EXISTENCE AND UNIQUENESS OF SOLUTIONS ON BOUNDED DOMAINS TO A FITZHUGH–NAGUMO TYPE ELLIPTIC SYSTEM

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In this paper we prove the existence and uniqueness of the boundary layer solution to a semilinear eigenvalue problem consisting of a coupled system of two elliptic partial differential equations. Although the system is not quasimonotone, there exists a transformation to a quasimonotone system. For the transformed system we may and will use maximum (sweeping) principle arguments to derive pointwise estimates. A degree argument completes the uniqueness proof.

1. Introduction.

We consider the following nonlinear eigenvalue problem:

\[
\begin{cases}
-\Delta u = \lambda (f(u) - v) & \text{in } \Omega, \\
-\Delta v = \lambda (\delta u - \gamma v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \Gamma = \partial \Omega,
\end{cases}
\]

(P\(_\lambda\))

with \(\lambda, \delta, \gamma > 0\) and where \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain. As usual, a domain is an open connected set. The nonlinearity \(f\) is assumed to be smooth and like a third order polynomial. We prove the existence of a curve of positive solutions \((u_\lambda, v_\lambda)\) to (P\(_\lambda\)) for \(\lambda\) large enough. These solutions are shown to be, except for a boundary layer of width \(O(\lambda^{-1/2})\), close to \((\rho, (\delta/\gamma)\rho)\) where \(\rho\) a positive zero of \(f(s) - (\delta/\gamma)s\) and \(f'(\rho) < 0\). The stability of these solutions as equilibria of the parabolic system

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + \lambda (f(u) - v) & \text{in } \mathbb{R}^+ \times \Omega, \\
\frac{\partial v}{\partial t} = \Delta v + \lambda (\delta u - \gamma v) & \text{in } \mathbb{R}^+ \times \Omega, \\
u = v = 0 & \text{on } \mathbb{R}^+ \times \Gamma,
\end{cases}
\]

(1)

with appropriate initial conditions is also proven. Finally it is shown that these solutions are unique in an appropriate order interval.

The question of existence of solutions to (P\(_\lambda\)) with \(\lambda = 1\) and with different kinds of nonlinearities was studied by Klaasen and Mitidieri [9] and De Figueiredo and Mitidieri [7], see also Rothe [21] and Lazer and McKenna [12]. The fact that the second equation can be inverted to solve \(v\) in terms
of $u$ and that the problem can then be written as a single equation in $u$
was used extensively. In particular this single equation can be treated by
variational techniques. Using this approach it was shown for example in [9]
with $f(u) = u(u-1)(a-u)$, $0 < a < 1/2$ and in [7] for more general $f$
of the same type, that there exist at least two nontrivial solutions, under the
assumptions that $\delta/\gamma$ is small enough and $\Omega$ contains a large enough ball.
By rescaling, this implies that there exist nontrivial solutions to $(P_\lambda)$ if $\lambda$
is large and $\delta/\gamma$ small.

Our treatment of the problem differs from the variational approach men-
tioned above. By imposing some natural restrictions on the parameters,
which are satisfied if $\delta/\gamma$ is small, it is possible to make a transformation of
$(P_\lambda)$ and a modification of $f$ to obtain a quasimonotone system. Solutions
to the quasimonotone system in a certain range correspond to solutions to
the original problem. This approach was also used in [19] as well as in [14]
for other systems of equations. The advantage of working with a quasimonoto-
ne system is that for such systems a comparison principle holds. From this
follows the existence of solutions between an ordered pair of sub- and super-
solutions. For such systems one also has an analogue of McNabb’s sweeping
principle, see [15], [2], [4] and [22]. This will be a main tool in many of the
proofs.

Using this quasimonotone approach we are able to give a complete qualita-
tive description of a specific solution to $(P_\lambda)$. This qualitative description
allows us to prove uniqueness and stability results. Results in this direction
were obtained by Lazer and McKenna [12] for a system with $\delta = \gamma$ and $f$
such that $f(s)/s$ is decreasing on $\mathbb{R}^+$. Existence and positivity of solutions
were considered in [9] and [7].

If we set $\delta = 0$ in $(P_\lambda)$ then the problem reduces to the well studied scalar
problem

$$(S_\lambda) \quad \begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma. \end{cases}$$

There is an extensive literature on such kind of problems. We just mention
[2], [4], [13] and more recently [5]. We note that our treatment of the quasimonotone system is similar to the treatment of problem $(S_\lambda)$ as was
done in [2] and [4]. The results of the present paper were announced in [20].

The structure of the paper is as follows. In the next section the precise
assumptions on the nonlinearity $f$ are stated, as well as the conditions which
we impose on the parameters $\gamma$ and $\delta$. It is then shown how $(P_\lambda)$ can be
transformed to a quasimonotone system. The main results are also stated
in this section. In Section 3 we prove several auxiliary results. The proofs
of the main theorems are given in Section 4. In Appendix A we define our
notion of sub- and supersolutions for quasimonotone systems and give some


related results. In particular we state a version of the sweeping principle for quasimonotone systems. This principle is used repeatedly in the proofs.

2. Assumptions and main results.

The assumptions on $f$ are the following.

**Condition A.** The function $f \in C^{1,1}(\mathbb{R})$, $f(0) \geq 0$ and there exists $\sigma_0 > 0$ such that for every $0 \leq \sigma < \sigma_0$ there exist $\rho^-_\sigma < \rho^+_\sigma > 0$ such that

1. $f(\rho^ \pm_\sigma) = \sigma \rho^ \pm_\sigma$ and $f(s) > \sigma s$ for $\rho^-_\sigma < s < \rho^+_\sigma$;
2. $f'(s) < 0$ for all $s \in (\rho^+_\sigma, \rho^+_0)$;
3. $J_\sigma(\rho) > 0$ on $(0, \rho^+_\sigma)$ for all $0 \leq \sigma < \sigma_0$ where

$$J_\sigma(\rho) := \int_{\rho}^{\rho^+_\sigma} (f(s) - \sigma s) \, ds.$$ 

See Figure 1.

![Figure 1](image.png)

**Example 1.** The function $f(u) = au - u^3$ with $a > 0$, see [12] and [7], satisfies Condition A above with $\sigma_0 = 2a/3$.

**Example 2.** Consider the function $f(u) = u(u - a)(u - 1)$ with $a > 0$. Condition A holds if $a < 1/2$. In this case $\sigma_0 = (2a^2 - 5a + 2)/9$. With this nonlinearity problem $(P_\lambda)$ is an extension of the FitzHugh-Nagumo equations, see [9] and [10].

As was said in the introduction, an important step in our analysis is to transform $(P_\lambda)$ and to modify $f$ in order to obtain a quasimonotone system. For the definition of a quasimonotone system and some results for such
systems we refer to Appendix A. In order to transform system \((P_\lambda)\) we need the following assumption on the parameters \(\delta\) and \(\gamma\):

**Condition B1.** Let \(M := \max \{-f'(s) ; 0 \leq s \leq \rho_0^+\}\) and suppose that
\[
\gamma - 2\sqrt{\delta} > M.
\]

We define \(\beta\) and \(\alpha\) by
\[
\beta := \frac{1}{2}(\gamma - M) - \frac{1}{2}\sqrt{(\gamma - M)^2 - 4\delta},
\]
(3)
\[
\alpha = \gamma - \beta.
\]
(4)

If Condition B1 holds then \(\beta \in \mathbb{R}\) and \(\alpha, \beta > 0\). Note that
\[
\vartheta := 1 - \frac{\delta}{\gamma \beta} > 0.
\]
(5)

One may verify that \((u, w)\) is a solution to
\[
\begin{aligned}
-\Delta u &= \lambda(f(u) - \beta u + \beta w) \quad \text{in } \Omega, \\
-\Delta w &= \lambda(f(u) + Mu - \alpha w) \quad \text{in } \Omega, \\
u &= w = 0 \quad \text{on } \Gamma,
\end{aligned}
\]
(Q\(\lambda\))

if and only if \((u, \beta u - \beta w)\) is a solution to \((P_\lambda)\).

Let \(\tilde{f} \in C^{1,1}(\mathbb{R})\) be a function satisfying \(\tilde{f}(s) = f(s)\) for all \(s \in [0, \rho_0^+]\) with \(\tilde{f}, \tilde{f}'\) bounded on \(\mathbb{R}\) and with \(\tilde{f}'(s) + M \geq 0\) for all \(s \in \mathbb{R}\). If we replace \(f\) in \((Q_\lambda)\) by \(\tilde{f}\) the system becomes quasimonotone. Since we are interested in solutions \((u, v)\) to \((P_\lambda)\) with \(u\) positive and \(\max u < \rho_0^+\) we can assume without loss of generality the following:

**Condition A*. The function \(f\) satisfies Condition A with \(f\) and \(f'\) bounded and \(f'(s) + M \geq 0\) for all \(s \in \mathbb{R}\) and \(f(s) \leq 0\) for \(s \geq \rho_0^+\).

Another condition which we impose is:

**Condition B2.** The constant \(\beta\) defined in (3) satisfies \(\beta < \sigma_0\).

Under this condition one has for \(\lambda\) large enough a positive nontrivial solution to the scalar problem
\[
\begin{aligned}
-\Delta u &= \lambda(f(u) - \beta u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]
which has its maximum in the interval \((\rho_0^-, \rho_0^+)\), see [4]. This solution will be used to obtain a nontrivial subsolution to \((Q_\lambda)\) for \(\lambda\) large enough. The definition of sub- and supersolutions is given in Appendix A.
We make some remarks on Conditions B1 and B2. Both conditions are satisfied if \( \frac{\delta}{\gamma} \) is small enough. More precisely, for fixed \( \delta > 0 \), B1 and B2 are satisfied if

\[
\begin{aligned}
\gamma &> \begin{cases} 
M + 2\sqrt{\delta} & \text{if } 0 \leq \delta < \sigma_0^2; \\
M + \sigma_0 + \delta/\sigma_0 & \text{if } \delta \geq \sigma_0^2.
\end{cases}
\end{aligned}
\]

In the first theorem we prove the existence of a curve of positive solutions to \((P_\lambda)\).

