DECOMPOSITION OF DOUBLE COSETS IN $p$-ADIC GROUPS

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Let $G$ be a split group over a locally compact field $F$ with non-trivial discrete valuation. Employing the structure theory of such groups and the theory of Coxeter groups, we obtain a general formula for the decomposition of double cosets $P_1\sigma P_2$ of subgroups $P_1, P_2 \subset G(F)$ containing an Iwahori subgroup into left cosets of $P_2$. When $P_1$ and $P_2$ are the same hyperspecial subgroup, we use this result to derive a formula of Iwahori for the degrees of elements of the spherical Hecke algebra.

1. Introduction.

Let $G$ be a semisimple algebraic group which is split over a locally compact field $F$ with non-trivial discrete valuation and let $I$ be an Iwahori subgroup of $G(F)$. In [6], Iwahori and Matsumoto show that the double cosets in $I\backslash G(F)/I$ are indexed by the elements of the extended affine Weyl group $\tilde{W}$ of $G$, and for $w$ in $\tilde{W}$, they exhibit an explicit set of representatives for the left cosets of $I$ in $IwI/I$. They also show that the number of single cosets of $I$ in $IwI/I$ is $q^{l(w)}$, where $q$ is the cardinality of the residue field of $F$ and $l$ is the standard combinatorial length function on $\tilde{W}$.

Let $O_F$ be the ring of integers of $F$ and let $K \subset G(F)$ be a hyperspecial subgroup, that is, a subgroup isomorphic to $G(O_F)$, where $G$ is a smooth group scheme over $O_F$ with general fiber $G$. In [5], Iwahori gives a formula for the number of left cosets of $K$ contained in a double coset in $K\backslash G(F)/K$ (i.e., the degree of the characteristic function of this double coset as an element of the spherical Hecke algebra), implicitly making use of the decomposition in [6] and the fact that $K$ contains an Iwahori subgroup. Suppose that $\pi$ is a uniformizer of $F$. The double cosets in $K\backslash G(F)/K$ are indexed by the dominant co-characters of a maximal torus of $G$ via the bijection $\lambda \mapsto K\lambda(\pi)K$. Let $W_0$ be the Weyl group of $G$ and $W_0^\lambda$ the stabilizer of $\lambda$ in $W_0$. Then there exists a unique set $[W_0/W_0^\lambda] \subset W_0$ of representatives of cosets of $W_0^\lambda$ of minimal length. Iwahori states that the index

$$[K\lambda(\pi)K : K] = q^{l(\sigma\lambda)} \sum_{\tau \in [W_0/W_0^\lambda]} q^{l(\tau)},$$

where $\lambda$ is a dominant co-character of a maximal torus of $G$.
where $\sigma_{\lambda}$ is a certain element of $\tilde{W}$ associated with the co-character $\lambda$ (see Section 6).

In this paper, we give a summary of the above results and generalize them, finding representatives for the left cosets of $P_2$ in $P_1\sigma P_2$, where $P_1$ and $P_2$ are subgroups of $G(F)$ containing an Iwahori subgroup $I$. To accomplish this, we make use of the structure theory of groups over fields with discrete valuation (as summarized in Section 3). Because of the close connection between subgroups of $G(F)$ of the above type and subgroups of the affine Weyl group (as given in [6]), the proof of the coset decomposition formula necessitates the use of the theory of Coxeter groups to prove certain results about the additivity of lengths of elements of $\tilde{W}$ (Section 4). In Section 5, we find the coset representatives mentioned above and give a formula for their number when the groups $P_1$ and $P_2$ are compact. In addition, we give several examples of this coset decomposition and an explanation of how our general results imply Iwahori’s in the case $P_1 = P_2 = K$ (Section 6).

This information on the decomposition of double cosets is useful in computing the action of Hecke operators on spaces of modular forms as defined in [3]. In fact, the results of this paper were used in [8] when $G$ is a compact form of $G_2$ or PGSp$_4$ over $\mathbb{Q}$ and $P_1 = P_2 = K$ to compute the action on certain spaces of forms of the spherical Hecke algebra of functions on $G(\mathbb{Q}_p)$ bi-invariant by the subgroup $K$ for several primes $p$ at which $G$ is split. The action of a function in this algebra is given by integrating it against a form $f$ with respect to a Haar measure on $G(\mathbb{Q}_p)$. This integral turns out to be a finite sum. Indeed, when the Hecke operator is the characteristic function of a double coset $K\sigma K$ of $K$, the integral is simply the sum over the right translations of $f$ by a set of representatives of the left cosets of $K$ inside $K\sigma K$. Since the Hecke algebra is generated by such characteristic functions, the decomposition of double cosets is therefore fundamental to the explicit determination of the actions these algebras.

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2. Notation.

In the following, we will denote by $G$ a connected semisimple algebraic group that is split over a locally compact field $F$ with non-trivial discrete valuation. Let $O_F$ be the ring of integers of $F$ and let $p$ be the prime ideal.

We choose a uniformizing parameter $\pi$ in $p$, and denote by $k$ the residue field $O_F/p$. Let $q$ be the (finite) cardinality of $k$ and let $R \subset O_F$ be a set of representatives for $k$ containing 0. The group $G$ is the general fiber of a Chevalley group scheme $\underline{G}$ over $O_F$ whose special fiber is semisimple. We let
\( K = G(\mathcal{O}_F) \subset G(F) = G(F) \) be the set of integral points of \( G \). \( K \) is then a hyperspecial maximal compact subgroup of \( G(F) \) (cf. [9, 3.8.1, 3.8.2]).

Let \( T \subset G \) be a split maximal torus scheme, and let \( T \) be its general fiber. We define \( N_T \) to be the normalizer of \( T \) in \( G \). Denote by \( X^*(T) \) the character module \( \text{Hom}(T, \mathbb{G}_m) \) of \( T \) and by \( X_*(T) \) the co-character module \( \text{Hom}(\mathbb{G}_m, T) \) of \( T \). Let \( \Phi \subset X^*(T) \) be the set of roots of \( T \), \( \Phi^+ \subset \Phi \) a subset of positive roots, and \( \Delta \subset \Phi^+ \) the corresponding set of simple roots. Also, let \( \Phi^\vee \subset X_*(T) \) be the coroots of \( T \) and \( \alpha \mapsto \alpha^\vee \) the standard bijection between \( \Phi \) and \( \Phi^\vee \). For each \( \alpha \in \Phi \) let \( U_\alpha \) be the one-dimensional unipotent subgroup scheme of \( G \) corresponding to \( \alpha \). Denote the general fiber of \( U_\alpha \) by \( U_\alpha \). We choose for each \( \alpha \) an isomorphism

\[
x_{\alpha}: \mathbb{G}_a \longrightarrow U_\alpha.
\]

When considered as a map \( F \longrightarrow U_\alpha(F) \), \( x_{\alpha} \) restricts to an isomorphism of \( \mathcal{O}_F \) with \( U_\alpha(\mathcal{O}_F) = U_\alpha(F) \cap K \).

We will denote by \( W_0 \) the Weyl group \( N_T/T = (N_T(F) \cap K)/T(\mathcal{O}_F) \) of \( G \) and by \( \tilde{W} \) the extended affine Weyl group \( N_T(F)/T(\mathcal{O}_F) \). Then \( W_0 \) and \( \tilde{W} \) act as groups of affine transformations on the space \( X_*(T) \otimes \mathbb{R} \). The stabilizer in \( \tilde{W} \) of \( 0 \in X_*(T) \otimes \mathbb{R} \) is \( W_0 \), and there is an isomorphism \( \tilde{W} \cong X_*(T) \rtimes W_0 \), where \( X_*(T) \) is embedded in \( \tilde{W} \) as the group of elements acting as translations on \( X_*(T) \otimes \mathbb{R} \). We denote by \( e \) the identity element of \( \tilde{W} \) and by \( t(\lambda) \) the element of \( \tilde{W} \) corresponding to \( \lambda \in X_*(T) \). In this notation, if \( w \in W_0 \) and \( \lambda \in X_*(T) \) then

\[
wt(\lambda)w^{-1} = t(w\lambda).
\]

Denote by \( w_{\alpha} \) the reflection in \( W_0 \) through the vanishing hyperplane in \( X_*(T) \otimes \mathbb{R} \) of the root \( \alpha \). The Weyl group \( W_0 \) is a Coxeter group with \( S_0 = \{ w_{\alpha} | \alpha \in \Delta \} \) as its set of involutive generators. Let \( \Phi = \Phi_1 \cup \cdots \cup \Phi_m \) be the decomposition of \( \Phi \) into irreducible root systems. (Each \( \Phi_i \) corresponds to the root system of an almost simple factor of \( G \).) Also, let \( \Delta_i = \Delta \cap \Phi_i \), and put \( l_i = \#\Delta_i \). Then \( l_1 + \cdots + l_m = l \), the dimension of \( T \), i.e., the rank of \( G \). Let \( \alpha_{0,i} \) be the highest root of \( \Phi_i \) with respect to the basis of simple roots \( \Delta_i \). Then the Coxeter group with set of involutive generators

\[
\tilde{S} = S_0 \cup \{ t(\alpha_{0,i}^\vee)w_{\alpha_{0,i}} | 1 \leq i \leq m \}
\]

is isomorphic to the affine Weyl group \( W_{\text{af}} \) of \( \Phi \) ([6, Prop. 1.1]). Via this isomorphism, we will view \( W_{\text{af}} \) as a subgroup of \( \tilde{W} \).

