Pacific Journal of Mathematics

TRILINEAR FORMS OF $\mathfrak{gl}_2$

Hung Yean Loke

Volume 197 No. 1 January 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 197, No. 1, 2001

TRILINEAR FORMS OF $\mathfrak{gl}_2$

Hung Yean Loke

Let G be either $GL_2(\mathbb{R})$ or $GL_2(\mathbb{C})$ with maximal compact subgroup K. Let $\mathfrak{g}$ be its complexified Lie algebra. In this paper, we will construct $(\mathfrak{g}, K)$-invariant forms on $\otimes_{i=1}^3 \pi_i$ where π_i is an infinitesimal principal series representation.

1. Introduction

1.1. In this paper we study the invariant linear forms on the tensor products of three principal series representations of $GL_2(F)$ where F is an archimedean field.

When F is a p-adic field, the existence of invariant trilinear forms is known through the work of Prasad [**Pa1**]. He shows that the space of invariant forms is at most one dimensional and that it exists if and only if a certain epsilon factor is 1. His work was partly motivated by [**Re**]. He also considers the case when $F = \mathbb{R}$ and we will describe his result in more detail below.

Let $F = \mathbb{R}$ or $\mathbb{C}$ and let $G = GL_2(F)$ with maximal compact subgroup K. Let $\mathfrak{g}$ be its Lie algebra. Let π_i $(i = 1, 2, 3)$ be an irreducible infinite dimensional Harish-Chandra module of G. Assume that the product of three central characters of π_i is trivial.

If $F = \mathbb{R}$ then π_i is either a principal series or discrete series representation. Let $\mathbb{H}$ be the quaternion division algebra over $\mathbb{R}$ and we identify its subset of non-zero elements $\mathbb{H}^*$ with U_2. If π_i is a discrete series, we denote π_i' to be the irreducible finite dimensional representation of $\mathbb{H}^*$ with the same infinitesimal character and central character as π_i. When $F = \mathbb{C}$, π_i is always in the principal series.

We recall that π_i corresponds to a representation σ_i of the Weil group W_F of F. For a non-trivial character ψ of F and a representation σ of W_F, we associate an epsilon factor [**JL**]

$$\epsilon(\sigma) := \epsilon\left(\sigma, \psi, s = \frac{1}{2}\right).$$

We note some facts about the epsilon factor (See Prop. 8.4, Thm. 9.5 of [**Pa1**]):

(i) $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \pm 1$.

(ii) $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$ if at least one of the representations π_i is a principal series representation.

(iii) $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ if and only if π_1, π_2 and π_3 are discrete series representations and $\pi_1' \otimes \pi_2' \otimes \pi_3'$ has a non-zero $\mathbb{H}^*$-invariant form.

The following result is due to Prasad [**Pa1**].

Theorem 1.1. *Suppose $F = \mathbb{R}$ and π_1 is a discrete series representation or a limit of discrete series representation. Then $\pi_1 \otimes \pi_2 \otimes \pi_3$ exhibits a $(\mathfrak{g}, K)$-invariant form if and only if $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$. In this case the invariant form is unique up to scalars.*

This paper completes the project by studying the remaining cases when all the representations π_i are principal representations.

Recall that an infinitesimal reducible principal series representation of $GL_2(\mathbb{R})$ either has a unique finite dimensional submodule, or a unique finite dimensional quotient. We say that the principal series is reducible of type I or II respectively. The main result of this paper is the following theorem:

Theorem 1.2. *Suppose $F = \mathbb{R}$ or $\mathbb{C}$ and π_1, π_2 and π_3 are $(\mathfrak{g}, K)$-modules belonging to the principal series representations. We make the following assumptions:*

(1) *If $F = \mathbb{R}$, then π_i is either irreducible or reducible of type I.*
(2) *If $F = \mathbb{C}$, then π_i is irreducible.*
(3) *The product of central characters of the three representations is trivial.*

Under a further mild assumption if $F = \mathbb{C}$ (see §4.6), $\pi_1 \otimes \pi_2 \otimes \pi_3$ exhibits a $(\mathfrak{g}, K)$-invariant form and it is unique up to scalars.

The proofs are given in §2.7 for $F = \mathbb{R}$ and §4.7 for $F = \mathbb{C}$.

Note that our result is consistent with those in the non-archimedean case (cf. Thm. 1.2 and Thm. 1.4 of [**Pa1**]).

1.2. In a related paper [**Pa2**], Prasad considers the invariant linear forms of $GL_2(F_1) \times GL_2(F_2)$ where F_1 is a quadratic extension of a non-archimedean local field F_2. In §2 of this paper, we investigate the case when $F_1 = \mathbb{C}$ and $F_2 = \mathbb{R}$ and we obtain the following theorem (cf. Thm. A, Thm. B [**Pa2**]):

Theorem 1.3. *Let π_1 be an irreducible infinite dimensional Harish-Chandra module of $GL_2(\mathbb{C})$. Let π_2 be an infinite dimensional Harish-Chandra module of $GL_2(\mathbb{R})$. Suppose the product of the central characters is trivial on $GL_2(\mathbb{R})$. We assume that π_2 satisfies one of the following conditions.*

(1) *π_2 is irreducible.*
(2) *π_2 is a reducible principal series representation with a finite dimensional submodule.*
(3) *π_2 is a reducible principal series representation with a finite dimensional quotient of dimension n and π_1 contains an irreducible K-type of dimension n.*

Then the dimension of $(\mathfrak{gl}_2(\mathbb{C}), O_2)$*-invariant forms on* $\pi_1 \otimes \pi_2$ *is at most one. The dimension is zero if and only if* π_2 *is in the discrete series and the restriction of the dual representation of* π_2' *to* SU_2 *is a* K*-type of* π_1*.*

The proofs of the above theorems are given in §3.4 and §3.9. Let $GL_2^+(\mathbb{R})$ denote the subgroup of $GL_2(\mathbb{R})$ with positive determinant. Using a similar argument we will prove the following proposition in §3.13: (Cf. Thm. 8.4.4 [**Pa2**], [**F**].)

Proposition 1.4. *Let* $\pi_1 = \mathcal{B}(\mu_1, \mu_2)$ *denote the infinitesimal principal series representation of* $GL_2(\mathbb{C})$ *where* μ_1, μ_2 *are characters of* $\mathbb{C}^*$ *(see* §6 [**JL**]*). Suppose* π_1 *is irreducible with trivial infinitesimal character, then it will exhibit a* $GL_2^+(\mathbb{R})$*-invariant form* ϕ *if and only if one of the following statement is true.*

(i) *There exists* $s \in \mathbb{C}$ *such that* $2s$ *is not an integer and* $\mu_1(z) = |z|^s$, $\mu_2(z) = |z|^{-s}$ *for all* $z \in \mathbb{C}^*$*.* π_1 *is spherical and* ϕ *is non-zero on the spherical vector.*

(ii) *There exists* $l \in \mathbb{Z}$ *such that* $\mu_1(z) = |z|^l z^{-l}$ *and* $\mu_2(z) = |z|^l \overline{z}^{-l}$*.* ϕ *is non-zero on the minimal* K*-types.*

In other words, π_1 *is a base change from a representation of* $GL_2(\mathbb{R})$*. The invariant form is unique up to scalars. In case* (i) *or* (ii) *such that* l *is even, the invariant form extends to a* $GL_2(\mathbb{R})$*-invariant form. Otherwise when* l *is odd in* (ii)*, the invariant form extends to the sign character of* $GL_2(\mathbb{R})$*.*

Theorem 1.3 will enable us to prove the following corollary in §3.12.

Corollary 1.5. *Let* π_1 *be an irreducible infinitesimal principal series representation of* $GL_2(\mathbb{C})$*. Let* π_f *be an irreducible finite dimensional representation of* $GL_2(\mathbb{R})$ *of dimension* n*. Assume that the product of the central characters is trivial on* $GL_2(\mathbb{R})$*. Then:*

(i) *The dimension of* $(\mathfrak{gl}_2(\mathbb{C}), O_2)$*-invariant forms on* $\pi_1 \otimes \pi_f$ *is at most one.*

(ii) *The dimension is one if* π_1 *contains an* n *dimensional irreducible* K*-type.*

The converse statement of Corollary 1.5(ii) is false (see Proposition 1.4(ii)). However we will show in Theorem 3.7(iii) that it is true for 'generic' π_1.

1.3. We will give vectors where the invariant forms in Theorem 1.2 and 1.3 take non-zero values. These are recorded in Corollary 2.2, Propositions 3.3 and 3.5, and Corollary 4.6.

1.4. The organization of this paper is as follows: §2 and §4 are mainly devoted to the proofs of Theorem 1.2 for $GL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$ respectively. In §3 we investigate invariant forms of representations of $GL_2(\mathbb{R}) \times GL_2(\mathbb{C})$ and we give the proofs of Theorem 1.3, Proposition 1.4 and Corollary 1.5.

The proofs in all the three sections are conceptually straightforward but rather tedious to achieve. First we ignore the central characters and we work with representations of $SL_2^{\pm}(\mathbb{R})$ or $SL_2(\mathbb{C})$. Next we write down a basis for the representations. An invariant form ϕ on a tensor product of representations will give rise to a system of equations derived from the actions of the Lie algebras and the maximal compact subgroups. Using these equations, we will show that the value of ϕ on a certain distinguish vector uniquely determines the invariant form. The main difficulty is to show existence and this is done by finding a non-trivial solution to the system of equations. The equations in §4 are especially long and we have omitted the details of the calculations. We have also recruited the help of the computer and the software Mathematica©.

1.5. Towards the end of §2, we show that if at most two of the three principal series representations of $GL_2(\mathbb{R})$ are of type II, then the tensor product the three representations will exhibit an invariant form for 'most' of the time. See Theorem 2.3.

In §3.11 we give a counter example to show that the third assumption in Theorem 1.3 is necessary in order for the theorem to hold. However if π_2 is a reducible principal series of type II which fails to satisfy the assumption, we will prove Theorem 3.7 in §3.14 which states that Theorem 1.3 remains true for 'almost' all π_1.

1.6. Tensor products of unitary representations of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ have been studied in [**Re**] and [**W**] respectively.

After the completion of this paper, the author was notified of an unpublished result in Tohru Uzawa's thesis where he proved Theorem 1.2 for $F = \mathbb{R}$ and π_i irreducible principal series representation using hyperfunction sections. See §3.5 of [**Uz**]. The proof given in this paper is comparatively more elementary.

Finally we recall that Gross and Prasad have a general multiplicity one statement in the category of smooth, Fréchet representations of moderate growth [**GP**], [**W**]. Our results suggest that perhaps it is enough to work in the algebraic category of $(\mathfrak{g}, K)$-modules.

