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EXISTENCE AND UNIQUENESS OF SOLUTIONS ON BOUNDED DOMAINS TO A FITZHUGH–NAGUMO TYPE ELLIPTIC SYSTEM

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In this paper we prove the existence and uniqueness of the boundary layer solution to a semilinear eigenvalue problem consisting of a coupled system of two elliptic partial differential equations. Although the system is not quasimonotone, there exists a transformation to a quasimonotone system. For the transformed system we may and will use maximum (sweeping) principle arguments to derive pointwise estimates. A degree argument completes the uniqueness proof.

1. Introduction.

We consider the following nonlinear eigenvalue problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u &= \lambda(f(u) - v) & \text{in } \Omega, \\ -\Delta v &= \lambda(\delta u - \gamma v) & \text{in } \Omega, \\ \quad u &= v = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

with $\lambda, \delta, \gamma > 0$ and where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. As usual, a domain is an open connected set. The nonlinearity f is assumed to be smooth and like a third order polynomial. We prove the existence of a curve of positive solutions (u_λ, v_λ) to (P_λ) for λ large enough. These solutions are shown to be, except for a boundary layer of width $O(\lambda^{-1/2})$, close to $(\rho, (\delta/\gamma)\rho)$ where ρ a positive zero of $f(s) - (\delta/\gamma)s$ and $f'(\rho) < 0$. The stability of these solutions as equilibria of the parabolic system

$$(1) \qquad \begin{cases} u_t &= \Delta u + \lambda(f(u) - v) & \text{in } \mathbb{R}^+ \times \Omega, \\ v_t &= \Delta v + \lambda(\delta u - \gamma v) & \text{in } \mathbb{R}^+ \times \Omega, \\ \ u &= v = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases}$$

with appropriate initial conditions is also proven. Finally it is shown that these solutions are unique in an appropriate order interval.

The question of existence of solutions to (P_λ) with $\lambda = 1$ and with different kinds of nonlinearities was studied by Klaasen and Mitidieri [**9**] and De Figueiredo and Mitidieri [**7**], see also Rothe [**21**] and Lazer and McKenna [**12**]. The fact that the second equation can be inverted to solve v in terms of u and that the problem can then be written as a single equation in u was used extensively. In particular this single equation can be treated by variational techniques. Using this approach it was shown for example in [**9**] with $f(u) = u(u-1)(a-u)$, $0 < a < 1/2$ and in [**7**] for more general f of the same type, that there exist at least two nontrivial solutions, under the assumptions that δ/γ is small enough and Ω contains a large enough ball. By rescaling, this implies that there exist nontrivial solutions to (P_λ) if λ is large and δ/γ small.

Our treatment of the problem differs from the variational approach mentioned above. By imposing some natural restrictions on the parameters, which are satisfied if δ/γ is small, it is possible to make a transformation of (P_λ) and a modification of f to obtain a quasimonotone system. Solutions to the quasimonotone system in a certain range correspond to solutions to the original problem. This approach was also used in [**19**] as well as in [**14**] for other systems of equations. The advantage of working with a quasimonotone system is that for such systems a comparison principle holds. From this follows the existence of solutions between an ordered pair of sub- and supersolutions. For such systems one also has an analogue of McNabb's *sweeping principle*, see [**15**], [**2**], [**4**] and [**22**]. This will be a main tool in many of the proofs.

Using this quasimonotone approach we are able to give a complete qualitative description of a specific solution to (P_λ). This qualitative description allows us to prove uniqueness and stability results. Results in this direction were obtained by Lazer and McKenna [**12**] for a system with $\delta = \gamma$ and f such that $f(s)/s$ is decreasing on $\mathbb{R}^+$. Existence and positivity of solutions were considered in [**9**] and [**7**].

If we set $\delta = 0$ in (P_λ) then the problem reduces to the well studied scalar problem

$$(\mathrm{S}_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

There is an extensive literature on such kind of problems. We just mention [**2**], [**4**], [**13**] and more recently [**5**]. We note that our treatment of the quasimonotone system is similar to the treatment of problem (S_λ) as was done in [**2**] and [**4**]. The results of the present paper were announced in [**20**].

The structure of the paper is as follows. In the next section the precise assumptions on the nonlinearity f are stated, as well as the conditions which we impose on the parameters γ and δ. It is then shown how (P_λ) can be transformed to a quasimonotone system. The main results are also stated in this section. In Section 3 we prove several auxiliary results. The proofs of the main theorems are given in Section 4. In Appendix A we define our notion of sub- and supersolutions for quasimonotone systems and give some related results. In particular we state a version of the sweeping principle for quasimonotone systems. This principle is used repeatedly in the proofs.

2. Assumptions and main results.

The assumptions on f are the following.

Condition A. *The function $f \in C^{1,1}(\mathbb{R})$, $f(0) \geq 0$ and there exists $\sigma_0 > 0$ such that for every $0 \leq \sigma < \sigma_0$ there exist $\rho_\sigma^- < \rho_\sigma^+$ with $\rho_\sigma^+ > 0$ such that*

(1) $f(\rho_\sigma^\pm) = \sigma\rho_\sigma^\pm$ *and* $f(s) > \sigma s$ *for* $\rho_\sigma^- < s < \rho_\sigma^+$;

(2) $f'(s) < 0$ *for all* $s \in (\rho_{\sigma_0}^+, \rho_0^+)$;

(3) $J_\sigma(\rho) > 0$ *on* $(0, \rho_\sigma^+)$ *for all* $0 \leq \sigma < \sigma_0$ *where*

$$J_\sigma(\rho) := \int_\rho^{\rho_\sigma^+} (f(s) - \sigma s)\, ds. \tag{2}$$

See Figure 1.

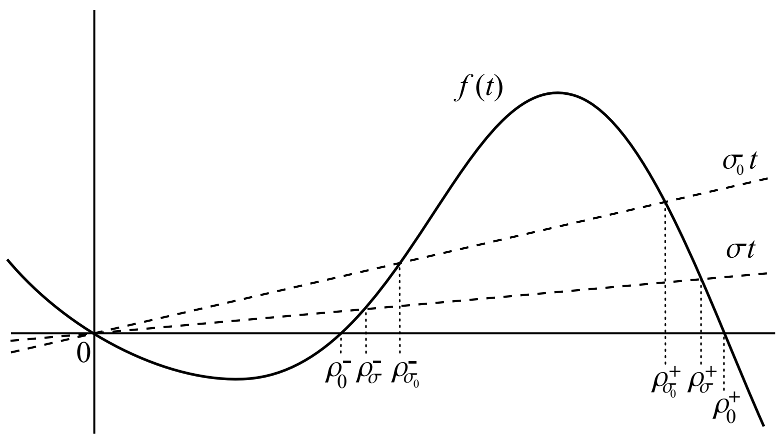

Figure 1.

Example 1. The function $f(u) = au - u^3$ with $a > 0$, see [**12**] and [**7**], satisfies Condition A above with $\sigma_0 = 2a/3$.

Example 2. Consider the function $f(u) = u(a-u)(u-1)$ with $a > 0$. Condition A holds if $a < 1/2$. In this case $\sigma_0 = \left(2a^2 - 5a + 2\right)/9$. With this nonlinearity problem (P_λ) is an extension of the FitzHugh-Nagumo equations, see [**9**] and [**10**].

As was said in the introduction, an important step in our analysis is to transform (P_λ) and to modify f in order to obtain a quasimonotone system. For the definition of a quasimonotone system and some results for such systems we refer to Appendix A. In order to transform system (P_λ) we need the following assumption on the parameters δ and γ:

Condition B1. *Let* $M := \max\left\{-f'(s)\ ;\ 0 \le s \le \rho_0^+\right\}$ *and suppose that*

$$\gamma - 2\sqrt{\delta} > M.$$

We define β and α by

$$\beta := \tfrac{1}{2}(\gamma - M) - \tfrac{1}{2}\sqrt{(\gamma - M)^2 - 4\delta}, \tag{3}$$

$$\alpha = \gamma - \beta. \tag{4}$$

If Condition B1 holds then $\beta \in \mathbb{R}$ and $\alpha, \beta > 0$. Note that $-\beta(\beta + M) = \delta - \gamma\beta$ and that

$$\vartheta := 1 - \frac{\delta}{\gamma\beta} > 0. \tag{5}$$

One may verify that (u, w) is a solution to

$$(\mathrm{Q}_\lambda) \qquad \left\{ \begin{array}{rcll} -\Delta u & = & \lambda(f(u) - \beta u + \beta w) & \text{in } \Omega, \\ -\Delta w & = & \lambda(f(u) + Mu - \alpha w) & \text{in } \Omega, \\ u & = & w \ = \ 0 & \text{on } \Gamma, \end{array} \right.$$

if and only if $(u, \beta u - \beta w)$ is a solution to (P_λ).

Let $\tilde{f} \in C^{1,1}(\mathbb{R})$ be a function satisfying $\tilde{f}(s) = f(s)$ for all $s \in [0, \rho_0^+]$ with $\tilde{f}, \tilde{f}'$ bounded on $\mathbb{R}$ and with $\tilde{f}'(s) + M \ge 0$ for all $s \in \mathbb{R}$. If we replace f in (Q_λ) by $\tilde{f}$ the system becomes quasimonotone. Since we are interested in solutions (u, v) to (P_λ) with u positive and $\max u < \rho_{\delta/\gamma}^+ \le \rho_0^+$ we can assume without loss of generality the following:

Condition A*. *The function f satisfies* Condition A *with f and f' bounded and $f'(s) + M \ge 0$ for all $s \in \mathbb{R}$ and $f(s) \le 0$ for $s \ge \rho_0^+$.*

Another condition which we impose is:

Condition B2. *The constant β defined in* (3) *satisfies $\beta < \sigma_0$.*

Under this condition one has for λ large enough a positive nontrivial solution to the scalar problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda(f(u) - \beta u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma, \end{array} \right.$$

which has its maximum in the interval $(\rho_\beta^-, \rho_\beta^+)$, see [**4**]. This solution will be used to obtain a nontrivial subsolution to (Q_λ) for λ large enough. The definition of sub- and supersolutions is given in Appendix A.

We make some remarks on Conditions B1 and B2. Both conditions are satisfied if δ/γ is small enough. More precisely, for fixed $\delta > 0$, B1 and B2 are satisfied if

$$\gamma > \begin{cases} M + 2\sqrt{\delta} & \text{if} \quad 0 \le \delta < \sigma_0^2; \\ M + \sigma_0 + \delta/\sigma_0 & \text{if} \quad \delta \ge \sigma_0^2. \end{cases}$$

In the first theorem we prove the existence of a curve of positive solutions to (P_λ).

Theorem 2.1 (Existence of a curve of solutions)**.** *Let f satisfy* Condition A*, let γ, δ be such that* Conditions B1 *and* B2 *hold and assume that Γ is C^3. Then there exist $\lambda^\star > 0$ and a function $\Lambda \in C^1([\lambda^\star, +\infty), C^2(\bar{\Omega}) \times C^2(\bar{\Omega}))$ such that $(u_\lambda, v_\lambda) := \Lambda(\lambda)$ is a positive solution, i.e., $u_\lambda, v_\lambda \ge 0$, to (P_λ) for all $\lambda \ge \lambda^\star$. Furthermore*

(1) $\max u_\lambda \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$ *and* $\max v_\lambda \in \dfrac{\delta}{\gamma}(\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$;

(2) $\lim_{\lambda\to\infty} \Lambda(\lambda) = \left(\rho^+_{\delta/\gamma}, \dfrac{\delta}{\gamma}\rho^+_{\delta/\gamma}\right)$ *uniformly on compact subsets of* Ω.

The stability of the solutions obtained in the theorem above will be considered in the space $X := C\left(\bar{\Omega}\right) \times C\left(\bar{\Omega}\right)$. For $\lambda > \lambda^\star$ we define the linear operator $A_\lambda : D(A_\lambda) \subset X \to X$ by

$$D(A_\lambda) := \{(u, v) \in X\, ;\, (\Delta u, \Delta v) \in X\} \tag{6}$$

and

$$A_\lambda \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \lambda \begin{pmatrix} f'(u_\lambda) & 1 \\ \delta & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

for $(u, v) \in D(A_\lambda)$. Here u_λ is the first component of $\Lambda(\lambda)$. In the definition of $D(A_\lambda)$, Δu and Δw are to be understood in distributional sense.