Let \( I \) be the Iwahori subgroup of \( G(F) \) generated by \( T(\mathcal{O}_F) \), the subgroups \( x_{\alpha}(\mathcal{O}_F) = U_\alpha(\mathcal{O}_F) \) for all \( \alpha \) in \( \Phi^+ \), and the subgroups \( x_{\alpha}(p) \) for all \( \alpha \) in \( \Phi^- \). If we denote by \( \overline{G} \) the semisimple algebraic group over \( k \) obtained by taking the special fiber of \( G \) then (as in [9, §3.5]) we have a surjective reduction mod \( p \) map \( K \longrightarrow \overline{G}(k) \), and \( I \) is the inverse image in \( K \) under this map of the Borel subgroup of \( \overline{G}(k) \) corresponding to \( \Phi^+ \). The triple
(G(F), I, N_T(F)) is a generalized Tits system in the sense of [5], a fact which will be used in Section 3 to study the structure of G(F).

Denote the normalizer of I in G(F) by \( \tilde{I} \) and set

\[ \Omega = (N_T(F) \cap \tilde{I})/\mathcal{T}(O_F) \subset \tilde{W} \]

The group \( \Omega \) is finite abelian and canonically isomorphic to \( X^*_{\mathfrak{T}}/\Lambda_r \), where \( \Lambda_r \) is the submodule of \( X^*_{\mathfrak{T}} \) generated by \( \Phi^\vee \) (cf. [5, §2]). Moreover, \( \Omega \) normalizes \( \tilde{W} \) and there is an isomorphism \( \tilde{W} \cong W_{\text{af}} \times \Omega \). If \( \rho \in \Omega \) and \( w \in W_{\text{af}} \) we will abbreviate the element \( pw\rho^{-1} \) of \( W_{\text{af}} \) by \( w^\rho \). If \( \Psi \) is a subset of \( W_{\text{af}} \), we will write \( \Psi^\rho \) for the set \( \{ w^\rho \mid w \in \Psi \} \).

For \( w \) in \( W_{\text{af}} \), let \( l(w) \) denote the length of \( w \) as an element of \( W_{\text{af}} \) with respect to \( \tilde{S} \). If \( w \in W_{\text{af}} \) and \( \rho \in \Omega \) then the length \( l(w^\rho) \) of \( w^\rho \) is defined to be \( l(w) \). If \( w' \in \tilde{W} \) then we can write \( w' = w_1 \cdots w_d \rho \) for some \( w_1, \ldots, w_d \) in \( \tilde{S} \) and \( \rho \) in \( \Omega \), and we say (by abuse of notation) that the expression \( w = w_1 \cdots w_d \rho \) is reduced if \( l(w) = d \). (Under this definition, the expression \( e = e \) is also to be considered reduced.)

Finally, if \( V \subset G(F) \), and \( \{ V_i \}_{i \in I} \) is a collection of subsets of \( G(F) \), then the notation \( V = \coprod_{i \in I} V_i \) will signify that \( V = \bigcup_{i \in I} V_i \) and that the \( V_i \) are pairwise disjoint.

\section{3. Structure Theory of p-adic Groups.}

We now state several results concerning the structure of \( G(F) \) which stem from the fact that the triple \((G(F), I, N_T(F))\) is a generalized Tits system (as defined in [5]). We also state a result of Iwahori and Matsumoto ([6, Cor. 2.7]) which gives a set of representatives for the left cosets of \( I \) contained in a double coset of \( I \).

We start with a summary of the structure theory pertaining to subgroups of \( G(F) \) containing \( I \). For any subgroup \( P \) of \( G(F) \) containing \( I \), we denote by \( W_P \) the subgroup \( (N_T(F) \cap P)/\mathcal{T}(O_F) \) of \( \tilde{W} \). In addition, we let \( S_P = W_P \cap \tilde{S} \) and \( \Omega_P = W_P \cap \Omega \).

\textbf{Proposition 3.1.} Let \( P, P_1, P_2 \subset G(F) \) be subgroups containing \( I \). Then

\[(i)\]

\[ P = IW_P I = \coprod_{w \in W_P} IwI. \]

Moreover, \( W_P \) is generated by \( S_P \) and \( S_P \) is stabilized under conjugation by elements of \( \Omega_P \). The map \( P \mapsto (S_P, \Omega_P) \) is a bijection from subgroups lying between \( I \) and \( G(F) \) to pairs \((S, \Omega')\) such that \( S \subset \tilde{S} \) and \( \Omega' \) is a subgroup of \( \Omega \) stabilizing \( S \).
Proposition 3.2. Let $W$ be compact if and only if $I = \{\nu \text{ and } \nu \text{ fixes } \tilde{s}\}$. This formula will follow easily from the above results. For each $\Phi$ number of irreducible root systems into which $W$ decomposes. Then

$$G(F) = \prod_{\sigma \in \Sigma_{P_1, P_2}} P_1 \sigma P_2.$$ 

In particular, if $\gamma, \gamma' \in \tilde{W}$ then $W_{P_1 \gamma W_{P_2}} = W_{P_1 \gamma' W_{P_2}}$ if and only if $P_1 \gamma P_2 = P_1 \gamma' P_2$.

Note that in Proposition 3.1 (i) if $P = K$ then $W_P$ is the Weyl group $W_0$, $S_P = \{w_\alpha \mid \alpha \in \Delta\}$, $\Omega_P = \{e\}$, and the decomposition given is essentially the standard Bruhat decomposition for the group $\mathcal{G}(k)$. When $P = G(F)$, Proposition 3.1 (i) yields the affine Bruhat decomposition of $G(F)$ with respect to $I$. Note that if $P$ is a subgroup of $G(F)$ containing $I$, then $P$ is compact if and only if $W_P$ is finite.

We also give the following summary of the structure of the coset space $I\backslash G(F)/I$ (see [6, Prop. 2.8, Theorem 3.3]).

**Proposition 3.3.** Suppose $\alpha \in \Delta$ and $i \in \{1, \ldots, m\}$, where $m$ is the number of irreducible root systems into which $\Phi$ decomposes. Then

(i) $I w_\alpha I = \prod_{\nu \in R} x_\alpha(\nu) w_\alpha I$

(ii) $I t(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} I = \prod_{\nu \in R} x_{-\alpha_{0,i}}(\pi \nu) t(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} I$.

We now develop notation which will allow us to give a formula for the representatives of the left cosets of $I$ in an arbitrary double coset in $I\backslash G(F)/I$. This formula will follow easily from the above results. For each $s$ in $\tilde{S}$, we fix a lifting $\tilde{s}$ of $s$ to $N_T(F)$. We define elements $g_s(\nu) \in G(F)$ for all $s$ in $\tilde{S}$ and $\nu$ in $R$ by setting

$$g_s(\nu) = \begin{cases} x_\alpha(\nu) \tilde{s} & \text{if } s = w_\alpha \text{ for some } \alpha \in \Delta \\ x_{-\alpha_{0,i}}(\pi \nu) \tilde{s} & \text{if } s = t(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} \text{ for some } i \text{ in } \{1, \ldots, m\}. \end{cases}$$
In this notation, Proposition 3.3 says that for each $s \in \tilde{S}$

$$I_{s} I = \prod_{\nu \in R} g_{s}(\nu) I.$$ 

For each $\rho$ in $\Omega$ we also choose some lifting $\bar{\rho}$ of $\rho$ to $N_{T}(F)$. 

For every $w$ in $\tilde{W}$ we fix an $(l(w) + 1)$-tuple $e(w) = (s_{w,1}, \ldots, s_{w,l(w)}, \rho_{w})$ in $\tilde{S}^{l(w)} \times \Omega$ such that $w = s_{w,1} \cdots s_{w,l(w)} \rho_{w}$. We define $g_{w} : R^{l(w)} \rightarrow G(F)$ to be the function which assigns to each $(\nu_{1}, \ldots, \nu_{l(w)})$ in $R^{l(w)}$ the element

$$g_{s_{w,1}}(\nu_{1}) \cdots g_{s_{w,l(w)}}(\nu_{l(w)}) \bar{\rho}_{w},$$ 

using the notation of the previous paragraph. Then we have the following fact concerning the coset space $I_\nu I / I$.