Acknowledgments. The author would like to thank D. Prasad for suggesting this problem and his many helpful comments. The proof of Proposition 4.2 is due to him.

2. $GL_2(\mathbb{R})$.

2.1. Throughout this paper all representations of $GL_2(\mathbb{C})$ or $GL_2(\mathbb{R})$ are infinitesimal representations unless otherwise stated.

In this section we will study $(\mathfrak{gl}_2(\mathbb{C}), O_2)$-invariant forms on tensor products of three principal series representations π_1, π_2 and π_3. It is assumed that the product of the central character is trivial so we will only work with $(\mathfrak{sl}_2(\mathbb{C}), O_2)$-modules.

2.2. Let

$$A = \frac{1}{\sqrt{2i}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C}).$$

We embed $\iota : GL_2(\mathbb{R}) \hookrightarrow GL_2(\mathbb{C})$ by $g \mapsto AgA^{-1}$. The image has maximal compact subgroup $K = K_0 \rtimes \omega$ where $K_0 = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R}\right\}$. For the rest of this paper $GL_2'(\mathbb{R})$ and $SL_2'(\mathbb{R})$ will refer to the images of $GL_2(\mathbb{R})$ and $SL_2(\mathbb{R})$ under ι. Let $\mathfrak{gl}_2'(\mathbb{R})$ and $\mathfrak{sl}_2'(\mathbb{R})$ denote their Lie algebras and $\mathfrak{h} := \mathfrak{sl}_2'(\mathbb{R}) \otimes \mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C})$. Let H, X, Y be the standard basis of $\mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{gl}_2(\mathbb{C})$. Note that $iH \in \mathrm{Lie}(K)$.

2.3. We recall some facts about principal series representations (see pp. 164-166 [**JL**]). A $(\mathfrak{h}, K)$-module $\pi = \pi(s, \epsilon, m)$ belonging to the principal series is parametized by $s \in \mathbb{C}$ and $\epsilon, m \in \{0, 1\}$. π is spanned by $\{w_n : n \in \mathbb{Z}, n \equiv \epsilon \pmod 2\}$ such that

$$\pi(H)w_n = nw_n, \ \pi(X)w_n = \frac{1}{2}(s+n+1)w_{n+2},$$
$$\pi(Y)w_n = \frac{1}{2}(s-n+1)w_{n-2}, \ \pi(\omega)w_n = (-1)^m w_{-n}.$$

$-1 \in K_0$ acts on π by $(-1)^\epsilon$. π is irreducible if and only if $s - \epsilon$ is not an odd integer.

If $s \geq 1$ and $s - \epsilon$ is an odd integer, then π contains a unique irreducible submodule d_s spanned by $\{w_n : |n| \geq s+1\}$. It is a self dual representation. When $s \geq 1$ it is called a discrete series representation. The quotient π/d_s is an irreducible finite dimensional representation.

If $s \leq -1$ and $s - \epsilon$ is an odd integer, then $\{w_n : |n| \geq -s-1\}$ is the unique submodule and the quotient is d_{-s+1}.

In the two reducible cases above ($s > 0$ and $s < 0$) we say that π is reducible of type I and II respectively.

If $s = 0$ and $\epsilon = 1$, $\pi = d_0$ is called a limit of discrete series.

2.4. For $i = 1, 2, 3$, let $\pi_i = \pi(s_i, \epsilon_i, m_i)$ be a principal series with basis $\{w^i_n : n \equiv \epsilon_i \pmod 2\}$. We assume that the product of the three central characters is trivial. Since $-1 \in K$ acts trivially,

$$\epsilon_1 + \epsilon_2 + \epsilon_3 \equiv 0 \pmod 2. \tag{1}$$

Suppose ϕ is a $(\mathfrak{h}, K)$-invariant form on $\pi := \pi_1 \otimes \pi_2 \otimes \pi_3$. The action of H gives

$$(n_1 + n_2 + n_3)\phi(w^1_{n_1} \otimes w^2_{n_2} \otimes w^3_{n_3}) = 0.$$

Hence $\phi(w^1_{n_1} \otimes w^2_{n_2} \otimes w^3_{n_3}) = 0$ unless $n_1 + n_2 + n_3 = 0$ so it suffices to find the values of

$$f(n_1, n_2) := \phi(w^1_{n_1} \otimes w^2_{n_2} \otimes w^3_{-n_1-n_2}).$$

Then the actions of $2X$, $2Y$ and ω on $\phi(w^1_{n_1} \otimes w^2_{n_2} \otimes w^3_{-n_1-n_2\pm 2})$ give

$$(s_3 - n_1 - n_2 - 1)f(n_1, n_2) = -(s_1 + n_1 + 1)f(n_1 + 2, n_2) + - (s_2 + n_2 + 1)f(n_1, n_2 + 2) \tag{2}$$

$$(s_3 + n_1 + n_2 - 1)f(n_1, n_2) = -(s_1 - n_1 + 1)f(n_1 - 2, n_2) + - (s_2 - n_2 + 1)f(n_1, n_2 - 2) \tag{3}$$

$$f(n_1, n_2) = (-1)^{m_1+m_2+m_3} f(-n_1, -n_2). \tag{4}$$

Suppose (4) is satisfied for all (n_1, n_2), then (2) is true at a point (n_1, n_2) if and only if (3) is true at $(-n_1, -n_2)$.

We will abuse notations and denote the various points in $\mathbb{Z}^2$ as well as their values of f by $a, b, \ldots, h$ in the following figure where d denotes the point (n_1, n_2) and the sides of the squares have length 2.

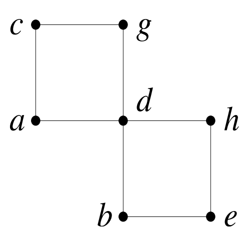

Figure 1.

By (2) and (3),

$$(s_3 - n_1 - n_2 + 1)a = -(s_1 + n_1 - 1)c - (s_2 + n_2 + 1)d \tag{5}$$

$$(s_3 - n_1 - n_2 + 1)b = -(s_1 + n_1 + 1)d - (s_2 + n_2 - 1)e \tag{6}$$

$$(s_3 + n_1 + n_2 - 1)d = -(s_1 - n_1 + 1)a - (s_2 - n_2 + 1)b. \tag{7}$$

Putting (5) and (6) into (7) we get

$$\begin{aligned} &(s_3^2 - s_1^2 - s_2^2 + 1 - 2n_1n_2)f(n_1, n_2) \\ &= (s_1 - n_1 + 1)(s_2 + n_2 + 1)f(n_1 - 2, n_2 + 2) \\ &\quad + (s_2 - n_2 + 1)(s_1 + n_1 + 1)f(n_1 + 2, n_2 - 2), \end{aligned} \tag{8}$$

We find the same relation about c, d, e if we use g, h instead of a, b. Suppose (4) is satisfied for all (n_1, n_2), then (8) holds at $d = (n_1, n_2)$ if and only if it holds at $d = (-n_1, -n_2)$.

2.5. Let $N \in \mathbb{Z}$ such that $N \equiv \epsilon_1 + \epsilon_2 \pmod 2$. Suppose we are given $f(n_1, n_2)$ where $n_1 + n_2 = N$ and they satisfy (8). In addition suppose $s_3 + N + 1 + 2k \neq 0$ for all non-negative integer k. We define $f(n_1, n_2)$ for $n_1 + n_2 = N + 2k$ $(k \geq 1)$ inductively using (3). We state a useful lemma.

Lemma 2.1. *$f(n_1, n_2)$ satisfies (2) for $n_1 + n_2 \geq N$.*

Proof. We will prove the lemma by induction on $n_1 + n_2$. We refer to Figure 2 where d is the point (n_1, n_2).

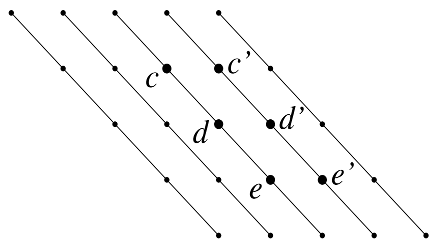

Figure 2.

By induction we assume that (2) is satisfied for all $f(l_1, l_2)$ such that $l_1 + l_2 \leq n_1 + n_2 - 2$. Hence $f(n_1, n_2)$ satisfies (8).

By (3) we have

$$tc' = -(s_1 - n_1 + 1)c - (s_2 - n_2 - 1)d \tag{9}$$

$$td' = -(s_1 - n_1 + 1)d - (s_2 - n_2 - 1)e \tag{10}$$

where $t = (s_3 + n_1 + n_2 + 1) \neq 0$ by assumption. We put (9) and (10) into the following expression.

$$\begin{aligned}
&-(s_1 + n_1 + 1)d' - (s_2 + n_2 + 1)c \\
&= t^{-1}((s_1 + n_1 + 1)(s_1 - n_1 - 1)d + (s_1 + n_1 + 1)(s_2 - n_2 + 1)e + \\
&\quad + (s_2 + n_2 + 1)(s_1 - n_1 + 1)c + (s_2 + n_2 + 1)(s_2 - n_2 - 1)d) \\
&\quad (\text{Substitute } (8)) \\
&= t^{-1}d(s_1^2 - (n_1 + 1)^2 + s_2^2 - (n_2 + 1)^2 + (s_3^2 - s_1^2 - s_2^2 + 1 - 2n_1n_2)) \\
&= t^{-1}d(s_3^2 - (n_1 + n_2 + 1)^2) \\
&= (s_3 - n_1 - n_2 - 1)d.
\end{aligned}$$

This proves the lemma. ∎

2.6. Non-Type II representations. We make the following assumptions about π_i.

(1) π_i is either irreducible or reducible of type I.
(2) $\epsilon_1 = \epsilon_2$ and $\epsilon_3 = 0$ (cf. (1)).

Suppose ϕ is an invariant form on $\pi_1 \otimes \pi_2 \otimes \pi_3$ and it gives rise to $f(n_1, n_2)$ as above.

If $\epsilon_1 = 0$, then $f(-2, 2) = (-1)^{m_1+m_2+m_3} f(2, -2)$ and by (8) we have

$$(11) \quad (s_3^2 - s_1^2 - s_2^2 + 1) f(0, 0) = (s_1 + 1)(s_2 + 1)(1 + (-1)^{m_1+m_2+m_3}) f(2, -2).$$

Since $f(0, 0) = (-1)^{m_1+m_2+m_3} f(0, 0)$, $f(0, 0) = 0$ if $m_1 + m_2 + m_3$ is odd. If $m_1 + m_2 + m_3$ is even, then (11) shows that $f(0, 0)$ determines $f(2, -2)$.