Theorem 2.2 (Stability)**.** *Assume that the conditions of* Theorem 2.1 *hold and let $\lambda^\star$ and Λ be as in that theorem. For every $\lambda \ge \lambda^\star$ the solution $\Lambda(\lambda) = (u_\lambda, v_\lambda)$ to (P_λ) is an exponentially stable equilibrium solution to the initial value problem* (1) *i.e., for every $\lambda \ge \lambda^\star$ there exists $\nu_\lambda > 0$ such that the spectrum $\sigma(A_\lambda)$ is contained in $\{\nu \in \mathbb{C}\, ;\, \mathrm{Re}\,\nu > \nu_\lambda\}$.*

Our last theorem is a result on the uniqueness, in a restricted sense, of solutions to (P_λ).

Theorem 2.3 (Uniqueness in order interval)**.** *Assume that the conditions of* Theorem 2.1 *hold and let $\lambda^\star$ and Λ be as in that theorem. For every function $z \in C_0(\Omega)$ with $z \ge 0$ and $\max z \in (\rho^-_\beta, \rho^+_{\delta/\gamma})$ there exists $\lambda_z > \lambda^\star$ such that if (u, v) is a solution to (P_λ) with $\lambda > \lambda_z$ and $u \in [z, \rho^+_{\delta/\gamma}]$ then $(u, v) = \Lambda(\lambda)$.*

In general one cannot expect uniqueness of solutions. Indeed it may for example be the case that the trivial solution is a stable solution to the problem. Then there will exist a third, unstable solution in $[0,\Lambda(\lambda)]$. This is the case when f is as in Example 2 and Conditions B1 and B2 hold, see [**19**].

We end this section with a summary of the notation that will be used.

Notation:

- Let $u_1, u_2 \in C(\bar\Omega)$. We write
$$u_1 \geq u_2 \text{ if } u_1(x) \geq u_2(x) \text{ for all } x \in \Omega;$$
$$u_1 \gneqq u_2 \text{ if } u_1 \geq u_2 \text{ and } u_1 \neq u_2;$$
$$u_1 > u_2 \text{ if } u_1(x) > u_2(x) \text{ for all } x \in \Omega.$$
- For $(u,w) \in C(\bar\Omega) \times C(\bar\Omega)$ we shall use $(u,w)(x) = (u(x), w(x))$.
- Let $(u_i, w_i) \in C(\bar\Omega) \times C(\bar\Omega)$, $i = 1,2$. We write
$$(u_1,w_1) \geq (u_2,w_2) \text{ if } u_1 \geq u_2 \text{ and } w_1 \geq w_2;$$
$$(u_1,w_1) \gneqq (u_2,w_2) \text{ if } (u_1,w_1) \geq (u_2,w_2) \text{ and } (u_1,w_1) \neq (u_2,w_2);$$
$$(u_1,w_1) > (u_2,w_2) \text{ if } u_1 > u_2 \text{ and } w_1 > w_2.$$
- If $(u_1,w_1) \geq (u_2,w_2)$ we denote by $[(u_1,w_1),(u_2,w_2)]$ the order interval
$$\left\{(u,w) \in C(\bar\Omega) \times C(\bar\Omega);\, (u_1,w_1) \leq (u,w) \leq (u_2,w_2)\right\}.$$
- By $\mathcal{D}^+(\Omega)$ we denote the set of $z \in C_0^\infty(\Omega)$ with $z \geq 0$ and $\mathcal{D}'(\Omega)$ denotes the usual space of distributions.
- For $u_1, u_2 \in C(\bar\Omega)$ we say $-\Delta u_1 \leq u_2$ in $\mathcal{D}'(\Omega)$-sense if
$$\int_\Omega u_1(-\Delta z)\,dx \leq \int_\Omega u_2 z\,dx$$
for all $z \in \mathcal{D}^+(\Omega)$.
- For a Banach space X we denote the bounded linear operators from X into X by $\mathcal{L}(X)$.

3. Preliminary results.

3.1. Estimates for positive solutions.

Proposition 3.1. *Let B be the unit ball in $\mathbb{R}^N$. Suppose that f satisfies* Condition A*. *Then there exists $\lambda_B > 0$ such that the problem*

$$\left\{\begin{array}{rcll} -\Delta u & = & \lambda_B(f(u) - \beta u + \beta w) & \text{in } B, \\ -\Delta w & = & \lambda_B(f(u) + Mu - \alpha w) & \text{in } B, \\ u & = & w \;=\; 0 & \text{on } \partial B, \end{array}\right. \tag{8}$$

has a solution (U_B, W_B) with the following properties:

(1) $0 \le (U_B, W_B) < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$, *with* $\vartheta = 1 - \delta/(\gamma\beta)$.

(2) U_B *and* W_B *are radially symmetric with*

$$U_B'(0) = W_B'(0) = 0 \quad \text{and} \quad U_B'(r), W_B'(r) < 0 \quad \text{on } (0,1].$$

(3) $(U_B(0), W_B(0)) > (\rho^-_{\delta/\gamma}, \vartheta\rho^-_{\delta/\gamma})$ *and* $W_B(0) \ge \vartheta\tau$ *where* $\tau := U_B(0)$.

Proof. Since Condition B2 holds, $J_\beta(\rho) > 0$ for all $0 \le \rho < \rho^+_\beta$. This implies that for λ large enough, say $\lambda = \lambda_B$, there exists a positive solution $\underline{u}$ to

$$\begin{cases} -\Delta u = \lambda(f(u) - \beta u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

with $\max \underline{u} \in (\rho^-_\beta, \rho^+_\beta)$, see [**4**]. Then $(\underline{u}, 0)$ is a subsolution to (8). Since $(\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ is a supersolution with $(\underline{u}, 0) < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ there exists a solution (U_B, W_B) with $\underline{u} < U_B < \rho^+_{\delta/\gamma}$ and $0 < W_B < \vartheta\rho^+_{\delta/\gamma}$ to (8), see Proposition A.3. Using an extension due to Troy, [**24**], of results of Gidas, Ni and Nirenberg, [**8**], to quasimonotone systems, we have that U_B and W_B are radially symmetric with $U_B'(0) = W_B'(0) = 0$ and $U_B'(r), W_B{}'(r) < 0$ on the interval $(0,1)$. Also $(-\Delta + \lambda_B\alpha) W_B = \lambda_B(f(U_B) + MU_B) \ge 0$ and by the strong maximum principle $W_B'(1) < 0$. Let $\tau := U_B(0)$. With $V_B = \beta(U_B - W_B)$ it also follows from the maximum principle that

$$\max V_B < (\delta/\gamma)\tau. \tag{9}$$

Indeed, $(-\Delta + \lambda_B\gamma)(V_B - \delta\tau/\gamma) = \lambda_B(U_B - \tau) \le 0$ in B, with $V_B = 0$ on ∂B and (9) follows. Since V_B is also radially symmetric and decreasing, $V_B(0) = \beta(\tau - W_B(0)) < (\delta/\gamma)\tau$ and hence

$$W_B(0) > (1 - \delta/(\gamma\beta))\tau = \vartheta\tau > \vartheta\rho^-_{\delta/\gamma}.$$

Also $V_B'(1) = \beta(U_B'(1) - W_B'(1)) < 0$ and hence $U_B'(1) < W_B{}'(1) < 0$. $\square$

Next we construct a family of subsolutions to (Q_λ) using the functions U_B and W_B. These subsolutions will be used to determine by sweeping the shape of the solutions to (Q_λ) in a certain order interval. We fix $z^* \in \Omega$ and let

$$\lambda^* := \lambda_B \operatorname{dist}(z^*, \Gamma)^{-2}. \tag{10}$$

Lemma 3.2. *For all* $\lambda \ge \lambda^*$ *we set*

$$Z_\lambda(x) := \begin{cases} (U_B, W_B)\left((\lambda/\lambda_B)^{1/2}(x - z^*)\right) & \text{for} \quad |x - z^*| \le (\lambda_B/\lambda)^{1/2}; \\ 0 & \text{for} \quad |x - z^*| > (\lambda_B/\lambda)^{1/2}, \end{cases}$$

with (U_B, W_B) *as in* Proposition 3.1. *Then* Z_λ *is a subsolution to* (Q_λ) *and*

$$Y := (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma}) \tag{11}$$

is a supersolution to (Q_λ) *with* $Z_\lambda < Y$.

Proof. It follows directly that Y is a supersolution. The function Z_λ is continuous and $Z_\lambda(x) = 0$ for $x \in \Gamma$. Denote by $Z_{\lambda,i}$, $i = 1, 2$, the two components of Z_λ. Let $z \in \mathcal{D}^+(\Omega)$. Then, with $B_\lambda = B(z^*, (\lambda_B/\lambda)^{1/2})$ and n denoting the outward normal, we obtain by the Green identity:

$$\begin{aligned}\int_\Omega Z_{\lambda,1}(-\Delta z)\,dx &= \int_{B_\lambda} Z_{\lambda,1}(-\Delta z)\,dx \\ &= -\int_{B_\lambda}(\Delta Z_{\lambda,1})z\,dx - \int_{\partial B_\lambda}\left(Z_{\lambda,1}\frac{\partial z}{\partial n} - z\frac{\partial Z_{\lambda,1}}{\partial n}\right)dS \\ &\le \lambda\int_\Omega \left(f(Z_{\lambda,1}) - \beta Z_{\lambda,1} + \beta Z_{\lambda,2}\right) z\,dx.\end{aligned}$$

A similar result holds for $Z_{\lambda,2}$. Finally $\max Z_{\lambda,1} = Z_{\lambda,1}(z^*) = \tau < \rho^+_{\delta/\gamma}$ and $\max Z_{\lambda,2} = Z_{\lambda,2}(z^*) = W_B(0) < \vartheta\rho^+_{\delta/\gamma}$. □

Since Z_λ is a subsolution to (Q_λ) and Y is a supersolution to (Q_λ) with $Z_\lambda < Y$ there exists at least one solution in the order interval $[Z_\lambda, Y]$. For every fixed $\lambda \ge \lambda^*$ we define for all $y \in \Omega$ satisfying $\operatorname{dist}(y, \Gamma) > (\lambda_B/\lambda)^{-1/2}$ the functions

$$Z^y_\lambda(x) := Z_\lambda(x + z^* - y)\,. \tag{12}$$

Repeating the proof of Lemma 3.2 one sees that for $\lambda \ge \lambda^*$,

$$\mathcal{S}_\lambda := \left\{Z^y_\lambda\,;\, y \in \Omega \text{ such that } \operatorname{dist}(y, \Gamma) > (\lambda_B/\lambda)^{1/2}\right\}$$

is a family of subsolutions. We shall use the sweeping principle with functions in $\mathcal{S}_\lambda$ to obtain, at least for λ large enough, estimates of solutions to (Q_λ) in the order interval $[Z_\lambda, Y]$. In order to estimate a solution in $[Z_\lambda, Y]$ in all of Ω as well as on the boundary we make the following assumption on Γ which holds if $\Gamma \in C^3$:

- Ω satisfies a uniform interior sphere condition, that is, there exists $\varepsilon_\Omega > 0$ such that $\Omega = \cup\{B(y, \varepsilon)\,;\, y \in \Omega \text{ and } \operatorname{dist}(y, \Gamma) > \varepsilon_\Omega\}$.

We may suppose that $\Omega_\varepsilon := \{y \in \Omega;\, \operatorname{dist}(y, \Gamma) > \varepsilon\}$ is connected for all $\varepsilon \le \varepsilon_\Omega$.

Lemma 3.3. *There exists $\lambda^\times > \lambda^*$ and $b > 0$ such that for all $\lambda > \lambda^\times$ we have the following estimate for every solution $(u, w) \in [Z_\lambda, Y]$ to (Q_λ):*

$$(u(x), w(x)) > \min\{b\lambda^{1/2}\operatorname{dist}(x, \Gamma), \tau\}\,(1, \vartheta)\,, \tag{13}$$

with $\vartheta = 1 - \delta/(\gamma\beta)$ and τ as in Proposition 3.1.

Proof. Let $\varepsilon_\lambda := (\lambda_B/\lambda)^{1/2}$ and $\lambda^\times := \max\{\lambda^*, \lambda_B\varepsilon_\Omega^{-2}\}$. Suppose that $(u, w) \in [Z_\lambda, Y]$ is a solution to (Q_λ) with $\lambda > \lambda^\times$. As in [**4**] there exists for every $y \in \Omega_{\varepsilon_\lambda}$ a curve in $\Omega_{\varepsilon_\lambda}$ connecting y with z^*. Using the sweeping principle, Proposition A.6, it follows that $(u,w) > Z_\lambda^y$ for all $y \in \Omega_{\varepsilon_\lambda}$. Using $(u(x),w(x)) \geq \sup_{y\in\Omega_{\varepsilon_\lambda}} Z_\lambda^y(x)$ one finds (13). $\square$

The next lemma improves the estimate we found in the previous one.