**Theorem 2.1 (Existence of a curve of solutions).** Let \( f \) satisfy Condition A, let \( \gamma, \delta \) be such that Conditions B1 and B2 hold and assume that \( \Gamma \) is \( C^3 \). Then there exist \( \lambda^* > 0 \) and a function \( \Lambda \in C^1([\lambda^*, +\infty), C^2(\bar{\Omega}) \times C^2(\bar{\Omega})) \) such that \((u_\lambda, v_\lambda) := \Lambda (\lambda)\) is a positive solution, i.e., \( u_\lambda, v_\lambda \geq 0 \), to \((P_\lambda)\) for all \( \lambda \geq \lambda^* \). Furthermore

1. \( \max u_\lambda \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma}) \) and \( \max v_\lambda \in \left( \frac{\delta}{\gamma} (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma}) \right) \);

2. \( \lim_{\lambda \to \infty} \Lambda (\lambda) = \left( \rho^+_{\delta/\gamma}, \frac{\delta}{\gamma} \rho^+_{\delta/\gamma} \right) \) uniformly on compact subsets of \( \Omega \).

The stability of the solutions obtained in the theorem above will be considered in the space \( X := C(\bar{\Omega}) \times C(\bar{\Omega}) \). For \( \lambda > \lambda^* \) we define the linear operator \( A_\lambda : D(A_\lambda) \subset X \to X \) by

\[
D(A_\lambda) := \{(u, v) \in X ; (\Delta u, \Delta v) \in X \}
\]

and

\[
A_\lambda \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \lambda \begin{pmatrix} f'(u_\lambda) & 1 \\ \delta & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]

for \((u, v) \in D(A_\lambda)\). Here \( u_\lambda \) is the first component of \( \Lambda (\lambda) \). In the definition of \( D(A_\lambda) \), \( \Delta u \) and \( \Delta v \) are to be understood in distributional sense.

**Theorem 2.2 (Stability).** Assume that the conditions of Theorem 2.1 hold and let \( \lambda^* \) and \( \Lambda \) be as in that theorem. For every \( \lambda \geq \lambda^* \) the solution \( \Lambda (\lambda) = (u_\lambda, v_\lambda) \) to \((P_\lambda)\) is an exponentially stable equilibrium solution to the initial value problem (1) i.e., for every \( \lambda \geq \lambda^* \) there exists \( \nu_\lambda > 0 \) such that the spectrum \( \sigma(A_\lambda) \) is contained in \( \{ \nu \in \mathbb{C} ; \text{Re} \nu > \nu_\lambda \} \).

Our last theorem is a result on the uniqueness, in a restricted sense, of solutions to \((P_\lambda)\).

**Theorem 2.3 (Uniqueness in order interval).** Assume that the conditions of Theorem 2.1 hold and let \( \lambda^* \) and \( \Lambda \) be as in that theorem. For every function \( z \in C_0(\Omega) \) with \( z \geq 0 \) and max \( z \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma}) \) there exists \( \lambda_2 > \lambda^* \) such that if \((u, v)\) is a solution to \((P_\lambda)\) with \( \lambda > \lambda_2 \) and \( u \in [z, \rho^+_{\delta/\gamma}] \) then \((u, v) = \Lambda (\lambda)\).
In general one cannot expect uniqueness of solutions. Indeed it may for example be the case that the trivial solution is a stable solution to the problem. Then there will exist a third, unstable solution in \([0, \Lambda (\lambda)]\). This is the case when \(f\) is as in Example 2 and Conditions B1 and B2 hold, see [19].

We end this section with a summary of the notation that will be used.

**Notation:**

- Let \(u_1, u_2 \in C(\Omega)\). We write
  \[
  u_1 \geq u_2 \text{ if } u_1(x) \geq u_2(x) \text{ for all } x \in \Omega;
  \]
  \[
  u_1 \preceq u_2 \text{ if } u_1(x) \geq u_2(x) \text{ and } u_1 \neq u_2;
  \]
  \[
  u_1 > u_2 \text{ if } u_1(x) > u_2(x) \text{ for all } x \in \Omega.
  \]
- For \((u, w) \in C(\Omega) \times C(\Omega)\) we shall use \((u, w)(x) = (u(x), w(x))\).
- Let \((u_i, w_i) \in C(\Omega) \times C(\Omega), i = 1, 2\). We write
  \[
  (u_1, w_1) \geq (u_2, w_2) \text{ if } u_1 \geq u_2 \text{ and } w_1 \geq w_2;
  \]
  \[
  (u_1, w_1) \preceq (u_2, w_2) \text{ if } (u_1, w_1) \geq (u_2, w_2) \text{ and } (u_1, w_1) \neq (u_2, w_2);
  \]
  \[
  (u_1, w_1) > (u_2, w_2) \text{ if } u_1 > u_2 \text{ and } w_1 > w_2.
  \]
- If \((u_1, w_1) \geq (u_2, w_2)\) we denote by \([([u_1, w_1], (u_2, w_2)]\) the order interval
  \[
  \{(u, w) \in C(\Omega) \times C(\Omega); (u_1, w_1) \leq (u, w) \leq (u_2, w_2)\}.
  \]
- By \(D^+(\Omega)\) we denote the set of \(z \in C_0^\infty(\Omega)\) with \(z \geq 0\); \(D'(\Omega)\) denotes the usual space of distributions.
- For \(u_1, u_2 \in C(\Omega)\) we say \(-\Delta u_1 \leq u_2\) in \(D'(\Omega)\)-sense if
  \[
  \int_{\Omega} u_1(-\Delta z) \, dx \leq \int_{\Omega} u_2 z \, dx
  \]
  for all \(z \in D^+(\Omega)\).
- For a Banach space \(X\) we denote the bounded linear operators from \(X\) into \(X\) by \(L(X)\).

### 3. Preliminary results.

#### 3.1. Estimates for positive solutions.

**Proposition 3.1.** Let \(B\) be the unit ball in \(\mathbb{R}^N\). Suppose that \(f\) satisfies Condition \(A^*\). Then there exists \(\lambda_B > 0\) such that the problem

\[
\begin{align*}
-\Delta u &= \lambda_B (f(u) - \beta u + \beta w) & \text{in } B, \\
-\Delta w &= \lambda_B (f(u) + Mu - \alpha w) & \text{in } B, \\
  u &= w = 0 & \text{on } \partial B,
\end{align*}
\]

has a solution \((U_B, W_B)\) with the following properties:
(1) $0 \leq (U_B, W_B) < (\rho^+_{\delta_j/\gamma}, \vartheta_{\rho^+_{\delta_j/\gamma}})$, with $\vartheta = 1 - \delta/(\gamma \beta)$.
(2) $U_B$ and $W_B$ are radially symmetric with $U_B(0) = W_B(0) = 0$ and $U_B'(r), W_B'(r) < 0$ on $(0,1]$.
(3) $(U_B(0), W_B(0)) > (\rho^+_{\delta_j/\gamma}, \vartheta_{\rho^+_{\delta_j/\gamma}})$ and $W_B(0) \geq \vartheta \tau$ where $\tau := U_B(0)$.

Proof. Since Condition B2 holds, $J_\delta(\rho) > 0$ for all $0 \leq \rho < \rho^+_{\delta_j}$. This implies that for $\lambda$ large enough, say $\lambda = \lambda_B$, there exists a positive solution $u$ to

$$\begin{cases} 
-\Delta u = \lambda (f(u) - \beta u) & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}$$

with $\max u \in (\rho^+_{\delta_j}, \rho^-_{\delta_j})$, see [4]. Then $(u, 0)$ is a subsolution to (8). Since $(\rho^+_{\delta_j/\gamma}, \vartheta_{\rho^+_{\delta_j/\gamma}})$ is a supersolution with $(u, 0) < (\rho^+_{\delta_j/\gamma}, \vartheta_{\rho^+_{\delta_j/\gamma}})$ there exists a solution $(U_B, W_B)$ with $u < U_B < \rho^+_{\delta_j/\gamma}$ and $0 < W_B < \vartheta_{\rho^+_{\delta_j/\gamma}}$ to (8), see Proposition A.3. Using an extension due to Troy, [24], of results of Gidas, Ni and Nirenberg, [8], to quasimonotone systems, we have that $U_B$ and $W_B$ are radially symmetric with $U_B(0) = W_B(0) = 0$ and $U_B'(r), W_B'(r) < 0$ on the interval $(0,1)$. Also $(-\Delta + \lambda_B \alpha) W_B = \lambda_B (f(U_B) + MU_B) \geq 0$ and by the strong maximum principle $W_B'(1) < 0$. Let $\tau := U_B(0)$. With $V_B = \beta (U_B - W_B)$ it also follows from the maximum principle that

$$\max V_B < (\delta/\gamma) \tau.$$  

Indeed, $(-\Delta + \lambda_B \gamma) (V_B - \delta \tau/\gamma) = \lambda_B (U_B - \tau) \leq 0$ in $B$, with $V_B = 0$ on $\partial B$ and (9) follows. Since $V_B$ is also radially symmetric and decreasing, $V_B(0) = \beta (\tau - W_B(0)) < (\delta/\gamma) \tau$ and hence

$$W_B(0) > (1 - \delta/(\gamma \beta)) \tau = \vartheta \tau > \vartheta \rho^-_{\delta_j/\gamma}.$$  

Also $V_B'(1) = \beta (U_B'(1) - W_B(1)) < 0$ and hence $U_B'(1) < W_B'(1) < 0$. □