**Corollary 3.4.** Suppose that $w \in \tilde{W}$ and that $w = s_1 \cdots s_d \rho$ is a reduced expression (i.e., $d = l(w)$), where $s_1, \ldots, s_d \in \tilde{S}$ and $\rho \in \Omega$. Then the index $[I_\nu I : I]$ is $q^{l(w)}$. In fact,

$$I_\nu I = \prod_{\nu \in R} g_{s_1}(\nu_1) \cdots g_{s_d}(\nu_d) \bar{\rho} I = \prod_{\nu \in R^{l(w)}} g_{w}(\nu) I.$$

**Proof.** For $U \subset G(F)$ let $\text{char}_U : G(F) \rightarrow \{0, 1\}$ be the characteristic function of $U$. Since $\text{char}_{I_{w'} I} \mapsto [I_{w'} I : I]$ ($w'$ in $\tilde{W}$) defines a character of the Iwahori Hecke algebra of $G$ with respect to $I$ [6, §3], it follows from Propositions 3.2 and 3.3 that

$$[I_\nu I : I] = [I_{s_1} \cdots s_d \rho I : I] = [I_{s_1} I] \cdots [I_{s_d} I : I] = q^{l(w)}$$

(cf. [6, Prop. 3.2]). To complete the proof it suffices to show that the union of the $q^{l(w)}$ cosets given above is all of $I_\nu I$. This also follows from Propositions 3.2 and 3.3 since

$$I_\nu I = I_{s_1} I_{s_2} \cdots I_{s_d} \rho I = I_{s_1} I_{s_2} I_{s_d} \rho I$$

$$= \bigcup_{\nu_1 \in R} g_{s_1}(\nu_1) I_{s_2} I_{s_d} \rho I$$

$$= \bigcup_{\nu_1, \ldots, \nu_d \in R} g_{s_1}(\nu_1) g_{s_2}(\nu_2) \cdots g_{s_d}(\nu_d) \bar{\rho} I.$$

\[\square\]


Let $W$ be a subgroup of the affine Weyl group $W_{af}$ which is generated by the set $S = W \cap \tilde{S}$. Then $W$ is a Coxeter group. If $W'$ is a subgroup of $W$ generated by $S \cap W'$, then $W'$ is also a Coxeter group, and we will refer to such a subgroup $W'$ as a special subgroup of $W$. Define $[W/W']$ to be the set

$$\{w \in W | l(ww') = l(w) + l(w') \text{ for all } w' \in W'\}.$$
The elements of \([W/W']\) are the representatives for \(W/W'\) of minimal length \([4, \S 5.12]\). We will have need of the following fact concerning \([W/W']\).

**Lemma 4.1.** Suppose that \(W\) is a Coxeter subgroup of \(W_d\) with set of generators \(S = \tilde{S} \cap W\). Let \(W'\) be a special subgroup of \(W\). If \(\tau\) in \([W/W']\) and \(s\) in \(S\) satisfy \(l(s\tau) < l(\tau)\), then \(s\tau\) is in \([W/W']\).

**Proof.** Since \(l(s\tau) < l(\tau)\) we know that \(l(s\tau) = l(\tau) - 1\). For \(w'\) in \(W'\) we therefore have that

\[
l(\tau w') - 1 = l(\tau) + l(w') - 1 = l(s\tau) + l(w') \geq l(s\tau w') \geq l(\tau w') - 1.
\]

Thus, \(l(s\tau w') = l(s\tau) + l(w')\) and \(s\tau\) is in \([W/W']\). \(\square\)

For the remainder of the section, we fix two special subgroups \(W_1\) and \(W_2\) of \(W_d\) with \(S_1 = W_1 \cap \tilde{S}\) and \(S_2 = W_2 \cap \tilde{S}\) as their respective sets of involutive generators. For \(\sigma\) in \(\tilde{W}\) define \(W_1^\sigma W_2\) to be the stabilizer under left multiplication of the coset \(\sigma W_2\) in \(W_1\), namely, \(W_1 \cap \sigma W_2\sigma^{-1}\). Let \([W_1 \setminus \tilde{W} / W_2] \subset \tilde{W}\) be a set of representatives for \(W_1 \setminus \tilde{W} / W_2\) of minimal length, i.e., each \(\sigma\) in \([W_1 \setminus \tilde{W} / W_2]\) is to be an element of shortest length in \(W_1\sigma W_2\). Our first order of business will be to show that \(W_1^\sigma W_2\) is a special subgroup of \(W_1\) for any \(\sigma\) in \([W_1 \setminus \tilde{W} / W_2]\). Our goal will then be to show that

\[
l(\tau \sigma w) = l(\tau) + l(\sigma) + l(w)
\]

for all \(w\) in \(W_2\), \(\sigma\) in \([W_1 \setminus \tilde{W} / W_2]\), and \(\tau\) in \([W_1 W_1^\sigma W_2]\)\. This fact is a simple generalization of a result of Howlett concerning finite reflection groups (cf. \([2, \S 2.7]\)). It will prove very important in our analysis of the decomposition of double cosets in Chapter 5.

In order to show that \(W_1^\sigma W_2\) is a special subgroup of \(W_1\), we will need the following result, which is a simple generalization of the exchange condition (cf. \([1, \S 2.3A]\)), one of several equivalent properties that distinguish Coxeter groups from among the more general class of groups generated by finitely many involutions.

**Proposition 4.2.** Suppose \(w\) in \(\tilde{W}\) has reduced expression \(w = s_1 \cdots s_d \rho\) for some \(s_1, \ldots, s_d\) in \(S\) and \(\rho\) in \(\Omega\). Then for all \(s\) in \(\tilde{S}\) either

(i) \(l(sw) = l(w) + 1\), or

(ii) \(w = ss_1 \cdots \hat{s}_i \cdots s_d \rho\) for some \(i\) in \(\{1, \ldots, d\}\).

The following is an easy consequence of Proposition 4.2.

**Lemma 4.3.** Suppose that \(w\) and \(w'\) are elements of \(\tilde{W}\) such that \(l(ww') = l(w) + l(w')\). If \(s\) in \(S\) satisfies \(l(sw) = l(w) + 1\), then either

(i) \(l(sw w') = l(ww') + 1\), or

(ii) \(ww' = sw \hat{w}'\) for some \(\hat{w}'\) in \(\tilde{W}\) with \(l(\hat{w}') < l(w')\).
In fact, in the second case, if \( s_1, \ldots, s_d \in \tilde{S}, \rho \in \Omega \) and \( w' = s_1 \cdots s_d \rho' \) is a reduced expression, then
\[
\hat{w}' = s_1 \cdots \hat{s}_i \cdots s_d \rho'
\]
for some \( i \) in \( \{1, \ldots, d\} \). In particular, if \( w' \) is an element of a special subgroup of \( W_\Omega \) then \( \hat{w}' \) is also an element of that subgroup.

**Proof.** \( l(sw') \) equals either \( l(w') + 1 \) or \( l(w') - 1 \) so suppose the latter is true. This is clearly impossible if \( l(w') = 0 \). Moreover, if \( l(w) = 0 \) then Proposition 4.2 implies that (ii) holds. Thus we may assume that \( l(w), l(w') > 0 \).

Let \( w = t_1 \cdots t_r \rho \) and \( w' = s_1 \cdots s_d \rho' \) be reduced expressions for some reflections \( t_1, \ldots, t_r, s_1, \ldots, s_d \) in \( \tilde{S} \) and some \( \rho, \rho' \) in \( \Omega \). Then we have the reduced expression
\[
ww' = t_1 \cdots t_r s_1 \cdots s_d \rho'
\]
for some \( i \) in \( \{1, \ldots, r\} \) or
\[
ww' = t_1 \cdots t_r s_1 \cdots t_i s_i \cdots s_d \rho'
\]
for some \( i \) in \( \{1, \ldots, d\} \). If the former holds then
\[
sww' = t_1 \cdots t_i s_i \cdots s_d \rho'
\]
which implies that \( l(sw) < l(w) \), a contradiction. Therefore, we must have
\[
ww' = st_1 \cdots t_r s_1 \cdots s_i s_i \cdots s_d \rho'
\]
Setting \( \hat{w}' = s_1 \cdots \hat{s}_i \cdots s_d \rho' \), the first and second statements follow since \( l(\hat{w}') \leq l(w' - 1) \). The third statement holds since if \( w' \) is in a special subgroup of \( W_\Omega \) then each of the generators \( s_1 \cdots s_d \) lies in that subgroup.

We are now able to state and prove our first result on the additivity of lengths for certain elements of \( \hat{W} \), which we will need to show that \( W_1^{\sigma W_2} \) is special.

**Lemma 4.4.** For all \( \sigma \) in \( [W_1 \setminus \hat{W}/W_2] \), \( w \) in \( W_1 \) and \( w' \) in \( W_2 \), we have \( l(w \sigma) = l(w) + l(\sigma) \) and \( l(\sigma w') = l(w') + l(\sigma) \).

**Proof.** We will prove the first statement; the second statement follows from the first by taking inverses. Suppose \( w_0 \in W_1 \). The result is trivial if \( l(w_0) \leq 1 \) as \( \sigma \in [W_1 \setminus \hat{W}/W_2] \). So assume \( l(w_0) > 1 \) and suppose by induction that \( l(w \sigma) = l(w) + l(\sigma) \) for all \( w \) in \( W_1 \) with \( l(w) < l(w_0) \).
We may write \( w_0 \) as \( sw'_0 \) where \( s \in S_1 \) and \( w'_0 \) in \( W_1 \) has length \( l(w_0) - 1 \). Then, by induction,

\[
l(sw_0) = l(w'_0) + l(\sigma) = l(sw) + l(\sigma) = l(w_0) + l(\sigma) - 1.
\]

We must therefore show that \( l(sw_0) = l(w_0) - 1 \). Since \( s \in S_1 \), either this is true or \( l(sw_0) = l(w_0) + 1 \). But if the latter holds then, by applying Lemma 4.3 (with \( w = sw_0 \) and \( w' = \sigma \)) we obtain that \( sw_0 = w_0\hat{\sigma} \) for some \( \hat{\sigma} \) of length less than \( l(\sigma) \). This, however, contradicts the fact that \( \sigma \in [W_1 \setminus \hat{W}/W_2] \) is of minimal length in its double coset. \( \Box \)

**Proposition 4.5.** If \( \sigma \) is an element of \( [W_1 \setminus \hat{W}/W_2] \), then \( W_1^{\sigma W_2} \) is a special subgroup of \( W_1 \).

