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Let G be a split group over a locally compact field F with non-trivial discrete valuation. Employing the structure theory of such groups and the theory of Coxeter groups, we obtain a general formula for the decomposition of double cosets $P_1\sigma P_2$ of subgroups $P_1, P_2 \subset G(F)$ containing an Iwahori subgroup into left cosets of P_2. When P_1 and P_2 are the same hyperspecial subgroup, we use this result to derive a formula of Iwahori for the degrees of elements of the spherical Hecke algebra.

1. Introduction.

Let G be a semisimple algebraic group which is split over a locally compact field F with non-trivial discrete valuation and let I be an Iwahori subgroup of $G(F)$. In [**6**], Iwahori and Matsumoto show that the double cosets in $I\backslash G(F)/I$ are indexed by the elements of the extended affine Weyl group $\tilde{W}$ of G, and for w in $\tilde{W}$, they exhibit an explicit set of representatives for the left cosets of I in IwI/I. They also show that the number of single cosets of I in IwI/I is $q^{l(w)}$, where q is the cardinality of the residue field of F and l is the standard combinatorial length function on $\tilde{W}$.

Let $\mathcal{O}_F$ be the ring of integers of F and let $K \subset G(F)$ be a hyperspecial subgroup, that is, a subgroup isomorphic to $\underline{G}(\mathcal{O}_F)$, where $\underline{G}$ is a smooth group scheme over $\mathcal{O}_F$ with general fiber G. In [**5**], Iwahori gives a formula for the number of left cosets of K contained in a double coset in $K\backslash G(F)/K$ (i.e., the degree of the characteristic function of this double coset as an element of the spherical Hecke algebra), implicitly making use of the decomposition in [**6**] and the fact that K contains an Iwahori subgroup. Suppose that π is a uniformizer of F. The double cosets in $K\backslash G(F)/K$ are indexed by the dominant co-characters of a maximal torus of G via the bijection $\lambda \mapsto K\lambda(\pi)K$. Let W_0 be the Weyl group of G and W_0^λ the stabilizer of λ in W_0. Then there exists a unique set $[W_0/W_0^\lambda] \subset W_0$ of representatives of cosets of W_0^λ of minimal length. Iwahori states that the index

$$[K\lambda(\pi)K : K] = q^{l(\sigma_\lambda)} \sum_{\tau \in [W_0/W_0^\lambda]} q^{l(\tau)},$$

where σ_λ is a certain element of $\tilde{W}$ associated with the co-character λ (see Section 6).

In this paper, we give a summary of the above results and generalize them, finding representatives for the left cosets of P_2 in $P_1 \sigma P_2$, where P_1 and P_2 are subgroups of $G(F)$ containing an Iwahori subgroup I. To accomplish this, we make use of the structure theory of groups over fields with discrete valuation (as summarized in Section 3). Because of the close connection between subgroups of $G(F)$ of the above type and subgroups of the affine Weyl group (as given in [**6**]), the proof of the coset decomposition formula necessitates the use of the theory of Coxeter groups to prove certain results about the additivity of lengths of elements of $\tilde{W}$ (Section 4). In Section 5, we find the coset representatives mentioned above and give a formula for their number when the groups P_1 and P_2 are compact. In addition, we give several examples of this coset decomposition and an explanation of how our general results imply Iwahori's in the case $P_1 = P_2 = K$ (Section 6).

This information on the decomposition of double cosets is useful in computing the action of Hecke operators on spaces of modular forms as defined in [**3**]. In fact, the results of this paper were used in [**8**] when G is a compact form of G_2 or PGSp_4 over $\mathbb{Q}$ and $P_1 = P_2 = K$ to compute the action on certain spaces of forms of the spherical Hecke algebra of functions on $G(\mathbb{Q}_p)$ bi-invariant by the subgroup K for several primes p at which G is split. The action of a function in this algebra is given by integrating it against a form f with respect to a Haar measure on $G(\mathbb{Q}_p)$. This integral turns out to be a finite sum. Indeed, when the Hecke operator is the characteristic function of a double coset $K\sigma K$ of K, the integral is simply the sum over the right translations of f by a set of representatives of the left cosets of K inside $K\sigma K$. Since the Hecke algebra is generated by such characteristic functions, the decomposition of double cosets is therefore fundamental to the explicit determination of the actions these algebras.

The idea for this paper arose from my collaboration with David Pollack while at Harvard University. I am grateful to him for his many valuable insights into the structure of $\mathfrak{p}$-adic groups. I would also like to thank Benedict Gross for introducing me to the theory of algebraic modular forms.

2. Notation.

In the following, we will denote by G a connected semisimple algebraic group that is split over a locally compact field F with non-trivial discrete valuation. Let $\mathcal{O}_F$ be the ring of integers of F and let $\mathfrak{p}$ be the prime ideal. We choose a uniformizing parameter π in $\mathfrak{p}$, and denote by k the residue field $\mathcal{O}_F/\mathfrak{p}$. Let q be the (finite) cardinality of k and let $R \subset \mathcal{O}_F$ be a set of representatives for k containing 0. The group G is the general fiber of a Chevalley group scheme $\underline{G}$ over $\mathcal{O}_F$ whose special fiber is semisimple. We let

$K = \underline{G}(\mathcal{O}_F) \subset \underline{G}(F) = G(F)$ be the set of integral points of G. K is then a hyperspecial maximal compact subgroup of $G(F)$ (cf. [9, 3.8.1, 3.8.2]).

Let $\underline{T} \subset \underline{G}$ be a split maximal torus scheme, and let T be its general fiber. We define N_T to be the normalizer of T in G. Denote by $X^*(T)$ the character module $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ of T and by $X_*(T)$ the co-character module $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, T)$ of T. Let $\Phi \subset X^*(T)$ be the set of roots of T, $\Phi^+ \subset \Phi$ a subset of positive roots, and $\Delta \subset \Phi^+$ the corresponding set of simple roots. Also, let $\Phi^\vee \subset X_*(T)$ be the coroots of T and $\alpha \mapsto \alpha^\vee$ the standard bijection between Φ and $\Phi^\vee$. For each $\alpha \in \Phi$ let $\underline{U}_\alpha$ be the one-dimensional unipotent subgroup scheme of $\underline{G}$ corresponding to α. Denote the general fiber of $\underline{U}_\alpha$ by U_α. We choose for each α an isomorphism

$$x_\alpha : \mathbb{G}_{\mathrm{a}} \longrightarrow \underline{U}_\alpha.$$

When considered as a map $F \longrightarrow \underline{U}_\alpha(F)$, x_α restricts to an isomorphism of $\mathcal{O}_F$ with $\underline{U}_\alpha(\mathcal{O}_F) = U_\alpha(F) \cap K$.

We will denote by W_0 the Weyl group $N_T/T = (N_T(F) \cap K)/\underline{T}(\mathcal{O}_F)$ of G and by $\tilde{W}$ the extended affine Weyl group $N_T(F)/\underline{T}(\mathcal{O}_F)$. Then W_0 and $\tilde{W}$ act as groups of affine transformations on the space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The stabilizer in $\tilde{W}$ of $0 \in X_*(T) \otimes \mathbb{R}$ is W_0, and there is an isomorphism $\tilde{W} \cong X_*(T) \rtimes W_0$, where $X_*(T)$ is embedded in $\tilde{W}$ as the group of elements acting as translations on $X_*(T) \otimes \mathbb{R}$. We denote by e the identity element of $\tilde{W}$ and by $t(\lambda)$ the element of $\tilde{W}$ corresponding to λ in $X_*(T)$. In this notation, if $w \in W_0$ and $\lambda \in X_*(T)$ then

$$wt(\lambda)w^{-1} = t(w\lambda).$$

Denote by w_α the reflection in W_0 through the vanishing hyperplane in $X_*(T) \otimes \mathbb{R}$ of the root α. The Weyl group W_0 is a Coxeter group with $S_0 = \{w_\alpha | \alpha \in \Delta\}$ as its set of involutive generators. Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ be the decomposition of Φ into irreducible root systems. (Each Φ_i corresponds to the root system of an almost simple factor of G.) Also, let $\Delta_i = \Delta \cap \Phi_i$, and put $l_i = \#\Delta_i$. Then $l_1 + \cdots + l_m = l$, the dimension of T, i.e., the rank of G. Let $\alpha_{0,i}$ be the highest root of Φ_i with respect to the basis of simple roots Δ_i. Then the Coxeter group with set of involutive generators

$$\tilde{S} = S_0 \cup \left\{ t(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} | 1 \leq i \leq m \right\}$$

is isomorphic to the affine Weyl group W_{af} of Φ ([6, Prop. 1.1]). Via this isomorphism, we will view W_{af} as a subgroup of $\tilde{W}$.