2.7. *Proof of Theorem 1.2 for $F = \mathbb{R}$.* We will construct $f(n_1, n_2)$ which satisfies (2), (3) and (4). Hence the function $f(\cdot, \cdot)$ will give rise to an invariant form ϕ.

First we assign arbitrary values to:

(i) $f(-1, 1)$ if $\epsilon_1 = 1$. We define $f(1, -1) = (-1)^{m_1+m_2+m_3} f(-1, 1)$.
(ii) $f(0, 0)$ if $\epsilon_1 = 0$ and $m_1 + m_2 + m_3$ is even. We define $f(2, -2)$ by (11).
(iii) $f(-2, 2)$ if $\epsilon_1 = 0$ and $m_1 + m_2 + m_3$ is odd. We set $f(0, 0) = 0$.

Using (8) repeatedly, we determine $f(n, -n)$ for all positive n. Note that this is possible because the coefficient of $f(n_1 + 2, n_2 - 2)$ in (8) does not vanish for positive $n = n_1 = -n_2$. By (4), we determine $f(n, -n)$ for all $n \leq 0$. Note that we could use (8) instead of (4) to get the same values for $f(n, -n)$.

Applying (3) inductively gives $f(n_1, n_2)$ for all $n_1 + n_2 > 0$. This is possible because $s_3 \geq 0$. Finally (4) gives $f(n_1, n_2)$ for $n_1 + n_2 < 0$. Again we may use (8) instead of (4) and base on the remark after (4) we will get the same values for $f(n_1, n_2)$.

We will show that f satisfies (2), (3) and (4). From the construction, f trivially satisfies (4), (3) if $n_1 + n_2 > 0$ and (2) if $n_1 + n_2 < 0$. Lemma 2.1 takes care of (2) when $n_1 + n_2 \geq 0$. By the remark after (4), f satisfies (3) for $n_1 + n_2 \leq 0$.

Finally we note that conversely an invariant form ϕ will give rise to a function f. The above construction shows that f is completely determined by its value at $f(0, 0)$, $f(-1, 1)$ or $f(-2, 2)$. This proves that ϕ is unique up to scalars. ☐

Corollary 2.2. (i) *The invariant form is non-trivial on the vector $w_1^1 \otimes w_{-1}^2 \otimes w_0^3$ if $\epsilon_1 = 1$.*

(ii) *If $\epsilon_1 = 0$ then the invariant form is non-zero on the vector $w_0^1 \otimes w_0^2 \otimes w_0^3$ if and only if $m_1 + m_2 + m_3$ is even. If $m_1 + m_2 + m_3$ is odd, then the invariant form is non-zero on $w_{-2}^1 \otimes w_2^2 \otimes w_0^3$.*

2.8. The proof can be modified to find $\mathfrak{gl}_2(\mathbb{R})$-invariants for π_i irreducible. In this case (1) is not necessary and we can show that the space of such invariants has dimension two.

2.9. Type II representations. Let $\pi_i = \pi_i(s_i, \epsilon_i, m_i)$ $(i = 1, 2, 3)$ be a principle series representation. We would like to investigate the situation when one or two out of the three representations are reducible of type II. For $\epsilon = 0, 1$ define

$$\mathcal{S}(\epsilon) := \{s \in \mathbb{Z} : s \equiv \epsilon - 1 \pmod 2, s < 0\}.$$

Without loss of generality we assume that $s_1 \in \mathcal{S}(\epsilon_1)$ and $s_3 \notin \mathcal{S}(\epsilon_3)$. In other words, π_1 is reducible of type II and π_3 is not. Note that if π_i $(i = 1, 2)$ is of type II, then it has a unique irreducible quotient d_{s_i}.

Define

$$S := \begin{cases} \max(-s_1, -s_2) & \text{if } \pi_1 \text{ and } \pi_2 \text{ are of type II} \\ -s_1 & \text{if only } \pi_1 \text{ is of type II.} \end{cases}$$

Theorem 2.3. *Given π_i $(i = 1, 2)$ and ϵ_3, m_3 as above. Then for all but finitely many $s_3 \in \mathbb{C} - \mathcal{S}(\epsilon_3)$*

$$\pi_1 \otimes \pi_2 \otimes \pi(s_3, \epsilon_3, m_3) \tag{12}$$

exhibits an invariant form unique up to scalars. The invariant form will filter through the quotient

$$d_{s_1} \otimes \pi_2 \otimes \pi(s_3, \epsilon_3, m_3). \tag{13}$$

Proof. First we assume that $\epsilon_1 = \epsilon_2$. We would like to apply the same method as in the proof of Theorem 1.2 where the three π_i's are not of type II. If π_1 or π_2 is of type II, the proof breaks down because the coefficient of $f(n_1 + 2, n_2 - 2)$ $(n_1 = -n_2 = S - 1)$ in (8) is zero. Fortunately Lemma 2.4 below shows that $f(n, -n) = 0$ if $0 \leq n \leq S - 1$. By applying (8) we show that $f(n, -n)$ $(n \geq S + 1)$ are determined by $f(-s + 1, s - 1)$ and (4) gives $f(-n, n)$. Moreover $f(n, -n)$ satisfies (8). Hence the conditions of Lemma 2.1 are satisfied and we may proceed to construct an invariant form as in §2.7.

The case $\epsilon_1 \neq \epsilon_2$ is similar and we leave the details to the reader.

By Theorem 1.1 there is an invariant form on (13) and it will pull back to a non-zero invariant form on (12) which is unique. This proves the last assertion. □

Lemma 2.4. *Suppose $\epsilon_1 = \epsilon_2$ and $f(n, -n)$ satisfies (4) and (8). Then for all but finitely many $s_3 \in \mathbb{C}$, $f(n, -n) = 0$ for $|n| \leq S - 1$.*

Proof. We will only consider the case $\epsilon_1 = \epsilon_2 = 0$. The case $\epsilon_1 = \epsilon_2 = 1$ is similar and we leave the proof to the reader.

We will solve $v = (f(0,0), f(2,-2), \dots, f(S-1, -S+1))$. Define a $(S+1)/2$ by $(S+1)/2$ matrix $A = (a_{ij})$ where $i, j = 0, \dots, (S-1)/2$ and

$$\begin{aligned} a_{jj} &= s_1^2 + s_2^2 - s_3^2 - 1 - 8j^2 \\ a_{j,j-1} &= (s_1 - 2j + 1)(s_2 - 2j + 1) \\ a_{j,j+1} &= (s_1 + 2j + 1)(s_2 + 2j + 1) \text{ if } j \neq 0 \\ a_{01} &= (s_1 + 1)(s_2 + 1)(1 + (-1)^{m_1 + m_2 + m_3}) \\ a_{ij} &= 0 \text{ if otherwise.} \end{aligned}$$

By (8) and (11), $Av = 0$. Hence $v = 0$ if and only if $A = (a_{ij})$ is invertible, that is, $\det A \neq 0$. Note that $\det A$ is a polynomial in s_3 of degree $S+1$. If s_3 is not a root of the polynomial, then $v = 0$. □

3. $GL_2(\mathbb{R})$ in $GL_2(\mathbb{C})$.

3.1. We retain the notations of §2.2 as well as the embedding $SL_2'(\mathbb{R}) \subset GL_2'(\mathbb{R}) \subset GL_2(\mathbb{C})$. Let π_1 and π_2 be irreducible Harish-Chandra modules of $GL_2(\mathbb{C})$ and $GL_2'(\mathbb{R})$ respectively. Suppose the product of the central characters is trivial on $GL_2'(\mathbb{R})$. Our goal of this section is to find $(\mathfrak{gl}_2'(\mathbb{R}) \otimes \mathbb{C}, O_2)$-invariant forms on $\pi_1 \otimes \pi_2$. Similar to §2.1 the central characters of π_1 and π_2 are not essential so we may replace $GL_2(\mathbb{C})$ and $GL_2'(\mathbb{R})$ by $SL_2(\mathbb{C}) \rtimes \omega$ and $SL_2'(\mathbb{R}) \rtimes \omega$ respectively.

3.2. We define some elements in $\mathfrak{sl}_2(\mathbb{C})$.

$$A_1 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad A_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A_3 = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$B_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B_2 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad B_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Next we define the following elements in $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$.

$$H_3 = A_3 \otimes i \quad H_+ = A_1 \otimes i - A_2 \otimes 1 \quad H_- = A_1 \otimes i + A_2 \otimes 1$$
$$F_3 = B_3 \otimes i \quad F_+ = B_1 \otimes i - B_2 \otimes 1 \quad F_- = B_1 \otimes i + B_2 \otimes 1$$

so that $A_2 = \frac{1}{2}(H_- - H_+)$. $\{A_1, A_2, A_3\}$ spans the Lie algebra of the maximal compact subgroup SU_2 and $\{H_3, H_+, H_-\}$ spans its split form. $\{2H_3, -iF_+, -iF_-\}$ forms the standard basis of $\mathfrak{h} := \mathfrak{sl}_2'(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

3.3. It is well-known that all irreducible infinitesimal representations of $SL_2(\mathbb{C})$ are either finite dimensional representations or principal series representations. We will follow the notation of §8.3 of [**Na**] and denote an irreducible representation by $\pi(k_0, c)$ where $2k_0$ is a non-negative integer and $c \in \mathbb{C}$. We recall Theorem 1 of §8.3 [**Na**].