**Proof.** We must show that if \( s_1, \ldots, s_d \in S_1 \) and \( s_1 \cdots s_d \in W_1^{\sigma W_2} \) then \( s_1, \ldots, s_d \in W_1^{\sigma W_2} \). Fix \( w \) in \( W_1^{\sigma W_2} \). We may write \( w = sw' \) where \( s \in S_1 \) and \( w' \in W_1 \) has length \( l(w) - 1 \). By induction, it suffices to show that \( w' \) and hence \( s \) are in \( W_1^{\sigma W_2} \).

Since \( w = sw' \in W_1^{\sigma W_2} \), \( w'\sigma W_2 = s\sigma W_2 \), so that \( w'\sigma = s\sigma w_0 \) for some \( w_0 \) in \( W_2 \). Therefore, in order to show that \( w' \) is in the stabilizer \( W_1^{\sigma W_2} \) of \( \sigma W_2 \) in \( W_1 \), it suffices to prove that \( s\sigma w_0 = \sigma w_0 \) for some \( \hat{w}_0 \) in \( W_2 \).

Now \( l(s\sigma w_0) \) is either equal to \( l(\sigma w_0) - 1 \) or \( l(\sigma w_0) + 1 \). Suppose the former is true. Then, by Lemma 4.3 (with \( w = \sigma \) and \( w' = w_0 \) in \( W_2 \)) we have that \( \sigma w_0 = s\sigma w_0 \) for some \( \hat{w}_0 \) in \( W_2 \) so that \( s\sigma w_0 = \sigma \hat{w}_0 \). Thus it suffices to rule out the case \( l(s\sigma w_0) = l(\sigma w_0) + 1 \).

If this holds then

\[
(3) \quad l(w') = l(s\sigma w_0) = l(\sigma w_0) + 1.
\]

But by Lemma 4.4, \( l(w') = l(w') + l(\sigma) \) and \( l(s\sigma w_0) = l(\sigma) + l(w_0) \). Thus, by (3), \( l(w') = l(w_0) + 1 \) and hence \( l(w) = l(w_0) + 2 \). On the other hand, \( w\sigma = sw'\sigma = \sigma w_0 \) so, by Lemma 4.4 again, \( l(w) = l(w_0) \). This contradiction implies that \( l(\sigma w_0) \) cannot equal \( l(\sigma w_0) + 1 \) and the proof is complete. \( \Box \)

We now state and prove the main result on length additivity.

**Theorem 4.6.** Suppose \( \sigma \in [W_1 \setminus \hat{W}/W_2] \), \( \tau \in [W_1/W_1^{\sigma W_2}] \) and \( w \in W_2 \). Then

\[
l(\tau \sigma w) = l(\tau) + l(\sigma) + l(w).
\]

**Proof.** We will prove the theorem by induction on \( l(\tau) \). The theorem is true for \( \tau = e \) by Lemma 4.4. So suppose \( l(\tau) > 0 \) and that the statement is true for all \( \tau' \) in \( [W_1/W_1^{\sigma W_2}] \) with \( l(\tau') < l(\tau) \). Let \( s \in S_1 \) be such that \( l(st) = l(\tau) - 1 \). Note that the element \( st \) is in \( [W_1/W_1^{\sigma W_2}] \) by Lemma 4.1. By Lemma 4.3 applied to \( (st)(\sigma w) \), we obtain that either

(i) \( l(s(st)(\sigma w)) = l((st)(\sigma w)) + 1 \), or

(ii) \( st\sigma w = \tau \gamma \) for some \( \gamma \) in \( \hat{W} \) with \( l(\gamma) < l(\sigma w) \).
In the first case, we obtain that 
\[ l(\tau \sigma w) = l(s(s\tau)(\sigma w)) = l((s\tau)(\sigma w)) + 1 \] 
by (i) 
\[ = l(s\tau) + l(\sigma) + l(w) + 1 \] 
by induction 
\[ = l(\tau) + l(\sigma) + l(w) \] 
so it remains to rule out the second case. 

We can assume that \( l(\sigma w) > 0 \) since the statement of the theorem is trivially true if \( l(\sigma w) = l(\sigma) + l(w) = 0 \). Since \( l(\sigma w) = l(\sigma) + l(w) \), we have a reduced expression 
\[ \sigma w = \rho s_1 \cdots s_t \] 
where \( s_1, \ldots, s_t \in \tilde{S}, \rho \in \Omega, \) and for some \( r \) in \( \{1, \ldots, t\} \), \( \rho \) together with the first \( r \) involutions in the product yield a reduced expression for \( \sigma \) while the next \( t - r \) involutions yield a reduced expression for \( w \). According to Lemma 4.3, the element \( \gamma \) in case (ii) is obtained from \( \sigma w \) by deleting one of the involutions \( s_i \) in the above reduced word. If \( i \leq r \) then \( \gamma = \hat{s}\tau \sigma \) for some for some \( \hat{s} \) with \( l(\hat{s}) < l(\sigma) \) so \( s\hat{s}\sigma w = \tau \hat{s}w \). Then \( W_1\sigma W_2 = W_1\hat{s}\sigma W_2 \), a contradiction as \( \tau, w \) is the shortest element in \( \tilde{W} \sigma W_2 \) as it lies in \( [W_1/W_1] \). 

**Corollary 4.7.** If \( \sigma \in [W_1/W_1] \) then \( \sigma \) is the unique element of minimal length in \( W_1\sigma W_2 \).

**Proof.** Let \( w \) and \( w' \) be elements of \( W_1 \) and \( W_2 \) respectively such that \( l(\sigma w') = l(\sigma) \). Write \( w = \tau \gamma \), where \( \tau \in [W_1/W_1] \) and \( \gamma \in W_1 \). Then 
\[ \gamma w' = \tau \gamma w' = \tau \gamma w'' \] 
for some \( w'' \) in \( W_2 \) since \( \gamma w' = \sigma w_2 = \sigma W_2 \). But then 
\[ l(\gamma) = l(\gamma w') = l(\tau \gamma w'') = l(\tau) + l(\gamma) + l(w''), \] 
which implies that \( \tau, w'' = e \) so that \( \sigma w' = \sigma \). 

**5. Double Coset Decomposition.**

Throughout this section we will use the notation developed in Sections 3 and 4. Fix two subgroups \( P_1 \) and \( P_2 \) of \( G(F) \) containing the Iwahori subgroup \( I \). The goal of this section is to find representatives for the left cosets of \( P_2 \) in a double coset \( P_1 \sigma P_2 \). We also give a formula for the number of left cosets in a double coset when \( P_1 \) and \( P_2 \) are compact. Using the notation
of Section 3, let $W_i = W_{P_i}$, the set of elements of $\tilde{W}$ with representatives in $P_i$ ($i = 1, 2$). We suppose for now that both $W_1$ and $W_2$ are contained in $W_{af}$. We will deal with general subgroups containing $I$ later on in the section. Let $S_i = S_{P_i}$ be the canonical set of involutive generators of $W_i$.

By Proposition 3.1,

$$G(F) = \coprod_{\sigma \in [W_1 \setminus \tilde{W} / W_2]} P_1 \sigma P_2.$$ 

Let us therefore fix $\sigma$ in $[W_1 \setminus \tilde{W} / W_2]$ and consider the coset $P_1 \sigma P_2$. We first decompose $P_1 \sigma P_2$ into a disjoint union of double cosets in $I \setminus G(F) / P_2$.

**Lemma 5.1.** The double coset $P_1 \sigma P_2$ is the disjoint union of the cosets $I \tau \sigma P_2$ as $\tau$ ranges over $[W_1 / W_1 \sigma W_2]$.

**Proof.** By Proposition 3.1 (i), we have that

$$P_1 = \coprod_{w \in W_1} IwI.$$ 

It follows that

$$P_1 \sigma P_2 = \bigcup_{w \in W_1} IwI \sigma P_2.$$ 

We claim that this last expression is equal to $\bigcup_{w \in W_1} Iw \sigma P_2$.

By Equation (2) in Proposition 3.2, if we write $w'$ in $W_1$ as a reduced expression $w' = s_1 \cdots s_d$ where $s_1, \ldots, s_d \in S_1$, we have that

$$Iw' I \sigma I = Is_1 \cdots s_d I \sigma I = Is_1 I \cdots Is_d I \sigma I.$$ 

But by Proposition 3.2 (i) applied repeatedly

$$Iw' I \sigma I = Is_1 \cdots s_d I \sigma I \subset Is_1 I \cdots Is_d I \sigma I = Iw' I \sigma I$$

and

$$Iw' I \sigma I = Is_1 I \cdots Is_d I \sigma I \subset \bigcup_{w \in W_1} IwI \sigma I.$$ 

Therefore

$$\bigcup_{w \in W_1} IwI \sigma I = \bigcup_{w \in W_1} IwI \sigma I$$

and the claim follows.

We must now determine which of the terms in the above union are the same. To this end, we apply Proposition 3.1 (ii) to the subgroups $I$ and $P_2$ of $G(F)$. Since $W_I = \langle e \rangle$ and $W_{P_2} = W_2$, it follows that for any $w, w'$ in $W_1$, $Iw \sigma P_2 = Iw' \sigma P_2$ if and only if $w \equiv w' \sigma \pmod{W_2}$, i.e., if and only if $w \equiv w' \pmod{W_1^{W_2}}$. Therefore, to obtain a disjoint union of cosets $Iw \sigma P_2$ we take the union over $w$ in the set of representatives $[W_1 / W_1 \sigma W_2]$. □
We now decompose $P_1 \sigma P_2$ into left cosets of $P_2$ by expressing each double coset $I \tau \sigma P_2$ as a union of such left cosets. Recall that $R$ is a set of representatives in $O_F$ for the residue field $k$ which contains 0.

