Clearly, p_N^{-1} (respectively p_S^{-1}) induces an analytic vector field X_N in $\mathcal{S}^2\setminus N$ (respectively $\mathcal{S}^2\setminus S$). The *Bendixson compactification* is the induced vector field on $\mathcal{S}^2$ which has the north pole as a singular point.

To study this singular point we define the map $p_S\circ p_N^{-1}$ from the plane $z=-\frac{1}{2}$ minus S to the plane $z=\frac{1}{2}$ minus N given by

$$u=\frac{x}{x^2+y^2}, \qquad v=\frac{y}{x^2+y^2},$$

where the (u,v) are the coordinates on the plane $z=1/2$.

Applying the above change of variables to the vector field (1) we get a new system on the uv-plane minus $(0,0)$ of the form

$$u'=\frac{1}{(u^2+v^2)^n}\overline{P}(u,v), \qquad v'=\frac{1}{(u^2+v^2)^n}\overline{Q}(u,v).$$

Finally, scaling the time we get a polynomial system of degree $m=n+2$ given by

$$u'=\overline{P}(u,v), \qquad v'=\overline{Q}(u,v), \tag{2}$$

and defined on the entire uv–plane. Moreover it has a singularity at $(0,0)$. So, the flow of system (2) (in a deleted neighborhood of $(0,0)$) is conjugate to the flow of system (1) in a neighborhood of infinity.

For $X \in \mathcal{P}_n(\mathbb{R}^2)$ the *Poincaré compactified vector field* $p(X)$ *corresponding to* X is an analytic vector field induced on $\mathcal{S}^2$ as follows (see, for instance [**8**]). We denote by $\mathcal{S}^2=\{y=(y_1,y_2,y_3)\in\mathbb{R}^3 : y_1^2+y_2^2+y_3^2=1\}$ the *Poincaré sphere* and by $T_y\mathcal{S}^2$ the tangent space to $\mathcal{S}^2$ at point y. Consider the central projection $f:\mathbb{R}^2=T_{(0,0,1)}\mathcal{S}^2\to\mathcal{S}^2$. Denote by X' the vector field $Df\circ X$ defined on $\mathcal{S}^2$ except on its equator $\mathcal{S}^1=\{y\in\mathcal{S}^2 : y_3=0\}$. Clearly $\mathcal{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Then $p(X)$ is the only analytic extension of $y_3^{n-1}X'$ to $\mathcal{S}^2$. The projection of the closed northern hemisphere of $\mathcal{S}^2$ on $y_3=0$ under $(y_1,y_2,y_3)\mapsto(y_1,y_2)$ is called *the Poincaré disc.* We compute the expression of $p(X)$ by using the local charts $U_i=\{y\in\mathcal{S}^2 : y_i>0\}$, and $V_i=\{y\in\mathcal{S}^2 : y_i<0\}$ where $i=1,2,3$, and the diffeomorphisms

$F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ which are the inverses of convenient central projections. We denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$, so z represents different things according to the local charts under consideration. Now some easy computations give for $p(X)$ the following expressions

$$
\begin{aligned}
z_2^n \cdot \Delta(z) \left[Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) \right] &\text{ in } U_1,\\
z_2^n \cdot \Delta(z) \left[P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right] &\text{ in } U_2,\\
\Delta(z) \left[P(z_1, z_2), Q(z_1, z_2) \right] &\text{ in } U_3,
\end{aligned}
$$

where $\Delta(z) = ({z_1}^2 + {z_2}^2 + 1)^{-\frac{n-1}{2}}$. Changing the reparametrization of the solutions of the vector fields $p(X)$ in what follows we will omit the factor $\Delta(z)$ in their local expressions. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$.

A singular point q of $p(X)$ is called an *infinite* (respectively *finite*) singular point if $q \in \mathcal{S}^1$ (respectively $q \in \mathcal{S}^2 \setminus \mathcal{S}^1$). So the infinite singular points $(z_1, 0)$ are given by the equations:

$$
\begin{aligned}
F(z_1) &= Q_n(1, z_1) - z_1 P_n(1, z_1) \quad \text{in} \quad U_1,\\
G(z_1) &= P_n(z_1, 1) - z_1 Q_n(z_1, 1) \quad \text{in} \quad U_2,
\end{aligned}
$$

where P_n and Q_n are the homogeneous parts of degree n of P and Q respectively. Eventually, we will use the equivalent notation $F(x, y) = xQ_n(x, y) - yP_n(x, y)$ and $G(x, y) = yP_n(x, y) - xQ_n(x, y)$ respectively.

As we said before a major problem we want to study is the maximum number of inseparable leaves for the polynomial foliations of degree n, $s(n)$. A key point in our approach is the study of the local phase portrait at the infinite singular points of $p(X)$. When those are *linearly zero* (i.e., linear part identically zero) we will use the *directional blow-up method* which can be described as follows (for more details see [**2**], [**6**] or [**13**]). Let us consider the polynomial system

$$
\begin{aligned}
\dot{x} &= X_n(x, y) + X_{n+1}(x, y) + \ldots = X(x, y),\\
\dot{y} &= Y_m(x, y) + Y_{m+1}(x, y) + \ldots = Y(x, y),
\end{aligned}
\tag{3}
$$

where $k = \min\{n, m\} \geq 2$ is called the *degree* of the linearly zero singular point at the origin.