Let I be the Iwahori subgroup of $G(F)$ generated by $\underline{T}(\mathcal{O}_F)$, the subgroups $x_\alpha(\mathcal{O}_F) = \underline{U}_\alpha(\mathcal{O}_F)$ for all α in Φ^+, and the subgroups $x_\alpha(\mathfrak{p})$ for all α in Φ^-. If we denote by $\overline{G}$ the semisimple algebraic group over k obtained by taking the special fiber of $\underline{G}$ then (as in [9, §3.5]) we have a surjective reduction mod $\mathfrak{p}$ map $K \longrightarrow \overline{G}(k)$, and I is the inverse image in K under this map of the Borel subgroup of $\overline{G}(k)$ corresponding to Φ^+. The triple

$(G(F), I, N_T(F))$ is a generalized Tits system in the sense of [5], a fact which will be used in Section 3 to study the structure of $G(F)$.

Denote the normalizer of I in $G(F)$ by $\tilde{I}$ and set

$$\Omega = (N_T(F) \cap \tilde{I})/\underline{T}(\mathcal{O}_F) \subset \tilde{W}.$$

The group Ω is finite abelian and canonically isomorphic to $X_*(T)/\Lambda_r$, where Λ_r is the submodule of $X_*(T)$ generated by $\Phi^\vee$ (cf. [5, §2]). Moreover, Ω normalizes W_{af} and there is an isomorphism $\tilde{W} \cong W_{\mathrm{af}} \rtimes \Omega$. If $\rho \in \Omega$ and $w \in W_{\mathrm{af}}$ we will abbreviate the element $\rho w \rho^{-1}$ of W_{af} by w^ρ. If Ψ is a subset of W_{af}, we will write Ψ^ρ for the set $\{w^\rho \mid w \in \Psi\}$.

For w in W_{af}, let $l(w)$ denote the length of w as an element of W_{af} with respect to $\tilde{S}$. If $w \in W_{\mathrm{af}}$ and $\rho \in \Omega$ then the length $l(w\rho)$ of $w\rho$ is defined to be $l(w)$. If $w' \in \tilde{W}$ then we can write $w' = w_1 \cdots w_d \rho$ for some $w_1, \ldots, w_d$ in $\tilde{S}$ and ρ in Ω, and we say (by abuse of notation) that the expression $w = w_1 \cdots w_d \rho$ is *reduced* if $l(w) = d$. (Under this definition, the expression $e = e$ is also to be considered reduced.)

Finally, if $V \subset G(F)$, and $\{V_i\}_{i \in I}$ is a collection of subsets of $G(F)$, then the notation $V = \coprod_{i \in I} V_i$ will signify that $V = \bigcup_{i \in I} V_i$ and that the V_i are pairwise disjoint.

3. Structure Theory of $\mathfrak{p}$-adic Groups.

We now state several results concerning the structure of $G(F)$ which stem from the fact that the triple $(G(F), I, N_T(F))$ is a generalized Tits system (as defined in [5]). We also state a result of Iwahori and Matsumoto ([6, Cor. 2.7]) which gives a set of representatives for the left cosets of I contained in a double coset of I.

We start with a summary of the structure theory pertaining to subgroups of $G(F)$ containing I. For any subgroup P of $G(F)$ containing I, we denote by W_P the subgroup $(N_T(F) \cap P)/\underline{T}(\mathcal{O}_F)$ of $\tilde{W}$. In addition, we let $S_P = W_P \cap \tilde{S}$ and $\Omega_P = W_P \cap \Omega$.

Proposition 3.1. *Let $P, P_1, P_2 \subset G(F)$ be subgroups containing I. Then*

(i)

$$P = IW_PI = \coprod_{w \in \tilde{W}_P} IwI.$$

Moreover, W_P is generated by S_P and S_P is stabilized under conjugation by elements of Ω_P. The map $P \mapsto (S_P, \Omega_P)$ is a bijection from subgroups lying between I and $G(F)$ to pairs (S, Ω') such that $S \subset \tilde{S}$ and Ω' is a subgroup of Ω stabilizing S.

(ii) *If* $\Sigma_{P_1,P_2} \subset \tilde{W}$ *is a set representatives for the double coset space* $W_{P_1}\backslash\tilde{W}/W_{P_2}$ *then*

$$G(F) = \coprod_{\sigma\in\Sigma_{P_1,P_2}} P_1\sigma P_2.$$

In particular, if $\gamma, \gamma' \in \tilde{W}$ *then* $W_{P_1}\gamma W_{P_2} = W_{P_1}\gamma' W_{P_2}$ *if and only if* $P_1\gamma P_2 = P_1\gamma' P_2$.

Note that in Proposition 3.1 (i) if $P = K$ then W_P is the Weyl group W_0, $S_P = \{w_\alpha \mid \alpha \in \Delta\}$, $\Omega_P = \{e\}$, and the decomposition given is essentially the standard Bruhat decomposition for the group $\overline{G}(k)$. When $P = G(F)$, Proposition 3.1 (i) yields the affine Bruhat decomposition of $G(F)$ with respect to I. Note that if P is a subgroup of $G(F)$ containing I, then P is compact if and only if W_P is finite.

We also give the following summary of the structure of the coset space $I\backslash G(F)/I$ (see [**6**, Prop. 2.8, Theorem 3.3]).

Proposition 3.2. *Let* w, w' *be elements of* $\tilde{W}$. *Then*

(i) *For all* $s \in \tilde{S}$
  a) $IsIwI = IswI$ *if* $l(sw) > l(w)$,
  b) $IsIwI = IswI \cup IwI$ *if* $l(sw) < l(w)$.

(ii) *If* $l(ww') = l(w) + l(w')$ *then*

$$IwIw'I = Iww'I. \tag{1}$$

In particular, if $s_1, \dots, s_d \in \tilde{S}$, $\rho \in \Omega$ *and* $w = s_1 \cdots s_d\rho$ *is a reduced expression, then*

$$Is_1I \cdots Is_dI\rho I = IwI. \tag{2}$$

In addition to the information resulting from the fact that the triple $(G(F), I, N_T(F))$ is a generalized Tits system, we will also need the following statement (cf. [**6**, Cor. 2.7]) concerning representatives for the left cosets of I inside the double cosets of I corresponding to the elements of $\tilde{S}$.

Proposition 3.3. *Suppose* $\alpha \in \Delta$ *and* $i \in \{1, \dots, m\}$, *where* m *is the number of irreducible root systems into which* Φ *decomposes. Then*

(i) $Iw_\alpha I = \coprod_{\nu\in R} x_\alpha(\nu)w_\alpha I$
(ii) $It(\alpha_{0,i}^\vee)w_{\alpha_{0,i}}I = \coprod_{\nu\in R} x_{-\alpha_{0,i}}(\pi\nu)t(\alpha_{0,i}^\vee)w_{\alpha_{0,i}}I$.

We now develop notation which will allow us to give a formula for the representatives of the left cosets of I in an arbitrary double coset in $I\backslash G(F)/I$. This formula will follow easily from the above results. For each s in $\tilde{S}$, we fix a lifting $\bar{s}$ of s to $N_T(F)$. We define elements $g_s(\nu) \in G(F)$ for all s in $\tilde{S}$ and ν in R by setting

$$g_s(\nu) = \begin{cases} x_\alpha(\nu)\bar{s} & \text{if } s = w_\alpha \text{ for some } \alpha \text{ in } \Delta \\ x_{-\alpha_{0,i}}(\pi\nu)\bar{s} & \text{if } s = t(\alpha_{0,i}^\vee)w_{\alpha_{0,i}} \text{ for some } i \text{ in } \{1, \dots, \text{m}\}. \end{cases}$$

In this notation, Proposition 3.3 says that for each $s \in \tilde{S}$

$$IsI = \coprod_{\nu \in R} g_s(\nu) I.$$

For each ρ in Ω we also choose some lifting $\bar{\rho}$ of ρ to $N_T(F)$.

For every w in $\tilde{W}$ we fix an $(l(w)+1)$-tuple $e(w) = (s_{w,1}, \dots, s_{w,l(w)}, \rho_w)$ in $\tilde{S}^{l(w)} \times \Omega$ such that $w = s_{w,1} \cdots s_{w,l(w)} \rho_w$. We define $g_w : R^{l(w)} \longrightarrow G(F)$ to be the function which assigns to each $(\nu_1, \dots, \nu_{l(w)})$ in $R^{l(w)}$ the element

$$g_{s_{w,1}}(\nu_1) \cdots g_{s_{w,l(w)}}(\nu_{l(w)}) \bar{\rho}_w,$$

using the notation of the previous paragraph. Then we have the following fact concerning the coset space IwI/I.