Next we construct a family of subsolutions to $(Q_\lambda)$ using the functions $U_B$ and $W_B$. These subsolutions will be used to determine by sweeping the shape of the solutions to $(Q_\lambda)$ in a certain order interval. We fix $z^* \in \Omega$ and let

$$\lambda^* := \lambda_B \dist(z^*, \Gamma)^{-2}.$$  

Lemma 3.2. For all $\lambda \geq \lambda^*$ we set

$$Z_\lambda(x) := \begin{cases} 
(U_B, W_B) \left(\frac{\lambda}{\lambda_B}\right)^{1/2} (x - z^*) & \text{for } |x - z^*| \leq (\lambda_B/\lambda)^{1/2}; \\
0 & \text{for } |x - z^*| > (\lambda_B/\lambda)^{1/2},
\end{cases}$$

with $(U_B, W_B)$ as in Proposition 3.1. Then $Z_\lambda$ is a subsolution to $(Q_\lambda)$ and

$$Y := (\rho^+_{\delta_j/\gamma}, \vartheta_{\rho^+_{\delta_j/\gamma}})$$

is a supersolution to $(Q_\lambda)$ with $Z_\lambda < Y$. 
Proof. It follows directly that \( Y \) is a supersolution. The function \( Z_\lambda \) is continuous and \( Z_\lambda(x) = 0 \) for \( x \in \Gamma \). Denote by \( Z_{\lambda,i}, i = 1,2, \) the two components of \( Z_\lambda \). Let \( z \in D^+(\Omega) \). Then, with \( B_\lambda = B(z^*,(\lambda_B/\lambda)^{1/2}) \) and \( n \) denoting the outward normal, we obtain by the Green identity:

\[
\int_\Omega Z_{\lambda,1}(-\Delta z) \, dx = \int_{B_\lambda} Z_{\lambda,1}(-\Delta z) \, dx = -\int_{B_\lambda} (\Delta Z_{\lambda,1}) z \, dx - \int_{\partial B_\lambda} \left( Z_{\lambda,1} \frac{\partial z}{\partial n} - z \frac{\partial Z_{\lambda,1}}{\partial n} \right) \, dS
\]

\[
\leq \lambda \int_{\Omega} (f(Z_{\lambda,1}) - \beta Z_{\lambda,1} + \beta Z_{\lambda,2}) z \, dx.
\]

A similar result holds for \( Z_{\lambda,2} \). Finally \( \max Z_{\lambda,1} = Z_{\lambda,1}(z^*) = \tau < \rho_{\delta/\gamma}^+ \) and \( \max Z_{\lambda,2} = Z_{\lambda,2}(z^*) = W_B(0) < \vartheta\rho_{\delta/\gamma}^+ \). \[\square\]

Since \( Z_\lambda \) is a subsolution to \( (Q_\lambda) \) and \( Y \) is a supersolution to \( (Q_\lambda) \) with \( Z_\lambda < Y \) there exists at least one solution in the order interval \([Z_\lambda, Y]\). For every fixed \( \lambda \geq \lambda^* \) we define for all \( y \in \Omega \) satisfying \( \text{dist}(y, \Gamma) > (\lambda_B/\lambda)^{-1/2} \) the functions

\[
Z_\lambda^y(x) := Z_\lambda(x + z^* - y).
\]

Repeating the proof of Lemma 3.2 one sees that for \( \lambda \geq \lambda^* \)

\[
S_\lambda := \left\{ Z_\lambda^y ; y \in \Omega \text{ such that } \text{dist}(y, \Gamma) > (\lambda_B/\lambda)^{1/2} \right\}
\]

is a family of subsolutions. We shall use the sweeping principle with functions in \( S_\lambda \) to obtain, at least for \( \lambda \) large enough, estimates of solutions to \( (Q_\lambda) \) in the order interval \([Z_\lambda, Y]\). In order to estimate a solution in \([Z_\lambda, Y]\) in all of \( \Omega \) as well as on the boundary we make the following assumption on \( \Gamma \) which holds if \( \Gamma \subset C^3 \):

- \( \Omega \) satisfies a uniform interior sphere condition, that is, there exists \( \varepsilon_\Omega > 0 \) such that \( \Omega = \cup \{ B(y, \varepsilon) ; y \in \Omega \text{ and } \text{dist}(y, \Gamma) > \varepsilon_\Omega \} \).

We may suppose that \( \Omega_\varepsilon := \{ y \in \Omega ; \text{dist}(y, \Gamma) > \varepsilon \} \) is connected for all \( \varepsilon \leq \varepsilon_\Omega \).

Lemma 3.3. There exists \( \lambda^x > \lambda^* \) and \( b > 0 \) such that for all \( \lambda > \lambda^x \) we have the following estimate for every solution \((u, w) \in [Z_\lambda, Y]\) to \((Q_\lambda)\):

\[
(u(x) , w(x)) > \min\{b\lambda^{1/2} \text{dist}(x, \Gamma), \tau\} \{1, \vartheta\},
\]

with \( \vartheta = 1 - \delta/(\gamma\beta) \) and \( \tau \) as in Proposition 3.1.

Proof. Let \( \varepsilon_\lambda := (\lambda_B/\lambda)^{1/2} \) and \( \lambda^x := \max \{ \lambda^*, \lambda_B\varepsilon_\Omega^{-2} \} \). Suppose that \((u, w) \in [Z_\lambda, Y]\) is a solution to \((Q_\lambda)\) with \( \lambda > \lambda^x \). As in [4] there exists for every \( y \in \Omega_\varepsilon \lambda \) a curve in \( \Omega_\varepsilon \lambda \) connecting \( y \) with \( z^* \). Using the sweeping
principle, Proposition A.6, it follows that \((u, w) > Z_\lambda^y\) for all \(y \in \Omega_{\epsilon, \lambda}\). Using \((u(x), w(x)) \geq \sup_{y \in \Omega_{\epsilon, \lambda}} Z_\lambda^y(x)\) one finds (13).

The next lemma improves the estimate we found in the previous one.

**Lemma 3.4.** For every \(\varepsilon > 0\) and \(\lambda > \lambda^\infty\) there exists a constant \(b(\varepsilon) > 0\), independent of \(\lambda\), such that for every solution \((u, w) \in [Z_\lambda, Y]\) to \((Q_\Lambda)\) it holds that

\[
(14) \quad (u(x), w(x)) > \min \left\{ b(\varepsilon) \lambda^{1/2} \text{dist} (x, \Gamma), \rho_{\delta/\gamma}^+ - \varepsilon \right\} (1, \vartheta),
\]
with \(\vartheta = 1 - \delta/(\gamma \beta)\). In particular there exists \(b_0 > 0\) such that

\[
(15) \quad (u(x), w(x)) > \min \left\{ b_0 \lambda^{1/2} \text{dist} (x, \Gamma), \rho_{\delta/\gamma}^+ \right\} (1, \vartheta).
\]

**Proof.** Let \(\lambda > \lambda^\infty\) be fixed and suppose \((u, w) \in [Z_\lambda, Y]\) is solution to \((Q_\Lambda)\). If \(\rho_{\delta/\gamma}^+ - \varepsilon \leq \tau\) then (13) holds with \(b(\varepsilon) = b\) and \(b\) as in the previous lemma.

Suppose \(\rho_{\delta/\gamma}^+ - \varepsilon > \tau\). Since \(f(s) - (\delta/\gamma)s > 0\) for all \(s \in (\rho_{\delta/\gamma}^+, \rho_{\delta/\gamma}^+)\) there exists \(\ell_\varepsilon > 0\) such that

\[ f(s) - (\delta/\gamma)s > \ell_\varepsilon (s - \tau) \quad \text{for all } s \in [\tau, \rho_{\delta/\gamma}^+ - \varepsilon]. \]

From Lemma 3.3 it follows that \((u(x), w(x)) > (\tau, \vartheta \tau)\) for all \(x \in \Omega\) such that \(\text{dist}(x, \Gamma) > \lambda^{-1/2} \tau / b\). For subsolutions we need the function \(e \geq 0\) satisfying

\[
(16) \quad \left\{ \begin{array}{ll}
-\Delta e = \mu e & \text{in } B_1 \\
e = 0 & \text{on } \partial B_1
\end{array} \right.
\]

where \(\mu\) is the principal eigenvalue and \(B_1\) the unit ball in \(\mathbb{R}^N\). We normalize \(e\) such that \(e(0) = 1\). Let \(\mu_\varepsilon = \mu/\ell_\varepsilon\) and

\[ \Omega'_\varepsilon := \left\{ y \in \Omega; \text{dist}(y, \Gamma) > (\sqrt{\mu_\varepsilon} + \tau/b)\lambda^{-1/2} \right\}. \]

We fix \(y \in \Omega'_\varepsilon\) and let \(B := B(y, (\mu_\varepsilon/\lambda)^{1/2})\). For every \(t \in [0, 1]\) we define the functions \((U_t, W_t)\) on \(\overline{B}\) by

\[
U_t(x) := \tau + t(\rho_{\delta/\gamma}^+ - \varepsilon - \tau)e (\lambda/\mu_\varepsilon)^{1/2} (y - x),
\]
\[
W_t(x) := \vartheta U_t(x).
\]

Then \(T := \{(U_t, W_t); t \in [0, 1]\}\) is a family of subsolutions to the problem

\[
(17) \quad \left\{ \begin{array}{ll}
-\Delta p = \lambda (f(p) - \beta p + \beta q) & \text{in } B, \\
-\Delta q = \lambda (f(p) + Mp - \alpha q) & \text{in } B, \\
p = u & \text{on } \partial B, \\
q = w & \text{on } \partial B.
\end{array} \right.
\]

Using the sweeping principle it follows that

\[(u(y), w(y)) > (U_1(y), W_1(y)) = (\rho_{\delta/\gamma}^+ - \varepsilon, \vartheta (\rho_{\delta/\gamma}^+ - \varepsilon)).\]
Since $y \in \Omega'$ was arbitrary we have that
\[(18) \quad (u(x), w(x)) > (\rho_{\delta/y}^+ - \varepsilon, \vartheta(\rho_{\delta/y}^+ - \varepsilon)) \text{ if } \text{dist}(x, \Gamma) > (\mu_{\varepsilon} + \tau/b)\lambda^{-1/2}.\]

Define $b(\varepsilon) := \lambda \left(\tau/b + \sqrt{\mu_{\varepsilon}}\right)^{-1}$, and note that \(\min\{b\lambda^{1/2}\text{dist}(x, \Gamma), \tau\} \geq b(\varepsilon) \lambda^{1/2}\text{dist}(x, \Gamma)\) if \(\text{dist}(x, \Gamma) \leq (\sqrt{\mu_{\varepsilon}} + \tau/b)\lambda^{-1/2}\). Hence by Lemma 3.3

\[(u(x), w(x)) > b(\varepsilon) \lambda^{1/2}\text{dist}(x, \Gamma)(1, \vartheta)\]

for all $x$ with \(\text{dist}(x, \Gamma) \leq (\sqrt{\mu_{\varepsilon}} + \tau/b)\lambda^{-1/2}\). This proves (14) while (15) follows by choosing $b_0 = b(\varepsilon)$ with $\varepsilon = \rho_{\delta/y}^+ - \rho_{\sigma_0}^+$. \(\square\)

The next lemma will be used in the proof of Theorem 2.3.