We consider the blow up change of variables (or simply, the blow up) $x = x$ and $y = xz$. Then, system (3) in the new variables x and z becomes

$$
\begin{aligned}
\dot{x} &= X(x, zx) = \overline{X}(x, z),\\
\dot{z} &= \frac{1}{x}(Y(x, zx) - zY(x, zx)) = \overline{Z}(x, z).
\end{aligned}
\tag{4}
$$

Since $k \geq 2$, $\overline{X}$ and $\overline{Z}$ have the common factor x^{k-1}, or perhaps some higher power of x. If x^r with $r \geq k-1$ is the maximum power of x which divides $\overline{X}$ and $\overline{Z}$, in order to remove the straight line of singular points $x = 0$, we change the time t to a new time τ such that $x^r dt = d\tau$. Due to this change, if necessary, we must take care with the orientation of the trajectories when we go back through the blow up's.

The system we obtain in the variables (x, z, τ) is a well defined polynomial system having $x = 0$ as invariant straight line. The blow up brings the origin of (3) to the straight line $x = 0$ of the system

$$x' = \frac{\overline{X}(x,z)}{x^r}, \quad \text{and} \quad z' = \frac{\overline{Z}(x,z)}{x^r}, \tag{5}$$

where the prime denotes derivative with respect τ. Then in order to control all trajectories in a neighborhood of the origin of (3) (except trajectories tangent to the y–axis which have to be controlled by using the corresponding blow up $x = zy$ and $y = y$) we must study the flow of (5) in a neighborhood of $x = 0$.

If there is a trajectory tending to the origin of (3) (in forward or backward time) with slope m, then we will have a singular point of (5) in $(0, m)$ with a simpler local flow than before the blow up change of variables. The polynomial $H(x, y) = yX_k(x, y) - xY_k(x, y)$ will be called the *characteristic polynomial*. It is well-known that the zeroes of this polynomial provide the *characteristic directions* for which the orbits tend to the origin (in forward or backward time) of system (3), see [**1**].

Perhaps we need more than one blow up in order to study completely the local phase portrait at a linearly zero singular point. But it can be proved that using a finite number of blow ups we can reduce the original linearly zero singular point to singular points having their linear part non-identically zero.

We finish this section by showing some well-known results on the upper bound of the maximum number of inseparable leaves and its relationship with the maximum number of hyperbolic sectors at infinity (on the Bendixson as well as on the Poincaré compactification). To prove these results we may apply, first, that the local phase portrait at a singular point of a planar analytic vector field is either a center, a focus, or it is formed by the union of a finite number of elliptic, hyperbolic or parabolic sectors, and second, the Poincaré–Hopf formula on the sum of the indices of the singular points of $p(X)$. We state them without proof (see for more details [**1**], [**23**] and [**18**]).

Lemma 2.1. *Let X be a planar polynomial foliation. The inseparable leaves of X correspond to the separatrices of hyperbolic sectors at the unique singular point of $b(X)$.*

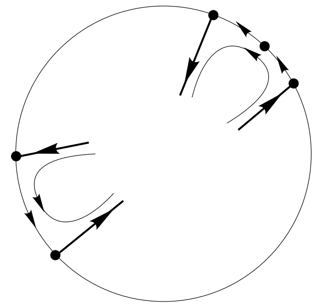


Figure 2. Relatively consecutive sectors on the Poincaré disc.

Theorem 2.2. *Let $X = (P, Q) \in \mathcal{PF}_n(\mathbb{R}^2)$. Then the maximum number of hyperbolic sectors at infinity of $b(X)$ is n. Therefore the maximum number of inseparable leaves is $2n$.*

These two results refer to the relationship between inseparable leaves and hyperbolic sectors at infinity of $b(X)$. However, most of our arguments will refer to hyperbolic sectors at the infinity of $p(X)$. Therefore we need to know which is the relation between hyperbolic sectors at infinity for both compactifications.

A hyperbolic sector of the Poincaré disc which has one boundary separatrix contained into $\mathcal{S}^1$ and one boundary separatrix outside $\mathcal{S}^1$ is called *adjacent to infinity*. Two adjacent sectors are called *relatively consecutive* if they are either consecutive on $\mathcal{S}^1$, or the unique sectors between them in counterclockwise or clockwise sense in $\mathcal{S}^1$ are hyperbolic sectors with the two boundaries on $\mathcal{S}^1$ (see Figure 2). Now, the proof of the following result is easy.

Lemma 2.3. *Let $X \in \mathcal{PF}_n(\mathbb{R}^2)$. If we identify the boundary $\mathcal{S}^1$ of the Poincaré disc to a point, then*

(a) *the hyperbolic sectors in the Poincaré disc whose two boundaries are not contained in $\mathcal{S}^1$ remain in the Bendixson sphere;*
(b) *the hyperbolic sectors with the two boundaries on $\mathcal{S}^1$ disappear;*
(c) *two relatively consecutive hyperbolic sectors give a unique hyperbolic sector in the Bendixson sphere.*

Remark 2.4. The problem of getting $s(n)$ is not a local problem. This is due to the fact that the local phase portraits at infinite singular points of X (in the Bendixson or the Poincaré sphere) are not enough to get the number of inseparable leaves of a planar polynomial foliation X. This claim becomes clear because of the possible connections between two separatrices of hyperbolic sectors as we show in Figure 3.