Lemma 3.4. *For every $\varepsilon > 0$ and $\lambda > \lambda^\times$ there exists a constant $b(\varepsilon) > 0$, independent of λ, such that for every solution $(u,w) \in [Z_\lambda, Y]$ to (Q_λ) it holds that*

$$(u(x), w(x)) > \min\left\{b(\varepsilon)\lambda^{1/2}\operatorname{dist}(x,\Gamma), \rho^+_{\delta/\gamma} - \varepsilon\right\}(1,\vartheta), \tag{14}$$

with $\vartheta = 1 - \delta/(\gamma\beta)$. In particular there exists $b_0 > 0$ such that

$$(u(x), w(x)) > \min\left\{b_0\lambda^{1/2}\operatorname{dist}(x,\Gamma), \rho^+_{\sigma_0}\right\}(1,\vartheta). \tag{15}$$

Proof. Let $\lambda > \lambda^\times$ be fixed and suppose $(u,w) \in [Z_\lambda, Y]$ is solution to (Q_λ). If $\rho^+_{\delta/\gamma} - \varepsilon \leq \tau$ then (13) holds with $b(\varepsilon) = b$ and b as in the previous lemma.

Suppose $\rho^+_{\delta/\gamma} - \varepsilon > \tau$. Since $f(s) - (\delta/\gamma)s > 0$ for all $s \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$ there exists $\ell_\varepsilon > 0$ such that

$$f(s) - (\delta/\gamma)s > \ell_\varepsilon(s-\tau) \qquad \text{for all } s \in [\tau, \rho^+_{\delta/\gamma} - \varepsilon].$$

From Lemma 3.3 it follows that $(u(x), w(x)) > (\tau, \vartheta\tau)$ for all $x \in \Omega$ such that $\operatorname{dist}(x,\Gamma) > \lambda^{-1/2}\tau/b$. For subsolutions we need the function $e \geq 0$ satisfying

$$\left\{\begin{array}{rcll} -\Delta e &=& \mu e & \text{in } B_1 \\ e &=& 0 & \text{on } \partial B_1 \end{array}\right. \tag{16}$$

where μ is the principal eigenvalue and B_1 the unit ball in $\mathbb{R}^N$. We normalize e such that $e(0) = 1$. Let $\mu_\varepsilon = \mu/\ell_\varepsilon$ and

$$\Omega' := \left\{y \in \Omega\,;\, \operatorname{dist}(y,\Gamma) > (\sqrt{\mu_\varepsilon} + \tau/b)\lambda^{-1/2}\right\}.$$

We fix $y \in \Omega'$ and let $B := B(y, (\mu_\varepsilon/\lambda)^{1/2})$. For every $t \in [0,1]$ we define the functions (U_t, W_t) on $\overline{B}$ by

$$\begin{aligned} U_t(x) &:= \tau + t(\rho^+_{\delta/\gamma} - \varepsilon - \tau)e\left((\lambda/\mu_\varepsilon)^{1/2}(y-x)\right), \\ W_t(x) &:= \vartheta U_t(x). \end{aligned}$$

Then $\mathcal{T} := \{(U_t, W_t)\,;\, t \in [0,1]\}$ is a family of subsolutions to the problem

$$\left\{\begin{array}{rcll} -\Delta p &=& \lambda(f(p) - \beta p + \beta q) & \text{in } B, \\ -\Delta q &=& \lambda(f(p) + Mp - \alpha q) & \text{in } B, \\ p &=& u & \text{on } \partial B, \\ q &=& w & \text{on } \partial B. \end{array}\right. \tag{17}$$

Using the sweeping principle it follows that

$$(u(y), w(y)) > (U_1(y), W_1(y)) = (\rho^+_{\delta/\gamma} - \varepsilon, \vartheta(\rho^+_{\delta/\gamma} - \varepsilon)).$$

Since $y \in \Omega'$ was arbitrary we have that

$$(18)\quad (u(x), w(x)) > (\rho^+_{\delta/\gamma} - \varepsilon, \vartheta(\rho^+_{\delta/\gamma} - \varepsilon)) \text{ if } \operatorname{dist}(x, \Gamma) > (\mu_\varepsilon + \tau/b)\lambda^{-1/2}.$$

Define $b(\varepsilon) := \tau\left(\tau/b + \sqrt{\mu_\varepsilon}\right)^{-1}$, and note that $\min\{b\lambda^{1/2}\operatorname{dist}(x,\Gamma), \tau\} \geq b(\varepsilon)\lambda^{1/2}\operatorname{dist}(x,\Gamma)$ if $\operatorname{dist}(x,\Gamma) \leq (\sqrt{\mu_\varepsilon} + \tau/b)\lambda^{-1/2}$. Hence by Lemma 3.3

$$(u(x), w(x)) > b(\varepsilon)\lambda^{1/2}\operatorname{dist}(x,\Gamma)(1,\vartheta)$$

for all x with $\operatorname{dist}(x,\Gamma) \leq (\sqrt{\mu_\varepsilon} + \tau/b)\lambda^{-1/2}$. This proves (14) while (15) follows by choosing $b_0 = b(\varepsilon)$ with $\varepsilon = \rho^+_{\delta/\gamma} - \rho^+_{\sigma_0}$. □

The next lemma will be used in the proof of Theorem 2.3.

Lemma 3.5. *Let $z_0 \in C_0(\Omega)$ be nonnegative with $\max z_0 \in (\rho^-_\beta, \rho^+_{\delta/\gamma})$. There exists $\lambda_{z_0} > 0$ such that if (u,w) is a solution to (Q_λ) with $u \in [z_0, \rho^+_{\delta/\gamma}]$ and $\lambda > \lambda_{z_0}$ then $(u,w) \in [Z_\lambda, Y]$.*

Proof. First note that if $u \in [z_0, \rho^+_{\delta/\gamma}]$ then $(u,w) \in [(z_0,0), Y]$. Let $x_0 \in \Omega$ be such that $z_0(x_0) = \max z_0$. Choose $\rho \in (\rho^-_\beta, \tau)$, where τ is as in Proposition 3.1, and $r_0 > 0$ such that $\rho < z_0(x) \leq u(x)$ for all $x \in B(x_0, r_0)$. Since $f(s) - \beta s > 0$ for all $s \in (\rho^-_\beta, \rho^+_\beta)$ there exists $\ell > 0$ such that

$$f(s) - \beta s > \ell(s-\rho) \qquad \text{for all } s \in [\rho, \tau].$$

Let e and μ be as (16) with $e(0) = 1$. Suppose that

$$(19)\qquad \lambda > (\mu/\ell)r_0^{-2}.$$

Then $r_\lambda := r_0 - \sqrt{\mu/(\lambda\ell)} > 0$. Let $y \in B(x_0, r_\lambda)$ be fixed and define on $B = B(y, \sqrt{\mu/(\ell\lambda)}) \subset B(x_0, r_0)$,

$$U_t(x) = \rho + t(\tau - \rho)e\left(\sqrt{\ell\lambda/\mu}(y - x)\right).$$

It holds that $\mathcal{T} := \{(U_t, 0)\,;\, t \in [0,1]\}$ is a family of subsolutions to (17) with u_λ, w_λ, instead of u, w. By a sweeping argument, starting with $(U_0, 0)$ one concludes that $(u(y), w(y)) > (U_1(y), W_1(y)) = (\tau, 0)$. Since $y \in B(x_0, r_\lambda)$ was arbitrary we have that $(u,w) \geq (\tau, 0)$ on $B(x_0, r_\lambda)$.

Let $Z^{x_0}_{\lambda,i}$, $i = 1,2$, denote the two components of $Z^{x_0}_\lambda$ defined in (12). The function $Z^{x_0}_{\lambda,1}$ has support $\overline{B}(x_0, \sqrt{\lambda_B/\lambda})$. Hence, if (19) is replaced by the stronger condition $\lambda > (\sqrt{\lambda_B} + \sqrt{\mu/\ell})^2 r_0^{-2}$, then $r_\lambda > \sqrt{\lambda_B/\lambda}$ and $(u,w)(x) > (Z^{x_0}_{\lambda,1}, 0)$ for all $x \in \Omega$.

From this it follows that $(u,w) \in \left[Z^{x_0}_\lambda, Y\right]$. Indeed, using the fact that $Z^{x_0}_\lambda$ is a subsolution one has that $-\Delta(w - Z^{x_0}_{\lambda,2}) + \alpha(w - Z^{x_0}_{\lambda,2}) \geq 0$ in $\mathcal{D}'(\Omega)$-sense.

As in the proof of Lemma 3.3 it now follows that $(u,w) \in [Z_\lambda, Y]$. □

3.2. The semilinear problem on the half space. In this section we consider the following problem

$$\begin{cases} -\Delta U &= f(U) - \beta U + \beta W & \text{in } \mathbb{R}^N_+, \\ -\Delta W &= f(U) + MU - \alpha W & \text{in } \mathbb{R}^N_+, \\ U &= W = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{20}$$

The main result which we prove is that there exists a positive solution (U, W) to (20) such that

$$\lim_{x_1 \to \infty} (U, W)\left(x_1, x'\right) = (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma}) \text{ uniformly in } x' \in \mathbb{R}^{N-1}, \tag{21}$$

with $\vartheta = 1 - \delta/(\gamma\beta)$. Moreover there exists only one such solution and $(U, W)(x_1, x') = (u, w)(x_1)$ where (u, w) a solution to the problem

$$\begin{cases} -u'' &= f(u) - \beta u + \beta w & \text{in } \mathbb{R}^+, \\ -w'' &= f(u) + Mu - \alpha w & \text{in } \mathbb{R}^+, \\ u(0) &= 0, \quad w(0) = 0, \\ u'(0) &= \kappa, \quad w'(0) = \nu, \end{cases} \tag{22}$$

for some appropriate initial data κ and ν. It is standard that we have for every pair $(\kappa, \nu) \in \mathbb{R}^2$ at least locally a unique solution to (22) which can be continued to some maximum interval. We denote such a solution by $(u, w)_{\kappa,\nu} = (u_{\kappa,\nu}, w_{\kappa,\nu})$. First we show that there exists a unique pair (κ, ν) such that the corresponding solution exists for all $r \in \mathbb{R}^+$, is positive and tends to $(\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ at infinity. Some properties of this solution that are needed later, are also proven.

Proposition 3.6. *Assume that f satisfies* Condition A*. *Then there exists a unique pair $(\bar\kappa, \bar\nu)$ such that the solution $(u, w)_{\bar\kappa,\bar\nu}$ to (22) is positive and satisfies*

$$\lim_{r \to \infty} (u, w)_{\bar\kappa,\bar\nu}(r) = (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma}), \tag{23}$$

with $\vartheta = 1 - \delta/(\beta\gamma)$. Moreover $\bar\kappa > \bar\nu > 0$ and $(u, w)_{\bar\kappa,\bar\nu}$ has the following properties:

(1) $0 < (u, w)_{\bar\kappa,\bar\nu}(r) < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ *for all $r > 0$;*

(2) $u_{\bar\kappa,\bar\nu}(r) > w_{\bar\kappa,\bar\nu}(r)$ *for all $r > 0$;*

(3) $(u', w')_{\bar\kappa,\bar\nu}(r) > (0, 0)$ *for all $r \in \mathbb{R}^+$ and $(u', w')_{\bar\kappa,\bar\nu}(r) \to (0, 0)$ as $r \to \infty$.*

The proof of this proposition consists of a number of lemmas. We also need to consider the following system

$$\text{(24)} \qquad \begin{cases} -u'' &= f(u) - v & \text{in } \mathbb{R}^+, \\ -v'' &= \delta u - \gamma v & \text{in } \mathbb{R}^+, \\ u(0) &= 0, \quad v(0) = 0, \\ u'(0) &= \kappa, \quad v'(0) = \eta. \end{cases}$$

Again we denote solutions to (24) by $(u,v)_{\kappa,\eta}$ with the understanding that the solutions are defined on a maximum interval. We point out the fact that for $\kappa,\nu \in \mathbb{R}$ it holds that the solution $(u,v)_{\kappa,\beta(\kappa-\nu)}$ to (24) is given by $(u_{\kappa,\nu}, \beta(u_{\kappa,\nu} - w_{\kappa,\nu}))$ where $(u,w)_{\kappa,\nu}$ is the solution to (22).