**Lemma 3.5.** Let $z_0 \in C_0(\Omega)$ be nonnegative with $\max z_0 \in (\rho_{\mu/\lambda}^+, \rho_{\mu/\lambda}^+)$. There exists $\lambda_{z_0} > 0$ such that if $(u, w)$ is a solution to $(Q_\lambda)$ with $u \in [z_0, \rho_{\mu/\lambda}^+]$ and $\lambda > \lambda_{z_0}$ then $(u, w) \in [Z_\lambda, Y]$.

**Proof.** First note that if $u \in [z_0, \rho_{\mu/\lambda}^+]$ then $(u, w) \in [(z_0, 0), Y]$. Let $x_0 \in \Omega$ be such that $z_0(x_0) = \max z_0$. Choose $\rho \in (\rho_{\mu/\lambda}^-, \tau)$, where $\tau$ is as in Proposition 3.1, and $r_0 > 0$ such that $\rho < z_0(x) \leq u(x)$ for all $x \in B(x_0, r_0)$. Since $f(s) - \beta s > 0$ for all $s \in (\rho_{\mu/\lambda}^-, \rho_{\mu/\lambda}^+)$ there exists $\ell > 0$ such that

\[f(s) - \beta s > \ell(s - \rho) \quad \text{for all } s \in [\rho, \tau].\]

Let $e$ and $\mu$ be as (16) with $e(0) = 1$. Suppose that

\[(19) \quad \lambda > (\mu/\ell)r_0^{-2}.\]

Then $r_\lambda := r_0 - \sqrt{\mu/(\ell \lambda)} > 0$. Let $y \in B(x_0, r_\lambda)$ be fixed and define on

\[B = B(y, \sqrt{\mu/(\ell \lambda)}) \subset B(x_0, r_0),\]

\[U_t(x) = \rho + t(\tau - \rho)e\left(\sqrt{\ell \lambda/\mu}(y - x)\right)\]

It holds that $T := \{(U_t, 0) ; t \in [0, 1]\}$ is a family of subsolutions to (17) with $u_\lambda, w_\lambda$, instead of $u, w$. By a sweeping argument, starting with $(U_0, 0)$ one concludes that $(u(y), w(y)) > (U_1(y), W_1(y)) = (\tau, 0)$. Since $y \in B(x_0, r_\lambda)$ was arbitrary we have that $(u, w) \geq (\tau, 0)$ on $B(x_0, r_\lambda)$.

Let $Z_{\lambda, i}^{x_0}$, $i = 1, 2$, denote the two components of $Z_{\lambda}^{x_0}$ defined in (12). The function $Z_{\lambda, 1}^{x_0}$ has support $B(x_0, \sqrt{\lambda B/\lambda})$. Hence, if (19) is replaced by the stronger condition $\lambda > (\sqrt{\lambda B} + \sqrt{\mu/\ell})^2r_0^2$, then $r_\lambda > \sqrt{\lambda B/\lambda}$ and $(u, w)(x) > (Z_{\lambda, 1}^{x_0}, 0)$ for all $x \in \Omega$.

From this it follows that $(u, w) \in [Z_{\lambda}^{x_0}, Y]$. Indeed, using the fact that $Z_{\lambda}^{x_0}$ is a subsolution one has that $-\Delta(w - Z_{\lambda, 2}^{x_0}) + \alpha(w - Z_{\lambda, 2}^{x_0}) \geq 0$ in $D'(\Omega)$-sense.

As in the proof of Lemma 3.3 it now follows that $(u, w) \in [Z_\lambda, Y]$. \(\square\)
3.2. The semilinear problem on the half space. In this section we consider the following problem

\[
\begin{align*}
-\Delta U &= f(U) - \beta U + \beta W & \text{in } \mathbb{R}_+^N, \\
-\Delta W &= f(U) + MU - \alpha W & \text{in } \mathbb{R}_+^N, \\
U &= W = 0 & \text{on } \partial \mathbb{R}_+^N.
\end{align*}
\]

The main result which we prove is that there exists a positive solution \((U, W)\) to (20) such that

\[
\lim_{x_1 \to \infty} (U, W)(x_1, x') = (\rho^{+}_{\beta/\gamma}, \theta\rho^{+}_{\beta/\gamma}) \quad \text{uniformly in } x' \in \mathbb{R}^{N-1},
\]

with \(\theta = 1 - \delta/(\gamma \beta)\). Moreover there exists only one such solution and \((U, W)(x_1, x') = (u, w)(x_1)\) where \((u, w)\) a solution to the problem

\[
\begin{align*}
-u'' &= f(u) - \beta u + \beta w & \text{in } \mathbb{R}^+, \\
-w'' &= f(u) + Mu - \alpha w & \text{in } \mathbb{R}^+, \\
u(0) &= 0, & w(0) = 0, \\
u'(0) &= \kappa, & w'(0) = \nu,
\end{align*}
\]

for some appropriate initial data \(\kappa\) and \(\nu\). It is standard that we have for every pair \((\kappa, \nu)\) \(\in \mathbb{R}^2\) at least locally a unique solution to (22) which can be continued to some maximum interval. We denote such a solution by \((u, w)_{\kappa, \nu} = (u_{\kappa, \nu}, w_{\kappa, \nu})\). First we show that there exists a unique pair \((\kappa, \nu)\) such that the corresponding solution exists for all \(r \in \mathbb{R}^+\), is positive and tends to \((\rho^{+}_{\beta/\gamma}, \theta\rho^{+}_{\beta/\gamma})\) at infinity. Some properties of this solution that are needed later, are also proven.

**Proposition 3.6.** Assume that \(f\) satisfies Condition A*. Then there exists a unique pair \((\bar{\kappa}, \bar{\nu})\) such that the solution \((u, w)_{\bar{\kappa}, \bar{\nu}}\) to (22) is positive and satisfies

\[
\lim_{r \to \infty} (u, w)_{\bar{\kappa}, \bar{\nu}}(r) = (\rho^{+}_{\beta/\gamma}, \theta\rho^{+}_{\beta/\gamma}),
\]

with \(\theta = 1 - \delta/(\beta \gamma)\). Moreover \(\bar{\kappa} > \bar{\nu} > 0\) and \((u, w)_{\bar{\kappa}, \bar{\nu}}\) has the following properties:

1. \(0 < (u, w)_{\bar{\kappa}, \bar{\nu}}(r) < (\rho^{+}_{\beta/\gamma}, \theta\rho^{+}_{\beta/\gamma})\) for all \(r > 0\);
2. \(u_{\bar{\kappa}, \bar{\nu}}(r) > w_{\bar{\kappa}, \bar{\nu}}(r)\) for all \(r > 0\);
3. \((u', w')(\bar{\kappa}, \bar{\nu})(r) > (0, 0)\) for all \(r \in \mathbb{R}^+\) and \((u', w')(r) \to (0, 0)\) as \(r \to \infty\).
The proof of this proposition consists of a number of lemmas. We also need to consider the following system

\[
\begin{align*}
-u'' &= f(u) - v & \text{in } \mathbb{R}^+, \\
-v'' &= \delta u - \gamma v & \text{in } \mathbb{R}^+, \\
u(0) &= 0, & v(0) &= 0, \\
u'(0) &= \kappa, & v'(0) &= \eta.
\end{align*}
\]

(24)

Again we denote solutions to (24) by \((u, v)_{\kappa, \eta}\) with the understanding that the solutions are defined on a maximum interval. We point out the fact that for \(\kappa, \nu \in \mathbb{R}\) it holds that the solution \((u, v)_{\kappa, \beta(\kappa-\nu)}\) to (24) is given by \((u_\kappa, \beta(u_\kappa - w_{\kappa, \nu}))\) where \((u, w)_{\kappa, \nu}\) is the solution to (22).

For a solution \((u, v) = (u, v)_{\kappa, \eta}\) to (24) we have the following identity for all \(r \geq 0\):

\[
((u')^2 - \kappa^2) - \frac{1}{\delta}((v')^2 - \eta^2) = -2 \int_0^u f(s) \, ds + 2uv - \frac{\gamma}{\delta}v^2. 
\]

(25)

Indeed, differentiating

\[
H(r) := u'(r)^2 - \frac{1}{\delta}v'(r)^2 + 2 \int_0^{u(r)} f(s) \, ds - 2u(r)v(r) + \frac{\gamma}{\delta}v(r)^2,
\]

(24) implies that \(H'(r) = 0\) for all \(r \geq 0\). Hence \(H(r) = H(0)\) for all \(r \geq 0\), which gives (25).

We shall often use the following one dimensional maximum principle, see e.g., [11, Theorem 2.9.2].

**Lemma 3.7.** If \(g \in C^2[0, +\infty)\) is bounded, \(g(0) \geq 0\) and \(-g'' + cg \geq 0\) with \(c \geq 0\), then \(g(r) > 0\) for all \(r > 0\). Moreover, if \(g(0) = 0\) then \(g'(0) > 0\).

Our first lemma is on the derivatives of solutions to (22).

