A PROBLEM OF MCMILLAN ON CONFORMAL MAPPINGS

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We answer one of two questions asked by McMillan in 1970 concerning distortion at the boundary by conformal mappings of the disk.

1. Introduction.

The purpose of this note is to answer a question of J.E. McMillan concerning boundary behavior of conformal mappings which was raised in the paper [4]. In that paper, McMillan gave a sufficient geometric condition for a subset of the boundary of a domain to have harmonic measure zero and used it to prove a result which we will describe below. A similar geometric lemma was the key to the original proof of the twist point theorem in [5]. The reader can refer to both of McMillan’s papers and to [6] for background on these problems and more generally to [1], [3] and [7] for the ideas used in this paper.

We will use \( \omega(z_0, F, \Omega) \) to denote the harmonic measure of the set \( F \) in the domain \( \Omega \) from the point \( z_0 \). Let \( \mathbb{D} \) denote the unit disk in the complex plane and let \( f : \mathbb{D} \to \Omega \) be a conformal map. Let \( A \) denote the set of all ideal accessible boundary points \( f(e^{i\theta}) \) of \( \Omega \) when \( f \) has the nontangential limit \( f(e^{i\theta}) \) at \( e^{i\theta} \). Note that points of \( A \) are prime ends of \( \Omega \) so that a single complex coordinate may represent more than one point of \( A \).

Let \( D(a, r) \) denote a disk with center \( a \) and radius \( r \). Choose \( r_0 < d(f(0), A) \) where \( d \) denotes Euclidean distance. For each \( a \in A \) and for each \( r < r_0 \) let \( \gamma(a, r) \subset \partial D(a, r) \) be the crosscut of \( \Omega \) separating \( a \) from \( f(0) \) which can be joined to \( a \) by a Jordan arc in \( \Omega \cap D(a, r) \). Let \( L(a, r) \) denote the Euclidean length of \( \gamma(a, r) \) and let \( U(a, r) = \bigcup_{r'<r} \gamma(a, r') \).

Let

\[
A(a, r) = \int_0^r L(a, \rho) \, d\rho
\]

denote the Lebesgue measure of \( U(a, r) \).

McMillan proved:
Theorem 1.1. The set of $a \in A$ such that
\[ \limsup_{r \to 0} \frac{A(a,r)}{\pi r^2} < \frac{1}{2} \]
has harmonic measure zero.

Notice that this theorem implies that the set of $a \in A$ such that
\[ \limsup_{r \to 0} \frac{L(a,r)}{2\pi r} < \frac{1}{2} \]
has harmonic measure zero.

McMillan also gave an example of a domain for which both
\[ \limsup_{r \to 0} \frac{A(a,r)}{\pi r^2} = 1 \quad \omega \ a.e. \]
and
\[ \liminf_{r \to 0} \frac{A(a,r)}{\pi r^2} = 0 \quad \omega \ a.e. \]
(implying the corresponding limits for $\frac{L(a,r)}{2\pi r}$) and conjectured that

\[ E_1 = \left\{ a \in A : \liminf_{r \to 0} \frac{A(a,r)}{\pi r^2} > \frac{1}{2} \right\} \]
and
\[ E_2 = \left\{ a \in A : \liminf_{r \to 0} \frac{L(a,r)}{2\pi r} > \frac{1}{2} \right\} \]
must be sets of harmonic measure zero.

Here, we will verify McMillan’s conjecture that the set $E_2$ must always have zero harmonic measure.

2. There are no points of density in $f^{-1}(E_2)$.

With the notations and definitions of the introduction we prove:

Theorem 2.1. The harmonic measure of the set $E_2$ is zero.

Proof. For any positive integers $m$ and $k$, let
\[ E_{m,k} = \left\{ a \in A : L(a,r) > \left( \frac{1}{2} + \frac{1}{m} \right) 2\pi r \quad \forall r < \frac{1}{k} \right\}. \]
Since $E_2$ is the countable union of sets $E_{m,k}$, it suffices to show that each $E_{m,k}$ has harmonic measure zero.

We will require the following lemma (see [7], p. 142) which is a consequence of results of Beurling, [2].
Lemma 2.1. Let \( f \) map \( \mathbb{D} \) conformally into \( \mathbb{C} \) and let \( 0 < \delta < 1 \). If \( z \in \mathbb{D} \) and \( I \) is an arc of \( \mathbb{T} \) with \( \omega(z, I) \geq \alpha > 0 \) then there exists a Borel set \( B \subset I \) such that
\[
|f(\xi) - f(z)| \leq \Lambda(f(S)) < K(\delta, \alpha)d_f(z) \quad \text{for} \quad \xi \in B
\]
where \( \Lambda \) denotes linear measure, \( d_f(z) \) is the euclidean distance from \( f(z) \) to the boundary of \( f(\mathbb{D}) \), \( S \) is the non-euclidean segment from \( z \) to \( \xi \) and where \( K(\delta, \alpha) \) depends only on \( \delta \) and \( \alpha \).