**Theorem 5.2.** Suppose that $P_1$ and $P_2$ are subgroups of $G(F)$ containing $I$. If $W_{P_1}, W_{P_2} \subset W_{af}$ and $\sigma \in W_{P_1} \setminus \tilde{W}/W_{P_2}$, then the double coset $P_1 \sigma P_2$ is equal to the disjoint union

$$P_1 \sigma P_2 = \bigcap_{\tau \in [W_{P_1}/W_{P_1}^\sigma W_{P_2}]} \bigcap_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)P_2.$$  

**Proof.** By Corollary 3.4, we have the decomposition

$$I \tau \sigma P_2 = I \tau \sigma IP_2 = \left( \bigcap_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)I \right)P_2 = \bigcup_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)P_2.$$  

Because of Lemma 5.1, the theorem will follow if we show that the cosets in the union (4) are distinct. So suppose that $g_{\tau \sigma}(\nu)P_2 = g_{\tau \sigma}(\nu')P_2$ for some $\tau \in [W_1/W_1^\sigma W_2]$ and $\nu, \nu' \in R^{(\tau \sigma)}$. We will show that $\nu = \nu'$. The main idea of the argument is to transfer the problem from $P_2$-cosets in $G(F)$ to $W_2$-cosets in $\tilde{W}$ and then to bring to bear our results on Coxeter groups from Section 4.

First we note that by Proposition 3.1 (i) and Corollary 3.4,

$$g_{\tau \sigma}(\nu)P_2 = \bigcap_{w \in W_2} g_{\tau \sigma}(\nu)IwI = \bigcap_{w \in W_2} \bigcap_{\nu'' \in R^{(w)}} g_{\tau \sigma}(\nu)g_w(\nu'')I$$

and similarly

$$g_{\tau \sigma}(\nu')P_2 = \bigcap_{w \in W_2} \bigcap_{\nu'' \in R^{(w)}} g_{\tau \sigma}(\nu')g_w(\nu'')I.$$  

Since these two $P$-cosets are equal, there must exist some $w$ in $W_2$ and $\nu''$ in $R^{(w)}$ such that $g_{\tau \sigma}(\nu')g_w(\nu'')I$ equals $g_{\tau \sigma}(\nu)g_w(0)I = g_{\tau \sigma}(\nu)I$. We will show that this equality can only hold if $w = e$. Then we will have that $g_{\tau \sigma}(\nu)I = g_{\tau \sigma}(\nu')I$, which immediately implies that $\nu = \nu'$ by Corollary 3.4.

So suppose that $g_{\tau \sigma}(\nu)I = g_{\tau \sigma}(\nu')g_w(\nu''I)$, where $w \in W_2$ and $\nu'' \in R^{(w)}$. By the definition of $g_{\tau \sigma}(\nu)$ and Proposition 3.2 (ii), we have that

$$g_{\tau \sigma}(\nu)I \subset I \tau \sigma I.$$  

Proof. By Corollary 3.4, we have the decomposition

$$I \tau \sigma P_2 = I \tau \sigma IP_2 = \left( \bigcap_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)I \right)P_2 = \bigcup_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)P_2.$$  

Because of Lemma 5.1, the theorem will follow if we show that the cosets in the union (4) are distinct. So suppose that $g_{\tau \sigma}(\nu)P_2 = g_{\tau \sigma}(\nu')P_2$ for some $\tau \in [W_1/W_1^\sigma W_2]$ and $\nu, \nu' \in R^{(\tau \sigma)}$. We will show that $\nu = \nu'$. The main idea of the argument is to transfer the problem from $P_2$-cosets in $G(F)$ to $W_2$-cosets in $\tilde{W}$ and then to bring to bear our results on Coxeter groups from Section 4.

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$$g_{\tau \sigma}(\nu)P_2 = \bigcap_{w \in W_2} g_{\tau \sigma}(\nu)IwI$$

and similarly

$$g_{\tau \sigma}(\nu')P_2 = \bigcap_{w \in W_2} \bigcap_{\nu'' \in R^{(w)}} g_{\tau \sigma}(\nu')g_w(\nu'')I.$$  

Since these two $P$-cosets are equal, there must exist some $w$ in $W_2$ and $\nu''$ in $R^{(w)}$ such that $g_{\tau \sigma}(\nu')g_w(\nu'')I$ equals $g_{\tau \sigma}(\nu)g_w(0)I = g_{\tau \sigma}(\nu)I$. We will show that this equality can only hold if $w = e$. Then we will have that $g_{\tau \sigma}(\nu)I = g_{\tau \sigma}(\nu')I$, which immediately implies that $\nu = \nu'$ by Corollary 3.4.

So suppose that $g_{\tau \sigma}(\nu)I = g_{\tau \sigma}(\nu')g_w(\nu''I)$, where $w \in W_2$ and $\nu'' \in R^{(w)}$. By the definition of $g_{\tau \sigma}(\nu)$ and Proposition 3.2 (ii), we have that

$$g_{\tau \sigma}(\nu)I \subset I \tau \sigma I.$$  

Proof. By Corollary 3.4, we have the decomposition

$$I \tau \sigma P_2 = I \tau \sigma IP_2 = \left( \bigcap_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)I \right)P_2 = \bigcup_{\nu \in R^{(\tau \sigma)}} g_{\tau \sigma}(\nu)P_2.$$  

Because of Lemma 5.1, the theorem will follow if we show that the cosets in the union (4) are distinct. So suppose that $g_{\tau \sigma}(\nu)P_2 = g_{\tau \sigma}(\nu')P_2$ for some $\tau \in [W_1/W_1^\sigma W_2]$ and $\nu, \nu' \in R^{(\tau \sigma)}$. We will show that $\nu = \nu'$. The main idea of the argument is to transfer the problem from $P_2$-cosets in $G(F)$ to $W_2$-cosets in $\tilde{W}$ and then to bring to bear our results on Coxeter groups from Section 4.

First we note that by Proposition 3.1 (i) and Corollary 3.4,

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and similarly

$$g_{\tau \sigma}(\nu')P_2 = \bigcap_{w \in W_2} \bigcap_{\nu'' \in R^{(w)}} g_{\tau \sigma}(\nu')g_w(\nu'')I.$$  

Since these two $P$-cosets are equal, there must exist some $w$ in $W_2$ and $\nu''$ in $R^{(w)}$ such that $g_{\tau \sigma}(\nu')g_w(\nu'')I$ equals $g_{\tau \sigma}(\nu)g_w(0)I = g_{\tau \sigma}(\nu)I$. We will show that this equality can only hold if $w = e$. Then we will have that $g_{\tau \sigma}(\nu)I = g_{\tau \sigma}(\nu')I$, which immediately implies that $\nu = \nu'$ by Corollary 3.4.

So suppose that $g_{\tau \sigma}(\nu)I = g_{\tau \sigma}(\nu')g_w(\nu''I)$, where $w \in W_2$ and $\nu'' \in R^{(w)}$. By the definition of $g_{\tau \sigma}(\nu)$ and Proposition 3.2 (ii), we have that

$$g_{\tau \sigma}(\nu)I \subset I \tau \sigma I.$$
Similarly, for each \( \nu'' \) in \( R^l(w) \),
\[
g_{\tau\sigma}(\nu')g_{w}(\nu'')I \subset I\tau\sigma wI.
\]

We are now able to use Section 4 since \( \sigma \in [W_1\backslash \tilde{W}/W_2] \), \( \tau \in [W_1/W_1^{\sigma W_2}] \) and \( w \in W_2 \). By Theorem 4.6, we conclude that \( l(\tau\sigma w) = l(\tau\sigma) + l(w) \). This implies via Equation (1) in Proposition 3.2 that \( I\tau\sigma w I = I\tau\sigma w I \).

Hence,
\[
(6) \quad g_{\tau\sigma}(\nu')g_{w}(\nu'')I \subset I\tau\sigma wI.
\]

Since the double cosets in (5) and (6) both contain the left \( I \)-coset \( g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')g_{w}(\nu'')I \), we conclude that they must be equal. But \( I\tau\sigma I = I\tau\sigma w I \) implies \( w = e \) since \( I \backslash G(F)/I \) is represented by \( \tilde{W} \) (Proposition 3.1).

\[ \square \]

**Remark 5.3.** If we take as a representative of \( P_1\sigma P_2 \) an element \( \sigma' \) of \( \sigma W_2 \) not equal to \( \sigma \) then
\[
P_1\sigma P_2 = \bigsqcup_{\tau \in [W_1/W_1^{\sigma W_2}]} \bigcup_{\nu \in R^l(\tau\sigma')} g_{\tau\sigma'}(\nu)P_2,
\]
but this union is no longer disjoint. For the number of cosets in the preceding union is larger than than the number of cosets given in Theorem 5.2 as \( l(\tau\sigma') = l(\tau) + l(\sigma') \geq l(\tau) + l(\tau) = l(\tau\sigma) \) for any \( \tau \in [W_1/W_1^{\sigma W_2}] \) by Theorem 4.6.