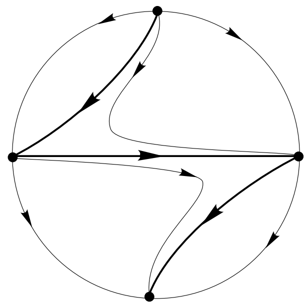


Figure 3. The straight line $y = 0$ connects two separatrices of hyperbolic sectors at infinity. This topological phase portrait is realizable for a quadratic polynomial foliation (see [**7**]).

3. New lower and upper bounds for s(n).

We start this section by giving an example of a planar polynomial foliation of degree $n \geq 4$ with $2n-4$ inseparable leaves. This example extends the result of [**23**] which gave an example when n is even. In fact, our example comes from Pluvinage family [**20**].

Proposition 3.1. *Let $B(x)$ be a polynomial of degree $n-1$ with $n-2$ extremes such that the values of $B(x)$ in its extreme are all different. If $A(x) = B'(x)$, then the planar polynomial foliation of degree $n \geq 4$ given by the flow of the system*

$$\dot{x} = 1, \qquad \dot{y} = y^2 A(x), \tag{6}$$

has $2n-4$ inseparable leaves.

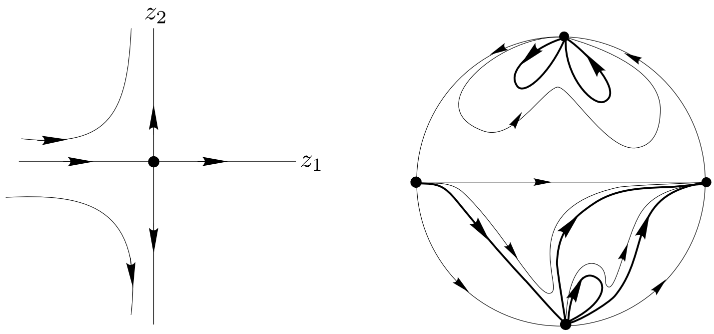
Proof. From Lemmas 2.1 and 2.3 in order to study the inseparable leaves we must study the infinite hyperbolic sectors of system (6).

If $A(x) = a_0 + a_1 x + \ldots a_{n-2}x^{n-2}$, then the infinite singular points are given by the polynomial $F(x,y) = a_{n-2}x^{n-1}y^2$. So, the singular points at infinity are the origin of the local charts U_k and V_k for $k = 1, 2$.

First we analyze the origin of the local chart U_2 given by system

$$\dot{z}_1 = z_2^n - z_1 z_2^{n-2} A\left(\frac{z_1}{z_2}\right), \qquad \dot{z}_2 = -z_2^{n-1} A\left(\frac{z_1}{z_2}\right).$$

The origin is linearly zero with identically zero characteristic polynomial. So, we apply Theorem 5.1 of [**1**]. The singular directions are given by the real roots of the polynomial $A(\cdot)$ which we can assume different from zero. So, after doing the blow up $z_1 = uz_2,\ z_2 = z_2$ and dividing by the common



(a) (b)

Figure 4. (a) The local phase portrait at the origin of U_1 for system (6). (b) The phase portrait of system (6) when $A(y) = 4x^3 - 14x + 6$.

factor z_2^{n-1} we obtain

$$\dot{u} = 1, \qquad \dot{z}_2 = -A(u).$$

Its solution curves are $z_2 = -B(u) + C$ where $C \in \mathbb{R}$. There are exactly $n-2$ solution curves of this form which have a tangent point with the u–axis. Going back to the coordinates (z_1, z_2) we get for each one of these $n-2$ curves two separatrices of a hyperbolic sector at the origin of U_2. Moreover we also obtain the $n-2$ corresponding elliptic sectors in between. Therefore, it has index 1 and there is no any connection between separatrices of hyperbolic sectors, because the extremes of $B(\cdot)$ are different. Hence, the origin of U_2 contributes with $2n-4$ inseparable leaves.

Second we consider the origin of the local chart U_1 given by

$$\dot{z}_1 = -z_1 z_2^n - z_1^2 z_2^{n-2} A\left(\frac{z_1}{z_2}\right), \qquad \dot{z}_2 = -z_2^{n+1}. \tag{7}$$

The origin is linearly zero with the z_1–axis and z_2–axis being invariant. Moreover it is immediately to see that it has index 0 because of the Poincaré-Hopf Theorem and the fact that the origin of U_2 has index 1. Finally an easy computation shows that its characteristic polynomial is

$$H(z_1, z_2) = z_1^2 z_2 a_{n-2}.$$

Hence it is a singular point of degree two (i.e., system (7) starts with terms of second order at the origin). See [**13**] for a general classification of the local phase portraits of singular points of degree two.

As above we may use the blow up method to show that the origin of U_1 is given by Figure 4(a). We omit the details.