Theorem 3.1. *A basis of* $\pi(k_0, c)$ *is*

$$\{f_v^k | k = k_0, k_0+1, \dots, k_{00},\ v = -k, -k+1, \dots k\}.$$

If $c^2 = (k_0+n)^2$ *for some positive integer* n*, then* $k_{00} = |c| - 1$ *and* $\pi(k_0, c)$ *is a finite dimensional representation. Otherwise* $k_{00} = \infty$ *and* $\pi(k_0, c)$ *is a principal series representation.*

The actions of the Lie algebra are as follows:

$$\begin{aligned}
H_+ f_v^k &= \sqrt{(k+v+1)(k-v)} f_{v+1}^k \\
H_- f_v^k &= \sqrt{(k+v)(k-v+1)} f_{v-1}^k \\
H_3 f_v^k &= v f_v^k
\end{aligned}$$

$$\begin{aligned}
F_+ f_v^k &= R_{k-v-1} C_k f_{v+1}^{k-1} \\
&\quad - \sqrt{(k-v)(k+v+1)} A_k f_{v+1}^k + R_{k+v+1} C_{k+1} f_{v+1}^{k+1}
\end{aligned} \tag{14}$$

$$\begin{aligned}
F_- f_v^k &= -R_{k+v-1} C_k f_{v-1}^{k-1} \\
&\quad - \sqrt{(k+v)(k-v+1)} A_k f_{v-1}^k - R_{k-v+1} C_{k+1} f_{v-1}^{k+1}
\end{aligned}$$

$$\begin{aligned}
F_3 f_v^k &= \sqrt{k^2 - v^2} C_k f_v^{k-1} \\
&\quad - v A_k f_v^k - \sqrt{(k+1)^2 - v^2} C_{k+1} f_v^{k+1}
\end{aligned} \tag{15}$$

where

$$R_k = \sqrt{(k+1)k}, \quad A_k = \frac{ik_0 c}{k(k+1)}, \quad C_k = \frac{i}{k}\sqrt{\frac{(k^2-k_0^2)(k^2-c^2)}{4k^2-1}}.$$

We refer to the definition of the infinitesimal irreducible principal series representation $\mathcal{B}(\mu_1, \mu_2)$ of $GL_2(\mathbb{C})$ in §6 [**JL**] where μ_1 and μ_2 are characters of $\mathbb{C}^*$. We write

$$\begin{aligned}
\mu_1(z) &= |z|^{2s_1 - a_1 - b_1} z^{a_1} \overline{z}^{b_1} \\
\mu_1(z) &= |z|^{2s_2 - a_2 - b_2} z^{a_2} \overline{z}^{b_2} \\
\mu_1 \mu_2^{-1}(z) &= |z|^{2s - a - b} z^{a} \overline{z}^{b}
\end{aligned}$$

where a_i, b_i, a, b are non-negative integers such that $a_i b_i = ab = 0$. Then $\pi(k_0, c)$ is the restriction of $\mathcal{B}(\mu_1, \mu_2)$ to $(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}, SU_2)$ such that $s = (\mathrm{sign}(b-a))c$ and $2k_0 = |b-a|$. The action of $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is given by

$$\omega f_v^k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f_v^k = i^{b_1 + b_2 - a_1 - a_2 - 2k} f_{-v}^k = (-1)^{m_0 + (k - k_0)} f_{-v}^k$$

where $m_0 = \min(b_1 - a_1, b_2 - a_2)$. We will denote the restriction of $\mathcal{B}(\mu_1, \mu_2)$ to $(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}, SU_2 \rtimes \omega)$ by $\pi(k_0, c, m_0)$.

3.4. We recall the definitions of the discrete series representation and its limit d_s in §2.3. We will give an alternative description. Let $\overline{\mathfrak{b}}$ be the Borel subalgebra of $\mathfrak{h} := \mathfrak{sl}_2'(\mathbb{R}) \otimes \mathbb{C}$ spanned by H_3 and F_-. Let χ_0 be the fundamental character of K_0 given by

$$\chi_0 : \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mapsto e^{i\theta}.$$

Let $n \geq 1$, then

$$d_{n-1} = \mathrm{Ind}_{(\mathfrak{h},K_0)}^{(\mathfrak{h},K)}(\mathcal{U}(\mathfrak{h}) \otimes_{\overline{\mathfrak{b}}} \chi_0^n).$$

We refer to Theorem 3.1 and suppose $\pi_1 = \pi(k_0, c)$ is a principal series representation and $\pi_2 = d_{n-1}$. Since $\pi_1 \otimes \pi_2$ has trivial central character, the action of $-1 \in K$ gives $2k_0 \equiv n \pmod 2$.

$$\begin{aligned} &\mathrm{Hom}_{(\mathfrak{h},K)}(\pi_1 \otimes \pi_2, \mathbb{C}) \\ &= \mathrm{Hom}_{(\mathfrak{h},K)}(\pi_1, \pi_2) \quad (\pi_2 \text{ is self-dual}) \\ &= \mathrm{Hom}_{(\mathfrak{h},K_0)}(\pi_1, \mathcal{U}(\mathfrak{h}) \otimes_{\overline{\mathfrak{b}}} \chi_0^n) \quad (\text{Frobenius reciprocity}) \\ &= \mathrm{Hom}_{K_0}((\pi_1)_{F_+}, \chi_0^n) \end{aligned}$$

where $(\pi_1)_{F_+} = \pi_1 / \{\pi_1(F_+)v : v \in \pi_1\}$ is the space of F_+ coinvariants. The next lemma proves Theorem 1.3 when π_2 is a discrete series representation.

Lemma 3.2.

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Hom}_{K_0}((\pi(k_0,c))_{F_+}, \chi_0^n) \\ &= \begin{cases} 1 & \text{if } -2k_0 + 2 \leq n \leq 2k_0 \text{ and } n \equiv 2k_0 \pmod 2 \\ 2 & \text{if } n \leq -2k_0 \text{ and } n \equiv 2k_0 \pmod 2 \\ 0 & \text{if otherwise.} \end{cases} \end{aligned}$$

Proof. Define

$$\begin{aligned} V_l &= \text{Span of } \{f_v^k : k_0 \leq k \leq l\} \\ W_l &= \text{Span of } \{f_v^k : k_0 < k \leq l, v = -k, -k+1\}. \end{aligned} \tag{16}$$

The lemma follows from (14) and the fact

$$W_l \oplus \pi(F_+)V_{l-1} = V_l.$$

□

3.5. Let $\pi_1 = \pi(k_0, c, m_0)$ and $\pi_2 = \pi(s, \epsilon, m)$ be infinitesimal principal series representations of $SL_2(\mathbb{C}) \rtimes \omega$ and $SL_2'(\mathbb{R}) \rtimes \omega$ respectively (cf. §3.3 and §2.3). We will construct a $(\mathfrak{h}, K)$-invariant linear form on $\pi_1 \otimes \pi_2$. Since $-1 \in K$ is assumed to act trivially, we have $2k_0 \equiv \epsilon \pmod 2$.

Note that the three assumptions in Theorem 1.3 is equivalent to the following statements.

1) If $\epsilon = 1$, then $s+1 \notin \{-1, -3, -5, -7, \dots, -2k_0+2\}$.
2) If $\epsilon = 0$, then $s+1 \notin \{0, -2, -4, -6, \dots, -2k_0+2\}$.

3.6. Let ϕ be an invariant form, then via the action of $H = 2H_3$ we get

$$\pi(H)\phi(f_v^k \otimes \omega_n) = (2v+n)\phi(f_v^k \otimes \omega_n) = 0$$

so $\phi = 0$ unless $2v + n = 0$. Now set $\phi_v^k = \phi(f_v^k \otimes \omega_{-2v})$ and the actions of F_+, $-F_-$ and ω give

$$\begin{aligned} E_{kv}^+ \quad &: \quad R_{k-v-1}C_k\phi_{v+1}^{k-1} - \sqrt{(k-v)(k+v+1)}A_k\phi_{v+1}^k + \\ &\qquad + R_{k+v+1}C_{k+1}\phi_{v+1}^{k+1} + \frac{i}{2}(s-2v-1)\phi_v^k = 0 \\ E_{kv}^- \quad &: \quad R_{k+v-1}C_k\phi_{v-1}^{k-1} + \sqrt{(k+v)(k-v+1)}A_k\phi_{v-1}^k + \\ &\qquad + R_{k-v+1}C_{k+1}\phi_{v-1}^{k+1} + \frac{i}{2}(s+2v-1)\phi_v^k = 0 \\ E_{kv}^0 \quad &: \quad \phi_v^k = (-1)^{m_0+(k-k_0)+m}\phi_{-v}^k. \end{aligned}$$

Note that the action of ω sends E_{kv}^- to $i^{m_1+m_2+2k}E_{k,-v}^+$. Therefore if E_{kv}^0 holds for all k, v then E_{kv}^- determines $E_{k,-v}^+$ and vice versa.

As in Figure 1, let $h, a, b \dots$ denote the points $(k,v), (k,v+1), (k-1,v) \dots$ respectively as in the following diagram

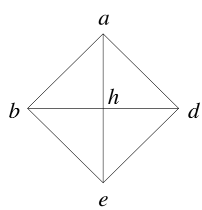

Figure 3.

We will abuse notations and use $a, b, \dots$ to denote the corresponding values of ϕ at those points. Let $s_v = \frac{i}{2}(s-2v+1)$ and $r_v = \frac{i}{2}(s+2v+1)$. $E_{kv}^{\pm}$ shows that the values of a, b, h determine that of d and the values of b, h, e determine that of d. We have the following equalities:

$$\begin{aligned} S_e = S_{bhde} = E_{k,v-1}^+ \ (k \geq v-1) \\ : R_{k-v}C_kb - \sqrt{(k-v+1)(k+v)}A_kh \\ + R_{k+v}C_{k+1}d + s_ve = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} N_a = N_{bhda} = E_{k,v+1}^- \ (-k \leq v+1) \\ : R_{k+v}C_kb + \sqrt{(k+v+1)(k-v)}A_kh \\ + R_{k-v}C_{k+1}d - r_va = 0 \end{aligned} \tag{18}$$

$$W_b = W_{aheb} = R_{k-v}S_{bhde} - R_{k+v}N_{bhda}\ (k+1 \geq |v|) \tag{19}$$
$$: -2v(2k+1)C_k b - 2(k+1)\sqrt{k^2-v^2}A_k h + R_{k-v}s_v e + R_{k+v}r_v a = 0$$

$$E_d = E_{ahed} = R_{k+v}S_{bhde} - R_{k-v}N_{bhda}\ (k+1 \geq |v|) \tag{20}$$
$$: -2k\sqrt{(k+1)^2-v^2}A_k h + 2v(2k+1)C_{k+1} + R_{k+v}s_v e + R_{k-v}r_v a = 0$$

S, N, W, E denote South, North, West and East respectively.