We now give a decomposition of double cosets into left cosets for arbitrary subgroups of \( G(F) \) containing \( I \). Adjusting our notation slightly, we let \( P'_1 \) and \( P'_2 \) be two such subgroups. Set \( W'_1 = W'_{P'_1} \) and \( W'_2 = W'_{P'_2} \), \( \tilde{S} I \). As before let \( S_i = W_i \cap \tilde{S} \). Recall that \( S_i \) is stabilized by \( \Omega P_i' \) under conjugation (Proposition 3.1). For \( i = 1, 2 \), let \( P_i \subset P'_i \) be the subgroup \( IW_i I \). Then
\[
P'_i = \bigsqcup_{\rho \in \Omega P_i'} P_i \rho = \bigsqcup_{\rho \in \Omega P_i'} \rho P_i
\]
by Proposition 3.1.

Let \([W_1\backslash \tilde{W}/W_2'] = [W'_{P'_1} \backslash \tilde{W}/W'_{P'_2}]\) be a set of representatives of smallest possible length for the double cosets in \( W'_1 \backslash \tilde{W}/W_2' \). (Note that this set of representatives is no longer unique in contrast to \([W_1\backslash \tilde{W}/W_2]\).)

**Lemma 5.4.** The set \([W_1\backslash \tilde{W}/W_2]\) is equal to the set of all products of the form \( \rho_1\sigma \rho_2 \) as \( \rho_i \) ranges over \( \Omega P_i' \) \((i = 1, 2)\) and \( \sigma \) ranges over \([W_1\backslash \tilde{W}/W_2']\).

In particular, if \( \rho_i \in \Omega P_i' \) \((i = 1, 2)\) and \( \sigma \in [W_1\backslash \tilde{W}/W_2] \), then \( \rho_1\sigma \rho_2 \) is also in \([W_1\backslash \tilde{W}/W_2]\).
Proof. The elements $\rho_1 \sigma \rho_2$ clearly exhaust $W_1 \pipe \tilde{W} / W_2$ since
\[
\tilde{W} = \bigsqcup_{\sigma \in [W_1 \pipe \tilde{W} / W_2]} W_1' \sigma W_2' = \bigsqcup_{\sigma \in [W_1' \pipe \tilde{W} / W_2']} \bigcup_{\rho_1 \in \Omega_{P_1'}} \bigcup_{\rho_2 \in \Omega_{P_2'}} W_1 \rho_1 \sigma \rho_2 W_2.
\]
Moreover, $\rho_1 \sigma \rho_2$ is of minimal length in $W_1 \rho_1 \sigma \rho_2 W_2$ since $\sigma$ is of minimal length in $W_1' \sigma W_2'$. The second statement follows trivially from the first. \[\Box\]

Let us now fix an element $\sigma$ in $[W_1' \pipe \tilde{W} / W_2']$ and consider the double coset $P_1' \sigma P_2'$. Denote by $\Omega_{P_1}'$ the stabilizer $\{ \rho \in \Omega_{P_1} | \rho \sigma = \sigma \}$ of $\sigma$ in $\Omega_{P_1}$ and let $\Omega_{P_1', P_2'} = \Omega_{P_1} \cap \Omega_{P_2'}$. Also, let $J_{P_1', P_2'}^\sigma$ be a set of representatives for $\Omega_{P_1} / \Omega_{P_1', P_2'}$.

Lemma 5.5. If $\rho \in \Omega_{P_1}'$, then
\[
\rho [W_1 / W_1^{\sigma W_2}] \rho^{-1} = [W_1 / W_1^{\sigma W_2}].
\]
(Recall that $W_2^\rho$ denotes the set $\{ w^\rho \mid w \in W_2 \}$.) In particular, if $\rho \in \Omega_{P_1', P_2'}^\sigma$, then $\rho [W_1 / W_1^{\sigma W_2}] \rho^{-1} = [W_1 / W_1^{\sigma W_2}]$.

Proof. The first statement is true since $\Omega_{P_1}'$ stabilizes $W_1$ and since
\[
\rho W_1^{\sigma W_2} \rho^{-1} = \rho (W_1 \cap \sigma W_2 \sigma^{-1}) \rho^{-1} = W_1 \cap \sigma \rho W_2^\rho (\sigma^\rho)^{-1} = W_1^{\sigma W_2}.
\]
The second statement follows from the first and the fact that if $\rho \in \Omega_{P_1', P_2'}^\sigma$, then $\rho$ stabilizes $\sigma$ and $W_2$. \[\Box\]

Following our procedure in the beginning of the section, we first decompose $P_1' \sigma P_2'$ into a union of cosets in $I \Gamma(G(F)) / P_2'$.

Lemma 5.6. The double coset $P_1' \sigma P_2'$ is the disjoint union
\[
\bigsqcup_{\gamma \in J_{P_1', P_2'}^\sigma} \bigsqcup_{\tau \in [W_1 / W_1^{\sigma W_2}]} I \gamma \tau \sigma P_2'.
\]

Proof. Since $\sigma \in [W_1 \pipe \tilde{W} / W_2]$ and $W_{P_i} = W_i \in W_{af} (i = 1, 2)$, we have that
\[
P_1 \sigma P_2 = \bigsqcup_{\tau \in [W_1 / W_1^{\sigma W_2}]} I \tau \sigma P_2,
\]
by Lemma 5.1. Therefore,
\[
P_1' \sigma P_2' = \bigcup_{\rho_1 \in \Omega_{P_1'}} \rho_1 P_1 \sigma P_2 \rho_2
\]
\[
= \bigcup_{\rho_1 \in \Omega_{P_1'}} \bigcup_{\tau \in [W_1 / W_1^{\sigma W_2}]} \rho_1 I \tau \sigma P_2 \rho_2
\]
Using the definitions of $\Omega_{P_1}$ and $J_{P_1,P_2)'$, we can eliminate some of the repetition of cosets in the union (7). For (7) is equal to

\begin{align*}
&= \bigcup_{\rho_1 \in \Omega_{P_1}} \bigcup_{\tau \in [W_1/W_1^{\gamma W_2}]} I_{\rho_1 \sigma \rho_2 P_2} \\
&= \bigcup_{\rho_1 \in \Omega_{P_1}} \bigcup_{\tau \in [W_1/W_1^{\gamma W_2}]} I_{\tau^\rho \sigma \rho_1 \rho_2 P_2} \\
&= \bigcup_{\rho_1 \in \Omega_{P_1}} \bigcup_{\tau \in [W_1/W_1^{\gamma W_2}]} I_{\tau^\rho \sigma \rho_1 P_2'}.
\end{align*}

By Lemma 5.5 and the fact that $\Omega_{P_1,P_2}'$ stabilizes $\sigma$ and $W_2$, this last expression is equal to

\begin{align*}
&= \bigcup_{\gamma \in J_{P_1,P_2}^\sigma} \bigcup_{\rho \in \Omega_{P_1,P_2}'} \bigcup_{\tau \in [W_1/W_1^{\gamma W_2}]} I_{\tau^\rho \sigma \gamma P_2'}.
\end{align*}

As in the proof of Lemma 5.1, we now determine whether the terms in the union (10) are distinct. Let $\gamma, \gamma' \in J_{P_1,P_2}^\sigma$, and let $\tau \in [W_1/W_1^{\gamma W_2}]$, $\tau' \in [W_1/W_1^{\gamma' W_2}]$. We will show that the terms $I_{\tau^\sigma \gamma P_2'}$ and $I_{\tau'^\sigma \gamma' P_2'}$ in (10) are equal only if $\gamma = \gamma'$ and $\tau = \tau'$. By Proposition 3.1 applied to $I$ and $P_2'$, we have that the two double cosets are equal if and only if

\begin{align*}
\tau \sigma^\gamma W_2' &= \tau \sigma^\gamma' W_2'.
\end{align*}

Since $W_2' = \prod_{\gamma \in \Omega_{P_2}'} \gamma W_2$, this is equivalent to the condition that

\begin{align*}
\tau \sigma^\gamma \rho W_2 &= \tau' \sigma^\gamma' \gamma W_2
\end{align*}

for some $\rho \in \Omega_{P_2}'$. But $\tau, \tau' \in W_1$ so this means that

\begin{align*}
W_1 \sigma^\gamma \gamma \rho W_2 = W_1 \sigma^\gamma' \gamma W_2.
\end{align*}

Now both $\sigma^\gamma \gamma \rho = \gamma \sigma \rho$ and $\sigma^\gamma' \gamma = \gamma' \sigma$ are in $[W_1 \setminus \tilde{W} / W_2]$ by Lemma 5.4 since $\sigma$ is an element of $[W_1 \setminus \tilde{W} / W_2]$. As a consequence of the uniqueness of the coset representatives of shortest length of $W_1 \setminus \tilde{W} / W_2$ (Corollary 4.7), Equation (12) can hold only if $\sigma^\gamma \gamma \rho = \sigma^\gamma' \gamma$. Since $\tilde{W} = W_2 \times \Omega$, it follows
easily from this that $\gamma \rho = \gamma'$ and hence that $\sigma^\gamma = \sigma'$. Thus $\gamma^{-1} \gamma' = \rho \in \Omega_{P_2}'$ and $\gamma^{-1} \gamma' \equiv \Omega_{P_1}'$ so that $\gamma \equiv \gamma'$ (mod $\Omega_{P_1}' \Omega_{P_2}'$). Since $\gamma, \gamma' \in J_{P_1}^\sigma$, we must have $\gamma = \gamma'$ and $\rho = e$.