Corollary 3.4. *Suppose that $w \in \tilde{W}$ and that $w = s_1 \cdots s_d \rho$ is a reduced expression (i.e., $d = l(w)$), where $s_1, \dots, s_d \in \tilde{S}$ and $\rho \in \Omega$. Then the index $[IwI : I]$ is $q^{l(w)}$. In fact,*

$$IwI = \coprod_{\nu_i \in R} g_{s_1}(\nu_1) \cdots g_{s_d}(\nu_d) \bar{\rho} I = \coprod_{\nu \in R^{l(w)}} g_w(\nu) I.$$

Proof. For $U \subset G(F)$ let $\mathrm{char}_U : G(F) \to \{0,1\}$ be the characteristic function of U. Since $\mathrm{char}_{Iw'I} \mapsto [Iw'I : I]$ (w' in $\tilde{W}$) defines a character of the Iwahori Hecke algebra of G with respect to I [**6**, §3], it follows from Propositions 3.2 and 3.3 that

$$[IwI : I] = [Is_1 \cdots s_d \rho I : I] = [Is_1 I] \cdots [Is_d I : I] = q^{l(w)}$$

(cf. [**6**, Prop. 3.2]). To complete the proof it suffices to show that the union of the $q^{l(w)}$ cosets given above is all of IwI. This also follows from Propositions 3.2 and 3.3 since

$$\begin{aligned} IwI = Is_1 s_2 \cdots s_d \rho I &= Is_1 I s_2 I \cdots I s_d I \rho I \\ &= \bigcup_{\nu_1 \in R} g_{s_1}(\nu_1) I s_2 I \cdots I s_d I \rho I \\ &= \bigcup_{\nu_1, \dots, \nu_d \in R} g_{s_1}(\nu_1) g_{s_2}(\nu_2) \cdots g_{s_d}(\nu_d) \bar{\rho} I. \end{aligned}$$

□

4. Coxeter Subgroups of the Extended Affine Weyl Group.

Let W be a subgroup of the affine Weyl group W_{af} which is generated by the set $S = W \cap \tilde{S}$. Then W is a Coxeter group. If W' is a subgroup of W generated by $S \cap W'$, then W' is also a Coxeter group, and we will refer to such a subgroup W' as a *special subgroup of* W. Define $[W/W']$ to be the set

$$\{w \in W | l(ww') = l(w) + l(w') \text{ for all } w' \in W'\}.$$

The elements of $[W/W']$ are the representatives for W/W' of minimal length [**4**, §5.12]. We will have need of the following fact concerning $[W/W']$.

Lemma 4.1. *Suppose that W is a Coxeter subgroup of W_{af} with set of generators $S = \tilde{S} \cap W$. Let W' be a special subgroup of W. If τ in $[W/W']$ and s in S satisfy $l(s\tau) < l(\tau)$, then $s\tau$ is in $[W/W']$.*

Proof. Since $l(s\tau) < l(\tau)$ we know that $l(s\tau) = l(\tau) - 1$. For w' in W' we therefore have that

$$l(\tau w') - 1 = l(\tau) + l(w') - 1 = l(s\tau) + l(w') \geq l(s\tau w') \geq l(\tau w') - 1.$$

Thus, $l(s\tau w') = l(s\tau) + l(w')$ and $s\tau$ is in $[W/W']$. □

For the remainder of the section, we fix two special subgroups W_1 and W_2 of W_{af} with $S_1 = W_1 \cap \tilde{S}$ and $S_2 = W_2 \cap \tilde{S}$ as their respective sets of involutive generators. For σ in $\tilde{W}$ define $W_1^{\sigma W_2}$ to be the stabilizer under left multiplication of the coset σW_2 in W_1, namely, $W_1 \cap \sigma W_2 \sigma^{-1}$. Let $[W_1\backslash\tilde{W}/W_2] \subset \tilde{W}$ be a set of representatives for $W_1\backslash\tilde{W}/W_2$ of minimal length, i.e., each σ in $[W_1\backslash\tilde{W}/W_2]$ is to be an element of shortest length in $W_1\sigma W_2$. Our first order of business will be to show that $W_1^{\sigma W_2}$ is a special subgroup of W_1 for any σ in $[W_1\backslash\tilde{W}/W_2]$. Our goal will then be to show that

$$l(\tau\sigma w) = l(\tau) + l(\sigma) + l(w)$$

for all w in W_2, σ in $[W_1\backslash\tilde{W}/W_2]$, and τ in $[W_1/W_1^{\sigma W_2}]$. This fact is a simple generalization of a result of Howlett concerning finite reflection groups (cf. [**2**, §2.7]). It will prove very important in our analysis of the decomposition of double cosets in Chapter 5.

In order to show that $W_1^{\sigma W_2}$ is a special subgroup of W_1, we will need the following result, which is a simple generalization of the exchange condition (cf. [**1**, §2.3A]), one of several equivalent properties that distinguish Coxeter groups from among the more general class of groups generated by finitely many involutions.

Proposition 4.2. *Suppose w in $\tilde{W}$ has reduced expression $w = s_1 \cdots s_d\rho$ for some $s_1, \ldots, s_d$ in $\tilde{S}$ and ρ in Ω. Then for all s in $\tilde{S}$ either*

(i) *$l(sw) = l(w) + 1$, or*
(ii) *$w = ss_1 \cdots \widehat{s_i} \cdots s_d\rho$ for some i in $\{1, \ldots, d\}$.*

The following is an easy consequence of Proposition 4.2.

Lemma 4.3. *Suppose that w and w' are elements of $\tilde{W}$ such that $l(ww') = l(w) + l(w')$. If s in S satisfies $l(sw) = l(w) + 1$, then either*

(i) *$l(sww') = l(ww') + 1$, or*
(ii) *$ww' = sw\hat{w}'$ for some $\hat{w}'$ in $\tilde{W}$ with $l(\hat{w}') < l(w')$.*

In fact, in the second case, if $s_1, \dots, s_d \in \tilde{S}$, $\rho \in \Omega$ *and* $w' = s_1 \cdots s_d \rho'$ *is a reduced expression, then*

$$\hat{w}' = s_1 \cdots \widehat{s_i} \cdots s_d \rho'$$

for some i *in* $\{1, \dots, d\}$*. In particular, if* w' *is an element of a special subgroup of* W_{af} *then* $\hat{w}'$ *is also an element of that subgroup.*

Proof. $l(sww')$ equals either $l(ww') + 1$ or $l(ww') - 1$ so suppose the latter is true. This is clearly impossible if $l(w') = 0$. Moreover, if $l(w) = 0$ then Proposition 4.2 implies that (ii) holds. Thus we may assume that $l(w), l(w') > 0$.

Let $w = t_1 \cdots t_r \rho$ and $w' = s_1 \cdots s_d \rho'$ be reduced expressions for some reflections $t_1, \dots, t_r, s_1, \dots, s_d$ in $\tilde{S}$ and some ρ, ρ' in Ω. Then we have the reduced expression

$$ww' = t_1 \cdots t_r \rho s_1 \cdots s_d \rho' = t_1 \cdots t_r s_1^\rho \cdots s_d^\rho \rho \rho'.$$

Since $s_i^\rho \in \tilde{S}$, it follows from Proposition 4.2, that either

$$ww' = st_1 \cdots \widehat{t_i} \cdots t_r s_1^\rho \cdots s_d^\rho \rho \rho'$$

for some i in $\{1, \dots, r\}$ or

$$ww' = st_1 \cdots t_r s_1^\rho \cdots \widehat{s_i^\rho} l \cdots s_d^\rho \rho \rho'$$

for some i in $\{1, \dots, d\}$. If the former holds then

$$sww' = t_1 \cdots \widehat{t_i} \cdots t_r \rho w'$$

which implies that $l(sw) < l(w)$, a contradiction. Therefore, we must have

$$ww' = st_1 \cdots t_r s_1^\rho \cdots \widehat{s_i^\rho} \cdots s_d^\rho \rho \rho' = sws_1 \cdots \widehat{s_i} \cdots s_d \rho'.$$

Setting $\hat{w}' = s_1 \cdots \widehat{s_i} \cdots s_d \rho'$, the first and second statements follow since $l(\hat{w}') \leq l(w') - 1$. The third statement holds since if w' is in a special subgroup of W_{af}, then each of the generators $s_1 \dots s_d$ lies in that subgroup. □

We are now able to state and prove our first result on the additivity of lengths for certain elements of $\tilde{W}$, which we will need to show that $W_1^{\sigma W_2}$ is special.

Lemma 4.4. *For all* σ *in* $[W_1 \backslash \tilde{W} / W_2]$*,* w *in* W_1 *and* w' *in* W_2*, we have* $l(w\sigma) = l(w) + l(\sigma)$ *and* $l(\sigma w') = l(w') + l(\sigma)$*.*

Proof. We will prove the first statement; the second statement follows from the first by taking inverses. Suppose $w_0 \in W_1$. The result is trivial if $l(w_0) \leq 1$ as $\sigma \in [W_1 \backslash \tilde{W} / W_2]$. So assume $l(w_0) > 1$ and suppose by induction that $l(w\sigma) = l(w) + l(\sigma)$ for all w in W_1 with $l(w) < l(w_0)$.