3.7. We will deal with $2k_0$ being odd and even separately.

Proposition 3.3. *Suppose $2k_0$ is odd, then ϕ is uniquely determined by its value at $f^{k_0}_{1/2} \otimes w_{-1}$.*

Proof. $E^0_{k_0,1}$ gives

$$\phi^{k_0}_{1/2} = (-1)^{m_0+m}\phi^{k_0}_{-1/2}. \tag{21}$$

Applying W_b repeatedly, we determine $\phi^{k_0}_v$ for $0 \leq v \leq k_0$ since the coefficient of e is non-zero by Assumptions 1 in §3.5. Using $E^0_{k_0,v}$ we get $\phi^{k_0}_v$ for $v < 0$. We will deduce the rest of ϕ^k_v inductively in the following way. Suppose we have determined ϕ^k_v for $k \leq k_1$. We define $\phi^{k_1+1}_v$ for $v \geq 0$ by $E^+_{k_1,v-1}$. Note that this is possible because the coefficient of d in (17) is non-zero. By $E^0_{k_1+1,v}$ we determine $\phi^{k_1+1}_{-v}$ $(v < 0)$. □

3.8. Suppose $2k_0$ is even, then $E^0_{k_0,0}$ gives $\phi^{k_0}_0 = (-1)^{m_0+m}\phi^{k_0}_0$.

Lemma 3.4. *ϕ is zero on $f^{k_0}_0 \otimes w_0$ if m_0+m is odd.*

We relax Assumption 2 in §3.5 slightly by allowing $s = -1$. If $k_0 \geq 1$, W_b for $(k,v) = (k_0, 0)$ gives

$$-2(k_0+1)k_0 A_{k_0}\phi^{k_0}_0 + R_{k_0}(s_0\phi^{k_0}_{-1} + r_0\phi^{k_0}_1) = 0. \tag{22}$$

By $E^0_{k_0,1}$, $\phi^{k_0}_{-1} = (-1)^{m_0+m}\phi^{k_0}_1$, and (22) becomes

$$R_{k_0}(s+1)(1+(-1)^{m_0+m})\phi^{k_0}_1 = 4k_0 c\phi^{k_0}_0. \tag{23}$$

We further subdivide into two subcases depending on whether m_0+m is even or odd.

Case 1: $k_0 \geq 1$ and m_0+m is even. If $s \neq -1$ then $\phi^{k_0}_1$ is determined by $\phi^{k_0}_0$. If $s = -1, c = 0$, then (23) is trivial. If $s = -1, c \neq 0$, then $\phi^{k_0}_0 = 0$.

Case 2: $m_0 + m$ is odd. Then Lemma 3.4 says that $\phi_0^{k_0} = 0$ and (23) is trivially satisfied. If $k_0 = 0$, then $E_{0,0}^{\pm}$ implies that $\phi_1^1 = \phi_{-1}^1 = 0$. W_b for $(k, v) = (1, 0)$ always holds for all values of ϕ_0^1.

Proposition 3.5. *Suppose $2k_0$ is even, then ϕ is uniquely determined by its value at the following vectors.*

(i) $f_0^0 \otimes w_0$ *if* $k_0 = 0$, $m_0 + m$ *is even.*
(ii) $f_0^1 \otimes w_0$ *if* $k_0 = 0$, $m_0 + m$ *is odd.*
(iii) $f_0^{k_0} \otimes w_0$ *if* $k_0 \geq 1$, $m_0 + m$ *is even,* $s \neq -1$.
(iv) $f_0^{k_0} \otimes w_0$ *and* $f_1^{k_0} \otimes w_{-2}$ *if* $k_0 \geq 1$, $m_0 + m$ *is even,* $s = -1$, $c = 0$.
(v) $f_1^{k_0} \otimes w_{-2}$ *if* $k_0 \geq 1$, $m_0 + m$ *is even,* $s = -1$, $c \neq 0$.
(vi) $f_1^{k_0} \otimes w_{-2}$ *if* $k_0 \geq 1$, $m_0 + m$ *is odd.*

Proof. By the discussions above, the values of ϕ on these vectors determine $\phi_0^{k_0}$ and $\phi_1^{k_0}$ which satisfy Lemma 3.4 and (23). Applying W_b repeatedly, we determine $\phi_v^{k_0}$ for $0 \leq v \leq k_0$ since the coefficient of e is non-zero by Assumptions 2 in §3.5. We proceed as in the proof of Proposition 3.3 to determine the rest of the values of ϕ_v^k. □

3.9. *Proof of Theorem* 1.3. Suppose we are given the value of ϕ at $f_{1/2}^{k_0} \otimes w_{-1}$ or any of the vectors in Proposition 3.5, then the proofs of Propositions 3.3 and 3.5 give a construction of ϕ_v^k. We will show that ϕ_v^k satisfies $E_{kv}^{\pm}$ and E_{kv}^0 and hence it gives rise to a $(\mathfrak{h}, K)$-invariant form. Note that (iv) and (v) of Proposition 3.5 do not satisfy Assumption 2 of §3.5 and they will not be considered.

By induction, suppose ϕ_v^l $(l \leq k)$ satisfies (17) to (20) and E_{lv}^0. By definition E_{kv}^+ holds for $v \geq 0$. Since ω sends E_{kv}^+ to $E_{k,-v}^-$, E_{kv}^- is satisfied for $v < 0$. By induction and W_b at $b = (k-1, v)$ imply that $E_{k,v-1}^+$ holds if and only if $E_{k,v+1}^-$ holds. We have thus shown that $E_{kv}^{\pm}$ holds for all $|v| \leq k$. By definition $E_{k+1,v}^0$ is true for all except possibly at $v = 0$. For $v = 0$, $E_{k+1,v}^0$ follows from the fact that E_{kv}^+ and E_{kv}^- are compatible with the action of ω. E_d at $d = (k+1, v)$ holds because it is consequence of $E_{k,v}^+$ and $E_{k,v}^-$. It remains to show that W_h holds at $h = (k, v)$ and this is proven in Lemma 3.6 below. The induction process is therefore completed and this proves the theorem. □

3.10. Consider the following diagram where h represents the point (k, v) as before.

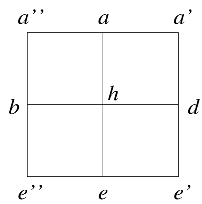

Figure 4.

Lemma 3.6. $W_h = 0$.

Proof. Note that ω sends W_h at $h = (k+1, v)$ to W_h at $h = (k+1, -v)$. Therefore it is enough to check for $v \leq 0$. Put N_h and S_h into W_h to get rid of e' and a' respectively and we get

$$\begin{aligned} W_h = &-2v(2k+3)C_{k+1}h - 2(k+2)\sqrt{(k+1)^2 - v^2}A_{k+1}d \\ &+ s_v C_{k+1}^{-1}(r_{v-1}h - R_{k+v-1}C_k e'' - \sqrt{(k+v)(k-v+1)}A_k e) \\ &+ r_v C_{k+1}^{-1}(-s_{v+1}h - R_{k-v-1}C_k a'' + \sqrt{(k-v)(k+v+1)}A_k a). \end{aligned}$$

Similarly N_a will get rid of d

$$\begin{aligned} W_h = &-2v(2k+3)C_{k+1}h \\ &- \frac{2(k+2)\sqrt{(k+1)^2 - v^2}A_{k+1}}{R_{k-v}C_{k+1}}(r_v a - R_{k+v}C_k b \\ &\qquad - \sqrt{(k+v+1)(k-v)}A_k h) \\ &+ s_v C_{k+1}^{-1}(r_{v-1}h - R_{k+v-1}C_k e'' - \sqrt{(k+v)(k-v+1)}A_k e) \\ &+ r_v C_{k+1}^{-1}(-s_{v+1}h - R_{k-v-1}C_k a'' + \sqrt{(k-v)(k+v+1)}A_k a). \end{aligned}$$

Substituting E_h to get rid of $s_v e''$ we have

$$\begin{aligned} C_{k+1}W_h = &-2v(2k+3)C_{k+1}^2 h \\ &- \frac{2(k+2)\sqrt{(k+1)^2 - v^2}A_{k+1}}{R_{k-v}}(r_v a - R_{k+v}C_k b \\ &\qquad - \sqrt{(k+v+1)(k-v)}A_k h) - (s_{v+1}r_v - s_v r_{v-1})h \\ &+ 2v(2k-1)C_k^2 h - 2(k-1)\sqrt{k^2 - v^2}C_k A_{k-1} b \\ &- s_v\sqrt{(k+v)(k-v+1)}A_k e + r_v\sqrt{(k-v)(k+v+1)}A_k a. \end{aligned}$$

Notice that a'' does not appear in the last equation. Substituting W_b to get rid of $s_v e$ we have

$$
\begin{aligned}
&C_{k+1}W_h \\
&= -2v(2k+3)C_{k+1}^2 h \\
&\quad - 2(k+2)A_{k+1}\sqrt{\frac{k+1+v}{k-v}}(r_v a - R_{k+v}C_k b \\
&\qquad\qquad - \sqrt{(k+v+1)(k-v)}A_k h) \\
&\quad + C_k(-2(k-1)\sqrt{k^2-v^2}A_{k-1}b + 2v(2k-1)C_k h) \\
&\quad + \sqrt{\frac{k+v}{k-v}}A_k(-2v(2k+1)C_k b - 2(k+1)\sqrt{k^2-v^2}A_k h + R_{k+v}r_v a) \\
&\quad + r_v\sqrt{(k-v)(k+v+1)}A_k a + (s_v r_{v-1} - s_{v+1}r_v)h \\
&= -2v(2k+3)C_{k+1}^2 h + \\
&\quad - 2(k+2)A_{k+1}\sqrt{\frac{k+1+v}{k-v}}(r_v a - R_{k+v}C_k b) \\
&\quad + 2(k+2)A_{k+1}A_k(k+1+v)h \\
&\quad + C_k(-2(k-1)\sqrt{k^2-v^2}A_{k-1}b + 2v(2k-1)C_k h) \\
&\quad + \sqrt{\frac{k+v}{k-v}}A_k(-2v(2k+1)C_k b - 2(k+1)\sqrt{k^2-v^2}A_k h + R_{k+v}r_v a) \\
&\quad + r_v\sqrt{(k-v)(k+v+1)}A_k a - 2vc \\
&= 0.
\end{aligned}
$$

□

3.11. Suppose $\pi_2 = \pi(s=-1,\epsilon,m)$, $k_0 \geq 1$ and k_0+m is even. If we apply the proof in §3.9 to (iv) and (v) of Proposition 3.5, then we can show that the dimension of invariant forms on $\pi_1 \otimes \pi_2$ is one if $c \neq 0$ and two if $c = 0$. This shows that Assumption 3 of Theorem 1.3 is necessary.

3.12. *Proof of Corollary* 1.5. Let π_2 be the principal series representation with finite quotient π_f of dimension n. π_2 contains the discrete series representation d_n:

(i) An invariant form on $\pi_1 \otimes \pi_f$ will pull back to an invariant form on $\pi_1 \otimes \pi_2$ which is unique.

(ii) If π_1 contains a n dimensional K-type, then by Theorem 1.3 $\pi_1 \otimes d_n$ does not have an invariant form. The invariant form on the tensor product $\pi_1 \otimes \pi_2$ must vanish on $\pi_2 \otimes d_n$ and so it filters through the quotient $\pi_1 \otimes \pi_f$. □

3.13. *Proof of Proposition* 1.4. Note that Proposition 1.4(i) (resp. (ii)) is equivalent to the condition that $k_0 = 0$ (resp. $c = 0$, $2k_0$ even). If $k_0 = 0$, then the invariant form exists by Corollary 1.5(ii).