Since $\gamma = \gamma'$, it follows from (11) that $\tau \gamma \sigma W_2^{-1} \gamma^{-1} = \tau' \gamma \sigma W_2^{-1} \gamma^{-1}$. Thus $\tau$ and $\tau'$ are elements of $[W_1/W_1^\sigma \gamma W_2^\sigma]$ which lie in the same left coset of $W_1^\sigma W_2^\sigma$. This forces $\tau = \tau'$. Therefore, we see that the union in (10) is disjoint; i.e.,

$$P_1' \sigma P_2' = \bigcup_{\gamma \in J_{P_1}^\sigma, \rho \in [W_1/W_1^\sigma W_2^\sigma]} \bigcup_{\tau \in \gamma W_1^\sigma \gamma W_2^\sigma} \bigcup_{\nu \in \Omega_{P_2}'} \bigcup_{\tau \gamma \sigma P_2'} g_{\gamma \tau \sigma} \nu P_2'.
$$

(Note that $l(\gamma \tau \sigma) = l(\tau \sigma)$ since $\gamma \in \Omega$.) In order to prove that these $P_2'$-cosets are distinct, we note that the argument in Theorem 5.2 will still work if $P_2$ is replaced by $P_2'$ and $W_2$ by $W_2'$ provided that $l(\gamma \tau \sigma w') = l(\gamma \tau \sigma) + l(w')$ for all $\gamma \in J_{P_1}^\sigma, \tau \in [W_1/W_1^\sigma W_2^\sigma]$ and $w'$ in $W$. This condition is easily proved to hold—write $w'$ as $w'' \rho$ where $w'' \in W_2$ and $\rho \in \Omega_{P_2}'$. Then, since $\sigma \in [W_1/W_1^\sigma W_2^\sigma]$ and $\tau \in [W_1/W_1^\sigma W_2^\sigma]$, we have by Theorem 4.6 that

$$l(\gamma \tau \sigma w') = l(\gamma \tau \sigma w'' \rho) = l(\tau \sigma w'') = l(\tau \sigma) + l(w'') = l(\gamma \tau \sigma) + l(w'' \rho) = l(\gamma \tau \sigma) + l(w').$$

Thus the $P_2'$-cosets appearing in the union are distinct and we have proved the following theorem.

**Theorem 5.7.** Let $P_1', P_2'$ be subgroups of $G(F)$ containing the Iwahori subgroup $I$. Let $\sigma$ be an element of $[W_1/W_1^\sigma] \\bigcap \\sigma W_2^\sigma$. Then

$$P_1' \sigma P_2' = \bigcup_{\gamma \in J_{P_1}^\sigma, \rho \in [W_1/W_1^\sigma W_2^\sigma]} \bigcup_{\tau \in \gamma W_1^\sigma \gamma W_2^\sigma} \bigcup_{\nu \in \Omega_{P_2}'} \bigcup_{\tau \gamma \sigma P_2'} g_{\gamma \tau \sigma} \nu P_2'.
$$

The number of terms in the disjoint union in Theorem 5.7 is calculated in the following corollary when $W_1$ and $W_2$ are finite (i.e., $P_1$ and $P_2$ are compact).
Corollary 5.8. Let \( \sigma \) be an element of \([W_1 \backslash \hat{W} / W_2']\). Suppose that \( W_1 \) is finite. Then the number of left cosets of \( P'_1 \) in \( P'_1 \sigma P'_2 \) is

\[
\#(J_{P'_1, P'_2}^{\sigma}) \cdot q^{l(\sigma)} \sum_{\tau \in [W_1 / W_1^{\sigma} W_2']} q^{l(\tau)} = \prod_{\gamma \in W_1 \sigma W_2'} \sum_{\gamma \in W_1 \sigma W_2} q^{l(\gamma)} / \sum_{w \in W_2} q^{l(w)}.
\]

Proof. That the number of cosets is equal to the first expression is immediate from the theorem and the definition of \( g_{\gamma \tau \sigma}^{\gamma \tau \sigma} \). To prove that the two expressions are equal, first note that \( \#(J_{P'_1, P'_2}^{\sigma}) = [\Omega_{P'_1} : \Omega_{P'_1, P'_2}^{\sigma}] \). Also, observe that \( \{ \tau \sigma \mid \tau \in [W_1 / W_1^{\sigma} W_2] \} \) is a set of coset representatives in \( W_1 \sigma W_2 \) for \( W_1 \sigma W_2 / W_2 \) so

\[
\sum_{\gamma \in W_1 \sigma W_2} q^{l(\gamma)} = \sum_{\tau \in [W_1 / W_1^{\sigma} W_2']} \sum_{w \in W_2} q^{l(\tau w)}.
\]

On the other hand, since \( l(\tau \sigma w) = l(\tau) + l(\gamma) + l(w) \), we obtain

\[
\left( q^{l(\sigma)} \sum_{\tau \in [W_1 / W_1^{\sigma} W_2']} q^{l(\tau)} \right) \left( \sum_{w \in W_2} q^{l(w)} \right)
= \sum_{\tau \in [W_1 / W_1^{\sigma} W_2']} \sum_{w \in W_2} q^{l(\tau) + l(\gamma) + l(w)}
= \sum_{\tau \in [W_1 / W_1^{\sigma} W_2']} \sum_{w \in W_2} q^{l(\tau \sigma w)}.
\]

\[\square\]

Example 5.9. Let \( G \) be the group \( \text{PGSp}_4 \). The rank of \( G \) is 2, and we choose a set of simple roots consisting of a short root \( \alpha_1 \) and a long root \( \alpha_2 \). We also let \( \alpha_0 \) be the highest root corresponding to this basis. The Weyl group \( W_0 \) is generated by the reflections \( w_1 = w_{\alpha_1} \) and \( w_2 = w_{\alpha_2} \), while \( W_{af} \) is generated by these reflections and \( w_0 = \varepsilon(\alpha_0)w_{\alpha_0} \). The group \( \Omega \) is cyclic of order 2. We denote the generator of \( \Omega \) by \( \rho \) and note that it interchanges \( w_0 \) and \( w_2 \) but fixes \( w_1 \).

We consider the Coxeter groups \( W_1 = \langle w_1 \rangle \) and \( W_2 = \langle w_0, w_2 \rangle \) inside \( W_{af} \). The reflections \( w_0 \) and \( w_2 \) commute so that \( W_2 \) consists of the four elements \( e, w_0, w_2 \) and \( w_0 w_2 \). Let \( W'_i = W_i \Omega \) and let \( P'_i \) be the compact subgroup \( IW'_i I \) of \( G(F) \) \((i = 1, 2)\). We will use the results of this to decompose double cosets in \( P'_1 \backslash G(F) / P'_2 \) and \( P'_1 / G(F) / P'_2 \).

Let \( \sigma = w_1 w_0 \). It is easily shown that \( \sigma \in [W'_1 \backslash \hat{W} / W'_2] \). Consider the double coset \( P'_1 \sigma P'_2 \). The group \( W'_2 = W_1 \cap \sigma W_2 \sigma^{-1} \) is trivial so that \([W'_1 / W'_2] = [W_1 / W'_2] = \{ e, w_1 \} \). The stabilizer \( \Omega_{P'_1, P'_2}^{\sigma} \) is trivial as well.
which means that \( J_{F_1}^{\sigma} P_1^{\prime} = \{ e, \rho \} \). By Lemma 5.6, \( P_1^{\prime} \sigma P_2^{\prime} \) is the disjoint union of the four double cosets

\[
\begin{align*}
I w_0 w_1 P_1^{\prime} \\
I w_1 w_0 w_1 P_1^{\prime} \\
I \rho w_0 w_1 P_1^{\prime} = I w_2 w_1 P_2^{\prime} \\
I \rho w_1 w_0 w_1 P_1^{\prime} = I w_1 w_2 w_1 P_2^{\prime}.
\end{align*}
\]

Thus by Theorem 5.7, \( P_1^{\prime} \sigma P_2^{\prime} \) is the disjoint union of the \( 2q^2 + 2q^3 \) cosets

\[
\begin{align*}
g_{w_0 w_1} (\nu_1) P_2^{\prime} & \quad (\nu_1 \in R^2) \\
g_{w_1 w_0} (\nu_2) P_2^{\prime} & \quad (\nu_2 \in R^3) \\
g_{w_2 w_1} (\nu_3) P_2^{\prime} & \quad (\nu_3 \in R^2) \\
g_{w_1 w_2 w_1} (\nu_4) P_2^{\prime} & \quad (\nu_4 \in R^3).
\end{align*}
\]

Now let \( \sigma = w_1 w_0 w_1 \in [W_2^{\prime} \setminus \tilde{W} / W_2^{\prime}] \) and consider the double coset \( P_2^{\prime} \sigma P_2^{\prime} \). Here the group \( W_2^{\prime} = W_2 \cap \sigma W_2 \sigma^{-1} = \langle w_0 \rangle \) since we have the braid relation \( w_1 w_0 w_1 w_0 w_1 w_0 w_1 w_1 = w_0 \). Thus \([W_2^{\prime} W_2^{\prime}] = \{ e, w_2 \}\). Also, \( J_{P_2^{\prime}}^{\sigma} = \{ e, \rho \} \). By Lemma 5.6, \( P_2^{\prime} \sigma P_2^{\prime} \) is the disjoint union of the four double cosets

\[
\begin{align*}
I w_1 w_0 w_1 P_2^{\prime} \\
I w_2 w_1 w_0 w_1 P_2^{\prime} \\
I \rho w_0 w_1 P_2^{\prime} = I w_1 w_2 w_1 P_2^{\prime} \\
I \rho w_1 w_0 w_1 P_2^{\prime} = I w_0 w_1 w_2 w_1 P_2^{\prime}.
\end{align*}
\]