For a solution $(u,v) = (u,v)_{\kappa,\eta}$ to (24) we have the following identity for all $r \geq 0$:

$$\text{(25)} \qquad \left((u')^2 - \kappa^2\right) - \frac{1}{\delta}((v')^2 - \eta^2) = -2\int_0^u f(s)\,ds + 2uv - \frac{\gamma}{\delta}v^2.$$

Indeed, differentiating

$$H(r) := u'(r)^2 - \frac{1}{\delta}v'(r)^2 + 2\int_0^{u(r)} f(s)\,ds - 2u(r)v(r) + \frac{\gamma}{\delta}v(r)^2,$$

(24) implies that $H'(r) = 0$ for all $r \geq 0$. Hence $H(r) = H(0)$ for all $r \geq 0$, which gives (25).

We shall often use the following one dimensional maximum principle, see e.g., [**11**, Theorem 2.9.2].

Lemma 3.7. *If $g \in C^2[0,+\infty)$ is bounded, $g(0) \geq 0$ and $-g'' + cg \gneqq 0$ with $c > 0$, then $g(r) > 0$ for all $r > 0$. Moreover, if $g(0) = 0$ then $g'(0) > 0$.*

Our first lemma is on the derivatives of solutions to (22).

Lemma 3.8. *Suppose that $(u,w)_{\kappa,\nu}$ is a solution to (22) with $\kappa,\nu > 0$ and $(u,w)_{\kappa,\nu}(r) > (0,0)$ for all $r > 0$. Then $(u',w')_{\kappa,\nu}(r) > (0,0)$ for all $r \geq 0$.*

Proof. Since the system is quasimonotone this follows from a moving plane argument, similar to the method used by Gidas, Ni and Nirenberg [**8**]. See also [**2**] where a similar argument is used for a scalar equation. □

Lemma 3.9. *For a bounded solution $(u,w)_{\kappa,\nu}$ to (22) with $u_{\kappa,\nu} \gneqq 0$ it holds that $0 < \nu < \kappa$ and $0 < w_{\kappa,\nu}(r) < u_{\kappa,\nu}(r)$ for all $r > 0$.*

Proof. Denote by (u,w) the solution $(u,w)_{\kappa,\nu}$. Since w is bounded and satisfies $-w'' + \alpha w = f(u) + Mu \gneqq 0$ with $w(0) = 0$ we have by Lemma 3.7 that $\nu > 0$ and $w(r) > 0$ for all $r > 0$. Let $\eta = \beta(\kappa - \nu)$. As observed earlier, the solution $(u,v) = (u,v)_{\kappa,\eta}$ to (24) is given by $(u, \beta(u-w))$. Since v is bounded with $v(0) = 0$ and $-v'' + \gamma v = \delta u \gneqq 0$ it holds again by Lemma 3.7 that $\eta > 0$ and $v(r) > 0$ for $r > 0$. Hence $\kappa > \nu$ and $w(r) < u(r)$ for $r > 0$. □

Lemma 3.10. *If* $(u,w)_{\kappa,\nu}$ *is a positive solution to* (22) *such that* (23) *holds then*

$$\lim_{r\to\infty}(u',w')_{\kappa,\nu}(r)=0.$$

Proof. Define $u_K(r):=\rho^+_{\delta/\gamma}-u_{\kappa,\nu}(r+K)$ and $w_K(r):=\vartheta\rho^+_{\delta/\gamma}-w_{\kappa,\nu}(r+K)$. It holds that u_K and w_K converge uniformly to zero on $[0,1]$ as $K\to\infty$. From (22) we have that they remain bounded in $C^2[0,1]$. By interpolation u_K, w_K converge to zero in $C^1[0,1]$. Therefore $(u',w')_{\kappa,\nu}(K)=(u'_K,w'_K)(0)\to(0,0)$, as $K\to\infty$. □

Let $(u,w)_{\kappa,\nu}$ be a solution to (22) for which (23) holds. Then $(u,v)_{\kappa,\eta}$ with $\eta=\beta(\kappa-\nu)$ is a solution to (24) and $v_{\kappa,\eta}=\beta(u_{\kappa,\nu}-w_{\kappa,\nu})\to(\delta/\gamma)\rho^+_{\delta/\gamma}$ as $r\to\infty$. Using Lemma 3.10 and letting $r\to\infty$ in (25) we obtain the following relationship between κ and ν:

$$\kappa^2-\frac{\beta^2}{\delta}(\kappa-\nu)^2=2\int_0^{\rho^+_{\delta/\gamma}}\left(f(s)-\frac{\delta}{\gamma}s\right)ds. \tag{26}$$

This will be used to prove the uniqueness of such solutions. Next we show that there exists initial data $(\bar\kappa,\bar\nu)$ for which the corresponding solution to (22) is positive and satisfies (23).

Lemma 3.11. *There exists* $\bar\kappa,\bar\nu\in\mathbb{R}$ *such that the solution* $(u,w)_{\bar\kappa,\bar\nu}$ *to* (22) *satisfies* (23). *Moreover this solution is positive and* $0<\bar\nu<\bar\kappa$.

Proof. We shall use super- and subsolutions and Lemma A.4 to find a positive solution to

$$\begin{cases}-u''&=&f(u)-\beta u+\beta w & \text{in } \mathbb{R}^+,\\ -w''&=&f(u)+Mu-\alpha w & \text{in } \mathbb{R}^+,\\ u(0)&=&w(0)=0, &\end{cases} \tag{27}$$

satisfying (23). As a supersolution we take $(\rho^+_{\delta/\gamma},\vartheta\rho^+_{\delta/\gamma})$. We have to construct a nonzero subsolution. From a phaseplane analysis one sees that the initial value problem

$$\begin{cases}-u''&=&f(u)-\beta u & \text{in } \mathbb{R}^+,\\ u(0)&=&0, &\\ u'(0)&=&(2J_\beta(0))^{1/2}, &\end{cases}$$

with $J_\beta(0)>0$ defined in (2), has a solution $\tilde u$ with $\lim_{r\to\infty}\tilde u(r)=\rho^+_\beta$ and $\tilde u'(r)>0$ for all $r\ge 0$. Then $(\tilde u,0)$ is a subsolution. By Lemma A.4 there exists a solution (u,w) to (27) such that $(0,\tilde u)<(u,w)<(\rho^+_{\delta/\gamma},\vartheta\rho^+_{\delta/\gamma})$. At this stage we may choose either the maximal or minimal solution. In the next lemma we shall prove that they are equal. Let $(\bar\kappa,\bar\nu):=(u'(0),w'(0))$. Then (u,w) is the solution to (22) with $(\kappa,\nu)=(\bar\kappa,\bar\nu)$.

It holds that $u(r), w(r) > 0$ for all $r > 0$. Indeed, $u(r) \geq \tilde{u}(r) > 0$ and since w is bounded with $-w'' + \alpha w = f(u) + Mu \gneqq 0$ and $w(0) = 0$ we have by Lemma 3.7 that $w(r) > 0$ for $r > 0$ and that $w'(0) = \nu > 0$. By Lemma 3.9, $\bar{\kappa} > \bar{\nu} > 0$.

Lemma 3.8 shows that $u'(r), w'(r) > 0$ for all $r > 0$. In particular $\lim_{r\to\infty} u(r) = \rho$ and $\lim_{r\to\infty} w(r) = \tilde{\rho}$ exist. From the equations we find that

$$-f(\rho) + \beta\rho - \beta\tilde{\rho} = -f(\rho) - M\rho + \alpha\tilde{\rho} = 0.$$

From this one gets that $\tilde{\rho} = \vartheta\rho$ and that $f(\rho) = (\delta/\gamma)\rho$. Since $\rho > \rho_\beta^+$ we have that $(\rho, \tilde{\rho}) = (\rho_{\delta/\gamma}^+, \vartheta\rho_{\delta/\gamma}^+)$. □

Our last lemma concerns the uniqueness part of Proposition 3.6.

Lemma 3.12. *Let $(u, w)_{\kappa,\nu}$ be a positive solution to* (22) *such that* (23) *holds. Then $(\kappa, \nu) = (\bar{\kappa}, \bar{\nu})$ with $(\bar{\kappa}, \bar{\nu})$ as in* Lemma 3.11.

Proof. Let $(\tilde{u}, 0)$ be the subsolution of the previous lemma. First we show that the minimum and maximum solutions to (27) in the order interval

$$[(\tilde{u}, 0), (\rho_{\delta/\gamma}^+, \vartheta\rho_{\delta/\gamma}^+)]$$

are equal.

Let $(u, w)_{\kappa,\nu}$ be the minimal solution and $(u, w)_{\bar{\kappa},\bar{\nu}}$ the maximal solution. It must hold that $\kappa \leq \bar{\kappa}$ and $\nu \leq \bar{\nu}$. If the solutions are not equal at least one of these inequalities must be strict. Suppose $\nu < \bar{\nu}$. By Lemma 3.9 we also have that $\kappa > \nu$ and $\bar{\kappa} > \bar{\nu}$ and by (26) that

$$\frac{\delta - \beta^2}{\delta}\kappa^2 + \frac{2\beta^2}{\delta}\kappa\nu - \frac{\beta^2}{\delta}\nu^2 = \frac{\delta - \beta^2}{\delta}\bar{\kappa}^2 + \frac{2\beta^2}{\delta}\bar{\kappa}\bar{\nu} - \frac{\beta^2}{\delta}\bar{\nu}^2. \tag{28}$$

The function $x \mapsto (1 - (\beta^2/\delta))x^2 + 2(\beta^2/\delta)x\nu - (\beta^2/\delta)\nu^2$ is strictly increasing on $[\kappa, \bar{\kappa}]$ because it has derivative $2\left(\delta - \beta^2\right)x/\delta + 2\beta^2\nu/\delta$ which is strictly positive for $x \in [\kappa, \bar{\kappa}]$ since $\delta > \beta^2$. Hence

$$\frac{\delta - \beta^2}{\delta}\kappa^2 + \frac{2\beta^2}{\delta}\kappa\nu - \frac{\beta^2}{\delta}\nu^2 \leq \frac{\delta - \beta^2}{\delta}\bar{\kappa}^2 + \frac{2\beta^2}{\delta}\bar{\kappa}\nu - \frac{\beta^2}{\delta}\nu^2.$$

The function $x \mapsto (1 - (\beta^2/\delta))\bar{\kappa} + 2(\beta^2/\delta)\bar{\kappa}x - (\beta^2/\delta)x^2$ has derivative $2\beta^2\bar{\kappa}\delta - 2\beta^2 x/\delta$. Since the derivative is strictly positive on $[\nu, \bar{\nu}]$ it follows that

$$\frac{\delta - \beta^2}{\delta}\kappa^2 + \frac{2\beta^2}{\delta}\kappa\nu - \frac{\beta^2}{\delta}\nu^2 < \frac{\delta - \beta^2}{\delta}\bar{\kappa}^2 + \frac{2\beta^2}{\delta}\bar{\kappa}\bar{\nu} - \frac{\beta^2}{\delta}\bar{\nu}^2,$$

contradicting (28). If $\kappa < \bar{\kappa}$ we find a contradiction by the same argument. We conclude that $\kappa = \bar{\kappa}$ and $\nu = \bar{\nu}$ and that $(u, w)_{\kappa,\nu} = (u, w)_{\bar{\kappa},\bar{\nu}}$.

It remains to show that any positive solution $(u, w)_{\kappa,\nu}$ for which (23) holds is in the order interval $[(\tilde{u}, 0), (\rho_{\delta/\gamma}^+, \vartheta\rho_{\delta/\gamma}^+)]$. First we show that $u_{\kappa,\nu} > \tilde{u}$.

Suppose that $u_{\kappa,\nu}(r) > \rho^+_\beta$ for all $r \geq R$. We define u^*_t for $0 \leq t \leq R$ on $[0,R]$ by

$$u^*_t\,(r) := \begin{cases} \tilde{u}(r-t) & \text{for } t \leq r \leq R; \\ 0 & \text{for } 0 \leq r < t, \end{cases}$$

with $\tilde{u}$ as in Lemma 3.11. Applying the sweeping principle with the subsolutions $\{(u^*_t, 0)\,;\, 0 \leq t \leq R\}$ one finds that $(u,w)_{\kappa,\nu}(r) > (\tilde{u}(r), 0) = (u^*_0(r), 0)$ for $r \in (0,R)$. Hence $u_{\kappa,\nu}(r) > \tilde{u}(r)$ for $r > 0$ and $(u,w)_{\kappa,\nu} \geq (\tilde{u}, 0)$. On the other hand, since $u_{\kappa,\nu}$, $w_{\kappa,\nu}$ are increasing by Lemma 3.8, it holds that $(u,w)_{\kappa,\nu} < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$. Since there is only one solution in the order interval $[(\tilde{u},0), (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})]$ the uniqueness is proved. □

Our main result concerning system (20) is the following.