**Lemma 3.8.** Suppose that \((u, w)_{\kappa, \nu}\) is a solution to (22) with \(\kappa, \nu > 0\) and \((u, w)_{\kappa, \nu}(r) > (0, 0)\) for all \(r > 0\). Then \((u', w')_{\kappa, \nu}(r) > (0, 0)\) for all \(r \geq 0\).

**Proof.** Since the system is quasimonotone this follows from a moving plane argument, similar to the method used by Gidas, Ni and Nirenberg [8]. See also [2] where a similar argument is used for a scalar equation.

**Lemma 3.9.** For a bounded solution \((u, w)_{\kappa, \nu}\) to (22) with \(u_{\kappa, \nu} \geq 0\) it holds that \(0 < \nu < \kappa\) and \(0 < w_{\kappa, \nu}(r) < u_{\kappa, \nu}(r)\) for all \(r > 0\).

**Proof.** Denote by \((u, w)\) the solution \((u, w)_{\kappa, \nu}\). Since \(w\) is bounded and satisfies \(-w'' + \alpha w = f(u) + Mu \geq 0\) with \(w(0) = 0\) we have by Lemma 3.7 that \(\nu > 0\) and \(w(r) > 0\) for all \(r > 0\). Let \(\eta = \beta(\kappa - \nu)\). As observed earlier, the solution \((u, v) = (u, v)_{\kappa, \eta}\) to (24) is given by \((u, \beta(u - w))\). Since \(v\) is bounded with \(v(0) = 0\) and \(-v'' + \gamma v = \delta u \geq 0\) it holds again by Lemma 3.7 that \(\eta > 0\) and \(v(r) > 0\) for \(r > 0\). Hence \(\kappa > \nu\) and \(w(r) < u(r)\) for \(r > 0\).
Lemma 3.10. If \((u, w)_{\kappa, \nu}\) is a positive solution to (22) such that (23) holds then
\[
\lim_{r \to \infty} (u', w')_{\kappa, \nu}(r) = 0.
\]

Proof. Define \(u_K(r) := \rho_{\delta/\gamma}^+ - u_{\kappa, \nu}(r+K)\) and \(w_K(r) := \partial \rho_{\delta/\gamma}^+ - w_{\kappa, \nu}(r+K)\). It holds that \(u_K\) and \(w_K\) converge uniformly to zero on \([0, 1]\) as \(K \to \infty\). From (22) we have that they remain bounded in \(C^2[0, 1]\). By interpolation \(u_K, w_K\) converge to zero in \(C^1[0, 1]\). Therefore \((u', w')_{\kappa, \nu}(K) = (u_K', w_K')(0) \to (0, 0)\), as \(K \to \infty\). □

Let \((u, w)_{\kappa, \nu}\) be a solution to (22) for which (23) holds. Then \((u', v')_{\kappa, \eta}\) with \(\eta = \beta(\kappa - \nu)\) is a solution to (24) and \(v_{\kappa, \eta} = \beta(u_{\kappa, \nu} - w_{\kappa, \nu}) \to (\delta/\gamma)\rho_{\delta/\gamma}^+\) as \(r \to \infty\). Using Lemma 3.10 and letting \(r \to \infty\) in (25) we obtain the following relationship between \(\kappa\) and \(\nu\):
\[
\kappa^2 - \beta^2 (\kappa - \nu)^2 = 2 \int_0^{\rho_{\delta/\gamma}^+} \left( f(s) - \frac{\delta}{\gamma} s \right) \, ds.
\]
This will be used to prove the uniqueness of such solutions. Next we show that there exists initial data \((\bar{\kappa}, \bar{\nu})\) for which the corresponding solution to (22) is positive and satisfies (23).

Lemma 3.11. There exists \(\bar{\kappa}, \bar{\nu} \in \mathbb{R}\) such that the solution \((u, w)_{\bar{\kappa}, \bar{\nu}}\) to (22) satisfies (23). Moreover this solution is positive and \(0 < \bar{\nu} < \bar{\kappa}\).

Proof. We shall use super- and subsolutions and Lemma A.4 to find a positive solution to
\[
\begin{aligned}
- u'' &= f(u) - \beta u + \beta w \quad \text{in } \mathbb{R}^+, \\
- w'' &= f(u) + Mu - \alpha w \quad \text{in } \mathbb{R}^+, \\
u(0) &= w(0) = 0,
\end{aligned}
\]
satisfying (23). As a supersolution we take \((\rho_{\delta/\gamma}^+, \partial \rho_{\delta/\gamma}^+)\). We have to construct a nonzero subsolution. From a phaseplane analysis one sees that the initial value problem
\[
\begin{aligned}
- u'' &= f(u) - \beta u \quad \text{in } \mathbb{R}^+, \\
u(0) &= 0, \\
w'(0) &= (2J_{\beta}(0))^{1/2},
\end{aligned}
\]
with \(J_{\beta}(0) > 0\) defined in (2), has a solution \(\bar{u}\) with \(\lim_{r \to \infty} \bar{u}(r) = \rho_{\delta/\gamma}^+\) and \(\bar{u}'(r) > 0\) for all \(r \geq 0\). Then \((\bar{u}, 0)\) is a subsolution. By Lemma A.4 there exists a solution \((u, w)\) to (27) such that \((0, \bar{u}) < (u, w) < (\rho_{\delta/\gamma}^+, \partial \rho_{\delta/\gamma}^+)\). At this stage we may choose either the maximal or minimal solution. In the next lemma we shall prove that they are equal. Let \((\bar{\kappa}, \bar{\nu}) := (u'(0), w'(0))\). Then \((u, w)\) is the solution to (22) with \((\kappa, \nu) = (\bar{\kappa}, \bar{\nu})\).
It holds that \( u(r), w(r) > 0 \) for all \( r > 0 \). Indeed, \( u(r) \geq \tilde{u}(r) > 0 \) and since \( w \) is bounded with \( -w'' + \alpha w = f(u) + Mu \geq 0 \) and \( w(0) = 0 \) we have by Lemma 3.7 that \( w(r) > 0 \) for \( r > 0 \) and that \( w'(0) = \nu > 0 \). By Lemma 3.9, \( \bar{\kappa} > \bar{\nu} > 0 \).

Lemma 3.8 shows that \( u'(r), w'(r) > 0 \) for all \( r > 0 \). In particular \( \lim_{r \to \infty} u(r) = \rho \) and \( \lim_{r \to \infty} w(r) = \bar{\rho} \) exist. From the equations we find that

\[
-f(\rho) + \beta \rho - \beta \bar{\rho} = -f(\rho) - M \rho + \alpha \bar{\rho} = 0.
\]

From this one gets that \( \bar{\rho} = \partial \rho \) and that \( f(\rho) = (\delta/\gamma) \rho \). Since \( \rho > \rho^+ \) we have that \( (\rho, \bar{\rho}) = (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma}) \).

Our last lemma concerns the uniqueness part of Proposition 3.6.

**Lemma 3.12.** Let \( (u,w)_{\kappa,\nu} \) be a positive solution to (22) such that (23) holds. Then \( (\kappa,\nu) = (\bar{\kappa},\bar{\nu}) \) with \( (\bar{\kappa},\bar{\nu}) \) as in Lemma 3.11.

**Proof.** Let \( (\tilde{u},0) \) be the subsolution of the previous lemma. First we show that the minimum and maximum solutions to (27) in the order interval

\[ [(\tilde{u},0), (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma})] \]

are equal.

Let \( (u,w)_{\kappa,\nu} \) be the minimal solution and \( (u,w)_{\bar{\kappa},\bar{\nu}} \) the maximal solution. It must hold that \( \kappa \leq \bar{\kappa} \) and \( \nu \leq \bar{\nu} \). If the solutions are not equal at least one of these inequalities must be strict. Suppose \( \nu < \bar{\nu} \). By Lemma 3.9 we also have that \( \kappa > \nu \) and \( \bar{\kappa} > \bar{\nu} \) and by (26) that

\[
\frac{\delta - \beta^2}{\delta - \kappa^2} + \frac{2\beta^2}{\delta - \kappa \nu} - \frac{\beta^2}{\delta - \nu^2} = \frac{\delta - \beta^2}{\delta - \bar{\kappa}^2} + \frac{2\beta^2}{\delta - \bar{\kappa} \bar{\nu}} - \frac{\beta^2}{\delta - \bar{\nu}^2}.
\]

The function \( x \mapsto (1 - (\beta^2/\delta))x^2 + 2(\beta^2/\delta)x - (\beta^2/\delta)\nu^2 \) is strictly increasing on \([\kappa, \bar{\kappa}]\) because it has derivative \( 2(\delta - \beta^2)x/\delta + 2\beta^2\nu/\delta \) which is strictly positive for \( x \in [\kappa, \bar{\kappa}] \) since \( \delta > \beta^2 \). Hence

\[
\frac{\delta - \beta^2}{\delta - \kappa^2} + \frac{2\beta^2}{\delta - \kappa \nu} - \frac{\beta^2}{\delta - \nu^2} \leq \frac{\delta - \beta^2}{\delta - \bar{\kappa}^2} + \frac{2\beta^2}{\delta - \bar{\kappa} \bar{\nu}} - \frac{\beta^2}{\delta - \bar{\nu}^2}.
\]

The function \( x \mapsto (1 - (\beta^2/\delta))\bar{\kappa} + 2(\beta^2/\delta)\bar{\kappa}x - (\beta^2/\delta)x^2 \) has derivative \( 2\beta^2\bar{\kappa}\delta - 2\beta^2x/\delta \). Since the derivative is strictly positive on \([\nu, \bar{\nu}]\) it follows that

\[
\frac{\delta - \beta^2}{\delta - \kappa^2} + \frac{2\beta^2}{\delta - \kappa \nu} - \frac{\beta^2}{\delta - \nu^2} < \frac{\delta - \beta^2}{\delta - \bar{\kappa}^2} + \frac{2\beta^2}{\delta - \bar{\kappa} \bar{\nu}} - \frac{\beta^2}{\delta - \bar{\nu}^2},
\]

contradicting (28). If \( \kappa < \bar{\kappa} \) we find a contradiction by the same argument. We conclude that \( \kappa = \bar{\kappa} \) and \( \nu = \bar{\nu} \) and that \( (u,w)_{\kappa,\nu} = (u,w)_{\bar{\kappa},\bar{\nu}} \).