The basic idea of the proof of Theorem 2.1 is that since points of \( E_{m,k} \) are separated from \( f(0) \) by circular arcs of wide angle and large radius, if \( f^{-1}(E_{m,k}) \) has a point of density then Lemma 2.1 will provide enough wide angled circular arcs of a fixed radius to wrap around on themselves and disconnect the domain \( \Omega \).

Suppose then that \( \eta \in \mathbb{T} \) is a point of density of \( f^{-1}(E_{m,k}) \) and let \( I \) denote an arc of \( \mathbb{T} \) centered at \( \eta \).

Given \( \delta_1 > 0 \) we can choose \( I \) such that
\[
\frac{|f^{-1}(E_{m,k}) \cap I|}{|I|} > (1 - \delta_1).
\]

Given \( \delta_2 > 0 \) we can find \( 0 < r(I, \delta_2) < 1 \) such that
\[
\omega((1 - r(I, \delta_2))\eta, I, \mathbb{D}) = 1 - \delta_2
\]
and this determines the point \( z_I = (1 - r(I, \delta_2))\eta \).

If we are given \( \delta_3 > 0 \) then if \( \delta_1 \) is sufficiently small, (1) implies that
\[
\omega(f^{-1}(E_{m,k}), z_I, \mathbb{D}) > (1 - \delta_3).
\]

By Lemma 2.1, if we are given \( \delta_4 > 0 \) then there is a Borel set \( B \subset I \) such that
\[
\omega(z_I, B, \mathbb{D}) > (1 - \delta_4)(1 - \delta_2)
\]
and such that
\[
|f(\xi) - f(z_I)| < K(\delta_4, (1 - \delta_2))d_f(z_I) \quad \forall \xi \in B.
\]

It follows that
\[
\omega(f^{-1}(E_{m,k}) \cap B, z_I, \mathbb{D}) > 1 - (\delta_2 + \delta_3 + \delta_4 - \delta_2\delta_4)
\]
and that (2) holds for all \( \xi \in f^{-1}(E_{m,k}) \cap B \). Notice that the constant \( K \) only depends on \( \delta_2 \) and \( \delta_4 \).

Since \( f(\eta) \in A \) we can choose \( I \) so that \( Kd_f(z_I) << \frac{1}{k} \) where \( k \) is the integer in the definition of \( E_{m,k} \). The finite number of steps required to get a contradiction in the construction to follow will only depend on the number \( m \) in the definition of \( E_{m,k} \). By choosing a sufficiently small arc \( I \), we can arrange that in each step of our construction, the positive number
\[
\delta \equiv \delta_2 + \delta_3 + \delta_4 - \delta_2\delta_4
\]
is small enough so that the construction can proceed to the next step. We assume that these conditions hold on the size of the interval $I$.

Let $w_0 = f(0), \ w_1 = f(z_I), \ d_0 = d_f(z_I)$ and let $x_1$ be a point of $\partial \Omega$ such that $|x_1 - w_1| = d_0$. Let the letters $c_1, c_2, \ldots$ denote positive constants which will be assumed to be sufficiently small in each step of the construction but will ultimately depend only on the number $m$ in the definition of the set $E_{m,k}$ and not on $f$, $\Omega$, or $\delta$. Let $C_1, C_2, \ldots$ denote other constants which may be purely numerical or which may depend only on the number $m$.

First let $0 < c_0 \ll 1$ and $c_1 \ll \frac{\pi}{m} c_0$. We will see that these choices allow for rotation by a fixed positive angle of certain separating circular arcs in consecutive steps of the construction to follow. The arc of $\partial D(x_1, c_1 d_0)$ which intersects the interior of $D(w_1, d_0)$ extends to a crosscut of $\Omega$ and determines a unique subdomain $U_1 \subset \Omega$ not containing $w_1$. We proceed to find a point close to $x_1$ which is contained in $E_{m,k}$. By Harnack’s inequality,

$$\omega(w_1, \partial U_1 \cap \partial \Omega \cap D(x_1, c_1 d_0), \Omega) \geq C_1 \omega(w_1', \partial U_1 \cap \partial \Omega \cap D(x_1, c_1 d_0), \Omega)$$

where $w_1'$ is the point on the line between $w_1$ and $x_1$ such that $|x_1 - w_1'| = \frac{c_1 d_0}{2}$. By the comparison principle for harmonic measure and the Beurling projection theorem, ([1], p. 43),

$$\omega(w_1', \partial U_1 \cap \partial \Omega \cap D(x_1, c_1 d_0), \Omega) \geq C_2 > 0.$$ 