Therefore, there are no further inseparable leaves coming from the origin of U_1 and the proof follows (see Figure 4(b) for a qualitative phase portrait in the Poincaré disc of system (6) when $n = 5$). □

Remark 3.2. It could seem natural from Proposition 3.1 that, in order to get a polynomial foliation of degree n with $2n$ inseparable leaves, we can change $\dot{y} = y^2 A(x)$ by $\dot{y} = A(x)$ being $A(x)$ of degree n and same characteristics. However, it is not difficult to show that the resultant polynomial foliation has no inseparable leaves.

We give now an example of a planar polynomial foliation of even degree n with $2n - \frac{n}{2}$ inseparable leaves. These examples improve the above lower bound when $n = 4$ and $n = 6$. In the case $n = 8$ we get a new example with 12 inseparable leaves. We give first a technical lemma.

Lemma 3.3. *Let n be an even integer and let $r_1, \ldots, r_{n/2}$ be non-zero real numbers such that $r_1 < r_2 < \ldots < r_{n/2}$. Then there exist polynomials $A(y)$ and $B(y)$ of degrees $n-1$ and n respectively such that for all $j = 1, \ldots, \frac{n}{2}$ satisfy*

(a) *$A(r_j) = 0$, $A'(r_j) > 0$;*
(b) *$B(r_j) = B'(r_j) = 0$ and $B''(r_j) < 0$; and*
(c) *$2A'(r_j)/B''(r_j) < -1$.*

Proof. Consider the following two polynomials

$$\overline{B}(y) = -\prod_{i=1}^{n/2}(y - r_i)^2 \quad \text{and} \quad A(y) = \left[\prod_{i=1}^{n/2}(y - r_i)\right]\left[\prod_{j=1}^{(n/2)-1}(y - s_j)\right],$$

with $0 < r_1 < s_1 < r_2 < \ldots < s_{(n/2)-1} < r_{n/2}$. Clearly polynomials A and $\overline{B}$ satisfy conditions (a) and (b) respectively. If they do not satisfy condition (c), then we take a sufficiently small positive constant k and we consider $B = k\overline{B}$ in such away that A and B satisfy the three statements of the lemma. □

Proposition 3.4. *Let n be an even integer number. Let $A(y) = a_0 + a_1 y + \ldots + a_{n-1}y^{n-1}$ and $B(y) = b_0 + b_1 y + \ldots + b_n y^n$ be polynomials of degree $n-1$ and n respectively satisfying the conditions of Lemma* 3.3*. Then the polynomial foliation in the plane of even degree $n \geq 2$ given by*

$$\dot{x} = xA(y) + 1, \qquad \dot{y} = B(y), \tag{8}$$

has $2n - \frac{n}{2}$ inseparable leaves.

Proof. First we note that, from statements (a) and (b) of Lemma 3.3, system (8) is a planar polynomial foliation of degree n. Therefore we study its inseparable leaves by using the Poincaré compactification.

Choosing $b_n - a_{n-1} \neq 0$, the linear factors of the polynomial $F(x, y) = (b_n - a_{n-1})xy^n$ are x and y. So, the unique singular points at infinity are the origin of the local charts U_k and V_k for $k = 1, 2$.

On one hand, easy computations show that the origin of the local chart U_2 is a hyperbolic node. On the other the expression of system (8) in the local chart U_1 is given by system

$$\begin{aligned} \dot{z}_1 &= -z_1 z_2^n - z_1 z_2^{n-1} A\left(\frac{z_1}{z_2}\right) + z_2^n B\left(\frac{z_1}{z_2}\right), \\ \dot{z}_2 &= -z_2^n A\left(\frac{z_1}{z_2}\right) - z_2^{n+1}. \end{aligned} \tag{9}$$

Hence, the origin is linearly zero with characteristic polynomial given by

$$H(z_1, z_2) = z_2^{n+1} B\left(\frac{z_1}{z_2}\right).$$

So, the characteristic directions correspond to $z_2 = 0$ and $z_1 = r_j z_2$. The characteristic direction $z_2 = 0$ can be studied by doing the blow up $z_1 = z_1,\ z_2 = vz_1$. Easy computations and Lemma 3.3 show that the singular point $(0, 0)$ of the system $(\dot{z}_1, \dot{v})$ is a saddle.

On the other hand, doing the blow up $z_1 = u_1 z_2,\ z_2 = z_2$ and dividing by the common factor z_2^{n-1} we get

$$\dot{u}_1 = B(u_1), \qquad \dot{z}_2 = -z_2 A(u_1) - z_2^2.$$

Clearly the singular points on $z_2 = 0$ are the points r_j where $B(r_j) = 0$ for $j = 1, \ldots, \frac{n}{2}$. To study these singular points, first, we move them at the origin through the linear change of variables $u_2 = u_1 - r_j,\ z_2 = z_2,\ j = 1, \ldots, \frac{n}{2}$, and second we consider their Taylor series expansion. We have

$$\begin{aligned} \dot{u}_2 &= B(r_j + u_2) = \frac{1}{2} B''(r_j) u_2^2 + O(u_2^3), \\ \dot{z}_2 &= -z_2 A(r_j + u_2) - z_2^2 = -A'(r_j) u_2 z_2 - z_2^2 + O(z_2 u_2^2), \end{aligned} \tag{10}$$

for each point $(r_j, 0)$, $j = 1, \ldots, \frac{n}{2}$. The origin of system (10) has linear part identically zero. The characteristic directions are given by the characteristic polynomial

$$H(u_2, z_2) = u_2 z_2 \left[\left(\frac{1}{2} B''(r_j) + A'(r_j)\right) u_2 + z_2\right].$$