We may write w_0 as sw_0' where $s \in S_1$ and w_0' in W_1 has length $l(w_0)-1$. Then, by induction,

$$l(sw_0\sigma) = l(w_0'\sigma) = l(w_0') + l(\sigma) = l(sw_0) + l(\sigma) = l(w_0) + l(\sigma) - 1.$$

We must therefore show that $l(sw_0\sigma) = l(w_0\sigma) - 1$. Since $s \in S_1$, either this is true or $l(sw_0\sigma) = l(w_0\sigma) + 1$. But if the latter holds then, by applying Lemma 4.3 (with $w = sw_0$ and $w' = \sigma$) we obtain that $sw_0\sigma = w_0\hat{\sigma}$ for some $\hat{\sigma}$ of length less than $l(\sigma)$. This, however, contradicts the fact that $\sigma \in [W_1\backslash\tilde{W}/W_2]$ is of minimal length in its double coset. □

Proposition 4.5. *If σ is an element of $[W_1\backslash\tilde{W}/W_2]$, then $W_1^{\sigma W_2}$ is a special subgroup of W_1.*

Proof. We must show that if $s_1, \ldots, s_d \in S_1$ and $s_1 \cdots s_d \in W_1^{\sigma W_2}$ then $s_1, \ldots, s_d \in W_1^{\sigma W_2}$. Fix w in $W_1^{\sigma W_2}$. We may write w as sw' where $s \in S_1$ and $w' \in W_1$ has length $l(w)-1$. By induction, it suffices to show that w' and hence s are in $W_1^{\sigma W_2}$.

Since $w = sw' \in W_1^{\sigma W_2}$, $w'\sigma W_2 = s\sigma W_2$, so that $w'\sigma = s\sigma w_0$ for some w_0 in W_2. Therefore, in order to show that w' is in the stabilizer $W_1^{\sigma W_2}$ of σW_2 in W_1, it suffices to prove that $s\sigma w_0 = \sigma\hat{w}_0$ for some $\hat{w}_0$ in W_2.

Now $l(s\sigma w_0)$ is either equal to $l(\sigma w_0) - 1$ or $l(\sigma w_0) + 1$. Suppose the former is true. Then, by Lemma 4.3 (with $w = \sigma$ and $w' = w_0$ in W_2) we have that $\sigma w_0 = s\sigma\hat{w}_0$ for some $\hat{w}_0$ in W_2 so that $s\sigma w_0 = \sigma\hat{w}_0$. Thus it suffices to rule out the case $l(s\sigma w_0) = l(\sigma w_0) + 1$.

If this holds then

$$l(w'\sigma) = l(s\sigma w_0) = l(\sigma w_0) + 1. \tag{3}$$

But by Lemma 4.4, $l(w'\sigma) = l(w') + l(\sigma)$ and $l(\sigma w_0) = l(\sigma) + l(w_0)$. Thus, by (3), $l(w') = l(w_0) + 1$ and hence $l(w) = l(w_0) + 2$. On the other hand, $w\sigma = sw'\sigma = \sigma w_0$ so, by Lemma 4.4 again, $l(w) = l(w_0)$. This contradiction implies that $l(\sigma w_0)$ cannot equal $l(\sigma w_0) + 1$ and the proof is complete. □

We now state and prove the main result on length additivity.

Theorem 4.6. *Suppose $\sigma \in [W_1\backslash\tilde{W}/W_2]$, $\tau \in [W_1/W_1^{\sigma W_2}]$ and $w \in W_2$. Then*

$$l(\tau\sigma w) = l(\tau) + l(\sigma) + l(w).$$

Proof. We will prove the theorem by induction on $l(\tau)$. The theorem is true for $\tau = e$ by Lemma 4.4. So suppose $l(\tau) > 0$ and that the statement is true for all τ' in $[W_1/W_1^{\sigma W_2}]$ with $l(\tau') < l(\tau)$. Let s in S_1 be such that $l(s\tau) = l(\tau) - 1$. Note that the element $s\tau$ is in $[W_1/W_1^{\sigma W_2}]$ by Lemma 4.1. By Lemma 4.3 applied to $(s\tau)(\sigma w)$, we obtain that either

(i) $l(s(s\tau)(\sigma w)) = l((s\tau)(\sigma w)) + 1$, or

(ii) $s\tau\sigma w = \tau\gamma$ for some γ in $\tilde{W}$ with $l(\gamma) < l(\sigma w)$.

In the first case, we obtain that

$$\begin{aligned} l(\tau\sigma w) &= l(s(s\tau)(\sigma w)) \\ &= l((s\tau)(\sigma w)) + 1 && \text{by (i)} \\ &= l((s\tau)\sigma w) + 1 \\ &= l(s\tau) + l(\sigma) + l(w) + 1 && \text{by induction} \\ &= l(\tau) + l(\sigma) + l(w) \end{aligned}$$

so it remains to rule out the second case.

We can assume that $l(\sigma w) > 0$ since the statement of the theorem is trivially true if $l(\sigma w) = l(\sigma) + l(w) = 0$. Since $l(\sigma w) = l(\sigma) + l(w)$, we have a reduced expression

$$\sigma w = \rho s_1 \cdots s_t$$

where $s_1, \ldots, s_t \in \tilde{S}$, $\rho \in \Omega$, and for some r in $\{1, \ldots, t\}$, ρ together with the first r involutions in the product yield a reduced expression for σ while the next $t - r$ involutions yield a reduced expression for w. According to Lemma 4.3, the element γ in case (ii) is obtained from σw by deleting one of the involutions s_i in the above reduced word. If $i \le r$ then $\gamma = \hat{\sigma} w$ for some for some $\hat{\sigma}$ with $l(\hat{\sigma}) < l(\sigma)$ so $s\tau\sigma w = \tau\hat{\sigma} w$. Then $W_1 \sigma W_2 = W_1 \hat{\sigma} W_2$, a contradiction as $\sigma \in [W_1 \backslash \tilde{W} / W_2]$ is an element of minimal length in $W_1 \sigma W_2$. On the other hand, if $i > r$ then $\gamma = \sigma \hat{w}$ for some $\hat{w}$ in W_2 with $l(\hat{w}) < l(w)$ so $s\tau\sigma w = \tau\sigma\hat{w}$. But then $\tau\sigma W_2 = s\tau\sigma W_2$ so $\tau \equiv s\tau \pmod{W_1^{\sigma W_2}}$, a contradiction since τ is the shortest element in $\tau W_1^{\sigma W_2}$ as it lies in $[W_1 / W_1^{\sigma W_2}]$. □

Corollary 4.7. *If $\sigma \in [W_1 \backslash \tilde{W} / W_2]$ then σ is the unique element of minimal length in $W_1 \sigma W_2$.*

Proof. Let w and w' be elements of W_1 and W_2 respectively such that $l(w\sigma w') = l(\sigma)$. Write $w = \tau\gamma$, where $\tau \in [W_1 / W_1^{\sigma W_2}]$ and $\gamma \in W_1^{\sigma W_2}$. Then

$$w\sigma w' = \tau\gamma\sigma w' = \tau\sigma w''$$

for some w'' in W_2 since $\gamma\sigma W_2 = \sigma W_2$. But then

$$l(\sigma) = l(w\sigma w') = l(\tau\sigma w'') = l(\tau) + l(\sigma) + l(w''),$$

which implies that $\tau, w'' = e$ so that $w\sigma w' = \sigma$. □

5. Double Coset Decomposition.

Throughout this section we will use the notation developed in Sections 3 and 4. Fix two subgroups P_1 and P_2 of $G(F)$ containing the Iwahori subgroup I. The goal of this section is to find representatives for the left cosets of P_2 in a double coset $P_1 \sigma P_2$. We also give a formula for the number of left cosets in a double coset when P_1 and P_2 are compact. Using the notation of Section 3, let $W_i = W_{P_i}$, the set of elements of $\tilde{W}$ with representatives in P_i ($i = 1, 2$). We suppose for now that both W_1 and W_2 are contained in W_{af}. We will deal with general subgroups containing I later on in the section. Let $S_i = S_{P_i}$ be the canonical set of involutive generators of W_i.

By Proposition 3.1,

$$G(F) = \coprod_{\sigma \in [W_1 \backslash \tilde{W} / W_2]} P_1 \sigma P_2.$$

Let us therefore fix σ in $[W_1 \backslash \tilde{W} / W_2]$ and consider the coset $P_1 \sigma P_2$. We first decompose $P_1 \sigma P_2$ into a disjoint union of double cosets in $I \backslash G(F) / P_2$.