Proposition 3.13. *Assume that f satisfies* Condition A**. Then there exists a unique positive solution (U,W) to* (20) *satisfying* (21)*. This solution is given by*

$$(U,W)(x_1,x') := (u,w)_{\bar{\kappa},\bar{\nu}}(x_1) \quad \textit{for } (x_1,x') \in \mathbb{R}^{N-1}, \tag{29}$$

with $(u,w)_{\bar{\kappa},\bar{\nu}}$ as in Lemma 3.11.

Proof. Clearly (29) defines a positive solution to (20) satisfying (21). Suppose that (U,W) is any positive solution satisfying (21). Define the functions

$$\begin{aligned} (\underline{u}, \underline{w})\,(x_1) &:= (\sup_{x'\in\mathbb{R}^{N-1}} U(x_1,x'), \sup_{x'\in\mathbb{R}^{N-1}} W(x_1,x')); \\ (\overline{u}, \overline{w})\,(x_1) &:= (\inf_{x'\in\mathbb{R}^{N-1}} U(x_1,x'), \inf_{x'\in\mathbb{R}^{N-1}} W(x_1,x')). \end{aligned}$$

By Lemma A.5 $(\underline{u}, \underline{w})$ is a subsolution and $(\overline{u}, \overline{w})$ is a supersolution to (27). Moreover $(\underline{u}, \underline{w}) < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$. This follows by sweeping using the family of supersolutions $\{(t, \vartheta t) \,:\, t \geq \rho^+_{\delta/\gamma}\}$.

Since $(\underline{u}, \underline{w})$ is a subsolution there exists a positive solution $(u,w)_*$ to (27) with $(\underline{u}, \underline{w}) \leq (u,w)_* < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$. By Lemma 3.12 we have that $(u,w)_* = (u,w)_{\bar{\kappa},\bar{\nu}}$.

Using a sweeping argument as in the proof of Lemma 3.12 it follows that $(\overline{u}, \overline{w}) > (\tilde{u}, 0)$ with $\tilde{u}$ as in the proof of Lemma 3.11. Hence there exists a positive solution $(u,w)_\circ$ to (27) with $(\tilde{u}, 0) < (u,w)_\circ \leq (\overline{u}, \overline{w})$ and by Lemma 3.12, $(u,w)_\circ = (u,w)_{\bar{\kappa},\bar{\nu}}$. Hence $(\overline{u}, \overline{w}) = (\underline{u}, \underline{w})$ which proves the uniqueness claim in the proposition. □

3.3. The linearized problem on the halfspace. Let $(u,w)_{\bar{\kappa},\bar{\nu}}$ be as in Proposition 3.6. In this paragraph we consider the following linear system:

$$\begin{cases} -\bar{r}\Delta\Phi = (f'(u_{\bar{\kappa},\bar{\nu}}) - \beta + \omega)\Phi + \beta\Psi - \bar{r}\omega\Phi & \text{in } \mathbb{R}^N_+, \\ -\bar{r}\Delta\Psi = (f'(u_{\bar{\kappa},\bar{\nu}}) + M)\Phi + (\omega - \alpha)\Psi - \bar{r}\omega\Psi & \text{in } \mathbb{R}^N_+, \\ \Phi = \Psi = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{30}$$

Here α, β, M are as in (4), (3) and B1 respectively, $\omega > \max\{\alpha, M\}$ and $\bar{r} \in \mathbb{R}$. For this problem we have the following result of Liouville type.

Proposition 3.14. *Suppose that* $\bar{r} \geq 1$ *and that* (Φ, Ψ) *is a bounded positive solution to* (30). *Then* $(\Phi, \Psi) \equiv (0, 0)$.

This proposition will be a consequence of the following lemma.

Lemma 3.15. *Suppose* $\varphi, \psi \in C[0, +\infty)$ *are bounded with* $\varphi, \psi \geq 0$, $\varphi(0) = \psi(0) = 0$ *and it holds that*

$$-\varphi'' \leq f'(u_{\bar{\kappa},\bar{\nu}})\varphi - \beta\varphi + \beta\psi, \tag{31}$$

$$-\psi'' \leq f'(u_{\bar{\kappa},\bar{\nu}})\varphi + M\varphi - \alpha\psi, \tag{32}$$

in $\mathcal{D}'(0, \infty)$*-sense. Then* $\varphi(x_1) = \psi(x_1) = 0$ *for all* $x_1 \geq 0$.

Proof. We set $(p, q) := (u', w')_{\bar{\kappa},\bar{\nu}}$ and recall that $p, q > 0$ on $[0, \infty)$. Without loss of generality we assume that $\varphi, \psi \leq 1$. We argue by contradiction and suppose that $(\varphi, \psi) \neq (0, 0)$. First we observe that if there exists $K > 0$ such that $\varphi(x_1) = \psi(x_1) = 0$ for all $x_1 \geq K$ then by a sweeping argument on $[0, K]$ with the family $\{(tp, tq)\,;\, t \geq 0\}$ of supersolutions it follows that $\varphi(x_1) = \psi(x_1) = 0$ for all $x_1 \in [0, K]$. This is in contradiction with our assumption.

Now let $K > 0$ and $\varepsilon > 0$ be such that that

$$f'(u_{\bar{\kappa},\bar{\nu}}(x_1)) < -\varepsilon \quad \text{for all } x_1 \geq K,$$

and note that also

$$f'\left(u_{\bar{\kappa},\bar{\nu}}(x_1)\right) + M - \alpha < -\varepsilon \quad \text{for all } x_1 \geq K.$$

By our first observation we may assume that

$$R(K) := \max\left\{\varphi(K)/p(K), \psi(K)/q(K)\right\} > 0.$$

We define the following functions on $[K, \infty)$:

$$\begin{aligned} S_t(x_1) &= \varphi(x_1) - e^{\sqrt{\varepsilon}(x_1 - t)}, \\ T_t(x_1) &= \psi(x_1) - e^{\sqrt{\varepsilon}(x_1 - t)}, \\ R_t(x_1) &= \max\left\{S_t(x_1)/p(x_1), T_t(x_1)/q(x_1)\right\} \end{aligned}$$

for $t \geq K$. It holds that

$$-S_t'' \leq (f'(u_{\bar{\kappa},\bar{\nu}}) - \beta)S_t + \beta T_t,$$

and

$$-T_t'' \leq (f'(u_{\bar{\kappa},\bar{\nu}}) + M)S_t - \alpha T_t,$$

in $\mathcal{D}'(K, \infty)$-sense. For $t > K$ let $m_t = \sup_{x_1 \in [K,t]} R_t(x_1)$. By the maximum principle one has that $m_t = R_t(K)$ for t large enough. Indeed, since for ω large enough, it holds in $\mathcal{D}'(K, t)$-sense that

$$-(S_t - m_t p)'' + \omega(S_t - m_t p) \leq 0,$$

and

$$-(T_t - m_t q)'' + \alpha(T_t - m_t q) \leq 0,$$

we see that m_t must be attained in K or in t. Since $R_t(t) \leq 0$ and $R_t(K) > 0$, if t is large enough, we conclude that $m_t = R_t(K)$. Now let $x_1 \in [K, \infty)$ be fixed. Then for all $t > x_1$ large we have

$$\begin{aligned} R(x_1) &= \max\{\varphi(x_1)/p(x_1), \psi(x_1)/q(x_1)\} \\ &\leq R_t(K) + \max\left\{e^{\sqrt{\varepsilon}(x_1-t)}/p(x_1), e^{\sqrt{\varepsilon}(x_1-t)}/q(x_1)\right\}. \end{aligned}$$

Letting $t \to \infty$ we deduce that $R(x_1) \leq R(K)$ and hence R attains its maximum on $[K, \infty)$ in $x_1 = K$. Consequently $\sup_{x_1 \in [0,\infty)} R(x_1)$ is attained in some point $r_0 \in (0, K]$. But this is in contradiction to the maximum principle. Indeed, in a similar way as above, one sees that $R(x_1)$ must attain its maximum on $[0, K+1]$ either in 0 or in $K+1$ and not in K. □

To see how Proposition 3.14 follows from this lemma, we define

$$\begin{aligned} \varphi(x_1) &:= \sup\left\{\Phi(x_1, x') \,;\, x' \in \mathbb{R}^{N-1}\right\}, \\ \psi(x_1) &:= \sup\left\{\Psi(x_1, x') \,;\, x' \in \mathbb{R}^{N-1}\right\}. \end{aligned}$$

Then by Lemma A.5, $\varphi, \psi \in C[0, +\infty)$ with $\varphi(0) = \psi(0) = 0$ and in $\mathcal{D}'(\mathbb{R}^N_+)$-sense

$$\begin{aligned} -\varphi'' &\leq \frac{1}{\bar{r}}(f'(u_{\bar{\kappa},\bar{\nu}}) - \beta + \omega)\varphi + \frac{1}{\bar{r}}\beta\psi - \omega\varphi \\ -\psi'' &\leq \frac{1}{\bar{r}}(f'(u_{\bar{\kappa},\bar{\nu}}) + M)\varphi + \frac{1}{\bar{r}}(\omega - \alpha)\psi - \omega\psi. \end{aligned}$$

Since $\bar{r} \geq 1$ we deduce that

$$\begin{aligned} -\varphi'' &\leq (f'(u_{\bar{\kappa},\bar{\nu}}) - \beta + \omega)\varphi + \beta\psi - \omega\varphi \\ &= (f'(u_{\bar{\kappa},\bar{\nu}}) - \beta + \omega)\varphi + \beta\psi \end{aligned}$$

and

$$\begin{aligned} -\psi'' &\leq (f'(u_{\bar{\kappa},\bar{\nu}}) + M)\varphi + (\omega - \alpha)\psi - \omega\psi \\ &= (f'(u_{\bar{\kappa},\bar{\nu}}) + M)\varphi - \alpha\psi. \end{aligned}$$

By the lemma $(\varphi, \psi)(x_1) = 0$ for $x_1 \geq 0$ and hence also $(\Phi, \Psi)(x) = (0, 0)$ on $\mathbb{R}^N_+$.

4. Proofs of the main results.

4.1. Proof of Theorem 2.1. From now on we assume that Γ is C^3. We begin by defining some operators. Recall that X denotes the space $C\left(\bar{\Omega}\right) \times C\left(\bar{\Omega}\right)$ and let $C^1_0\left(\bar{\Omega}\right) = \left\{u \in C^1\left(\bar{\Omega}\right) \,;\, u(x) = 0 \text{ for } x \in \Gamma\right\}$. For $k, \lambda > 0$

define $\left(-\lambda^{-1}\Delta+k\right)_0^{-1} : C\left(\bar{\Omega}\right) \to C_0^1\left(\bar{\Omega}\right)$ by $u = \left(-\lambda^{-1}\Delta+k\right)_0^{-1} g$ with $u \in C_0^1\left(\bar{\Omega}\right)$ the unique function satisfying

$$\begin{cases} -\lambda^{-1}\Delta u + ku &= g \quad \text{in } \mathcal{D}'(\Omega)\text{-sense,} \\ u &= 0 \quad \text{on } \Gamma. \end{cases}$$

Let j be the embedding of $C_0^1\left(\bar{\Omega}\right) \times C_0^1\left(\bar{\Omega}\right)$ in X and define the operator $K_{k,\lambda} : X \to X$ by

$$K_{k,\lambda}\begin{pmatrix} g \\ h \end{pmatrix} = j \circ \begin{pmatrix} \left(-\lambda^{-1}\Delta+k\right)_0^{-1} & 0 \\ 0 & \left(-\lambda^{-1}\Delta+k\right)_0^{-1} \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix}.$$

Since j is compact and $\left(-\lambda^{-1}\Delta+k\right)_0^{-1}$ is continuous, $K_{k,\lambda}$ is a compact linear map on X. We shall also use the fact that $\|K_{k,\lambda}\|_{\mathcal{L}(X)}$ is uniformly bounded in λ. This follows from the fact that

$$\left\|\left(-\lambda^{-1}\Delta+k\right)_0^{-1} g\right\|_\infty \leq \frac{1}{k}\,\|g\|_\infty \tag{33}$$

for every $g \in C\left(\bar{\Omega}\right)$. We fix $\omega > \max\{\alpha, M\}$. For a function $u \in C\left(\bar{\Omega}\right)$ and $\lambda > 0$ the we define the operators $M_u, T_{u,\lambda} \in \mathcal{L}(X)$ by

$$M_u\begin{pmatrix} h \\ g \end{pmatrix} = \begin{pmatrix} f'(u)-\beta & \beta \\ f'(u)+M & -\alpha \end{pmatrix}\begin{pmatrix} h \\ g \end{pmatrix}, \tag{34}$$

and

$$T_{u,\lambda} := K_{\omega,\lambda}(M_u + \omega I). \tag{35}$$

Operators of this kind were studied extensively in [**23**]. If $u \in [0, \rho^+_{\delta/\gamma}]$ then $T_{u,\lambda}$ is a positive irreducible compact operator on X, see [**23**, Lemma 1.3]. Moreover, $T_{u,\lambda}$ has a positive spectral radius (see e.g., [**18**]) which we denote by $r(T_{u,\lambda})$. By the Krein-Rutman Theorem (see e.g., [**1**, Theorem 3.1]), $r(T_{u,\lambda})$ is an eigenvalue of $T_{u,\lambda}$ to which a positive eigenfunction pertains. In the next lemma we prove that for λ large enough it holds for every solution $(u,w) \in [Z_\lambda, Y]$ that $r(T_{u,\lambda}) < 1$.