So by Lemma 2.1 and Equation (3), if $\delta << C_1 C_2$, there is a constant $C_3$ such that

$$\omega(w_1, \partial U_1 \cap \partial \Omega \cap D(x_1, c_1 d_0) \cap E_{m,k}, \Omega) \geq C_3 > 0.$$ 

Choose a point $x_1^* \in \partial U_1 \cap \partial \Omega \cap D(x_1, c_1 d_0) \cap E_{m,k}$. If $c_0$ is sufficiently small then the arc of $\{z \in \mathbb{C} : |x_1^* - z| = c_0 d_0\}$ which intersects $D(w_1, d_0)$ has an angle greater than $\pi(1 - \frac{1}{2m})$. This arc must therefore be part of the crosscut whose length is $L(x_1^*, c_0 d_0) > \pi(1 + \frac{1}{m})$. Denote by $\overline{ab}$ the segment with endpoints $a \in \mathbb{C}$ and $b \in \mathbb{C}$. Let $w_1^*$ be the point on $\overline{x_1^* w_1}$ with $|x_1 - w_1^*| = c_0 d_0$ and consider the annulus

$$R_1 = \{z \in \mathbb{C} : (1 - c_2)|x_1^* - w_1^*| < |x_1^* - z| < (1 + c_2)|x_1^* - w_1^*|\}$$

where $c_2 \ll \frac{\pi}{m} c_0$. Let $S_1$ be the component of $R_1 \cap \Omega$ which intersects $D(w_1, d_0)$ and let $x_2$ be a point of $\partial \Omega \cap S_1$ such that $x_2^* x_2$ has minimal angle clockwise from $\overline{x_1^* w_1}$.

Let $S_1^*$ denote the sector of $R_1$ clockwise between $\overline{x_1^* w_1^*}$ and $\overline{x_1^* x_2}$. The circular arc $\partial D(x_2, c_2 d_0) \cap S_1^*$ is part of a crosscut of $\Omega$ which determines a unique subdomain $U_2 \subset \Omega$ not containing $w_1^*$. By an argument similar to the previous one using Harnack’s inequality, the comparison principle for harmonic measure and the Beurling projection theorem but now in the annular sector $S_1$, it follows that

$$\omega(w_1, \partial \Omega \cap \partial U_2 \cap D(x_2, c_2 d_0), \Omega) > C_4 > 0.$$
We remark that $C_4$ depends on $c_0, c_1, c_2$ and therefore only on $m$ and that the remaining constants $C_j$ may have similar dependence on $m$.

A simple geometric argument shows that there is a point $x^*_2$ in $D(x_2, c_2d_0) \cap E_{m,k}$ and a constant $c_3 > 0$ determined by the diameter of the $E_{m,k} \cap D(x_2, c_2d_0)$ such that the set of distances

$$\{|x^*_2 - w| : w \in D(x_1, c_1d_0) \cap \partial \Omega\}$$

contains an interval $J_1$ of length greater than $c_3d_0$.

Let $R_2 = \{ w \in \mathbb{C} : |w - x^*_2| \in J_1 \}$ and let $S_2$ be the component of $R_2 \cap \Omega$ which intersects $S_1$. Each of the circular arcs of $S_2$ centered at $x^*_2$ is a crosscut of $\Omega$. If there is such a crosscut $L_1 \subset S_2 \cap \Omega$ which does not separate $x^*_2$ from $w_0$ then we repeat the above construction of $S_2$ but in the counterclockwise direction from $\overline{x_1^2w_1^*}$. Then any circular arc $L_2 \subset S_2 \cap \Omega$ centered at $x^*_2$ which intersects $S_1$, separates $x^*_2$ from $w_0$. For otherwise, $w_0$ is contained in both subdomains of $\Omega$ determined by the concave sides of $L_1$ and $L_2$. Since $w_0$ lies on the convex side of any circular arc which defines $L(a, r)$ for some $a \in A$ and $r > 0$ and therefore of any arc of $S_1$, this is impossible. If one choice of $S_2$, clockwise or counterclockwise from $\overline{x_1^2w_1^*}$, fails to separate $x^*_2$ from $w_0$ we choose the other. Otherwise, the construction can continue, as described below, in both directions until the non-separating case occurs and after that point, a topological argument similar to the above allows the construction to continue in the remaining direction.