Lemma 5.1. *The double coset $P_1 \sigma P_2$ is the disjoint union of the cosets $I \tau \sigma P_2$ as τ ranges over $[W_1 / W_1^{\sigma W_2}]$.*

Proof. By Proposition 3.1 (i), we have that

$$P_1 = \coprod_{w \in W_1} IwI.$$

It follows that

$$P_1 \sigma P_2 = \bigcup_{w \in W_1} IwI\sigma P_2.$$

We claim that this last expression is equal to $\bigcup_{w \in W_1} Iw\sigma P_2$.

By Equation (2) in Proposition 3.2, if we write w' in W_1 as a reduced expression $w' = s_1 \cdots s_d$ where $s_1, \ldots, s_d \in S_1$, we have that

$$Iw'I\sigma I = Is_1 \cdots s_d I \sigma I = Is_1 I \cdots I s_d I \sigma I.$$

But by Proposition 3.2 (i) applied repeatedly

$$Iw'\sigma I = Is_1 \cdots s_d \sigma I \subset Is_1 I \cdots Is_d I \sigma I = Iw'I\sigma I$$

and

$$Iw'I\sigma I = Is_1 I \cdots Is_d I \sigma I \subset \bigcup_{w \in W_1} Iw\sigma I.$$

Therefore

$$\bigcup_{w \in W_1} Iw\sigma I = \bigcup_{w \in W_1} IwI\sigma I$$

and the claim follows.

We must now determine which of the terms in the above union are the same. To this end, we apply Proposition 3.1 (ii) to the subgroups I and P_2 of $G(F)$. Since $W_I = \langle e \rangle$ and $W_{P_2} = W_2$, it follows that for any w, w' in W_1, $Iw\sigma P_2 = Iw'\sigma P_2$ if and only if $w\sigma \equiv w'\sigma \pmod{W_2}$, i.e., if and only if $w \equiv w' \pmod{W_1^{\sigma W_2}}$. Therefore, to obtain a disjoint union of cosets $Iw\sigma P_2$ we take the union over w in the set of representatives $[W_1 / W_1^{\sigma W_2}]$. □

We now decompose $P_1\sigma P_2$ into left cosets of P_2 by expressing each double coset $I\tau\sigma P_2$ as a union of such left cosets. Recall that R is a set of representatives in $\mathcal{O}_F$ for the residue field k which contains 0.

Theorem 5.2. *Suppose that P_1 and P_2 are subgroups of $G(F)$ containing I. If $W_{P_1}, W_{P_2} \subset W_{af}$ and $\sigma \in W_{P_1}\backslash\tilde{W}/W_{P_2}$, then the double coset $P_1\sigma P_2$ is equal to the disjoint union*

$$P_1\sigma P_2 = \coprod_{\tau\in[W_{P_1}/W_{P_1}^{\sigma W_{P_2}}]} \ \coprod_{\nu\in R^{l(\tau\sigma)}} g_{\tau\sigma}(\nu)P_2.$$

Proof. By Corollary 3.4, we have the decomposition

$$(4)\qquad \begin{aligned} I\tau\sigma P_2 = I\tau\sigma I P_2 &= \left(\coprod_{\nu\in R^{l(\tau\sigma)}} g_{\tau\sigma}(\nu)I\right)P_2 \\ &= \bigcup_{\nu\in R^{l(\tau\sigma)}} g_{\tau\sigma}(\nu)P_2. \end{aligned}$$

Because of Lemma 5.1, the theorem will follow if we show that the cosets in the union (4) are distinct. So suppose that

$$g_{\tau\sigma}(\nu)P_2 = g_{\tau\sigma}(\nu')P_2$$

for some $\tau \in [W_1/W_1^{\sigma W_2}]$ and $\nu, \nu' \in R^{l(\tau\sigma)}$. We will show that $\nu = \nu'$. The main idea of the argument is to transfer the problem from P_2-cosets in $G(F)$ to W_2-cosets in $\tilde{W}$ and then to bring to bear our results on Coxeter groups from Section 4.

First we note that by Proposition 3.1 (i) and Corollary 3.4,

$$\begin{aligned} g_{\tau\sigma}(\nu)P_2 &= \coprod_{w\in W_2} g_{\tau\sigma}(\nu)IwI \\ &= \coprod_{w\in W_2} \ \coprod_{\nu''\in R^{l(w)}} g_{\tau\sigma}(\nu)g_w(\nu'')I \end{aligned}$$

and similarly

$$g_{\tau\sigma}(\nu')P_2 = \coprod_{w\in W_2} \ \coprod_{\nu''\in R^{l(w)}} g_{\tau\sigma}(\nu')g_w(\nu'')I.$$

Since these two P-cosets are equal, there must exist some w in W_2 and ν'' in $R^{l(w)}$ such that $g_{\tau\sigma}(\nu')g_w(\nu'')I$ equals $g_{\tau\sigma}(\nu)g_e(0)I = g_{\tau\sigma}(\nu)I$. We will show that this equality can only hold if $w = e$. Then we will have that $g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')I$, which immediately implies that $\nu = \nu'$ by Corollary 3.4.

So suppose that $g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')g_w(\nu'')I$, where $w \in W_2$ and $\nu'' \in R^{l(w)}$. By the definition of $g_{\tau\sigma}(\nu)$ and Proposition 3.2 (ii), we have that

$$(5)\qquad g_{\tau\sigma}(\nu)I \subset I\tau\sigma I.$$

Similarly, for each ν'' in $R^{l(w)}$,

$$g_{\tau\sigma}(\nu')g_w(\nu'')I \subset I\tau\sigma IwI.$$

We are now able to use Section 4 since $\sigma \in [W_1\backslash\tilde{W}/W_2]$, $\tau \in [W_1/W_1^{\sigma W_2}]$ and $w \in W_2$. By Theorem 4.6, we conclude that $l(\tau\sigma w) = l(\tau\sigma) + l(w)$. This implies via Equation (1) in Proposition 3.2 that $I\tau\sigma IwI = I\tau\sigma wI$. Hence,

$$g_{\tau\sigma}(\nu')g_w(\nu'')I \subset I\tau\sigma wI. \tag{6}$$

Since the double cosets in (5) and (6) both contain the left I-coset $g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')g_w(\nu'')I$, we conclude that they must be equal. But $I\tau\sigma I = I\tau\sigma wI$ implies $w = e$ since $I\backslash G(F)/I$ is represented by $\tilde{W}$ (Proposition 3.1). □

Remark 5.3. If we take as a representative of $P_1\sigma P_2$ an element σ' of σW_2 not equal to σ then

$$P_1\sigma P_2 = \coprod_{\tau\in[W_1/W_1^{\sigma' W_2}]} \bigcup_{\nu\in R^{l(\tau\sigma')}} g_{\tau\sigma'}(\nu)P_2,$$

but this union is no longer disjoint. For the number of cosets in the preceeding union is larger than than the number of cosets given in Theorem 5.2 as $l(\tau\sigma') = l(\tau) + l(\sigma') > l(\tau) + l(\sigma) = l(\tau\sigma)$ for any $\tau \in [W_1/W_1^{\sigma' W_2}]$ by Theorem 4.6.

We now give a decomposition of double cosets into left cosets for arbitrary subgroups of $G(F)$ containing I. Adjusting our notation slightly, we let P_1' and P_2' be two such subgroups. Set $W_i' = W_{P_i'}$ and $W_i = W_i' \cap W_{\mathrm{af}}$. As before let $S_i = W_i \cap \tilde{S}$. Recall that S_i is stabilized by $\Omega_{P_i'}$ under conjugation (Proposition 3.1). For $i = 1, 2$, let $P_i \subset P_i'$ be the subgroup IW_iI. Then

$$P_i' = \coprod_{\rho\in\Omega_{P_i'}} P_i\rho = \coprod_{\rho\in\Omega_{P_i'}} \rho P_i$$

by Proposition 3.1.

Let $[W_1'\backslash\tilde{W}/W_2'] = [W_{P_1'}\backslash\tilde{W}/W_{P_2'}]$ be a set of representatives of smallest possible length for the double cosets in $W_1'\backslash\tilde{W}/W_2'$. (Note that this set of representatives is no longer unique in contrast to $[W_1\backslash\tilde{W}/W_2]$.)