Clearly $u_2 = 0$ as well as $z_2 = 0$ are characteristic directions. Therefore, we apply to the origin of system (10) the directional blow ups $u_2 = u_3 z_2,\ z_2 = z_2$ and $u_2 = u_2\ z_2 = w_1 u_2$. After omitting the common factors z_2 and u_2

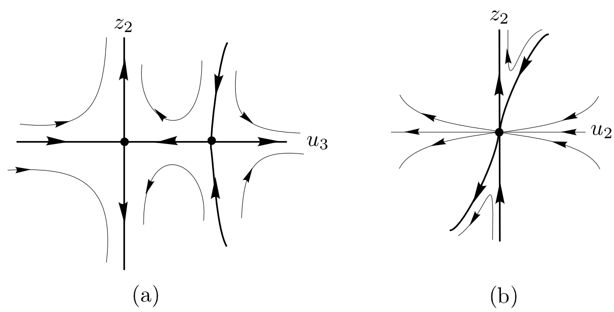


Figure 5. (a) Two saddles on $z_2 = 0$ for system $(\dot{u}_3, \dot{z}_2)$. (b) The local phase portrait at any singular point $(r_j, 0)$ (translated to the origin) for system $(\dot{u}_2, \dot{z}_2)$.

respectively we get

$$(11)\qquad \begin{aligned} \dot{u}_3 &= u_3 + \left(A'(r_j) + \frac{1}{2}B''(r_j)\right) u_3^2 + O(u_3^3 z_2), \\ \dot{z}_2 &= -z_2 - A'(r_j)u_3 z_2 + O(u_3^2 z_2^2), \end{aligned}$$

and

$$(12)\qquad \begin{aligned} \dot{u}_2 &= \frac{1}{2}B''(r_j)u_2 + O(u_2^2), \\ \dot{w}_1 &= -\left(\frac{1}{2}B''(r_j) + A'(r_j)\right) w_1 + O(u_2 w_1). \end{aligned}$$

From Lemma 3.3 the origin of system (12) is an attracting node. On the other hand, the singular points of system (11) on $z_2 = 0$ are $(0,0)$ and $(-\left(A'(r_j) + \frac{1}{2}B''(r_j)\right)^{-1}, 0)$. The linear part at these singular points is given by the matrices

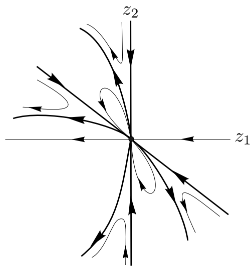
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & * \\ 0 & -\frac{B''(r_j)}{B''(r_j)+2A'(r_j)} \end{pmatrix},$$

respectively. Moreover, from condition (c) of Lemma 3.3 it follows easily that

$$-\left(\frac{2A'(r_j)}{B''(r_j)} + 1\right)^{-1} > 0,$$

then both singular points are saddles. In Figure 5(b) we show the local phase portrait at each singular point $(r_j, 0)$ of system $(\dot{u}_2, \dot{z}_2)$.

Finally, by considering all singular points $(r_j, 0)$, $j = 1, \dots, \frac{n}{2}$ and going back through the blow up it is easy to show that the origin of system (9) is a singular point with n hyperbolic sectors whose separatrices are not tangent



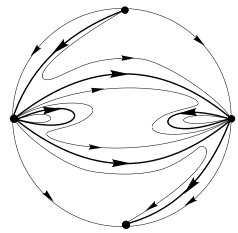


Figure 6. (a) The origin of system (9) when $n = 4$. (b) The phase portrait of system 8 in the Poincaré sphere when $n = 4$.

to $z_2 = 0$. Precisely, Figure 6(a) gives the local phase portrait at the origin of system (9) when $n = 4$.

Therefore the origin of U_1 has n hyperbolic sectors with their separatrices not contained in $\mathcal{S}^1$. Indeed, the $n/2$ invariant lines $y = r_j$ by the flow of system (8) are separatrices of such hyperbolic sectors connecting the origins of the local charts U_1 and V_1. Hence the number of inseparable leaves is $2n - \frac{n}{2}$. In Figure 6(b) we show the phase portrait of system (8) in the Poincaré disc when $n = 4$ and so the number of inseparable leaves is 6. $\square$

From the previous results it follows immediately the next corollary.

Corollary 3.5. *The following statements hold.*

(a) $s(0) = s(1) = 0$.
(b) $s(2) = s(3) = 3$.
(c) $6 \leq s(4) \leq 8$.
(d) $9 \leq s(6) \leq 12$.
(e) $2n - 4 \leq s(n) \leq 2n$ *if* $n = 5$ *or* $n \geq 7$.

The last part of this section is devoted to study the maximum number of inseparable leaves for a planar polynomial foliation whose Poincaré compactification has the equator fulfilled of singular points. In other words, its polynomial $F(z_1)$ is identically zero.

Proposition 3.6. *Let* $X \in \mathcal{PF}_n(\mathbb{R}^2)$. *Assume that the infinity* $\mathcal{S}^1 \subset \mathcal{S}^2$ *is fulfilled of singular points of* $p(X)$. *Then the maximum number of inseparable leaves of* X *is at most* $2n - 2$.