Suppose ϕ is an infinitesimal $GL_2^+(R)$-invariant form and we denote $\phi_v^k := \phi(f_v^k)$. The action of $-1 \in K$ implies that $2k_0$ is even. The action of H_3 shows that $\phi_v^k = 0$ unless $v = 0$. The actions of F_+ and F_- give ($k \geq 1$)

$$C_k\phi_0^{k-1} - A_k\phi_0^k + C_{k+1}\phi_0^{k+1} = 0$$
$$C_k\phi_0^{k-1} + A_k\phi_0^k + C_{k+1}\phi_0^{k+1} = 0.$$

Solving the two equations gives

$$C_k\phi_0^{k-1} = -C_{k+1}\phi_0^{k+1}, \quad A_k\phi_0^k = 0. \tag{24}$$

If $c \neq 0$ and $k_0 \neq 0$, then $A_k \neq 0$ and $\phi_0^k = 0$ for all $k \geq k_0$.

If $c = 0$ or $k_0 = 0$, then $A_k = 0$. The first equation in (24) inductively implies that $\phi_0^{k_0}$ determines $\phi_0^{k_0+2n}$. Similarly $\phi_0^{k_0+2n+1} = 0$ because $C_{k_0} = 0$. This gives rise to a non-trivial invariant form which is uniquely determined by $\phi_0^{k_0}$.

Finally the action of ω gives $\phi_0^{k_0+2n} = (-1)^{m_0+m}\phi_0^{k_0+2n}$. Here we set $m = 0$ (resp. $m = 1$) if we are considering $GL_2(\mathbb{R})$-invariant form (resp. the sign character of $GL_2(\mathbb{R})$). Clearly ϕ_0^k has a non-trivial solution if and only if $m_0 + m \equiv l + m \equiv 0 \pmod 2$. Hence the invariant form will extend to a $GL_2(\mathbb{R})$-invariant form if and only if l is even. Otherwise we get the sign character. ☐

3.14. Generic statements. Given $m_0 \in \mathbb{Z}$ and k_0 a non-negative half integer, define

$$\mathcal{C} = \{c \in \mathbb{C} : c^2 \neq (k_0 + n)^2 \text{ for all positive integer } n\}.$$

Let $\pi_2 = \pi(s, \epsilon, m)$ ($s \neq 1$) be a reducible principal series representation of $(\mathfrak{h}, K)$. We will denote the finite dimensional quotient or submodule of π_2 by π_f. Note that π_f has dimension $|s| - 1$. We assume that:

1) $s + 1 \equiv \epsilon \equiv 2k_0 \pmod 2$.
2) $|s| + 1 \leq 2k_0$.

Theorem 3.7. *Suppose m_0, k_0 and π_2 as above. Then for all but finitely many $c \in \mathcal{C}$, the following statements are true.*

(i) $\pi(k_0, c, m_0) \otimes \pi_2$ *exhibits a $(\mathfrak{h}, K)$-invariant form ϕ and it is unique up to scalars.*
(ii) *The invariant form ϕ is non-zero on the vector $f_{v_0+1}^{k_0} \otimes w_{-2v_0-2}$ where $v_0 = \frac{|s|-1}{2}$.*
(iii) $\pi(k_0, c, m_0) \otimes \pi_f$ *does not exhibit a $(\mathfrak{h}, K)$-invariant form.*

Note that (i) complements Theorem 1.3 for reducible principal series representation of type II and (iii) is a generic converse statement of Corollary 1.5(ii).

Before proving the theorem we need a lemma.

Lemma 3.8. *Let* $v_0 = \frac{|s|-1}{2}$. *Then for all but finitely many* c, $(s+1+2v_0)\phi^{k_0}_{v_0+1} = 0$ *implies that* $\phi^{k_0}_v = 0$ *for all* $|v| \le v_0$.

Proof. We will only prove the case when $s+1$ is negative even. The other cases are similar and we will leave them to the reader.

Let $w = (\phi^{k_0}_0, \phi^{k_0}_1, \dots, \phi^{k_0}_{v_0})$ and we want to show that $w = 0$. First we define a (v_0+1) by (v_0+1) matrix $A = (a_{ij})$ where $i, j = 0, \dots, v_0$ and

$$\begin{aligned} a_{11} &= 4k_0c, \quad a_{12} = -R_{k_0}(s+1)(1+(-1)^{m_0+m}), \\ a_{t,t-1} &= R_{k_0-t}, \quad a_{t,t+1} = R_{k_0+t}, \\ a_{tt} &= -2ic\sqrt{k_0^2 - t^2}, \\ a_{ij} &= 0 \text{ otherwise} \end{aligned}$$

and $t \neq 0$. Then W_h and (23) implies that $Aw = 0$. Since $\det A$ is a polynomial in c of degree v_0+1, $w = 0$ for all but finitely many $c \in \mathbb{C}$. □

Proof of Theorem 3.7. (i) We only have to deal with $s < 0$. Lemma 3.8 implies that given an arbitrary value of $\phi^{k_0}_{v_0+1}$, $\phi^k_v = 0$ for all $0 \le v \le v_0$. We can deduce $\phi^{k_0}_v$ using W_h and $E^0_{k_0,v}$. The same constructions as in the proofs of Propositions 3.3 and 3.5 give rise to an invariant form on $\pi(k_0, c, m_0) \otimes \pi_2$.

(ii) This follows from Lemma 3.8.

(iii) Let π'_2 be the principal series with finite quotient π_f. If $\pi(k_0, c) \otimes \pi_f$ exhibits an invariant form, then the form will pull back to an invariant form ϕ on $\pi_1 \otimes \pi'_2$. The form ϕ vanishes on the subspace $\pi_1 \otimes d_s$ by Theorem 1.3. In particular it is zero on the vector $f^{k_0}_{v_0+1} \otimes w_{-2v_0-2}$. This contradicts (ii). □

4. $GL_2(\mathbb{C})$.

4.1. In this section we investigate the invariants on the tensor products of three infinitesimal representations of $GL_2(\mathbb{C})$. Similar to the last two sections, it suffices to restrict our attention to infinitesimal representations of $SL_2(\mathbb{C})$. Let $K = SU_2(\mathbb{C})$ be the maximal compact subgroup of $SL_2(\mathbb{C})$ and let $\mathfrak{j}$ be the complexified Lie algebra of $SL_2(\mathbb{C})$. Note that since K is connected, its action is completely determined by its Lie algebra $\mathfrak{k} := \mathrm{Lie}(K) \otimes \mathbb{C}$.

4.2. Let V_i ($i = 1, 2, 3$) be the standard representation of $SU_2(\mathbb{C})$ with standard basis $\{x_i, y_i\}$. Let V_i^* be its dual space with dual basis $\{x_i^*, y_i^*\}$. We will denote $\mathrm{Sym}^n V$ by $S^n V$. Then the $SU_2(\mathbb{C})$ equivariant pairing of $S^n(V_i^*) \times S^n V_i$ is given by

$$\langle (x^*)^a (y^*)^{n-a}, x^b y^{n-b} \rangle = \frac{n!}{a!(n-a)!} \delta_{ab}.$$

Theorem 4.1. *$S^{n_1}(V_1) \otimes S^{n_2}(V_2) \otimes S^{n_3}(V_3)$ has a $SU_2(\mathbb{C})$-invariant linear form if and only if there exists non-negative integers $\alpha_1, \alpha_2, \alpha_3$ such that $n_1 = \alpha_2 + \alpha_3$, $n_2 = \alpha_3 + \alpha_1$ and $n_3 = \alpha_1 + \alpha_2$. In this case, the invariant form is a scalar multiple of*

$$\begin{vmatrix} x_1^* & x_2^* \\ y_1^* & y_2^* \end{vmatrix}^{\alpha_3} \begin{vmatrix} x_2^* & x_3^* \\ y_2^* & y_3^* \end{vmatrix}^{\alpha_1} \begin{vmatrix} x_3^* & x_1^* \\ y_3^* & y_1^* \end{vmatrix}^{\alpha_2}. \tag{25}$$

Proof. This is a consequence of the Clebsch-Gordan formula. □

We will denote the function in (25) by $\phi(k_1, k_2, k_3)$ where $k_i = n_i/2$ so that $\alpha_i = k_{i+1} + k_{i-1} - k_i$ for $i \in \mathbb{Z}/3\mathbb{Z}$.

4.3. We start with three representations $\pi_i = \pi(k_{0i}, c_i)$ ($i = 1, 2, 3$) as in Theorem 3.1 with basis $f^k_{v,i}$. Without loss of generality, we assume that $k_{01} \geq k_{02} \geq k_{03}$. We define

$$x_i^{k-v} y_i^{k+v} = (-1)^{k_i - v_i} \sqrt{(k_i - v_i)!(k_i + v_i)!} f^k_{v,i} \tag{26}$$

so that $H_+ = y_i \frac{\partial}{\partial x_i}$ and $H_- = x_i \frac{\partial}{\partial y_i}$. Note that

$$S^{2k}(V_i) = \text{Span of } \{x_i^a y_i^b : a + b = 2k\} \tag{27}$$

is an irreducible K-type of π_i.

By Theorem 4.1, $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ will exhibit an invariant form of $K = SU_2(\mathbb{C})$ if and only if $k_{01} + k_{02} + k_{03}$ is an integer.

4.4. Suppose ϕ is a $(\mathfrak{j}, K)$-invariant form on Π. Then by Theorem 4.1

$$\phi = \sum_{k_1, k_2, k_3} d(k_1, k_2, k_3) \phi(k_1, k_2, k_3) \tag{28}$$

where $d(k_1, k_2, k_3) \in \mathbb{C}$. Define

$$\begin{aligned}
&\phi(v_1, v_2, v_3; k_1, k_2, k_3) := \phi(f^{k_1}_{v_1,1} \otimes f^{k_2}_{v_2,2} \otimes f^{k_3}_{v_3,3}) \\
&\Phi(v_1, v_2, v_3; k_1, k_2, k_3) := \phi(x_1^{k_1 - v_1} y_1^{k_1 + v_1} \otimes x_2^{k_2 - v_2} y_2^{k_2 + v_2} \otimes x_3^{k_3 - v_3} y_3^{k_3 + v_3}) \\
&= (-1)^{k_1 + k_2 + k_3 - v_1 - v_2 - v_3} \prod_{j=1}^{3} \sqrt{(k_j + v_j)!(k_j - v_j)!} \phi(v_1, v_2, v_3; k_1, k_2, k_3).
\end{aligned}$$

It is relatively easy to show uniqueness of ϕ in some cases.