We have now arranged that each of the circular arcs of $S_2$ centered at $x^*_2$ separates $x^*_2$ from $w_0$ and can be joined to $x^*_2$ by a Jordan arc lying inside $S_1$. Therefore, since $x^*_2 \in E_{m,k}$, each circular arc of $S_2$ has an angular measure greater than $(1 + \frac{c}{n})\pi$. Let $w_2$ be a point of $S_2 \cap S_1$ and let $x_3$ be a point of $\partial \Omega \cap S_2$ which minimizes the clockwise angle from $\overline{x_2^2w_2}$ to $\overline{x_2^2x_3}$. Let $S^*$ denote the sector of $R_2$ clockwise between $\overline{x_2^2w_2}$ and $\overline{x_2^2x_3}$. As before the circular arc $\partial D(x_3, c_3d_0) \cap S^*$ extends to a crosscut of $\Omega$ which determines a unique subdomain of $U_3 \subset \Omega$ not containing $w_1$. The same harmonic measure argument as before but now done in the union of annular corridors $S_1 \cup S_2$ shows that

$$\omega(w_1, \partial \Omega \cap \partial U_3 \cap D(x_3, c_3d_0), \Omega) > C_6 > 0.$$ 

If $\delta > 0$ is sufficiently small, then as before, Lemma 2.1 and (3) imply that

$$\omega(w_1, \partial \Omega \cap \partial U_3 \cap D(x_3, c_3d_0) \cap E_{m,k}, \Omega) > C_7 > 0$$

and we find $x^*_3 \in \partial \Omega \cap \partial U_3 \cap D(x_3, c_3d_0) \cap E_{m,k}$ such that the set of distances

$$\{|x^*_3 - w| : w \in D(x_2^*, c_3d_0) \cap \partial \Omega\}$$

contain an interval $J_3$ of length greater than $c_4d_0$, where $c_4$ depends only on the previous $c_3$ and on $m$. Note that since the constants satisfy $c_3 \ll c_0 \frac{\pi}{m}$, there is a numerical constant $c > 0$ such that the clockwise angle from $\overline{x_1^2x_2}$
to \(x_j^*\). The construction continues in this way so that having found annular corridors \(S_1, \ldots, S_j\) with centers \(x_1^*, x_2^*, \ldots x_j^*\) we find \(x_{j+1}^* \in E_{m,k}\) so that there is an interval of distances \(J_j\) between \(x_{j+1}^*\) and the part of \(\partial \Omega\) in a disk of radius \(c_d\) centered at \(x_j^*\). The intersection of the annulus centered at \(x_{j+1}^*\) determined by \(J_j\) with \(\Omega\) contains a component \(S_{j+1}\) which intersects \(S_j\). Concentric circular arcs of this annular piece separate \(x_{j+1}^*\) from \(w_0\) (or else the construction continues in the other direction) and each such circular arc can be joined to \(x_{j+1}^*\) through the annular corridor \(S_j\) by a Jordan arc contained in the circle. Therefore, each such arc has an angle greater than \((1 + \frac{2}{m})\pi\). Let \(w_{j+1}\) be a point of \(S_{j+1} \cap S_j\) and find \(x_{j+2}\) which minimizes the clockwise angle between \(x_{j+1}^*w_{j+1}\) and \(x_{j+1}^*x_{j+2}\). The construction can continue if \(\delta > 0\) is sufficiently small since the harmonic measure of the end of \(S_{j+1}\) near \(x_{j+2}\) from \(w_1\) in \(S_1 \cup S_2 \cup \ldots S_{j+1} \cup D(w_1,d_0)\) is greater than some positive numerical constant.

But it is clear from the construction that the union of annular corridors \(S_1 \cup \cdots \cup S_j\) must wrap around on itself after a finite number of steps which only depends on \(m\). The union of annular corridors thus formed, being a subset of \(\Omega\), would contain a closed curve in \(\Omega\) whose interior component contains the points \(x_i^* \in \partial \Omega\). Since \(\Omega\) is simply connected, this contradiction shows that \(f^{-1}(E_{m,k})\) does not contain a point of density and therefore must have measure zero. Therefore \(E_{m,k}\) has harmonic measure zero in \(\Omega\). \(\square\)

**Note.** The authors have now answered the question left open here. The result will appear in a forthcoming paper.

**References**


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