Proof. Let $X = (P, Q) \in \mathcal{PF}_n(\mathbb{R}^2)$. Since $\mathcal{S}^1 \subset \mathcal{S}^2$ is fulfilled of singular points of $p(X)$, we get that

$$yP_n(x, y) - xQ_n(x, y) \equiv 0, \tag{13}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are the homogeneous parts of degree n of $P(x,y)$ and $Q(x,y)$, respectively.

We consider the Bendixson compactification $b(X)$ of X. Then the big circle $\Gamma = \{(x,y) : x^2+y^2 = a^2\}$ in the xy-plane is the small circle $\Sigma = \{(u,v) : u^2+v^2 = 1/a^2\}$ in the uv-plane (here we are using the notation of Section 2). If a is large enough then Σ intersects each elliptic and hyperbolic sector at $(0,0)$ in the uv-plane. Each of such sectors contains a point of Σ where the vector field is tangent to Σ. These points correspond to points (x,y) in the xy-plane where $x^2+y^2 = a^2$ and $xP(x,y)+yQ(x,y) = 0$. Replacing (13) in the above equations we get

$$a^2\frac{Q_n(x,y)}{y} + Z(x,y) = 0, \qquad x^2+y^2-a^2 = 0, \tag{14}$$

where $Z(x,y)$ is a polynomial of degree at most n and $Q_n(x,y)/y$ is a polynomial of degree at most $n-1$. Applying Bezout Theorem there are at most $2n$ solutions of (14). Then $e+h \le 2n$ (where e and h denote the number of elliptic and hyperbolic sectors at the unique singular point of $b(X)$, respectively). Finally, the Poincaré-Bendixson Index Formula and the Poincaré-Hopf Theorem show that $e-h = 2$. So $h \le n-1$, and consequently the maximum number of inseparable leaves of X is at most $2n-2$. $\square$

4. Structural stability.

In this section we shall show why the maximum number of inseparable leaves for a polynomial foliation of degree n, has to be reached by a "degenerate" system. That is, some (or all) infinite singular point must have non elementary linear part. We point out this claim by considering the notion of structural stability under polynomial perturbations. Indeed our notion of structural stability involves the Poincaré compactification of the polynomial foliation (and its polynomial neighbors). This definition comes from Santos [**22**], Pugh [**21**] and Sotomayor [**25**] where structural stability of polynomial systems (not necessarily foliations) is considered. We notice that other notions of structural stability for polynomial vector fields have been considered in the literature, see [**24**].

We consider on the set of all planar polynomial foliations the well–known coefficient topology. A vector field $X \in \mathcal{PF}_n(\mathbb{R}^2)$ is said to be *structurally stable with respect to perturbations in* $\mathcal{P}_n(\mathbb{R}^2)$ if there exists a neighborhood N of X in $\mathcal{P}_n(\mathbb{R}^2)$ such that $Y \in N$ implies that $p(X)$ and $p(Y)$ are topologically equivalent; that is, there exists a homeomorphism of $\mathcal{S}^2$ carrying orbits of the flow induced by $p(X)$ onto orbits of the flow induced by $p(Y)$, preserving sense but not necessarily parametrization.

We notice that, although the necessity (or not) that all periodic orbits must be hyperbolic for structural stability under polynomial perturbations is still an open problem (see Problem 1.1 in [**25**]), we can ignore it since our vector field X is a foliation and hence it has no periodic orbits. The following characterization follows then directly from [**25**] (see also Santos [**22**] and Pugh [**21**]).

Corollary 4.1. *A vector field $X \in \mathcal{PF}_n(\mathbb{R}^2)$ is structurally stable with respect to perturbations in $\mathcal{P}_n(\mathbb{R}^2)$ if and only if the following conditions hold:*

(a) *All its infinite singular points are either hyperbolic saddles or hyperbolic nodes; and*
(b) *it has no saddle connections outside infinity.*

Theorem 4.2. *Let $X \in \mathcal{PF}_n(\mathbb{R}^2)$. If X is structurally stable with respect to perturbations in $\mathcal{P}_n(\mathbb{R}^2)$, then n is even and the number s of its inseparable leaves satisfies $s \in \{0, 4, 6, 8, \dots, n\}$.*

Proof. Suppose X lies in $\mathcal{PF}_n(\mathbb{R}^2)$ and it is structurally stable with respect to perturbations in $\mathcal{P}_n(\mathbb{R}^2)$. By Corollary 4.1 all its infinite singular points are either hyperbolic nodes or hyperbolic saddles of index 1 or -1 respectively. On the other hand, since $p(X)$ is the compactification of a planar polynomial foliation, the sum of their indices has to be 2.

By rotating the coordinates, if necessary, we can assume that all singular points are in the local charts U_1 and V_1. Hence, if $p(X)$ has $k \geq 0$ hyperbolic saddles in U_1, then it has $k+1$ hyperbolic nodes in U_1. Therefore, the polynomial $F(z_1) = Q_n(1, z_1) - z_1 P_n(1, z_1)$ which gives the infinite singular points $(z_1, 0)$ of $p(X)$ in U_1, has exactly $2k+1 \leq n+1$ simple real roots.