Proposition 4.2. *Suppose $k_{01} \leq k_{02} + k_{03}$ and let ϕ be an invariant form on $\pi_1 \otimes \pi_2 \otimes \pi_3$ as given in (28). Then:*

(i) *ϕ is unique up to scalars.*

(ii) *It is non-zero if and only if it is non-zero on (cf. (27))*

$$S^{2k_{01}}(V_1) \otimes S^{2k_{02}}(V_2) \otimes S^{2k_{03}}(V_3). \tag{29}$$

Proof. By Theorem 4.1 (ii) implies (i). Consider the Cartan decomposition of $\mathfrak{j} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p}$ is spanned by $\{F_+, F_-, F_3\}$. The action of $\mathfrak{p}$ on the K-types of π_i defines the following maps of SU_2 representations.

$$\mathfrak{p} \otimes S^n(V_i) \simeq S^2(V_i) \otimes S^n(V_i) \xrightarrow{\varphi_i} S^{n+2}(V_i)$$

where φ_i denotes the multiplication of polynomials of degree 2 and n.

To prove (ii), we suppose ϕ is zero on (29). We will now prove that $\phi \equiv 0$ by induction. Suppose ϕ is zero on

$$S(a,b,c) := S^a(V_1) \otimes S^b(V_2) \otimes S^c(V_3)$$

for all $k_{01} + k_{02} + k_{03} \leq a + b + c \leq n$. The action of $\mathfrak{p}$ defines an action

$$\begin{aligned} &\mathfrak{p} \otimes S(a,b,c) \\ &\simeq S^2 \otimes S(a,b,c) \xrightarrow{\overline{\varphi}} S(a+2,b,c) \oplus S(a,b+2,c) \oplus S(a,b,c+2) \end{aligned}$$

where $\overline{\varphi} = \varphi_1 \otimes 1 \otimes 1 + 1 \otimes \varphi_2 \otimes 1 + 1 \otimes 1 \otimes \varphi_3$. The kernel of $\overline{\varphi}$ lies in $\sum S(a,b,c)$ where the sum is taken over all $a+b+c \leq n$. By induction, ϕ is zero on the kernel of $\overline{\varphi}$. Since ϕ is $\mathfrak{j}$-invariant, it is zero on the image of $\overline{\varphi}$. On the other hand, the restriction of ϕ on the codomain of $\overline{\varphi}$ is a linear combination L of functions $\phi(\frac{a}{2}+1, \frac{b}{2}, \frac{c}{2})$, $\phi(\frac{a}{2}, \frac{b}{2}+1, \frac{c}{2})$ and $\phi(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}+1)$. The following lemma completes the induction by showing that $\phi = 0$ on the codomain of $\overline{\varphi}$.

Lemma 4.3. *Let L be a linear combination of $\phi(\frac{a}{2}+1, \frac{b}{2}, \frac{c}{2})$, $\phi(\frac{a}{2}, \frac{b}{2}+1, \frac{c}{2})$ and $\phi(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}+1)$. Suppose L is zero on the image of $\overline{\varphi}$, then L is zero.*

Proof. We assume that $a \leq b \leq c$ and $c \leq a + b$. Let $\alpha_1 = \frac{1}{2}(b+c-a)$ and $\alpha_2 = \frac{1}{2}(a+c-b)$. Clearly $\alpha_1 \geq \alpha_2$ and we further assume that $\alpha_1 \geq 1$.

If e is a non-negative integer, we denote $\varrho(e) = \frac{1}{2}e(e+1)$. Suppose

$$\begin{aligned} L = &\frac{z_1}{\varrho(a+1)}\phi\left(\frac{a}{2}+1, \frac{b}{2}, \frac{c}{2}\right) + \frac{z_2}{\varrho(b+1)}\phi\left(\frac{a}{2}, \frac{b}{2}+1, \frac{c}{2}\right) \\ &+ \frac{z_3}{\varrho(c+1)}\phi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}+1\right) \end{aligned}$$

where $z_i \in \mathbb{C}$. If $\alpha_2 = 0$, then $a = 0$, $b = c$ and $\phi(\frac{a}{2}, \frac{b}{2}+1, \frac{c}{2}) = \phi(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}+1) = 0$. In this case we set $z_2 = z_3 = 0$.

Consider

$$v_1 := x^2 \otimes x_1^a x_2^{\alpha_1 - 1} y_2^{\alpha_3+1} y_3^c, \qquad v_2 := x^2 \otimes x_1^a y_2^b x_3^{\alpha_1-1} y_3^{\alpha_2+1}$$

in $S^2 \otimes S(a,b,c)$. By (25), $L(\overline{\varphi}(v_1)) = L(\overline{\varphi}(v_2)) = 0$ shows that the coefficients z_i must satisfy

$$\varrho(a+1)^{-1}z_1 + \varrho(\alpha_1)^{-1}z_2 + \varrho(\alpha_1)z_3 = 0 \tag{30}$$

$$\varrho(a+1)^{-1}z_1 + \varrho(\alpha_1)z_2 + \varrho(\alpha_1)^{-1}z_3 = 0. \tag{31}$$

If $\alpha_2 = 0$, then $z_2 = z_3 = 0$ and (30) implies that $z_1 = 0$.

From now on we suppose that $\alpha_2 \geq 1$. Solving (30) and (31) we get $z_2 = z_3$. By symmetry we have $z_1 = z_3$. Putting these back into (30) gives $(\varrho(a+1)^{-1} + \varrho(\alpha_1)^{-1} + \varrho(\alpha_1))z_1 = 0$. Since the coefficients are strictly positive, $z_1 = 0$. Hence $z_1 = z_2 = z_3 = 0$. This proves the lemma and the proposition. $\square$

4.5. Considering the action of F_3 (see (15)) on ϕ we have

$$\begin{aligned} 0 \;=\; & \sqrt{k_1^2 - v_1^2}C_{k_1}\phi(v_1,v_2,v_3;k_1-1,k_2,k_3) + \\ & \sqrt{k_2^2 - v_2^2}C_{k_2}\phi(v_1,v_2,v_3;k_1,k_2-1,k_3) + \\ & \sqrt{k_3^2 - v_3^2}C_{k_3}\phi(v_1,v_2,v_3;k_1,k_2,k_3-1) + \\ & -(v_1A_{k_1} + v_2A_{k_2} + v_3A_{k_3})\phi(v_1,v_2,v_3;k_1,k_2,k_3) + \\ & -\sqrt{(k_1+1)^2 - v_1^2}C_{k_1+1}\phi(v_1,v_2,v_3;k_1+1,k_2,k_3) + \\ & -\sqrt{(k_2+1)^2 - v_2^2}C_{k_2+1}\phi(v_1,v_2,v_3;k_1,k_2+1,k_3) + \\ & -\sqrt{(k_3+1)^2 - v_3^2}C_{k_3+1}\phi(v_1,v_2,v_3;k_1,k_2,k_3+1). \end{aligned}$$

If we perform a change of coordinates using (26), we get

$$\begin{aligned} 0 \;=\; & -(k_1^2 - v_1^2)C_{k_1}\Phi(v_1,v_2,v_3;k_1-1,k_2,k_3) + \\ & -(k_2^2 - v_2^2)C_{k_2}\Phi(v_1,v_2,v_3;k_1,k_2-1,k_3) + \\ & -(k_3^2 - v_3^2)C_{k_3}\Phi(v_1,v_2,v_3;k_1,k_2,k_3-1) + \\ & -(v_1A_{k_1} + v_2A_{k_2} + v_3A_{k_3})\Phi(v_1,v_2,v_3;k_1,k_2,k_3) + \\ & + C_{k_1+1}\Phi(v_1,v_2,v_3;k_1+1,k_2,k_3) + \\ & + C_{k_2+1}\Phi(v_1,v_2,v_3;k_1,k_2+1,k_3) + \\ & + C_{k_3+1}\Phi(v_1,v_2,v_3;k_1,k_2,k_3+1). \end{aligned} \tag{32}$$

Define the polynomial

$$\begin{aligned} & P(x_1,x_2,x_3,y_1,y_2,y_3) \\ := \; & \sum_{j=1}^{3} C_{k_j} d(k_{j-1},k_j-1,k_{j+1})x_jy_j\phi(k_{j-1},k_j-1,k_{j+1}) + \end{aligned}$$

$$- \frac{1}{2} A_{k_j} d(k_{j-1}, k_j - 1, k_{j+1}) \left(y_j \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial x_j} \right) \phi(k_1, k_2, k_3) +$$
$$- C_{k_j+1} d(k_{j-1}, k_j + 1, k_{j+1}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_j} \phi(k_{j-1}, k_j + 1, k_{j+1}).$$

Let $p = \prod_{i=1}^{3} (k_i + v_i)!(k_i - v_i)!$, then the left hand side of (32) is p times the coefficient of $x_i^{k_i - v_i} y_i^{k_i + v_i}$, $(|v_i| \le k_i)$ in P. Hence (32) is equivalent to the following polynomial being zero.

$$\begin{aligned}
(33) \quad & \phi(k_1 - 1, k_2 - 1, k_3 - 1)^{-1} P(x_1, x_2, x_3, y_1, y_2, y_3) \\
& = x_1 y_1 D_1^2 d_1 + x_2 y_2 D_2^2 d_2 + x_3 y_3 D_3^2 d_3 + \\
& \quad + (A_{k_1} D_1 (k_1 x_2 x_3 y_1^2 + (k_3 - k_2) x_1 y_1 D_1 - k_1 x_1^2 y_2 y_3) + \\
& \quad + A_{k_2} D_2 (k_2 x_3 x_1 y_2^2 + (k_1 - k_3) x_2 y_2 D_2 - k_2 x_2^2 y_3 y_1) + \\
& \quad + A_{k_3} D_3 (k_3 x_1 x_2 y_3^2 + (k_2 - k_1) x_3 y_3 D_3 - k_3 x_3^2 y_1 y_2)) d + \\
& \quad + (D_3^2 \alpha_2 \beta_2 x_3 y_3 + D_2^2 \alpha_3 \beta_3 x_2 y_2 - D_3 D_2 \beta_2 \beta_3 (x_3 y_2 + x_2 y_3)) l_1 + \\
& \quad + (D_1^2 \alpha_3 \beta_3 x_1 y_1 + D_3^2 \alpha_1 \beta_1 x_3 y_3 - D_1 D_3 \beta_3 \beta_1 (x_1 y_3 + x_3 y_1)) l_2 + \\
& \quad + (D_2^2 \alpha_1 \beta_1 x_2 y_2 + D_1^2 \alpha_2 \beta_2 x_1 y_1 - D_2 D_1 \beta_1 \beta_2 (x_2 y_1 + x_1 y_2)) l_3
\end{aligned}$$

where

$$l_i = C_{k_i+1} d(k_{i-1}, k_i + 1, k_{i+1}), \quad D_i = \begin{vmatrix} x_{i+1} & x_{i+2} \\ y_{i+1} & y_{i+2} \end{vmatrix},$$
$$\beta_i = \alpha_i + 1, \quad d_i = C_{k_i} d(k_{i-1}, k_i - 1, k_{i+1}), \quad d = d(k_1, k_2, k_3).$$