Lemma 4.1. *There exists $\lambda^\star > \lambda^\times$ such that for all $\lambda > \lambda^\star$ and every solution $(u,w) \in [Z_\lambda, Y]$ to* (Q_λ) *the corresponding operator $T_{u,\lambda}$ has spectral radius $r(T_{u,\lambda}) < 1$.*

Proof. We prove the lemma by a contradiction argument. Assume that it does not hold. Then there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ with $\lambda^\times < \lambda_n \to \infty$ and solutions $(u_n, w_n) := (u_{\lambda_n}, w_{\lambda_n}) \in [Z_{\lambda_n}, Y]$ to (Q_λ) with $\lambda = \lambda_n$ such that $r_n \geq 1$, with r_n denoting the spectral radius of $T_n := T_{u_n,\lambda_n}$. Let $(\varphi_n, \psi_n) \in X$ be the positive eigenfunction pertaining to r_n. We normalize the eigenfunction such that $\max \varphi_n = 1$. This can be done since $\varphi_n = 0$

implies that $\psi_n = 0$, a contradiction because (φ_n, ψ_n) is an eigenfunction. It holds that

$$\begin{cases} -r_n\lambda_n^{-1}\Delta\varphi_n &= (f'(u_n) + \omega - \beta)\varphi_n + \beta\psi_n - r_n\omega\varphi_n & \text{in } \Omega, \\ -r_n\lambda_n^{-1}\Delta\psi_n &= (f'(u_n) + M)\varphi_n + (\omega - \alpha)\psi_n - r_n\omega\psi_n & \text{in } \Omega, \\ \varphi_n &= \psi_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Because operator norms $\|T_n\|_{\mathcal{L}(X)}$ are uniformly bounded it follows from

$$r_n = \overline{\lim_{k\to\infty}} \left(\left\| T_n^k \right\|_{\mathcal{L}(X)} \right)^{1/k} \le \|T_n\|_{\mathcal{L}(X)}$$

that the sequence $\{r_n\}_{n=1}^{\infty}$ is bounded. By going over to a subsequence, still denoted by $\{r_n\}_{n=1}^{\infty}$, we can assume that $r_n \to \bar{r} \ge 1$. With $\theta_n := \beta(\varphi_n - \psi_n)$ one has that

$$\begin{cases} -r_n\lambda_n^{-1}\Delta\theta_n &= \delta\varphi_n - \gamma\theta_n + (1 - r_n)\,\omega\theta_n & \text{in } \Omega, \\ \theta_n &= 0 & \text{on } \partial\Omega. \end{cases}$$

This shows that $\varphi_n \ge \psi_n$ and hence $-r_n\lambda_n^{-1}\Delta\varphi_n \le f'(u_n)\,\varphi_n$. Using estimate (15) in Lemma 3.4 we have for all $x \in \Omega$ with $\operatorname{dist}(x, \Gamma) > b_0^{-1}\lambda_n^{-1/2}\rho_{\sigma_0}^{+}$ that $f'(u_n(x)) \le 0$ and consequently

$$-\Delta\varphi_n \le 0 \quad \text{in} \quad \left\{ x \in \Omega\,;\, \operatorname{dist}(x, \Gamma) > b_0^{-1}\lambda_n^{-1/2}\rho_{\sigma_0}^{+} \right\}. \tag{36}$$

Hence φ_n attains its maximum in a point $\bar{x}_n$ with $\operatorname{dist}(\bar{x}_n, \Gamma) \le b_0^{-1}\lambda_n^{-1/2}\rho_{\sigma_0}^{+}$. Let $\bar{x}_{\Gamma,n} \in \Gamma$ be such that $|\bar{x}_n - \bar{x}_{\Gamma,n}| = \operatorname{dist}(\bar{x}_{\Gamma,n}, \Gamma)$. By going over to a subsequence we can assume that $\bar{x}_{\Gamma,n} \to \bar{x} \in \Gamma$.

By a blow-up argument around $\bar{x}$, similar to the argument in [**4**], one constructs $U, W, \Phi, \Psi \in C^2\left(\mathbb{R}_+^N\right) \cap C\left(\overline{\mathbb{R}_+^N}\right)$ such that (U, W) satisfies

$$\begin{cases} -\Delta U &= f(U) - \beta U + \beta W & \text{in } \mathbb{R}_+^N, \\ -\Delta W &= f(U) + MU - \alpha W & \text{in } \mathbb{R}_+^N, \\ U &= W = 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

and (Φ, Ψ) satisfies

$$\begin{cases} -\bar{r}\Delta\Phi &= (f'(U) + \omega - \beta)\Phi + \beta\Psi - \bar{r}\omega\Phi & \text{in } \mathbb{R}_+^N, \\ -\bar{r}\Delta\Psi &= (f'(U) + M)\Phi + (\omega - \alpha)\Psi - \bar{r}\omega\Psi & \text{in } \mathbb{R}_+^N, \\ \Phi &= \Psi = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

The normalization $\max\phi_n = 1$ leads to $\sup\Phi = 1$. Furthermore, using the uniform estimate (14) it follows that

$$\lim_{x_1\to\infty}(U, W)(x_1, x') = (\rho_{\delta/\gamma}^{+}, \vartheta\rho_{\delta/\gamma}^{+}) \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}. \tag{37}$$

Hence, by Proposition 3.13, $(U,W)(x_1,x') = (u,w)_{\bar{\kappa},\bar{\nu}}(x_1)$. Then (Φ,Ψ) is a bounded positive solution to (30) with $\bar{r} \geq 1$. By Proposition 3.14, $(\Phi,\Psi) \equiv (0,0)$, in contradiction with $\sup \Phi = 1$. $\square$

We shall use this lemma to prove that there can be at most one solution to (Q_λ) in the order interval $[Z_\lambda, Y]$. First we define the operator $H_\lambda : X \to X$

$$H_\lambda := K_{\omega,\lambda}(F + \omega I),$$

where $F : X \to X$ is defined by

$$F\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} f(u) - \beta u + \beta w \\ f(u) + Mu - \alpha w \end{pmatrix}.$$

We shall show that H_λ has at most one fixed point in $[Z_\lambda, Y]$. In order to use the Leray-Schauder degree we have to consider the fixed point problem

$$H_\lambda(u,w) = (u,w)$$

in an appropriate space.

Let μ be the principal eigenvalue and $e \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ the corresponding eigenfunction to the problem

$$\begin{cases} -\Delta e &= \mu e & \text{in } \Omega, \\ \quad e &= 0 & \text{on } \Gamma. \end{cases}$$

We normalize e such that $\max e = 1$. Following Amann [**1**] we define

$$C_e(\bar{\Omega}) := \left\{ u \in C(\bar{\Omega})\ ;\ \exists\, t > 0 \text{ such that } |u| \leq te \right\}, \tag{38}$$

equipped with the norm $\|u\|_e = \inf\{t > 0\,;\, -te \leq u \leq te\}$. It holds that $C_e(\bar{\Omega})$ is a Banach space, in fact a Banach lattice, with closed unit ball $\{u \in C(\bar{\Omega})\ ;\ -e \leq u \leq e\}$. Let $X_e = C_e(\bar{\Omega}) \times C_e(\bar{\Omega})$. Order intervals in X_e will be denoted by $[\cdot,\cdot]_e$. Let j_1, j_2 be the embeddings of X_e in X and $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ in X_e respectively and define $H_\lambda^e : X_e \to X_e$ by

$$H_\lambda^e := j_2 \circ \begin{pmatrix} \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} & 0 \\ 0 & \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} \end{pmatrix} \circ (F + \omega I) \circ j_1.$$

We recall that $\left(-\lambda^{-1}\Delta + \omega\right)_0^{-1}$ was defined as an operator from $C(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$. We note that (u,w) is a fixed point of H_λ^e if and only if $j_1(u,w)$ is a fixed point of H_λ. Hence it suffices to show that H_λ^e has a unique fixed point in $[Z_\lambda, Y] \cap X_e$.

It also holds for $(u_i, w_i) \in X$ with $(u_1,w_1) < (u_2,w_2)$ that

$$H_\lambda(u_1,w_1) < H_\lambda(u_2,w_2). \tag{39}$$

In fact $H_\lambda(u_2,w_2) - H_\lambda(u_1,w_1)$ is an element of the interior of the positive cone of X_e, or equivalently, there exists $t > 0$ such that

$$H_\lambda(u_2,w_2) - H_\lambda(u_1,w_1) \geq (te,te). \tag{40}$$

Since neither Z_λ nor Y are fixed points of H_λ we find, using (39), that any fixed point $(u,w) \in [Z_\lambda, Y]$ satisfies

$$Z_\lambda^* := H_\lambda Z_\lambda < (u,w) < H_\lambda Y =: Y^*.$$

From (40) we even have the stronger result that $(u,w) \in \operatorname{int}[Z_\lambda^*, Y^*]_e$, the interior of $[Z_\lambda^*, Y^*]_e$ with respect to the $\|\cdot\|_e$-topology. The uniqueness of a fixed point of H_λ in $[Z_\lambda, Y]$ for $\lambda \geq \lambda^\star$ then follows from the next lemma.

Lemma 4.2. *For every $\lambda > \lambda^\star$ there exists a unique fixed point of H_λ^e in* $\operatorname{int}[Z_\lambda^*, Y^*]_e$.

Proof. We have for every $\lambda > \lambda^\star$ that there exists at least one solution to (Q_λ) in the order interval $[Z_\lambda, Y]$ which, as we observed, is a fixed point of H_λ^e and $(u,w) \in \operatorname{int}[Z_\lambda^*, Y^*]_e$. To show that this is the only solution, we shall use a degree argument.

Suppose $(u,w) \in \operatorname{int}[Z_\lambda^*, Y^*]_e$, with $\lambda > \lambda^\star$, is a fixed point of H_λ^e. The operator H_λ^e is differentiable and $T_{u,\lambda}^e := dH_\lambda^e(u,w) \in \mathcal{L}(X_e)$ given by

$$T_{u,\lambda}^e = j_2 \circ \begin{pmatrix} \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} & 0 \\ 0 & \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} \end{pmatrix} \circ (M_u + \omega I) \circ j_1,$$

with M_u as defined in (34). From Lemma 4.1 we have that the spectral radius $r(T_{u,\lambda}^e) < 1$. Indeed μ is an eigenvalue of $T_{u,\lambda}^e$ if and only if μ is an eigenvalue of $T_{u,\lambda}$. Since $r(T_{u,\lambda}) > 0$ it holds that $r(T_{u,\lambda}^e) > 0$. But $T_{u,\lambda}^e$ is a positive compact operator and hence $r(T_{u,\lambda}^e)$ is an eigenvalue of $T_{u,\lambda}^e$ to which a positive eigenfunction pertains. This implies that $r(T_{u,\lambda}^e) = r(T_{u,\lambda}) < 1$.