We will show that n has to be even. Let denote by $a_1 < a_2 < \dots < a_{2k+1}$ the infinite singular points in the local chart U_1, and by b_1 the symmetric (with respect to the origin) singular point to a_1 in the local chart V_1. We assume that the flow in a neighborhood of a_1 restricted to $z_2 = 0$ is a source. Easily, since $2k+1$ is odd, the flow in a neighborhood of a_{2k+1} restricted to $z_2 = 0$ is also a source. Consequently, the flow in a neighborhood of b_1 restricted to $z_2 = 0$ is an attractor. Therefore, since the flow in the local charts U_1 and V_1 is related by the factor $(-1)^{n-1}$, n has to be even.

If $k = 0$ then $p(X)$ has only two nodes in $\mathcal{S}^1$ and it has $s = 0$. Otherwise ($k > 0$), the hyperbolic sectors at infinity come from consecutive saddles, and the separatrices of these hyperbolic sectors will be inseparable leaves (see Lemma 2.1). Since an infinite saddle contributes only with one separatrix outside the infinity and there are no saddle connection, the symmetry between U_1 and V_1 force that the number of inseparable leaves lies in $\{0, 4, 6, 8, \dots, 2k\}$. Since $2k \leq n$, the theorem follows. □

We give now an example of a structurally stable planar polynomial foliation of degree n with n inseparable leaves. In fact from the definition of structural stability any system in a suitable neighborhood of it will be also a planar polynomial foliation of degree n with n inseparable leaves.

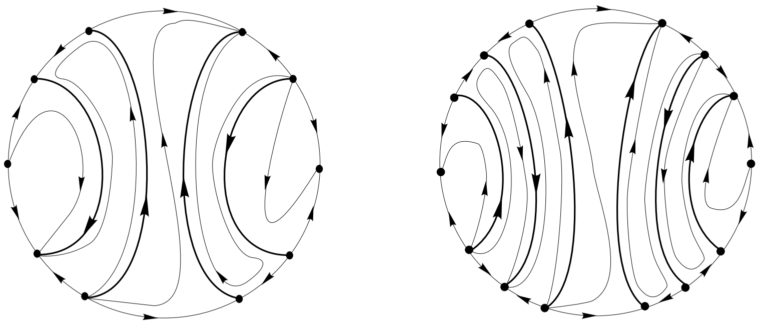


Figure 7. (a) A structurally stable planar polynomial foliation of degree four with four inseparable leaves. (b) A structurally stable planar polynomial foliation of degree six with six inseparable leaves.

As a corollary of our construction we can realize a structurally stable planar polynomial foliations of degree n with $s \in \{0, 4, 6, 8, \dots, n\}$ inseparable leaves.

We introduce first the definition of algebraic solution we will use later. An *invariant algebraic curve* of system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y)$$

is an algebraic curve $f(x,y) = 0$ with $f \in \mathbb{R}[x,y]$, such that for some polynomial $K(x,y)$ we have

$$\frac{\partial f}{\partial x}(x,y)P(x,y) + \frac{\partial f}{\partial y}(x,y)Q(x,y) = K(x,y)f(x,y).$$

We say that the curve $f(x,y) = 0$ is an *algebraic solution* of the system if it is an invariant algebraic curve and $f(x,y)$ is an irreductible polynomial over $\mathbb{R}[x,y]$.

Proposition 4.3. *Let n be even. Given $a \in \mathbb{R}$ sufficiently large, the system*

$$\dot{x} = aF(x,y) - x\frac{\partial F(x,y)}{\partial y}, \qquad \dot{y} = x\frac{\partial F(x,y)}{\partial x}, \tag{15}$$

where $F(x,y) = \prod_{i=1}^{\frac{n}{2}}(x^2 - i^2y^2 - i^2)$, is a structurally stable planar polynomial foliation of degree n with n inseparable leaves (*see Figure* 7).

Proof. System (15) can be written in the following way

$$
\begin{aligned}
\dot{x} &= aF(x,y) + 2xy \sum_{i=1}^{\frac{n}{2}} i^2 \prod_{j=1,\ j\neq i}^{\frac{n}{2}} (x^2 - j^2y^2 - j^2) = P(x,y), \\
\dot{y} &= 2x^2 \sum_{i=1}^{\frac{n}{2}} \prod_{j=1,\ j\neq i}^{\frac{n}{2}} (x^2 - j^2y^2 - j^2) = Q(x,y).
\end{aligned} \tag{16}
$$

We claim that system (15) has $\frac{n}{2}$ irreductible algebraic solutions given by $x^2 - k^2y^2 - k^2 = 0, \quad k = 1, \ldots, \frac{n}{2}$. The claim follows from the following equality

$$
\begin{aligned}
&2x\left(aF(x,y) + 2xy \sum_{i=1}^{\frac{n}{2}} i^2 \prod_{j=1,\ j\neq i}^{\frac{n}{2}} (x^2 - j^2y^2 - j^2)\right) \\
&\quad - 2k^2y\left(2x^2 \sum_{i=1}^{\frac{n}{2}} \prod_{j=1,\ j\neq i}^{\frac{n}{2}} (x^2 - j^2y^2 - j^2)\right) \\
&= (x^2 - k^2y^2 - k^2)\Bigg(2ax \prod_{i=1,\ i\neq k}^{\frac{n}{2}} (x^2 - i^2y^2 - i^2) \\
&\quad + 4xy \sum_{i=1,\ i\neq k}^{\frac{n}{2}} (i^2 - k^2) \prod_{j=1,\ j\neq i,k}^{\frac{n}{2}} (x^2 - j^2y^2 - j^2)\Bigg).
\end{aligned}
$$