Lemma 5.4. *The set $[W_1\backslash\tilde{W}/W_2]$ is equal to the set of all products of the form $\rho_1\sigma\rho_2$ as ρ_i ranges over $\Omega_{P_i'}$ $(i = 1, 2)$ and σ ranges over $[W_1'\backslash\tilde{W}/W_2']$. In particular, if $\rho_i \in \Omega_{P_i'}$ $(i = 1, 2)$ and $\sigma \in [W_1\backslash\tilde{W}/W_2]$, then $\rho_1\sigma\rho_2$ is also in $[W_1\backslash\tilde{W}/W_2]$.*

Proof. The elements $\rho_1\sigma\rho_2$ clearly exhaust $W_1\backslash\tilde{W}/W_2$ since

$$\tilde{W} = \coprod_{\sigma\in[W_1'\backslash\tilde{W}/W_2']} W_1'\sigma W_2' = \coprod_{\sigma\in[W_1'\backslash\tilde{W}/W_2']} \bigcup_{\rho_1\in\Omega_{P_1'}} \bigcup_{\rho_2\in\Omega_{P_2'}} W_1\rho_1\sigma\rho_2 W_2.$$

Moreover, $\rho_1\sigma\rho_2$ is of minimal length in $W_1\rho_1\sigma\rho_2W_2$ since σ is of minimal length in $W_1'\sigma W_2'$. The second statement follows trivially from the first. □

Let us now fix an element σ in $[W_1'\backslash\tilde{W}/W_2']$ and consider the double coset $P_1'\sigma P_2'$. Denote by $\Omega^\sigma_{P_1'}$ the stabilizer $\{\rho \in \Omega_{P_1'} | \sigma^\rho = \sigma\}$ of σ in $\Omega_{P_1'}$ and let $\Omega^\sigma_{P_1',P_2'} = \Omega^\sigma_{P_1'} \cap \Omega_{P_2'}$. Also, let $J^\sigma_{P_1',P_2'}$ be a set of representatives for $\Omega_{P_1'} \Big/ \Omega^\sigma_{P_1',P_2'}$.

Lemma 5.5. *If $\rho \in \Omega_{P_1'}$, then*

$$\rho[W_1/W_1^{\sigma W_2}]\rho^{-1} = [W_1/W_1^{\sigma^\rho W_2^\rho}].$$

(*Recall that W_2^ρ denotes the set $\{w^\rho \mid w \in W_2\}$.*) *In particular, if $\rho \in \Omega^\sigma_{P_1',P_2'}$, then $\rho[W_1/W_1^{\sigma W_2}]\rho^{-1} = [W_1/W_1^{\sigma W_2}]$.*

Proof. The first statement is true since $\Omega_{P_1'}$ stabilizes W_1 and since

$$\rho W_1^{\sigma W_2}\rho^{-1} = \rho(W_1 \cap \sigma W_2\sigma^{-1})\rho^{-1} = W_1 \cap \sigma^\rho W_2^\rho(\sigma^\rho)^{-1} = W_1^{\sigma^\rho W_2^\rho}.$$

The second statement follows from the first and the fact that if $\rho \in \Omega^\sigma_{P_1',P_2'}$ then ρ stabilizes σ and W_2. □

Following our procedure in the beginning of the section, we first decompose $P_1'\sigma P_2'$ into a union of cosets in $I\backslash G(F)/P_2'$.

Lemma 5.6. *The double coset $P_1'\sigma P_2'$ is the disjoint union*

$$\coprod_{\gamma\in J^\sigma_{P_1',P_2'}} \coprod_{\tau\in[W_1/W_1^{\sigma W_2}]} I\gamma\tau\sigma P_2'.$$

Proof. Since $\sigma \in [W_1\backslash\tilde{W}/W_2]$ and $W_{P_i} = W_i \in W_{\mathrm{af}}$ $(i = 1, 2)$, we have that

$$P_1\sigma P_2 = \coprod_{\tau\in[W_1/W_1^{\sigma W_2}]} I\tau\sigma P_2,$$

by Lemma 5.1. Therefore,

$$(7) \qquad \begin{aligned} P_1'\sigma P_2' &= \bigcup_{\rho_i\in\Omega_{P_i'}} \rho_1 P_1\sigma P_2\rho_2 \\ &= \bigcup_{\rho_i\in\Omega_{P_i'}} \bigcup_{\tau\in[W_1/W_1^{\sigma W_2}]} \rho_1 I\tau\sigma P_2\rho_2 \end{aligned}$$

$$= \bigcup_{\rho_i \in \Omega_{P_i'}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\rho_1\tau\sigma\rho_2 P_2$$
$$= \bigcup_{\rho_i \in \Omega_{P_i'}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\rho_1}\sigma^{\rho_1}\rho_1\rho_2 P_2$$
$$= \bigcup_{\rho_1 \in \Omega_{P_1'}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\rho_1}\sigma^{\rho_1}\rho_1 P_2'.$$

Using the definitions of $\Omega_{P_1'}$ and $J^{\sigma}_{P_1',P_2'}$, we can eliminate some of the repetition of cosets in the union (7). For (7) is equal to

$$\bigcup_{\rho_1 \in \Omega_{P_1'}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\rho_1}\sigma^{\rho_1}\rho_1 P_2' \tag{8}$$
$$= \bigcup_{\gamma \in J^{\sigma}_{P_1',P_2'}} \bigcup_{\rho \in \Omega^{\sigma}_{P_1',P_2'}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\gamma\rho}\sigma^{\gamma\rho}\gamma\rho P_2'$$
$$= \bigcup_{\gamma \in J^{\sigma}_{P_1',P_2'}} \bigcup_{\rho \in \Omega^{\sigma}_{P_1',P_2'}} \bigcup_{\tau' \in \gamma\rho[W_1/W_1^{\sigma W_2}](\gamma\rho)^{-1}} I\tau'\sigma^{\gamma\rho}\gamma P_2'. \tag{9}$$

By Lemma 5.5 and the fact that $\Omega^{\sigma}_{P_1',P_2'}$ stabilizes σ and W_2, this last expression is equal to

$$\bigcup_{\gamma \in J^{\sigma}_{P_1',P_2'}} \bigcup_{\tau' \in [W_1/W_1^{\sigma^{\gamma} W_2^{\gamma}}]} I\tau'\sigma^{\gamma}\gamma P_2'. \tag{10}$$

As in the proof of Lemma 5.1, we now determine whether the terms in the union (10) are distinct. Let $\gamma, \gamma' \in J^{\sigma}_{P_1',P_2'}$, and let $\tau \in [W_1/W_1^{\sigma^{\gamma}W_2^{\gamma}}]$, $\tau' \in [W_1/W_1^{\sigma^{\gamma'}W_2^{\gamma'}}]$. We will show that the terms $I\tau\sigma^{\gamma}\gamma P_2'$ and $I\tau'\sigma^{\gamma'}\gamma' P_2'$ in (10) are equal only if $\gamma = \gamma'$ and $\tau = \tau'$. By Proposition 3.1 applied to I and P_2', we have that the two double cosets are equal if and only if

$$\tau\sigma^{\gamma}\gamma W_2' = \tau\sigma^{\gamma'}\gamma' W_2'. \tag{11}$$

Since $W_2' = \coprod_{\gamma \in \Omega_{P_2'}} \gamma W_2$, this is equivalent to the condition that

$$\tau\sigma^{\gamma}\gamma\rho W_2 = \tau'\sigma^{\gamma'}\gamma' W_2$$

for some $\rho \in \Omega_{P_2'}$. But $\tau, \tau' \in W_1$ so this means that

$$W_1\sigma^{\gamma}\gamma\rho W_2 = W_1\sigma^{\gamma'}\gamma' W_2. \tag{12}$$

Now both $\sigma^{\gamma}\gamma\rho = \gamma\sigma\rho$ and $\sigma^{\gamma'}\gamma' = \gamma'\sigma$are in $[W_1\backslash\tilde{W}/W_2]$ by Lemma 5.4 since σ is an element of $[W_1'\backslash\tilde{W}/W_2']$. As a consequence of the uniqueness of the coset representatives of shortest length of $W_1\backslash\tilde{W}/W_2$ (Corollary 4.7), Equation (12) can hold only if $\sigma^{\gamma}\gamma\rho = \sigma^{\gamma'}\gamma'$. Since $\tilde{W} = W_{\mathrm{af}} \rtimes \Omega$, it follows easily from this that $\gamma\rho = \gamma'$ and hence that $\sigma^\gamma = \sigma^{\gamma'}$. Thus $\gamma^{-1}\gamma' = \rho \in \Omega_{P_2'}$ and $\gamma^{-1}\gamma' \in \Omega^\sigma_{P_1'}$ so that $\gamma \equiv \gamma' \pmod{\Omega^\sigma_{P_1',P_2'}}$. Since $\gamma, \gamma' \in J^\sigma_{P_1',P_2'}$ we must have $\gamma = \gamma'$ and $\rho = e$.