In particular 1 is not an eigenvalue of $T_{u,\lambda}^e$ and consequently the index of the fixed point (u,w) is well defined with

$$\operatorname{index}(u,w) = 1,$$

see [**17**, p. 66]. Using the homotopy invariance of the degree and the fact that $\operatorname{int}[Z_\lambda^*, Y^*]_e$ is convex, it follows that

$$\operatorname{degree}\left(I - H_{u,\lambda}^e, \operatorname{int}[Z_\lambda^*, Y^*]_e, 0\right) = 1.$$

Indeed, let $\bar{z} \in \operatorname{int}[Z_\lambda^*, Y^*]_e$ be arbitrary and define the homotopy

$$G_t = (1-t)(I - \bar{z}) + t(I - H_{u,\lambda}^e).$$

It holds that $G_t z = 0$ if and only if $z = (1-t)\bar{z} + tH_{u,\lambda}^e z$ and hence $z \in \operatorname{int}[Z_\lambda^*, Y^*]_e$. Since G_t has no zeros on the boundary $\operatorname{int}[Z_\lambda^*, Y^*]_e$ we have that

$$\operatorname{degree}(G_1, \operatorname{int}[Z_\lambda^*, Y^*]_e, 0) = \operatorname{degree}(G_0, \operatorname{int}[Z_\lambda^*, Y^*]_e, 0) = 1.$$

By the additivity property of the degree we see that H_λ^e can have at most one fixed point in $\operatorname{int}[Z_\lambda^*, Y^*]_e$. $\square$

The proof of Theorem 2.1 can now be completed. For all $\lambda > \lambda^\star$ we have a unique solution $\tilde{\Lambda}(\lambda) = (u_\lambda, w_\lambda) \in \operatorname{int}[Z_\lambda^*, Y^*]_e \subset [Z_\lambda, Y]$ to (Q_λ).

Using the Implicit Function Theorem we have that $\tilde{\Lambda} \in C^1((\lambda^\star, +\infty), X_e)$. Indeed the operator $(\lambda, (u,w)) \longmapsto (u,w) - H^e(\lambda,(u,w))$ is C^1 and the derivative with respect to (u,w) is given by $I - T^e_{u,\lambda}$. For fixed $\lambda_0 > \lambda^\star$ it holds that $I - T^e_{u,\lambda_0} \in \mathrm{Isom}(X_e)$. By the Implicit Function Theorem the solution set of $H^e(\lambda,(u,w)) = 0$ consists in a neighbourhood of λ_0 of a C^1-curve, parameterized by λ. By uniqueness of solutions in $\operatorname{int}[Z_\lambda^*, Y^*]_e$ we have that this curve is in a neighbourhood of $(\lambda_0, u_{\lambda_0}, w_{\lambda_0})$ given by $\tilde{\Lambda}$. Since this can be done for every $\lambda > \lambda^\star$ we have that $\tilde{\Lambda}$ in $C^1((\lambda^\star, +\infty), X_e)$. Using a bootstrap argument one proves that $\tilde{\Lambda} \in C^1\left((\lambda^\star, +\infty), C^2\left(\bar{\Omega}\right) \times C^2\left(\bar{\Omega}\right)\right)$.

Finally we define $\Lambda(\lambda) := (u_\lambda, \beta(u_\lambda - w_\lambda))$. Then $\Lambda(\lambda)$ is a solution to (P_λ) for all $\lambda > \lambda^\star$ and $\Lambda \in C^1\left((\lambda^\star, \infty), C^2\left(\bar{\Omega}\right) \times C^2\left(\bar{\Omega}\right)\right)$.

4.2. Proof of Theorems 2.2 and 2.3. In this section we assume that the conditions of Theorem 2.1 hold. We define the operator $B_\lambda : D(B_\lambda) \to X$ by

$$D(B_\lambda) := \{(u,w) \in X \,;\, (\Delta u, \Delta w) \in X\},$$

with Δu and Δw in distributional sense,

$$B_\lambda := L_\lambda - M_{u_\lambda},$$

with

$$L_\lambda := \begin{pmatrix} -\lambda^{-1}\Delta & 0 \\ 0 & -\lambda^{-1}\Delta \end{pmatrix}$$

and M_{u_λ} as defined in (34).

Lemma 4.3. *For all $\lambda > \lambda^\star$, with $\lambda^\star$ as in* Lemma 4.1*, the operator B_λ is invertible and $B_\lambda^{-1} \in \mathcal{L}(X)$ is a positive compact operator with a positive spectral radius $r_\lambda = r(B_\lambda^{-1})$. Moreover r_λ is an eigenvalue of B_λ^{-1} with a corresponding positive eigenfunction.*

Proof. Denote by T_λ and M_λ the operators T_{u_λ} and M_{u_λ} respectively. Since $\lambda > \lambda^\star$ the spectral radius $r(T_\lambda)$ of T_λ satisfies

$$0 < r(T_\lambda) < 1. \tag{41}$$

Hence $B_\lambda = L_\lambda + \omega I - (M_\lambda + \omega I) = \left(\lambda^{-1}L_1 + \omega I\right)(I - T_\lambda)$ is invertible with $B_\lambda^{-1} = (I - T_\lambda)^{-1}K_{\omega,\lambda}$. It holds that B_λ^{-1} is positive and compact since $(I - T_\lambda)^{-1}$ is a positive bounded operator and $K_{\omega,\lambda}$ is a positive compact operator. Moreover, since $r(T_\lambda) < 1$ it follows from [**23**, Lemma 1.4] that B_λ^{-1} is irreducible, and hence the spectral radius $r_\lambda = r(B_\lambda^{-1})$ is positive. By the Krein-Rutman Theorem r_λ corresponds to a positive eigenfunction. □

Lemma 4.4. *For all* $\nu > 0$ *the operator* $B_\lambda + \nu I$ *is invertible. The inverse* $(B_\lambda + \nu I)^{-1} \in \mathcal{L}(X)$ *is positive and compact and its spectral radius is given by* $r((B_\lambda + \nu I)^{-1}) = (r_\lambda^{-1} + \nu)^{-1}$.

Proof. Let $k = \omega + \nu$. Then

$$\begin{aligned} B_\lambda + \nu I &= L_\lambda + kI - (M_\lambda + \omega I) \\ &= (L_\lambda + kI)\,(I - K_{k,\lambda}(M_\lambda + \omega I)). \end{aligned}$$

The operator $K_{k,\lambda}(M_\lambda + \omega I)$ is also positive, compact and irreducible. Again using the Krein-Rutman Theorem we find that $r(K_{k,\lambda}(M_\lambda + \omega I))$ is an eigenvalue of the adjoint operator $(K_{k,\lambda}(M_\lambda + \omega I))^*$ pertaining to a positive functional, say $\Upsilon_{k,\lambda}$. Let h_λ be the positive eigenfunction of B_λ^{-1} corresponding to the eigenvalue r_λ as in Lemma 4.3. It holds that

$$h_\lambda = \left(r_\lambda^{-1} + k - \omega\right) K_{k,\lambda} h_\lambda + K_{k,\lambda}(M_\lambda + \omega I)\, h_\lambda. \tag{42}$$

Using (42) it follows that

$$\begin{aligned} \langle h_\lambda, \Upsilon_{k,\lambda}\rangle &= \left(r_\lambda^{-1} + k - \omega\right) \langle K_{k,\lambda} h_\lambda, \Upsilon_{k,\lambda}\rangle + \langle K_{k,\lambda}(M_\lambda + \omega I)\, h_\lambda, \Upsilon_{k,\lambda}\rangle \\ &\geq \langle K_{k,\lambda}(M_\lambda + \omega I)\, h_\lambda, \Upsilon_{k,\lambda}\rangle \\ &= r\left(K_{k,\lambda}(M_\lambda + \omega I)\right) \langle h_\lambda, \Upsilon_{k,\lambda}\rangle . \end{aligned}$$

Hence $r(K_{k,\lambda}(M_\lambda + \omega I)) < 1$ and the operator $B_\lambda + \nu I$ is invertible with

$$(B_\lambda + \nu I)^{-1} = (I - K_{k,\lambda}(M_\lambda + \omega I))^{-1}\,(L_\lambda + kI)^{-1}.$$

Moreover $(B_\lambda + \nu I)^{-1}$ is compact and positive and irreducible. Since

$$(B_\lambda + \nu I)^{-1} h_\lambda = (r_\lambda^{-1} + \nu)^{-1} h_\lambda$$

we see that $(r_\lambda^{-1} + \nu)^{-1}$ is an eigenvalue to which a positive eigenfunction pertains. It then follows from the irreducibility of $(B_\lambda + \nu I)^{-1}$ that the spectral radius of this operator must be $(r_\lambda^{-1} + \nu)^{-1}$. □

Lemma 4.5. *If* $\mu \in \mathbb{C}$ *is such that* $\operatorname{Re}\mu < r_\lambda^{-1}$ *then* μ *is in the resolvent set of* B_λ.

Proof. Let $h \in X$ be arbitrary and consider the equation

$$B_\lambda g - \mu g = h, \tag{43}$$

where $\operatorname{Re}\mu < r_\lambda^{-1}$. Choose $\nu \in \mathbb{R}$ large enough such that

$$(\operatorname{Re}\mu)^2 + 2\nu\left(\operatorname{Re}\mu - r_\lambda^{-1}\right) + (\operatorname{Im}\mu)^2 < r_\lambda^{-2}$$

and $\mu + \nu \neq 0$. Then

$$\begin{aligned} |\mu + \nu|^2 &= (\operatorname{Re}\mu)^2 + 2\nu \operatorname{Re}\mu + \nu^2 + (\operatorname{Im}\mu)^2 \\ &< r_\lambda^{-2} + 2\nu r_\lambda^{-1} + \nu^2, \end{aligned}$$

and hence $0 < |\mu + \nu| < r_\lambda^{-1} + \nu$. Equation (43) is equivalent with

$$(B_\lambda + \nu I)g - (\mu + \nu)g = h.$$

Using Lemma 4.4 we can rewrite this as

$$\left((B_\lambda + \nu I)^{-1} - (\mu + \nu)^{-1} I\right) g = -(\mu + \nu)^{-1}(B_\lambda + \nu I)^{-1} h. \tag{44}$$

Since $\left|(\mu + \nu)^{-1}\right| > (r_\lambda^{-1} + \nu)^{-1} = r\left((B_\lambda + \nu I)^{-1}\right)$ we have that $(\mu + \nu)^{-1}$ is in the resolvent set $(B_\lambda + \nu I)^{-1}$ and hence (44) has a unique solution. It follows from the closed graph theorem that μ is in the resolvent set of B_λ. □

Proof of Theorem 2.2. Consider the operator A_λ defined in (7). It holds that μ is in the resolvent set of A_λ if and only if μ/λ is in the resolvent set of B_λ. Indeed if μ/λ is in the resolvent set of B_λ the operator defined by

$$(u, v) \mapsto (\varphi, \beta\varphi - \beta\psi)$$

where $(\varphi, \psi) := \lambda^{-1}(B_\lambda - \mu/\lambda)^{-1}(u, u - \frac{1}{\beta}v)$ is directly seen to be $(A_\lambda - \mu)^{-1}$. Conversely, for μ in the resolvent set of A_λ the operator defined by

$$(u, w) \mapsto (\varphi, \varphi - (1/\beta)\theta)$$

where $(\varphi, \theta) := \lambda(A_\lambda - \mu I)^{-1}(u, \beta u - \beta w)$ is $(B_\lambda - \mu/\lambda)^{-1}$. Hence, using the last lemma we have that all $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu \leq \nu_\lambda := \lambda r_\lambda^{-1}$ that μ is in the resolvent set of A_λ. □

Proof of Theorem 2.3. The theorem follows directly from Lemma 3.5 and the fact that $\Lambda(\lambda)$ is the unique solution in $[Z_\lambda, Y]$. □

A. Appendix.

We recall some facts about quasimonotone systems. We remark that in this section Ω may be an unbounded domain.

Definition A.1. A system of elliptic equations

$$\begin{cases} -\Delta u &= F_1(x, u, w) \quad \text{in } \Omega, \\ -\Delta w &= F_2(x, u, w) \quad \text{in } \Omega, \end{cases} \tag{45}$$

with $F_i \in C^1\left(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}\right)$ is called *quasimonotone* if

$$\frac{\partial F_1}{\partial u}(x, u, w) \geq 0 \quad \text{and} \quad \frac{\partial F_2}{\partial w}(x, u, w) \geq 0 \text{ for all } (x, u, w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}.$$

This definition suffices for our purposes. For a more general definition we refer to [**16**].

Definition A.2. A pair $(u,w) \in C\left(\bar{\Omega}\right) \times C\left(\bar{\Omega}\right)$ is called a *subsolution* to the problem

$$\left\{\begin{array}{lcll} -\Delta u & = & F_1\left(x,u,w\right) & \text{in } \Omega, \\ -\Delta w & = & F_2\left(x,u,w\right) & \text{in } \Omega, \\ (u,w) & = & (\varphi,\psi) & \text{on } \Gamma = \partial\Omega, \end{array}\right. \tag{46}$$

with $\varphi, \psi \in C\left(\Gamma\right)$ if

(1) it holds in $\mathcal{D}'(\Omega)$-sense that

$$\begin{array}{rcl} -\Delta u & \leq & F_1\left(x,u,w\right), \\ -\Delta w & \leq & F_2\left(x,u,w\right); \end{array}$$

(2) $(u,w) \leq (\varphi,\psi)$ on Γ.