On the other hand we also claim that system (15) has no finite singular points. Here the key is to make the change of variables $u = x^2$ and $v = 1+y^2$. In these new variables the polynomial equations $\dot{x} = 0$ and $\dot{y} = 0$ become

$$
aF(u,v) + 2\sqrt{u(v-1)}\,\frac{\partial F(u,v)}{\partial v} = 0, \qquad 2u\frac{\partial F(u,v)}{\partial u} = 0, \tag{17}
$$

where $F(u,v) = \prod_{i=1}^{\frac{n}{2}} (u - i^2v),\ i = 1, \ldots, \frac{n}{2}$. Since both $u \geq 0$ and $v \geq 1$ system (17) is well defined. On the other hand, if we force $u = 0$ in the first equation of (17) we get $F(0,v) = 0$ which has no solutions because $v \geq 1$. So, we may assume $u > 0$. Since $F(u,v)$ is a homogeneous polynomial of degree $\frac{n}{2}$ the zeroes of the second equation of (17) not contained in $u = 0$ are on $\frac{n}{2} - 1$ straight lines $u = a_jv,\ j = 1, \ldots, \frac{n}{2} - 1$. We only are interested in the $a_j > 0,\ j = 1, \ldots, \frac{n}{2} - 1$ because $u \geq 0$ and $v \geq 1$. Substituting $u = a_jv,\ j = 1, \ldots, \frac{n}{2} - 1$ in the first equation of (17) we get

$$
v^{\frac{n}{2}} + M\sqrt{v(v-1)}\,v^{\frac{n}{2}-1} = 0, \tag{18}
$$

where $M \in \mathbb{R}$ depends on the a_j's. Since $v \geq 1$ we may divide the above equation by $v^{\frac{n}{2}-1}$, and then multiply by $av - M\sqrt{v(v-1)}$. Clearly, a positive real root of Equation (18) must be also a root of the new equation

$$(a^2 - M^2)v^2 + M^2 v = 0.$$

Easily, choosing a such that $a^2 - M^2 > 0$ (i.e., choosing a sufficiently large) we have no positive solutions of the last equation or, equivalently our system (16) has no (finite) singular points.

To finish the proof we show that all infinite singular points are hyperbolic saddles and nodes. More precisely there are $\frac{n}{2}+1$ consecutive nodes and $\frac{n}{2}$ consecutive saddles which separatrices are given by the algebraic solutions (hyperbolas) $x^2 - i^2y^2 - i^2 = 0$. We note first that by construction there are no saddle connections.

The infinite singular points are given by the real roots of the polynomial $F(z_1) = Q_n(1, z_1) - z_1 P_n(1, z_1)$ where $P_n(x, y)$ and $Q_n(x, y)$ are the homogeneous polynomials of degree n for system (16). Making some computations, we get

$$F(z_1) = (2 - az_1)\prod_{j=1}^{\frac{n}{2}}(1 - j^2 z_1^2).$$

The real roots of this polynomial are given by $z_1 = \pm\frac{1}{j}$, $j = 1, \dots, \frac{n}{2}$ and $z_1 = \frac{2}{a}$. So, there are $n+1$ infinite singular points. The eigenvalues at these singular points $(z_1, 0)$ are given (see Section 2) by $F'(z_1)$ and $-P_n(1, z_1)$. So, $z_1 = \frac{2}{a}$ is always a node (attracting if $a > 0$ and repulsive if $a < 0$). Some computations show that if $a > 0$ (respectively $a < 0$) we have $\frac{n}{2}$ consecutive nodes (respectively saddles) at the infinite singular points $(-\frac{1}{j}, 0)$, $j = 1 \dots \frac{n}{2}$ (respectively $(\frac{1}{j}, 0)$ $j = 1, \dots, \frac{n}{2}$). □

The next result follows immediately from Proposition 4.3.

Corollary 4.4. *Let $X = (P(x, y), Q(x, y))$ be the planar polynomial foliation of degree n described in Proposition 4.3. Then, for any number $m \in \mathbb{N}$, the planar polynomial foliation*

$$Y = \big((x^2 + y^2 + 1)^m P(x, y), (x^2 + y^2 + 1)^m Q(x, y)\big)$$

is a structurally stable planar polynomial foliation of even degree with n inseparable leaves.

The next result follows easily from Corollary 4.4.

Corollary 4.5. *Let $n \in \mathbb{N}$ even. If $n \geq 4$ and $s \in \{0, 4, 6, 8, \dots, n\}$, then there are structurally stable planar polynomial foliations of degree n with s inseparable leaves.*

We finish with two open problems:

1) For $X \in \mathcal{PF}_n(\mathbb{R}^2)$ and $n \geq 4$, what is the exact value of $s(n)$?
2) If we change the notion of structural stability in such away only polynomial foliation perturbations are allowed, what are the structurally stable planar polynomial foliations of degree n?

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