Since $\gamma = \gamma'$, it follows from (11) that $\tau\gamma\sigma W_2'\gamma^{-1} = \tau'\gamma\sigma W_2'\gamma^{-1}$. Thus τ and τ' are elements of $[W_1/W_1^{\sigma^\gamma W_2^\gamma}]$ which lie in the same left coset of $W_1^{\sigma^\gamma W_2^\gamma}$. This forces $\tau = \tau'$. Therefore, we see that the union in (10) is disjoint; i.e.,

$$P_1'\sigma P_2' = \coprod_{\gamma \in J^\sigma_{P_1',P_2'}} \coprod_{\tau \in \gamma[W_1/W_1^{\sigma W_2}]\gamma'} I\tau\sigma^\gamma\gamma P_2' = \coprod_{\gamma \in J^\sigma_{P_1',P_2'}} \coprod_{\tau \in [W_1/W_1^{\sigma W_2}]} I\gamma\tau\sigma P_2'.$$

□

Proceeding as in the proof of Theorem 5.2, we now use this decomposition of $P_1'\sigma P_2'$ to conclude, in analogy to (4), that

$$P_1'\sigma P_2' = \coprod_{\gamma \in J^\sigma_{P_1',P_2'}} \coprod_{\tau \in [W_1/W_1^{\sigma W_2}]} \bigcup_{\nu \in R^{l(\tau\sigma)}} g_{\gamma\tau\sigma}(\nu)P_2'.$$

(Note that $l(\gamma\tau\sigma) = l(\tau\sigma)$ since $\gamma \in \Omega$.) In order to prove that these P_2'-cosets are distinct, we note that the argument in Theorem 5.2 will still work if P_2 is replaced by P_2' and W_2 by W_2' provided that $l(\gamma\tau\sigma w') = l(\gamma\tau\sigma) + l(w')$ for all γ in $J^\sigma_{P_1',P_2'}$, τ in $[W_1/W_1^{\sigma W_2}]$ and w' in W_2'. This condition is easily proved to hold—write w' as $w''\rho$ where $w'' \in W_2$ and $\rho \in \Omega_{P_2'}$. Then, since $\sigma \in [W_1\backslash\tilde{W}/W_2]$ and $\tau \in [W_1/W_1^{\sigma W_2}]$, we have by Theorem 4.6 that

$$\begin{aligned} l(\gamma\tau\sigma w') &= l(\gamma\tau\sigma w''\rho) \\ &= l(\tau\sigma w'') \\ &= l(\tau\sigma) + l(w'') \\ &= l(\gamma\tau\sigma) + l(w''\rho) \\ &= l(\gamma\tau\sigma) + l(w'). \end{aligned}$$

Thus the P_2'-cosets appearing in the union are distinct and we have proved the following theorem.

Theorem 5.7. *Let P_1', P_2' be subgroups of $G(F)$ containing the Iwahori subgroup I. Let σ be an element of $[W_{P_1'}\backslash\tilde{W}/W_{P_2'}]$. Then*

$$P_1'\sigma P_2' = \coprod_{\gamma \in J^\sigma_{P_1',P_2'}} \coprod_{\tau \in [W_{P_1}/W_{P_1}^{\sigma W_{P_2}}]} \coprod_{\nu \in R^{l(\tau\sigma)}} g_{\gamma\tau\sigma}(\nu)P_2'.$$

The number of terms in the disjoint union in Theorem 5.7 is calculated in the following corollary when W_1 and W_2 are finite (i.e., P_1 and P_2 are compact).

Corollary 5.8. *Let σ be an element of $[W_1'\backslash\tilde{W}/W_2']$. Suppose that W_1 is finite. Then the number of left cosets of P_2' in $P_1'\sigma P_2'$ is*

$$\#(J^{\sigma}_{P_1',P_2'})\cdot q^{l(\sigma)}\cdot \sum_{\tau\in[W_1/W_1^{\sigma W_2}]} q^{l(\tau)} = [\Omega_{P_1'}:\Omega^{\sigma}_{P_1',P_2'}]\sum_{\gamma\in W_1\sigma W_2} q^{l(\gamma)} \Bigg/ \sum_{w\in W_2} q^{l(w)}\,.$$

Proof. That the number of cosets is equal to the first expression is immediate from the theorem and the definition of $g_{\gamma\tau\sigma}(\nu)$. To prove that the two expressions are equal, first note that $\#(J^{\sigma}_{P_1',P_2'}) = [\Omega_{P_1'}:\Omega^{\sigma}_{P_1',P_2'}]$. Also, observe that $\{\tau\sigma \mid \tau\in[W_1/W_1^{\sigma W_2}]\}$ is a set of coset representatives in $W_1\sigma W_2$ for $W_1\sigma W_2/W_2$ so

$$\sum_{\gamma\in W_1\sigma W_2} q^{l(\gamma)} = \sum_{\tau\in[W_1/W_1^{\sigma W_2}]}\ \sum_{w\in W_2} q^{l(\tau\sigma w)}.$$

On the other hand, since $l(\tau\sigma w) = l(\tau)+l(\sigma)+l(w)$, we obtain

$$\begin{aligned}\Bigg(q^{l(\sigma)}&\sum_{\tau\in[W_1/W_1^{\sigma W_2}]} q^{l(\tau)}\Bigg)\Bigg(\sum_{w\in W_2} q^{l(w)}\Bigg)\\ &= \sum_{\tau\in[W_1/W_1^{\sigma W_2}]}\ \sum_{w\in W_2} q^{l(\tau)+l(\sigma)+l(w)}\\ &= \sum_{\tau\in[W_1/W_1^{\sigma W_2}]}\ \sum_{w\in W_2} q^{l(\tau\sigma w)}.\end{aligned}$$

□

Example 5.9. Let G be the group PGSp_4. The rank of G is 2, and we choose a set of simple roots consisting of a short root α_1 and a long root α_2. We also let α_0 be the highest root corresponding to this basis. The Weyl group W_0 is generated by the reflections $w_1 = w_{\alpha_1}$ and $w_2 = w_{\alpha_2}$, while W_{af} is generated by these reflections and $w_0 = t(\alpha_0^{\vee})w_{\alpha_0}$. The group Ω is cyclic of order 2. We denote the generator of Ω by ρ and note that it interchanges w_0 and w_2 but fixes w_1.

We consider the Coxeter groups $W_1 = \langle w_1\rangle$ and $W_2 = \langle w_0, w_2\rangle$ inside W_{af}. The reflections w_0 and w_2 commute so that W_2 consists of the four elements e, w_0, w_2 and w_0w_2. Let $W_i' = W_i\Omega$ and let P_i' be the compact subgroup $IW_i'I$ of $G(F)$ $(i=1,2)$. We will use the results of this to decompose double cosets in $P_1'\backslash G(F)/P_2'$ and $P_2'\backslash G(F)/P_2'$.

Let $\sigma = w_1w_0$. It is easily shown that $\sigma\in[W_1'\backslash\tilde{W}/W_2']$. Consider the double coset $P_1'\sigma P_2'$. The group $W_1^{\sigma W_2} = W_1\cap\sigma W_2\sigma^{-1}$ is trivial so that $[W_1/W_1^{\sigma W_2}] = [W_1/W_1^{\sigma W_2}] = \{e, w_1\}$. The stabilizer $\Omega^{\sigma}_{P_1',P_2'}$ is trivial as well which means that $J^{\sigma}_{P_1',P_2'} = \{e, \rho\}$. By Lemma 5.6, $P_1'\sigma P_2'$ is the disjoint union of the four double cosets

$$\begin{aligned}&Iw_0w_1P'\\&Iw_1w_0w_1P_2'\\&I\rho w_0w_1P_2' = Iw_2w_1P_2'\\&I\rho w_1w_0w_1P_2' = Iw_1w_2w_1P_2'.\end{aligned}$$

Thus by Theorem 5.7, $P_1'\sigma P_2'$ is the disjoint union of the $2q^2+2q^3$ cosets

$$\begin{array}{ll}g_{w_0w_1}(\nu_1)P_2' & (\nu_1 \in R^2)\\ g_{w_1w_0w_1}(\nu_2)P_2' & (\nu_2 \in R^3)\\ g_{w_2w_1}(\nu_3)P_2' & (\nu_3 \in R^2)\\ g_{w_1w_2w_1}(\nu_4)P_2' & (\nu_4 \in R^3).\end{array}$$

Now let $\sigma = w_1w_0w_1 \in [W_2'\backslash\tilde{W}/W_2']$ and consider the double coset $P_2'\sigma P_2'$. Here the group $W_2^{\sigma W_2} = W_2 \cap \sigma W_2 \sigma^{-1} = \langle w_0 \rangle$ since we have the braid relation $w_1w_0w_1w_0w_1w_0w_1 = w_0$. Thus $[W_2/W_2^{\sigma W_2}] = \{e, w_2\}$. Also, $J^{\sigma}_{P_2',P_2'} = \{e, \rho\}$. By Lemma 5.6, $P_2'\sigma P_2'$ is the disjoint union of the four double cosets

$$\begin{aligned}&Iw_1w_0w_1P_2'\\&Iw_2w_1w_0w_1P_2'\\&I\rho w_1w_0w_1P_2' = Iw_1w_2w_1P_2'\\&I\rho w_2w_1w_0w_1P_2' = Iw_0w_1w_2w_1P_2'.\end{aligned}$$

It follows from Theorem 5.7 that $P_2'\sigma P_2'$ is the disjoint union of the $2q^3+2q^4$ cosets

$$\begin{array}{ll}g_{w_1w_0w_1}(\nu_1)P_2' & (\nu_1 \in R^3)\\ g_{w_2w_1w_0w_1}(\nu_2)P_2' & (\nu_2 \in R^4)\\ g_{w_1w_2w_1}(\nu_3)P_2' & (\nu_3 \in R^3)\\ g_{w_0w_1w_2w_1}(\nu_4)P_2' & (\nu_4 \in R^4).\end{array}$$

6. Degrees of Spherical Hecke Operators.

We now consider the special case when the subgroups P_1 and P_2 containing I are both equal to the hyperspecial subgroup K. Using the results of the previous section, we can determine the left cosets of K occurring in a given double coset $K\sigma K$. The number of such cosets is by definition the degree of the function $\mathrm{ch}_{K\sigma K}$ in the spherical Hecke algebra of K. With the coset decomposition of the preceeding section, we will derive the formula for the degree given in [**5**, §5].