There are seven non-zero coefficients in the polynomial (33) and three of them are

$$\begin{aligned}
(34) \quad & d_1 - \beta_2 \beta_3 l_1 + \beta_3 (1 + 2k_2) l_2 + \beta_2 (1 + 2k_3) l_3 + \\
& \quad + ((A_{k_2} - A_{k_1}) k_2 + (A_{k_1} - A_{k_3}) k_3) d = 0 \\
(35) \quad & d_2 + \beta_3 (1 + 2k_1) l_1 - \beta_1 \beta_3 l_2 + \beta_1 (1 + 2k_3) l_3 + \\
& \quad + ((A_{k_3} - A_{k_2}) k_3 + (A_{k_2} - A_{k_1}) k_1) d = 0 \\
(36) \quad & d_3 + \beta_2 (1 + 2k_1) l_1 + \beta_1 (1 + 2k_2) l_2 - \beta_1 \beta_2 l_3 + \\
& \quad + ((A_{k_1} - A_{k_3}) k_1 + (A_{k_3} - A_{k_2}) k_2) d = 0.
\end{aligned}$$

The rest of the coefficients are linear combinations of the above three equations. Solving for l_1, l_2, l_3 gives

$$\begin{aligned}
(37) \quad R_1 l_1 = & -\alpha_1 \beta_1 d_1 + \beta_2 (1 + 2k_2) d_2 + \beta_3 (1 + 2k_3) d_3 + \\
& + \alpha_1 (1 + k)(-A_{k_2} + A_{k_3} + A_{k_1} k_2 - A_{k_2} k_2 - A_{k_1} k_3 + A_{k_3} k_3) d \\
(38) \quad R_2 l_2 = & \beta_1 (1 + 2k_1) d_1 - \alpha_2 \beta_2 d_2 + \beta_3 (1 + 2k_3) d_3 + \\
& + \alpha_2 (1 + k)(A_{k_1} - A_{k_3} + A_{k_1} k_1 - A_{k_2} k_1 + A_{k_2} k_3 - A_{k_3} k_3) d \\
(39) \quad R_3 l_3 = & \beta_1 (1 + 2k_1) d_1 + \beta_2 (1 + 2k_2) d_2 - \alpha_3 \beta_3 d_3 +
\end{aligned}$$

$$+ \alpha_3(1+k)(-A_{k_1} + A_{k_2} - A_{k_1}k_1 + A_{k_3}k_1 + A_{k_2}k_2 - A_{k_3}k_2)d$$

where $k = k_1 + k_2 + k_3$ and $R_i = -(1+k)(2+k)\prod_{j\neq i}\beta_j$.

Lemma 4.4. *The linear form ϕ in (26) is $\mathfrak{j}$-invariant if and only if its coefficients $d(k_1,k_2,k_3)$ satisfy (37), (38) and (39).*

Proof. ϕ is K-invariant. Since $\mathfrak{k} = \mathrm{Lie}(K)\otimes\mathbb{C}$ and F_3 generates $\mathfrak{j}$, ϕ is $\mathfrak{j}$-invariant if and only if $F_3\phi = 0$. The latter is true if and only if (33) is zero if and only if (37), (38) and (39) are satisfied. □

4.6. For technical reasons which will be clear later, we assume that in addition to $k_{01} \geq k_{02} \geq k_{03}$, π_i satisfy the following condition:

If $k_a := k_{01} - k_{02} - k_{03} > 0$, then there does NOT exist non-negative integers r, s and $r + s = k_a - 1$ satisfying

$$c_1 = \frac{c_2}{k_{02}+r} + \frac{c_3}{k_{03}+s}. \tag{40}$$

Proposition 4.5. *Assuming §4.6, then the dimension of $(\mathfrak{j}, K)$-invariant trilinear form on $\pi_1\otimes\pi_2\otimes\pi_3$ is at most 1.*

Proof. Suppose $k_{01} \leq k_{02}+k_{03}$. Note that the leading coefficients of R_i of l_i in (37), (38) and (39) do not vanish. By induction $d(k_1,k_2,k_3)$ is determined by $d(k_{01},k_{02},k_{03})$.

Next if $k_{01} > k_{02}+k_{03}$, then by (37), (38) and (39) we deduce that $d(k_1,k_2,k_3)$ is determined by $d(k_{01},k_{02}+r',k_{03}+s)$ where $r'+s = k_{01} - k_{02} - k_{03}$. We will show that $d(k_{01},k_{02}+r',k_{03}+s)$ determines one another. Suppose $r' = r+1 \geq 1$, then (38) (resp. (39)) gives a linear relation between $d(k_{01},k_{02}+r+1,k_{03}+s)$ (resp. $d(k_{01},k_{02}+r,k_{03}+s+1)$) and $d' := d(k_{01},k_{02}+r+1,k_{03}+s+1)$. Hence $d(k_{01},k_{02}+r+1,k_{03}+s)$ and $d(k_{01},k_{02}+r,k_{03}+s+1)$ are linearly related provide $d'\neq 0$. By (37) and (38) $d' = 0$ if and only if (40) holds. □

Corollary 4.6. *The invariant ϕ is nonzero on the subspace:*

(i) $S(2k_{01},2k_{02},2k_{03})$ *if* $k_{01}\leq k_{02}+k_{03}$.

(ii) $S(2k_{01},a,b)$ *if* $k_{01} > k_{02}+k_{03}$, $a\equiv k_{02} \pmod{\mathbb{Z}}$, $b\equiv k_{03} \pmod{\mathbb{Z}}$ *and* $a+b = k_{01}$. (*Under the assumption in* §4.6.)

If (40) holds, then there are at most 2 solutions of (r,s). It follows from the last proof that the dimension of the trilinear form is at most 3. We conjecture that Proposition 4.5 is still true without the assumptions in §4.6.

4.7. *Proof of Theorem 1.2 for $F = \mathbb{C}$.* The proof of Proposition 4.5 provides a way of constructing ϕ. We will inductively construct $d(k_1,k_2,k_3)$ in the following way. Suppose we have already determined $d(k_1,k_2,k_3)$ for $k_1+k_2+k_3\leq n$ and (37), (38) and (39) are satisfied whenever the $d(k_1,k_2,k_3)$ in the equations have been defined.

We define $d(k_1, k_2, k_3)$ for $k_1 + k_2 + k_3 = n + 1$ using either (37), (38) or (39). It remains to show that $d(k_1, k_2, k_3)$ is independent of the equations used.

Consider Figure 5 below where the integral points (k_1, k_2, k_3) are labeled 1 to 14.

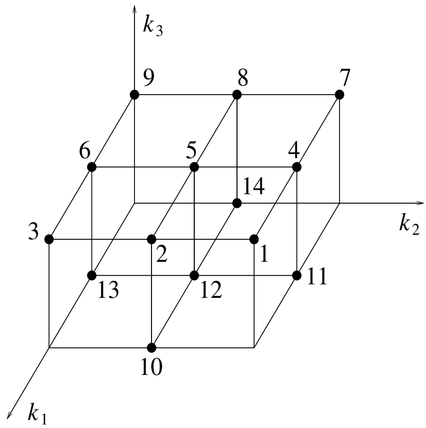

Figure 5.

We will denote the values of $d(k_1, k_2, k_3)$ at point s simply by d_s. Suppose d_s are defined except d_1. d_1 can be determined by d_2, d_3, d_5 and d_{10} using (38). d_3 is a linear combination of d_6, d_9, d_5 and d_{13}. d_2 is a linear combination of d_5, d_6, d_8 and d_{12}. d_{10} is a linear combination of d_{12}, d_{13}, d_{14} and d_5. Hence $d_1 = L_2(d_5, d_6, d_8, d_9, d_{12}, d_{13}, d_{14})$ where L_2 is a linear combination of its entries.

Alternatively d_1 can be determined by d_4, d_5, d_7 and d_{11} using (37). Substituting d_4, d_7 and d_{11} in a similar manner as in the last paragraph indicates that $d_1 = L_1(d_5, d_6, d_8, d_9, d_{12}, d_{13}, d_{14})$ where L_1 is a linear combination of its entries. The following lemma completes the proof of Theorem 1.2.

Lemma 4.7. *$L_1 = L_2$.*

Proof. The proof is simply achieved by writing out L_1 and L_2 in full, simplify and compare. However the equations are long and tedious and we omit the details. □

4.8. Question. We conjecture that Theorems 1.2 and 1.3 can be extended to include reducible infinitesimal principal representations of $GL_2(\mathbb{C})$.

References

[F] Y. Flicker, *On the distinguished representations*, Crelle, **418** (1991), 139-172.

[GP] B. Gross and D. Prasad, *On irreducible representations of* $SO_{2n+1} \times SO_{2m}$, Can. J. Math., **46(5)** (1994), 930-950.

[HS] M. Harris and A.J. Scholl, *A note on trilinear forms for reducible representations and Beilinson's conjectures*, preprint.

[JL] H. Jacquet and R. Langlands, *Automorphic forms on* GL_2, LNM, **114** (1970).

[Na] M.A. Naimark, *Linear representation of the Lorentz group*, **61**, Pergamon Press, 1964.

[Pa1] D. Prasad, *Trilinear forms for representations of* GL_2 *and local* ϵ *factors*, Compositio Mathematica, **75** (1990), 1-46.

[Pa2] ———, *Invariant forms for representation of* GL_2 *over a local field*, Amer. J. Math., **114** (1992), 1317-1363.

[Re] J. Repka, *Tensor products of unitary representations of* $SL_2(\mathbb{R})$, Amer. J. Math., **100(4)** (1978), 747-774.

[Uz] T. Uzawa, *Invariant hyperfunction sections of line bundles*, Ph.D thesis, Yale University, 1990.

[Wa] N. Wallach, *Real reductive groups* I *and* II, Academic Press, 1992.

[W] F.L. Williams, *Tensor products of principal series representations*, LNM, **358**, Springer Verlag, 1973.

Received April 6, 1999 and revised September 20, 1999.

Department of Mathematics
National University of Singapore
Singapore 119260
E-mail address: loke@math.nus.edu.sg