It remains to show that any positive solution \( (u,w)_{\kappa,\nu} \) for which (23) holds is in the order interval \([\tilde{u},0), (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma})]\). First we show that \( u_{\kappa,\nu} > \tilde{u} \).
Suppose that \( u_{\kappa,\nu}(r) > \rho^+_{\delta/\gamma} \) for all \( r \geq R \). We define \( u^*_t \) for \( 0 \leq t \leq R \) on \([0, R]\) by
\[
  u^*_t(r) := \begin{cases} 
    \bar{u}(r - t) & \text{for } t \leq r \leq R; \\
    0 & \text{for } 0 \leq r < t,
  \end{cases}
\]
with \( \bar{u} \) as in Lemma 3.11. Applying the sweeping principle with the subsolutions \( \{(u^*_t, 0) ; 0 \leq t \leq R\} \) one finds that \( (u, w)_{\kappa,\nu}(r) > (\bar{u}(r), 0) = (u^*_0(r), 0) \) for \( r \in (0, R) \). Hence \( u_{\kappa,\nu}(r) > \bar{u}(r) \) for \( r > 0 \) and \((u, w)_{\kappa,\nu} \geq (\bar{u}, 0)\). On the other hand, since \( u_{\kappa,\nu}, w_{\kappa,\nu} \) are increasing by Lemma 3.8, it holds that \((u, w)_{\kappa,\nu} < (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma})\). Since there is only one solution in the order interval \([\bar{u}, 0), (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma})]\) the uniqueness is proved. \(\square\)

Our main result concerning system (20) is the following.

**Proposition 3.13.** Assume that \( f \) satisfies Condition \( A^* \). Then there exists a unique positive solution \((U, W)\) to (20) satisfying (21). This solution is given by
\[
  (U, W)(x_1, x') := (u, w)_{\bar{\kappa}, \bar{\nu}}(x_1) \quad \text{for } (x_1, x') \in \mathbb{R}^{N-1},
\]
with \((u, w)_{\bar{\kappa}, \bar{\nu}}\) as in Lemma 3.11.

**Proof.** Clearly (29) defines a positive solution to (20) satisfying (21). Suppose that \((U, W)\) is any positive solution satisfying (21). Define the functions
\[
\begin{align*}
  (u, w)(x_1) & := \left( \sup_{x' \in \mathbb{R}^{N-1}} U(x_1, x'), \sup_{x' \in \mathbb{R}^{N-1}} W(x_1, x') \right); \\
  (\bar{u}, \bar{w})(x_1) & := \left( \inf_{x' \in \mathbb{R}^{N-1}} U(x_1, x'), \inf_{x' \in \mathbb{R}^{N-1}} W(x_1, x') \right).
\end{align*}
\]
By Lemma A.5 \((u, w)\) is a subsolution and \((\bar{u}, \bar{w})\) is a supersolution to (27). Moreover \((u, w) < (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma})\). This follows by sweeping using the family of supersolutions \(\{(t, \bar{u}) : t \geq \rho^+_{\delta/\gamma}\}\).

Since \((u, w)\) is a subsolution there exists a positive solution \((u, w)_s\) to (27) with \((u, w)_s \leq (u, w)_s < (\rho^+_{\delta/\gamma}, \partial \rho^+_{\delta/\gamma})\). By Lemma 3.12 we have that \((u, w)_s = (u, w)_{\bar{\kappa}, \bar{\nu}}\).

Using a sweeping argument as in the proof of Lemma 3.12 it follows that \((\bar{u}, \bar{w}) \geq (\bar{u}, 0)\) with \(\bar{u}\) as in the proof of Lemma 3.11. Hence there exists a positive solution \((u, w)_o\) to (27) with \((\bar{u}, 0) < (u, w)_o \leq (\bar{u}, \bar{w})\) and by Lemma 3.12, \((u, w)_o = (u, w)_{\bar{\kappa}, \bar{\nu}}\). Hence \((\bar{u}, \bar{w}) = (u, w)\) which proves the uniqueness claim in the proposition. \(\square\)

### 3.3. The linearized problem on the halfspace

Let \((u, w)_{\bar{\kappa}, \bar{\nu}}\) be as in Proposition 3.6. In this paragraph we consider the following linear system:
\[
\begin{align*}
  -\bar{r}\Delta \Phi &= (f'(u_{\bar{\kappa}, \bar{\nu}}) - \beta + \omega)\Phi + \beta \Psi - \bar{r}\omega \Phi \quad \text{in } \mathbb{R}^N_+; \\
  -\bar{r}\Delta \Psi &= (f'(u_{\bar{\kappa}, \bar{\nu}}) + M)\Phi + (\omega - \alpha)\Psi - \bar{r}\omega \Psi \quad \text{in } \mathbb{R}^N_+; \\
  \Phi &= \Psi = 0 \quad \text{on } \partial \mathbb{R}^N_+.
\end{align*}
\]
Here $\alpha, \beta, M$ are as in (4), (3) and B1 respectively, $\omega > \max \{\alpha, M\}$ and $\tilde{r} \in \mathbb{R}$. For this problem we have the following result of Liouville type.

**Proposition 3.14.** Suppose that $\tilde{r} \geq 1$ and that $(\Phi, \Psi)$ is a bounded positive solution to (30). Then $(\Phi, \Psi) \equiv (0,0)$.

This proposition will be a consequence of the following lemma.

**Lemma 3.15.** Suppose $\varphi, \psi \in C[0, +\infty)$ are bounded with $\varphi, \psi \geq 0$, $\varphi(0) = \psi(0) = 0$ and it holds that

\begin{align*}
(31) & \quad -\varphi'' \leq f'(u_{\varphi, \psi})\varphi - \beta\varphi + \beta\psi, \\
(32) & \quad -\psi'' \leq f'(u_{\varphi, \psi})\psi + M\psi - \alpha\psi,
\end{align*}

in $D'(0, \infty)$-sense. Then $\varphi(x) = \psi(x) = 0$ for all $x \geq 0$.

**Proof.** We set $(p, q) := (u', w')_{\varphi, \psi}$ and recall that $p, q > 0$ on $[0, \infty)$. Without loss of generality we assume that $\varphi, \psi \leq 1$. We argue by contradiction and suppose that $(\varphi, \psi) \neq (0,0)$. First we observe that if there exists $K > 0$ such that $\varphi(x_1) = \psi(x_1) = 0$ for all $x_1 \geq K$ then by a sweeping argument on $[0, K]$ with the family $\{(tp, tq) ; t \geq 0\}$ of supersolutions it follows that $\varphi(x_1) = \psi(x_1) = 0$ for all $x_1 \in [0, K]$. This is in contradiction with our assumption.

Now let $K > 0$ and $\varepsilon > 0$ be such that that

$$f'(u_{\varphi, \psi}(x_1)) < -\varepsilon \quad \text{for all } x_1 \geq K,$$

and note that also

$$f' (u_{\varphi, \psi}(x_1)) + M - \alpha < -\varepsilon \quad \text{for all } x_1 \geq K.$$

By our first observation we may assume that

$$R(K) := \max \{\varphi(K)/p(K), \psi(K)/q(K)\} > 0.$$

We define the following functions on $[K, \infty)$:

\begin{align*}
S_t(x_1) & = \varphi(x_1) - e^{\sqrt{\varepsilon}}(x_1 - t), \\
T_t(x_1) & = \psi(x_1) - e^{\sqrt{\varepsilon}}(x_1 - t), \\
R_t(x_1) & = \max \{S_t(x_1)/p(x_1), T_t(x_1)/q(x_1)\}
\end{align*}

for $t \geq K$. It holds that

$$-S_t'' \leq (f'(u_{\varphi, \psi}) - \beta)S_t + \beta T_t,$$

and

$$-T_t'' \leq (f'(u_{\varphi, \psi}) + M)S_t - \alpha T_t,$$

in $D'(K, \infty)$-sense. For $t > K$ let $m_t = \sup_{x_1 \in [K,t]} R_t(x_1)$. By the maximum principle one has that $m_t = R_t(K)$ for $t$ large enough. Indeed, since for $\omega$ large enough, it holds in $D'(K, t)$-sense that

$$-(S_t - m_t p)'' + \omega(S_t - m_t p) \leq 0,$$
and
\[-(T_t - mtq)'' + \alpha (T_t - mtq) \leq 0,\]
we see that \(m_t\) must be attained in \(K\) or in \(t\). Since \(R_t(t) \leq 0\) and \(R_t(K) > 0\), if \(t\) is large enough, we conclude that \(m_t = R_t(K)\). Now let \(x_1 \in [K, \infty)\) be fixed. Then for all \(t > x_1\) large we have
\[
R(x_1) = \max \left\{ \frac{\varphi(x_1)}{p(x_1)}, \frac{\psi(x_1)}{q(x_1)} \right\} \\
\leq R_t(K) + \max \left\{ e^{\sqrt{\varphi(x_1-t)}}/p(x_1), e^{\sqrt{\varphi(x_1-t)}}/q(x_1) \right\}.
\]
Letting \(t \to \infty\) we deduce that \(R(x_1) \leq R(K)\) and hence \(R\) attains its maximum on \([K, \infty)\) in \(x_1 = K\). Consequently \(\sup_{x_1 \in [0, K]} R(x_1)\) is attained in some point \(r_0 \in (0, K]\). But this is in contradiction to the maximum principle. Indeed, in a similar way as above, one sees that \(R(x_1)\) must attain its maximum on \([0, K+1]\) either in 0 or in \(K+1\) and not in \(K\). \(\Box\)