Since $W_K = W_0$, the elements of $W_0\backslash\tilde{W}/W_0$ index the double coset space $K\backslash G(F)/K$. Therefore, as a result of the fact that $\tilde{W} \cong X_*(T) \rtimes W_0$, we see that $K\backslash G(F)/K$ can be identified with the set of orbits of W_0 on $X_*(T)$. (This is the Cartan decomposition—cf. [**6**, 2.5].) A set of representatives in $X_*(T)$ for these orbits is the set X_+ of dominant co-characters, i.e., co-characters λ such that $\langle \alpha, \lambda \rangle \geq 0$ for all α in Φ^+. The isomorphism $\tilde{W} \cong X_*(T) \rtimes W_0$ can be chosen to satisfy the condition that the image in $\tilde{W}$ of the element $\lambda(\pi)$ of $T(F)$ is $t(\lambda)$. Thus we have a one-to-one correspondence

$$X_+ \longrightarrow K\backslash G(F)/K$$

given by the map $\lambda \mapsto Kt(\lambda)K = K\lambda(\pi)K$.

Now given $\lambda \in X_*(T)$, there is a unique element w_λ in W_0 such that $l(t(\lambda)w_\lambda) = \min_{w\in W_0} l(t(\lambda)w)$ (see [**6**, §1.9]). It can be shown (see [**7**, §2.2]) that the representative of the double coset $W_0 t(\lambda) W_0$ of shortest length is $t(\lambda)w_\lambda$. Thus $[W_0\backslash\tilde{W}/W_0]$ is equal to the set $\{t(\lambda)w_\lambda \mid \lambda \in X_+\}$.

Since W_0 acts on $X_*(T)$, we may consider the stabilizer W_0^λ in W_0 of an element λ of $X_*(T)$. It is easily seen that $W_0^\lambda = W_0^{t(\lambda)w_\lambda W_0}$ so that $[W_0/W_0^\lambda] = [W_0/W_0^{t(\lambda)W_0}]$.

Fix λ in X_+. Applying Theorem 5.2 with $P_1 = P_2 = K$ and $\sigma = t(\lambda)w_\lambda$ we arrive at the following results.

Proposition 6.1. *The double coset*

$$K\lambda(\pi)K = Kt(\lambda)K$$

is equal to the disjoint union

$$\coprod_{\tau\in[W_0/W_0^\lambda]} \ \coprod_{\nu\in R^{l(\tau t(\lambda)w_\lambda)}} g_{\tau t(\lambda)w_\lambda}(\nu)K.$$

Corollary 6.2. *The number of left* (*or right*) *cosets of K in $K\lambda(\pi)K$ is*

$$\begin{aligned} q^{l(t(\lambda)w_\lambda)} \sum_{\tau\in[W_0/W_0^\lambda]} q^{l(\tau)} &= q^{\min_{w\in W_0} l(t(\lambda)w)} \sum_{\tau\in[W_0/W_0^\lambda]} q^{l(\tau)} \\ &= \sum_{\gamma\in W_0 t(\lambda) W_0} q^{l(\gamma)} \Bigg/ \sum_{w\in W_0} q^{l(w)} . \end{aligned}$$

Proof. This follows from Corollary 5.8 and the definition of w_λ. □

Example 6.3. In [**8**], for compact forms G of PGSp_4 over $\mathbb{Q}$ corresponding to various quaternion algebras, we determined the actions of spherical Hecke algebras on $G(\mathbb{Q}_p)$ for certain primes p at which G is split. This necessitated computations of the above kind for the simple adjoint group $G = \mathrm{PGSp}_4$ of type $C_2 = B_2$. Examples of such computations follow.

We carry over the notation from the example in the previous section. Since PGSp_4 is of adjoint type (i.e., $X_*(T)$ is the lattice dual to the lattice generated by Φ), the fundamental co-weights $\check{\omega}_1, \check{\omega}_2$ in Λ (which satisfy $\langle \alpha_i, \check{\omega}_j \rangle = \delta_{ij}$) are in $X_*(T)$. Furthermore, any λ in X_+ is of the form $a_1\check{\omega}_1 + a_2\check{\omega}_2$ for some non-negative integers a_1 and a_2.

If λ in X_+ is 0 then W_0^λ is clearly all of W_0 and $[W_0/W_0^\lambda]$ is trivial. On the other hand, if $\lambda = a_1\check{\omega}_1 + a_2\check{\omega}_2$ for positive a_1, a_2, then $W_0^\lambda = \langle e \rangle$ and $[W_0/W_0^\lambda] = W_0$. Now suppose that λ is the long co-weight $\check{\omega}_1$. We have $W_0^{\check{\omega}_1} = \langle w_2 \rangle$ and $[W_0/W_0^{\check{\omega}_1}]$ is the set $\{e, w_1, w_2w_1, w_1w_2w_1\}$. Also, $t(\check{\omega}_1)$ can be shown to have the reduced expression $w_0w_1w_2w_1$. A reduced expression for $t(\check{\omega}_1)w_{\check{\omega}_1}$ can be obtained from $t(\check{\omega}_1)$ by dropping the last three involutions in this expression (which are contained in W_0) to yield w_0.

We therefore have that $K\check{\omega}_1(\pi)K$ is the disjoint union of the double cosets

$$
\begin{aligned}
&Iw_0K\\
&Iw_1w_0K\\
&Iw_2w_1w_0K\\
&Iw_1w_2w_1w_0K.
\end{aligned}
$$

It follows that $K\check{\omega}_1(\pi)K$ is the union of the $q + q^2 + q^3 + q^4 = q \cdot \frac{q^4-1}{q-1}$ cosets

$$
\begin{array}{ll}
g_{w_0}(\nu_1)K & (\nu_1 \in R)\\
g_{w_1w_0}(\nu_2)K & (\nu_2 \in R^2)\\
g_{w_2w_1w_0}(\nu_3)K & (\nu_3 \in R^3)\\
g_{w_1w_2w_1w_0}(\nu_4)K & (\nu_4 \in R^4).
\end{array}
$$

For the short co-weight $\check{\omega}_2$, we have $W_0^{\check{\omega}_2} = \langle w_1 \rangle$ and $[W_0/W_0^{\check{\omega}_2}]$ is the set $\{e, w_2, w_1w_2, w_2w_1w_2\}$. A reduced expression for $t(\check{\omega}_2)$ is $w_0w_1w_0\rho = \rho w_2w_1w_2$, and for $t(\check{\omega}_2)w_{\check{\omega}_2}$ is ρ. $K\check{\omega}_2K$ is the union of the cosets

$$
\begin{aligned}
&I\rho K\\
&Iw_2\rho K\\
&Iw_1w_2\rho K\\
&Iw_2w_1w_2\rho K.
\end{aligned}
$$

Therefore, $K\check{\omega}_2K$ is the union of the $1 + q + q^2 + q^3 = \frac{q^4-1}{q-1}$ left cosets

$$
\begin{array}{ll}
\rho K & \\
g_{w_2\rho}(\nu_1)K & (\nu_1 \in R)\\
g_{w_1w_2\rho}(\nu_2)K & (\nu_2 \in R^2)\\
g_{w_2w_1w_2\rho}(\nu_3)K & (\nu_3 \in R^3).
\end{array}
$$

Remark 6.4. If $\overline{G}$ is the group over k obtained by taking the special fiber of $\underline{G}$, denote by $\overline{B}_\lambda$ the standard parabolic subgroup of $\overline{G}$ corresponding to the co-character λ and the choice of positive roots Φ^+. In [8], it is shown using the formula in Corollary 6.2 that the index $[K\lambda(\pi)K : K]$ is equal to

$$\frac{\#\left(\overline{G}/\overline{B}_\lambda\right)(k)}{q^{\dim\left(\overline{G}/\overline{B}_\lambda\right)(k)}} \cdot q^{\langle 2\delta, \lambda \rangle},$$

where $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and $\langle\,,\rangle$ is the standard pairing between $X_*(T)$ and $X^*(T)$.

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