It follows from Theorem 5.7 that \( P_2^{\prime} \sigma P_2^{\prime} \) is the disjoint union of the \( 2q^3 + 2q^4 \) cosets

\[
\begin{align*}
g_{w_1 w_0 w_1} (\nu_1) P_2^{\prime} & \quad (\nu_1 \in R^3) \\
g_{w_2 w_1 w_0 w_1} (\nu_2) P_2^{\prime} & \quad (\nu_2 \in R^4) \\
g_{w_1 w_2 w_1} (\nu_3) P_2^{\prime} & \quad (\nu_3 \in R^3) \\
g_{w_1 w_2 w_1} (\nu_4) P_2^{\prime} & \quad (\nu_4 \in R^4).
\end{align*}
\]

6. Degrees of Spherical Hecke Operators.

We now consider the special case when the subgroups \( P_1 \) and \( P_2 \) containing \( I \) are both equal to the hyperspecial subgroup \( K \). Using the results of the previous section, we can determine the left cosets of \( K \) occurring in a given double coset \( K \sigma K \). The number of such cosets is by definition the degree of the function \( \text{ch}_{K \sigma K} \) in the spherical Hecke algebra of \( K \). With the coset decomposition of the preceding section, we will derive the formula for the degree given in [5, §5].

Since \( W_K = \tilde{W} \), the elements of \( \tilde{W} \) index the double coset space \( K \setminus G(F) / K \). Therefore, as a result of the fact that \( \tilde{W} \cong X_\ast(T) \times \tilde{W} \), we see that \( K \setminus G(F) / K \) can be identified with the set of orbits of \( \tilde{W} \) on \( X_\ast(T) \). (This is the Cartan decomposition—cf. [6, 2.5].) A set of representatives
in $X_+(T)$ for these orbits is the set $X_+$ of dominant co-characters, i.e., co-characters $\lambda$ such that $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha$ in $\Phi^+$. The isomorphism $\tilde{W} \cong X_+(T) \rtimes W_0$ can be chosen to satisfy the condition that the image in $\tilde{W}$ of the element $\lambda(\pi)$ of $T(F)$ is $t(\lambda)$. Thus we have a one-to-one correspondence

$$X_+ \rightarrow K\backslash G(F)/K$$

given by the map $\lambda \mapsto Kt(\lambda)K = K\lambda(\pi)K$.

Now given $\lambda \in X_+(T)$, there is a unique element $w_\lambda$ in $W_0$ such that $l(t(\lambda)w_\lambda) = \min_{w \in W_0} l(t(\lambda)w)$ (see [6, §1.9]). It can be shown (see [7, §2.2]) that the representative of the double coset $W_0t(\lambda)W_0$ of shortest length is $t(\lambda)w_\lambda$. Thus $[W_0/\tilde{W}W_0]$ is equal to the set $\{t(\lambda)w_\lambda \mid \lambda \in X_+\}$.

Since $W_0$ acts on $X_+(T)$, we may consider the stabilizer $W_0^\lambda$ in $W_0$ of an element $\lambda$ of $X_+(T)$. It is easily seen that $W_0^\lambda = W_0^{t(\lambda)w_\lambda}W_0$ so that $[W_0/\tilde{W}W_0^\lambda] = [W_0/W_0^{t(\lambda)W_0}]$.

Fix $\lambda$ in $X_+$. Applying Theorem 5.2 with $P_1 = P_2 = K$ and $\sigma = t(\lambda)w_\lambda$ we arrive at the following results.

**Proposition 6.1.** The double coset

$$K\lambda(\pi)K = Kt(\lambda)K$$

is equal to the disjoint union

$$\bigsqcup_{\tau \in [W_0/\tilde{W}W_0^\lambda]} \prod_{\nu \in R^{t(\lambda)w_\lambda}} g_{\tau t(\lambda)w_\lambda}(\nu)K.$$

**Corollary 6.2.** The number of left (or right) cosets of $K$ in $K\lambda(\pi)K$ is

$$q^{l(t(\lambda)w_\lambda)} \sum_{\tau \in [W_0/\tilde{W}W_0^\lambda]} q^{l(\tau)} = q^{\min_{w \in W_0} l(t(\lambda)w)} \sum_{\tau \in [W_0/\tilde{W}W_0^\lambda]} q^{l(\tau)}$$

$$= \sum_{\gamma \in W_0t(\lambda)W_0} q^{l(\gamma)} / \sum_{w \in W_0} q^{l(w)}.$$

**Proof.** This follows from Corollary 5.8 and the definition of $w_\lambda$. \qed

**Example 6.3.** In [8], for compact forms $G$ of PGSp$_4$ over $\mathbb{Q}$ corresponding to various quaternion algebras, we determined the actions of spherical Hecke algebras on $G(\mathbb{Q}_p)$ for certain primes $p$ at which $G$ is split. This necessitated computations of the above kind for the simple adjoint group $G = \text{PGSp}_4$ of type $C_2 = B_2$. Examples of such computations follow.

We carry over the notation from the example in the previous section. Since PGSp$_4$ is of adjoint type (i.e., $X_+(T)$ is the lattice dual to the lattice generated by $\Phi$), the fundamental co-weights $\tilde{\omega}_1, \tilde{\omega}_2$ in $\Lambda$ (which satisfy $\langle \alpha_i, \tilde{\omega}_j \rangle = \delta_{ij}$) are in $X_+(T)$. Furthermore, any $\lambda$ in $X_+$ is of the form $a_1\tilde{\omega}_1 + a_2\tilde{\omega}_2$ for some non-negative integers $a_1$ and $a_2$. 
If $\lambda$ in $X_+$ is 0 then $W_0^\lambda$ is clearly all of $W_0$ and $[W_0/W_0^\lambda]$ is trivial. On the other hand, if $\lambda = a_1\tilde{\omega}_1 + a_2\tilde{\omega}_2$ for positive $a_1, a_2$, then $W_0^\lambda = \langle e \rangle$ and $[W_0/W_0^\lambda] = W_0$. Now suppose that $\lambda$ is the long co-weight $\tilde{\omega}_1$. We have $W_0^\omega = \langle w_2 \rangle$ and $[W_0/W_0^\omega]$ is the set $\{ e, w_1, w_2w_1, w_1w_2w_1 \}$. Also, $t(\tilde{\omega}_1)$ can be shown to have the reduced expression $w_0w_1w_2w_1$. A reduced expression for $t(\tilde{\omega}_1)w_{\omega_1}$ can be obtained from $t(\tilde{\omega}_1)$ by dropping the last three involutions in this expression (which are contained in $W_0$) to yield $w_0$.

We therefore have that $K\tilde{\omega}_1(\pi)K$ is the disjoint union of the double cosets

$$
Iw_0K \\
Iw_1w_0K \\
Iw_2w_1w_0K \\
Iw_1w_2w_1w_0K.
$$

It follows that $K\tilde{\omega}_1(\pi)K$ is the union of the $q + q^2 + q^3 + q^4 = q \cdot \frac{q^4 - 1}{q - 1}$ cosets

$$
g_{w_0}(v_1)K \quad (v_1 \in R) \\
g_{w_1w_0}(v_2)K \quad (v_2 \in R^2) \\
g_{w_2w_1w_0}(v_3)K \quad (v_3 \in R^3) \\
g_{w_1w_2w_1w_0}(v_4)K \quad (v_4 \in R^4).
$$

For the short co-weight $\tilde{\omega}_2$, we have $W_0^{\tilde{\omega}_2} = \langle w_1 \rangle$ and $[W_0/W_0^{\tilde{\omega}_2}]$ is the set $\{ e, w_2, w_1w_2, w_2w_1w_2 \}$. A reduced expression for $t(\tilde{\omega}_2)$ is $w_0w_1w_0\rho = \rho w_2w_1w_2$, and for $t(\tilde{\omega}_2)w_{\omega_2}$ is $p$. $K\tilde{\omega}_2K$ is the union of the cosets

$$
I\rho K \\
Iw_2w_1\rho K \\
Iw_1w_2w_1\rho K \\
Iw_2w_1w_2w_1\rho K.
$$

Therefore, $K\tilde{\omega}_2K$ is the union of the $1 + q + q^3 + q^4 = \frac{q^4 - 1}{q - 1}$ left cosets

$$
\rho K \\
g_{w_2}(v_1)K \quad (v_1 \in R) \\
g_{w_1w_2}(v_2)K \quad (v_2 \in R^2) \\
g_{w_2w_1w_2}(v_3)K \quad (v_3 \in R^3).
$$

**Remark 6.4.** If $G$ is the group over $k$ obtained by taking the special fiber of $G$, denote by $B_\lambda$ the standard parabolic subgroup of $G$ corresponding to the co-character $\lambda$ and the choice of positive roots $\Phi^+$. In [8], it is shown using the formula in Corollary 6.2 that the index $[K\lambda(\pi)K : K]$ is equal to

$$
\# \left( \frac{G}{B_\lambda} \right)(k) \cdot q^{\dim(\frac{G}{B_\lambda})(k)}.
$$
where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\langle , \rangle$ is the standard pairing between $X_*(T)$ and $X^*(T)$.

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