Supersolutions are defined by reversing the inequality signs. If (u,w) is both a subsolution and a supersolution then it is called a C-solution.

We note that if Ω is a bounded smooth domain and F_1, F_2 are C^1 then a C-solution (u,w) is in $C^2\left(\bar{\Omega}\right) \times C^2\left(\bar{\Omega}\right)$. We often use the following results from [**16**, Theorem 1.3].

Proposition A.3. *Let Ω be a bounded smooth domain and assume that* (46) *is quasimonotone.*

(1) *If (u_i,w_i), $i=1,2$, are subsolutions to this system then (u,w) defined by*

$$\left(u\left(x\right), w\left(x\right)\right) := \left(\max_{1,2}\left\{u_i\left(x\right)\right\}, \max_{1,2}\left\{w_i\left(x\right)\right\}\right)$$

is again a subsolution to (46).

(2) *If $(\underline{u},\underline{w})$ is a subsolution and $(\overline{u},\overline{w})$ a supersolution to* (46) *then there exists a C-solution (u,w) to* (46) *with*

$$\left(\underline{u},\underline{w}\right) \leq (u,w) \leq \left(\overline{u},\overline{w}\right).$$

We give some results for $\Omega = \mathbb{R}^N_+ := \left\{(x_1,x')\ ;\ x_1 \in \mathbb{R},\ x' \in \mathbb{R}^{N-1}\right\}$. The first is that one has also for quasimonotone systems the existence of a minimal and maximal solutions between an ordered pair of sub- and supersolutions.

Lemma A.4. *Consider the following halfspace problem:*

$$\left\{\begin{array}{rcll} -\Delta u & = & F_1\left(u,w\right) & \text{in } \mathbb{R}^N_+, \\ -\Delta w & = & F_2\left(u,w\right) & \text{in } \mathbb{R}^N_+, \\ u & = & w \ = \ 0 & \text{on } \partial\mathbb{R}^N_+, \end{array}\right. \tag{47}$$

with $F_i \in C^{1,\alpha}\left(\mathbb{R}\times\mathbb{R}\right)$, $0<\alpha<1$, and suppose this system is quasimonotone. If there exists a bounded subsolution $(\underline{u},\underline{w})$ and bounded supersolution $(\overline{u},\overline{w})$ to this system with $(\underline{u},\underline{w}) \leq (\overline{u},\overline{w})$, then there exist a maximal and a minimal $C^{2,\alpha}$-solution in the order interval $[(\underline{u},\underline{w}),(\overline{u},\overline{w})]$ to this problem.

The proof of this lemma is almost the same as for bounded domains. We only observe that if $\omega > 0$ is such that $\frac{\partial}{\partial u} F_1(u,w) + \omega \geq 0$ and $\frac{\partial}{\partial w} F_2(u,w) + \omega \geq 0$ for $(\underline{u},\underline{w}) \leq (u,w) \leq (\overline{u},\overline{w})$ then one can define inductively

$$(u_0,w_0) = (\underline{u},\underline{w}), \qquad (u_{n+1},w_{n+1}) = T(u_n,w_n), \quad n = 0,1,2,\ldots$$

with $(u,w) = T(u_n,w_n)$ the unique solution to the linear problem

$$\begin{cases} (-\Delta + \omega_1)\, u &= F_1(u_n,w_n) + \omega u_n & \text{in } \mathbb{R}^N_+, \\ (-\Delta + \omega_2)\, w &= F_2(u_n,w_n) + \omega w_n & \text{in } \mathbb{R}^N_+, \\ u = w = 0 & & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

That this system has a unique solution follows from the fact that if $k > 0$ and $g \in L^\infty(\mathbb{R}^N_+)$ then there exists a unique $u \in L^\infty(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_+)$ such that $-\Delta u + ku = f$ in $\mathcal{D}'(\mathbb{R}^N_+)$-sense and $u = 0$ on $\partial\mathbb{R}^N_+$, see see e.g., [**6**, Proposition 27, p. 635]. Since the system is quasimonotone we have, see also [**16**], that

$$(\underline{u},\underline{w}) \leq (u_n,w_n) \leq (u_{n+1},w_{n+1}) \leq (\overline{u},\overline{w}) \quad \text{for } n = 0,1,2,\ldots.$$

Letting $n \to \infty$ one obtains a solution.

The next lemma is used to reduce the study of equations on $\mathbb{R}^N_+$ to the study of inequalities on $\mathbb{R}^+$.

Lemma A.5. *Suppose that* $(U,W) \in C^2(\mathbb{R}^N_+) \cap C\left(\overline{\mathbb{R}^N_+}\right)$ *is a bounded solution of*

$$\begin{cases} -\Delta U &= F_1(x_1,U,W) & \text{in } \mathbb{R}^N_+, \\ -\Delta W &= F_2(x_1,U,W) & \text{in } \mathbb{R}^N_+, \\ U = W = 0 & & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{48}$$

with $F_i(x_1,s,t) \in C^{1,\alpha}\left(\overline{\mathbb{R}^3_+}\right)$ *and* $0 < \alpha < 1$. *Assume* (48) *is quasimonotone and that* $|F_i(x_1,s,t)| \leq h(s,t)$ *with* h *a continuous function on* $\mathbb{R}^2$. *Define* (u,w) *by*

$$(u,w)(x_1) := \left(\sup\nolimits_{x' \in \mathbb{R}^{N-1}} U(x_1,x), \sup\nolimits_{x' \in \mathbb{R}^{N-1}} W(x_1,x)\right).$$

It holds that $u,w \in C[0,\infty)$ *with* $u(0) = w(0) = 0$ *and in* $\mathcal{D}'(\mathbb{R}^+)$*-sense that*

$$-u'' \leq F_1(x_1,u,w) \tag{49}$$

$$-w'' \leq F_2(x_1,u,w). \tag{50}$$

Proof. Since U and W are bounded, ΔU and ΔW are also bounded. From this and the fact that $U = W = 0$ on $\partial\mathbb{R}^N_+$ one obtains by standard regularity results that $U,W \in C^{2,\alpha}\left(\overline{\mathbb{R}^N_+}\right)$. In particular we have uniform bounds on the first order derivatives of U and W.

Let $\{q_j\,;\, j = 1, 2, \dots\}$ be a numbering of $\mathbb{Q}^{N-1}$ and define the functions U_j and W_j on $\mathbb{R}^N_+$ by

$$(U_j, W_j)(x) = (U, W)(x + (0, q_j)).$$

For $k = 1, 2, \dots$, we define (S_k, T_k) on $\mathbb{R}^N_+$ by

$$(S_k, T_k)(x) = (\sup_{1\le j\le k} U_j(x), \sup_{1\le j\le k} W_j(x)),$$

and let $(S, T)(x) := \lim_{k\to\infty} (S_k, T_k)(x)$. It follows from the uniform continuity of U and W that $(S(x), T(x)) = (u(x_1), w(x_1))$.

Since the system is quasimonotone it follows from Proposition A.3 and induction that in $\mathcal{D}'(\mathbb{R}^N_+)$-sense $-\Delta S_k \le F_1(x_1, S_k, T_k)z$ for every $k = 1, 2, \dots$. By dominated convergence it then follows that $-\Delta u \le F_1(x_1, u, w)$ in $\mathcal{D}'(\mathbb{R}^N_+)$-sense In particular if $z_1 \in \mathcal{D}^+(\mathbb{R}_+)$ we set $z := z_1 z_2$ with $z_2 \in \mathcal{D}^+\left(\mathbb{R}^{N-1}\right)$, $z_2 \ne 0$ one sees that this implies (49).

Since $F_1(x_1, u, w)$ is bounded there exists $M > 0$ such that $F_1(x_1, u, w) - 2M \le 0$ on $\mathbb{R}^+$. Then $-\left(u + Mx_1^2\right)'' \le 0$ in $\mathcal{D}'(\mathbb{R}_+)$-sense. Hence $x_1 \mapsto u(x_1) + Mx_1^2$ is convex and consequently continuous on $(0, \infty)$. Since $\frac{\partial}{\partial x_1}U$ is uniformly bounded and $U(0, x') = 0$ for all $x' \in \mathbb{R}^{N-1}$, it follows that u is continuous in 0 with $u(0) = 0$. The result for w is obtained mutatis mutandis. $\square$

Finally we prove a direct analogue for a quasimonotone system of the sweeping principle for scalar equations in [**15**]. Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1, \Gamma_2 \in C^2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Here Γ_i may be empty. Let $e \in C^1(\bar\Omega)$ be such that $e(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $e(x) = 0$, $\frac{\partial e}{\partial n}(x) < 0$ for $x \in \Gamma_2$ where n is the outward normal and let $C_e(\bar\Omega)$ be as in (38), see also [**1**].

Proposition A.6. *Suppose that* (46) *is quasimonotone. If* (u, w) *is a supersolution, and* $\{(u_t, w_t)\,;\, t \in [0, 1]\}$ *is a family of subsolutions such that*

(1) $(u_t, w_t) < (g_1, g_2)$ *on* Γ_1 *and* $(u_t, w_t) = (g_1, g_2)$ *on* Γ_2 *for all* $t \in [0, 1]$;
(2) $t \mapsto u_t - u_0$ *and* $t \mapsto w_t - w_0$ *is continuous from* $[0, 1]$ *into* $C_e(\bar\Omega)$;
(3) $(u_0, w_0) \le (u, w)$ *in* $\bar\Omega$;
(4) $u_t \ne u$ *and* $w_t \ne w$ *for all* $t \in [0, 1]$;

then there exists $r > 0$ *such that* $(u, w) - (u_t, w_t) > (re, re)$ *for all* $t \in [0, 1]$.

Proof. Let $S = \{t \in [0, 1]\,; (u_t, w_t) \le (u, w) \text{ in } \Omega\}$. By assumption $0 \in S$. Since convergence in $C_e(\bar\Omega)$ implies pointwise convergence it follows that S is closed. Let $t_0 \in S$. It holds with ω large enough in $\mathcal{D}'(\Omega)$-sense that

$$\begin{aligned}-\Delta(u - u_{t_0}) + \omega(u - u_{t_0}) &\ge F_1(u, w) + \omega u - F_1(u_{t_0}, w_{t_0}) + \omega u_{t_0} \\ &= F_1(u, w) + \omega u - F_1(u_{t_0}, w) + \omega u_{t_0} \\ &\quad + F_1(u_{t_0}, w) - F_1(u_{t_0}, w_{t_0}) \ge 0.\end{aligned}$$

Since $u \ne u_{t_0}$ there exists $s' > 0$ such that $u - u_{t_0} > s' e_0$ with e_0 a $C^1(\bar\Omega)$ function with $e(x) > 0$ for $x \in \Omega$, $e_0(x) = 0$ and $\frac{\partial e_0}{\partial n}(x) < 0$ for $x \in \Gamma$, see [**3**, Corollary p. 581]. Since $u(x) - u_{t_0}(x) > 0$ for $x \in \Gamma_1$ and Γ_1 is compact, there exists $s_1 > 0$ such $u - u_{t_0} \geq s_1 e$. In the same way there exists s_2 such that $w - w_{t_0} \geq s_2 e$. By hypothesis 2 there exists $\delta > 0$ such that $\left\|u_t - u_{t_0}\right\|_e, \|w_t - w_{t_0}\|_e < s/2$ for all $t \in [0,1]$ for which $|t - t_0| < \delta$. This implies that for all such t we have that $u_t - u_{t_0} \leq \frac{s}{2}e$ and hence $u - u_t = u - u_{t_0} - (u_{t_0} - u_t) \geq \frac{s}{2}e$ and in the same way $w - w_t \geq \frac{s}{2}e$. Hence S is open and we have that $S = [0,1]$. By the compactness of $[0,1]$ and by hypotheses (2) it follows that there exists $r > 0$ such that $u - u_t \geq re$ and $w - w_t \geq re$. $\square$

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Department of Mathematics
Delft University of Technology
2600 GA Delft
The Netherlands
E-mail address: WSKCJR@puknet.puk.ac.za

Department of Mathematics
Delft University of Technology
2600 GA Delft
The Netherlands
E-mail address: sweers@twi.tudelft.nl