To see how Proposition 3.14 follows from this lemma, we define
\[
\varphi(x_1) := \sup \left\{ \Phi(x_1, x') ; x' \in \mathbb{R}^{N-1} \right\}, \\
\psi(x_1) := \sup \left\{ \Psi(x_1, x') ; x' \in \mathbb{R}^{N-1} \right\}.
\]
Then by Lemma A.5, \(\varphi, \psi \in C([0, +\infty))\) with \(\varphi(0) = \psi(0) = 0\) and in \(D'(\mathbb{R}^N)\)-sense
\[
-\varphi'' \leq \frac{1}{\bar{r}} (f'(u_{\bar{r}, \bar{r}}) - \beta + \omega) \varphi + \frac{1}{\bar{r}} \beta \psi - \omega \varphi, \\
-\psi'' \leq \frac{1}{\bar{r}} (f'(u_{\bar{r}, \bar{r}}) + M) \varphi + \frac{1}{\bar{r}} (\omega - \alpha) \psi - \omega \psi.
\]
Since \(\bar{r} \geq 1\) we deduce that
\[
-\varphi'' \leq (f'(u_{\bar{r}, \bar{r}}) - \beta + \omega) \varphi + \beta \psi - \omega \varphi = (f'(u_{\bar{r}, \bar{r}}) - \beta + \omega) \varphi + \beta \psi
\]
and
\[
-\psi'' \leq (f'(u_{\bar{r}, \bar{r}}) + M) \varphi + (\omega - \alpha) \psi - \omega \psi = (f'(u_{\bar{r}, \bar{r}}) + M) \varphi - \alpha \psi.
\]
By the lemma \((\varphi, \psi)(x_1) = 0\) for \(x_1 \geq 0\) and hence also \((\Phi, \Psi)(x) = (0, 0)\) on \(\mathbb{R}^N_+\).

4. Proofs of the main results.

4.1. Proof of Theorem 2.1. From now on we assume that \(\Gamma\) is \(C^3\). We begin by defining some operators. Recall that \(X\) denotes the space \(C(\bar{\Omega}) \times C(\bar{\Omega})\) and let \(C^1_0(\Omega) = \{ u \in C^1(\Omega) ; u(x) = 0 \text{ for } x \in \Gamma \}\). For \(k, \lambda > 0\)
define \((-\lambda^{-1}\Delta + k)^{-1}_0 : C(\bar{\Omega}) \rightarrow C^1_0(\bar{\Omega})\) by \(u = (-\lambda^{-1}\Delta + k)^{-1}_0 g\) with \(u \in C^1_0(\bar{\Omega})\) the unique function satisfying
\[
\begin{cases}
-\lambda^{-1}\Delta u + ku = g & \text{in } D'(\Omega)\text{-sense}, \\
u = 0 & \text{on } \Gamma.
\end{cases}
\]
Let \(j\) be the embedding of \(C^0_0(\bar{\Omega}) \times C^1_0(\bar{\Omega})\) in \(X\) and define the operator \(K_{k,\lambda} : X \rightarrow X\) by
\[
K_{k,\lambda}\begin{pmatrix} g \\ h \end{pmatrix} = j \circ \begin{pmatrix} 0 & (-\lambda^{-1}\Delta + k)^{-1}_0 \\ (-\lambda^{-1}\Delta + k)^{-1}_0 & 0 \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}.
\]
Since \(j\) is compact and \((-\lambda^{-1}\Delta + k)^{-1}_0\) is continuous, \(K_{k,\lambda}\) is a compact linear map on \(X\). We shall also use the fact that \(\|K_{k,\lambda}\|_{\mathcal{L}(X)}\) is uniformly bounded in \(\lambda\). This follows from the fact that
\[
\|(-\lambda^{-1}\Delta + k)^{-1}_0 g\|_{\infty} \leq \frac{1}{\rho} \|g\|_{\infty}
\]
for every \(g \in C(\bar{\Omega})\). We fix \(\omega > \max\{\alpha, M\}\). For a function \(u \in C(\bar{\Omega})\) and \(\lambda > 0\) the we define the operators \(M_u, T_{u,\lambda} \in \mathcal{L}(X)\) by
\[
M_u\begin{pmatrix} h \\ g \end{pmatrix} = \begin{pmatrix} f'(u) - \beta & \beta \\ \beta & f'(u) + M - \alpha \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix},
\]
and
\[
T_{u,\lambda} := K_{\omega,\lambda}(M_u + \omega I).
\]
Operators of this kind were studied extensively in [23]. If \(u \in \lbrack 0, \rho^{-1}_{\delta/\gamma}\rbrack\) then \(T_{u,\lambda}\) is a positive irreducible compact operator on \(X\), see [23, Lemma 1.3]. Moreover, \(T_{u,\lambda}\) has a positive spectral radius (see e.g., [18]) which we denote by \(r(T_{u,\lambda})\). By the Krein-Rutman Theorem (see e.g., [1, Theorem 3.1]), \(r(T_{u,\lambda})\) is an eigenvalue of \(T_{u,\lambda}\) to which a positive eigenfunction pertains. In the next lemma we prove that for \(\lambda\) large enough it holds for every solution \((u, w) \in [Z_\lambda, Y]\) that \(r(T_{u,\lambda}) < 1\).

Lemma 4.1. There exists \(\lambda^* > \lambda^\times\) such that for all \(\lambda > \lambda^*\) and every solution \((u, w) \in [Z_\lambda, Y]\) to (Q_\lambda) the corresponding operator \(T_{u,\lambda}\) has spectral radius \(r(T_{u,\lambda}) < 1\).

Proof. We prove the lemma by a contradiction argument. Assume that it does not hold. Then there exist a sequence \(\{\lambda_n\}_{n=1}^\infty\) with \(\lambda^\times < \lambda_n \rightarrow \infty\) and solutions \((u_n, w_n) := (u_{\lambda_n}, w_{\lambda_n}) \in [Z_{\lambda_n}, Y]\) to (Q_\lambda) with \(\lambda = \lambda_n\) such that \(r_n \geq 1\), with \(r_n\) denoting the spectral radius of \(T_n \equiv T_{u_n,\lambda_n}\). Let \((\varphi_n, \psi_n) \in X\) be the positive eigenfunction pertaining to \(r_n\). We normalize the eigenfunction such that \(\max \varphi_n = 1\). This can be done since \(\varphi_n = 0\)
implies that \( \psi_n = 0 \), a contradiction because \((\varphi_n, \psi_n)\) is an eigenfunction. It holds that

\[
\begin{cases}
-r_n \lambda_n^{-1} \Delta \varphi_n &= (f'(u_n) + \omega - \beta) \varphi_n + \beta \psi_n - r_n \omega \varphi_n \quad \text{in } \Omega, \\
r_n \lambda_n^{-1} \Delta \psi_n &= (f'(u_n) + M) \varphi_n + (\omega - \alpha) \psi_n - r_n \omega \psi_n \quad \text{in } \Omega, \\
\varphi_n &= \psi_n = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Because operator norms \( \|T_n\|_{\mathcal{L}(X)} \) are uniformly bounded it follows from

\[
r_n = \lim_{k \to \infty} \left( \frac{\|T_n^k\|_{\mathcal{L}(X)}}{k} \right)^{1/k} \leq \|T_n\|_{\mathcal{L}(X)}
\]

that the sequence \( \{r_n\}_{n=1}^\infty \) is bounded. By going over to a subsequence, still denoted by \( \{r_n\}_{n=1}^\infty \), we can assume that \( r_n \to \bar{r} \geq 1 \). With \( \theta_n := \beta(\varphi_n - \psi_n) \) one has that

\[
\begin{cases}
-r_n \lambda_n^{-1} \Delta \theta_n &= \delta \varphi_n - \gamma \theta_n + (1 - r_n) \omega \theta_n \quad \text{in } \Omega, \\
\theta_n &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

This shows that \( \varphi_n \geq \psi_n \) and hence \( -r_n \lambda_n^{-1} \Delta \varphi_n \leq f'(u_n) \varphi_n \). Using estimate (15) in Lemma 3.4 we have for all \( x \in \Omega \) with \( \text{dist } (x, \Gamma) > b_0^{-1} \lambda_n^{-1/2} \rho_{\sigma_0}^+ \) that \( f'(u_n(x)) \leq 0 \) and consequently

\[
(36) \quad -\Delta \varphi_n \leq 0 \quad \text{in } \left\{ x \in \Omega \mid \text{dist } (x, \Gamma) > b_0^{-1} \lambda_n^{-1/2} \rho_{\sigma_0}^+ \right\}.
\]

Hence \( \varphi_n \) attains its maximum in a point \( \bar{x}_n \) with \( \text{dist}(\bar{x}_n, \Gamma) \leq b_0^{-1} \lambda_n^{-1/2} \rho_{\sigma_0}^+ \).

Let \( \bar{x}_{\Gamma,n} \in \Gamma \) be such that \( |\bar{x}_n - \bar{x}_{\Gamma,n}| = \text{dist}(\bar{x}_{\Gamma,n}, \Gamma) \). By going over to a subsequence we can assume that \( \bar{x}_{\Gamma,n} \to \bar{x} \in \Gamma \).

By a blow-up argument around \( \bar{x} \), similar to the argument in [4], one constructs \( U, W, \Phi, \Psi \in C^2(\mathbb{R}^N_+) \cap C\left(\overline{\mathbb{R}^N_+}\right) \) such that \((U, W)\) satisfies

\[
\begin{align*}
-\Delta U &= f(U - \beta U + \beta W) \quad \text{in } \mathbb{R}^N_+, \\
-\Delta W &= f(U) + MU - \alpha W \quad \text{in } \mathbb{R}^N_+, \\
U &= W = 0 \quad \text{on } \partial \mathbb{R}^N_+,
\end{align*}
\]

and \((\Phi, \Psi)\) satisfies

\[
\begin{align*}
-\tilde{r} \Delta \Phi &= (f'(U) + \omega - \beta) \Phi + \beta \Psi - \tilde{r} \omega \Phi \quad \text{in } \mathbb{R}^N_+, \\
-\tilde{r} \Delta \Psi &= (f'(U) + M) \Phi + (\omega - \alpha) \Psi - \tilde{r} \omega \Psi \quad \text{in } \mathbb{R}^N_+, \\
\Phi &= \Psi = 0 \quad \text{on } \partial \mathbb{R}^N_+.
\end{align*}
\]

The normalization \( \max \phi_n = 1 \) leads to \( \sup \Phi = 1 \). Furthermore, using the uniform estimate (14) it follows that

\[
(37) \quad \lim_{x_1 \to -\infty} (U, W)(x_1, x') = (\rho_{\delta/\gamma}^+, \rho_{\delta/\gamma}^+) \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}.
\]
Hence, by Proposition 3.13, \((U, W)(x_1, x') = (u, w)_{k, \rho}(x_1)\). Then \((\Phi, \Psi)\) is a bounded positive solution to (30) with \(\bar{r} \geq 1\). By Proposition 3.14, \((\Phi, \Psi) \equiv (0, 0)\), in contradiction with \(\sup \Phi = 1\).

We shall use this lemma to prove that there can be at most one solution to \((Q_\lambda)\) in the order interval \([Z_\lambda, Y]\). First we define the operator \(H_\lambda : X \rightarrow X\)

\[H_\lambda := K_{\omega, \lambda}(F + \omega I),\]

where \(F : X \rightarrow X\) is defined by

\[
F \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} f(u) - \beta u + \beta w \\ f(u) + Mu - \alpha w \end{pmatrix}.
\]

We shall show that \(H_\lambda\) has at most one fixed point in \([Z_\lambda, Y]\). In order to use the Leray-Schauder degree we have to consider the fixed point problem

\[H_\lambda(u, w) = (u, w)\]

in an appropriate space.

Let \(\mu\) be the principal eigenvalue and \(e \in C^1(\bar{\Omega}) \cap C^2(\Omega)\) the corresponding eigenfunction to the problem

\[
\begin{cases}
-\Delta e = \mu e & \text{in } \Omega, \\
\partial \Omega & = 0 \\
e & \text{on } \Gamma.
\end{cases}
\]

We normalize \(e\) such that \(\max e = 1\). Following Amann [1] we define

\[(38) \quad C_e(\bar{\Omega}) := \{ u \in C(\bar{\Omega}) ; \exists t > 0 \text{ such that } |u| \leq te \},\]

equipped with the norm \(\|u\|_e = \inf\{t > 0 ; -te \leq u \leq te\}\). It holds that \(C_e(\bar{\Omega})\) is a Banach space, in fact a Banach lattice, with closed unit ball \(\{ u \in C(\bar{\Omega}) ; -e \leq u \leq e \}\). Let \(X_e = C_e(\bar{\Omega}) \times C_e(\bar{\Omega})\). Order intervals in \(X_e\) will be denoted by \([\cdot, \cdot],_e\). Let \(j_1, j_2\) be the embeddings of \(X_e\) in \(X\) and \(C^1_0(\bar{\Omega}) \times C^1_0(\Omega)\) in \(X_e\) respectively and define \(H^e_\lambda : X_e \rightarrow X_e\) by

\[H^e_\lambda := j_2 \circ \begin{pmatrix} -\lambda^{-1} \Delta + \omega \\ 0 \end{pmatrix}^{-1}_{0} \circ (F + \omega I) \circ j_1.\]

We recall that \((-\lambda^{-1} \Delta + \omega)^{-1}_{0}\) was defined as an operator from \(C(\bar{\Omega})\) into \(C^1_0(\bar{\Omega})\). We note that \((u, w)\) is a fixed point of \(H^e_\lambda\) if and only if \(j_1(u, w)\) is a fixed point of \(H_\lambda\). Hence it suffices to show that \(H^e_\lambda\) has a unique fixed point in \([Z_\lambda, Y]\) \(\cap X_e\).

It also holds for \((u_i, w_i) \in X\) with \((u_1, w_1) < (u_2, w_2)\) that

\[(39) \quad H_\lambda(u_1, w_1) < H_\lambda(u_2, w_2).\]

In fact \(H_\lambda(u_2, w_2) - H_\lambda(u_1, w_1)\) is an element of the interior of the positive cone of \(X_e\), or equivalently, there exists \(t > 0\) such that

\[(40) \quad H_\lambda(u_2, w_2) - H_\lambda(u_1, w_1) \geq (te, te).\]
Since neither $Z_\lambda$ nor $Y$ are fixed points of $H_\lambda$ we find, using (39), that any fixed point $(u, w) \in [Z_\lambda, Y]$ satisfies

$$Z_\lambda^*: = H_\lambda Z_\lambda < (u, w) < H_\lambda Y =: Y^*.$$ 

From (40) we even have the stronger result that $(u, w) \in \text{int} [Z_\lambda^*, Y^*]_e$, the interior of $[Z_\lambda^*, Y^*]_e$ with respect to the $\|\cdot\|_e$-topology. The uniqueness of a fixed point of $H_\lambda$ in $[Z_\lambda, Y]$ for $\lambda \geq \lambda^*$ then follows from the next lemma.

**Lemma 4.2.** For every $\lambda > \lambda^*$ there exists a unique fixed point of $H_\lambda^e$ in $\text{int} [Z_\lambda^*, Y^*]_e$.

**Proof.** We have for every $\lambda > \lambda^*$ that there exists at least one solution to $(Q_\lambda)$ in the order interval $[Z_\lambda, Y]$ which, as we observed, is a fixed point of $H_\lambda^e$ and $(u, w) \in \text{int} [Z_\lambda^*, Y^*]_e$. To show that this is the only solution, we shall use a degree argument.

Suppose $(u, w) \in \text{int} [Z_\lambda^*, Y^*]_e$ with $\lambda > \lambda^*$, is a fixed point of $H_\lambda^e$. The operator $H_\lambda^e$ is differentiable and $T_{u, \lambda}^e := dH_\lambda^e (u, w) \in \mathcal{L}(X_e)$ given by

$$T_{u, \lambda}^e = j_2 \circ \begin{pmatrix} -\lambda^{-1} \Delta + \omega & 0 \\ 0 & -\lambda^{-1} \Delta + \omega \end{pmatrix} \circ (M_u + \omega I) \circ j_1,$$

with $M_u$ as defined in (34). From Lemma 4.1 we have that the spectral radius $r(T_{u, \lambda}^e) < 1$. Indeed $\mu$ is an eigenvalue of $T_{u, \lambda}^e$ if and only if $\mu$ is an eigenvalue of $T_{u, \lambda}$. Since $r(T_{u, \lambda}) > 0$ it holds that $r(T_{u, \lambda}^e) > 0$. But $T_{u, \lambda}^e$ is a positive compact operator and hence $r(T_{u, \lambda}^e)$ is an eigenvalue of $T_{u, \lambda}^e$ to which a positive eigenfunction pertains. This implies that $r(T_{u, \lambda}^e) = r(T_{u, \lambda}) < 1$.

In particular 1 is not an eigenvalue of $T_{u, \lambda}^e$ and consequently the index of the fixed point $(u, w)$ is well defined with

$$\text{index} (u, w) = 1,$$

see [17, p. 66]. Using the homotopy invariance of the degree and the fact that $\text{int} [Z_\lambda^*, Y^*]_e$ is convex, it follows that

$$\text{degree} \left( I - H_{u, \lambda}^e, \text{int} [Z_\lambda^*, Y^*]_e, 0 \right) = 1.$$ 

Indeed, let $\bar{z} \in \text{int} [Z_\lambda^*, Y^*]_e$ be arbitrary and define the homotopy

$$G_t = (1 - t) (I - \bar{z}) + t (I - H_{u, \lambda}^e).$$ 

It holds that $G_t \bar{z} = 0$ if and only if $z = (1 - t) \bar{z} + t H_{u, \lambda}^e z$ and hence $z \in \text{int} [Z_\lambda^*, Y^*]_e$. Since $G_1$ has no zeros on the boundary $\text{int} [Z_\lambda^*, Y^*]_e$ we have that

$$\text{degree} (G_1, \text{int} [Z_\lambda^*, Y^*]_e, 0) = \text{degree} (G_0, \text{int} [Z_\lambda^*, Y^*]_e, 0) = 1.$$ 

By the additivity property of the degree we see that $H_\lambda^e$ can have at most one fixed point in $\text{int} [Z_\lambda^*, Y^*]_e$. \(\square\)
The proof of Theorem 2.1 can now be completed. For all \( \lambda > \lambda^* \) we have a unique solution \( \bar{\Lambda} (\lambda) = (u_{\lambda}, w_{\lambda}) \in \text{int} [Z^*_\lambda, Y^*_\lambda] \subset [Z_\lambda, Y] \) to \((Q\lambda)\).

Using the Implicit Function Theorem we have that \( \bar{\Lambda} \in C^1 ((\lambda^*, +\infty), X_e). \) Indeed the operator \((\lambda, (u, w)) \mapsto (u, w) - H^e (\lambda, (u, w)) \) is \( C^1 \) and the derivative with respect to \((u, w)\) is given by \( I - T^e_{u,\lambda}. \) For fixed \( \lambda_0 > \lambda^* \) it holds that \( I - T^e_{u,\lambda_0} \in \text{Isom}(X_e). \) By the Implicit Function Theorem the solution set of \( H^e (\lambda, (u, w)) = 0 \) consists in a neighbourhood of \( \lambda_0 \) of a \( C^1 \)-curve, parameterized by \( \lambda. \) By uniqueness of solutions in int \([Z^*_\lambda, Y^*_\lambda]\) we have that \( \bar{\Lambda} \) in this curve is in a neighbourhood of \((\lambda_0, u_{\lambda_0}, w_{\lambda_0})\) given by \( \bar{\Lambda}. \) Since this can be done for every \( \lambda > \lambda^* \) we have that \( \bar{\Lambda} \) in \( C^1 ((\lambda^*, +\infty), X_e). \) Using a bootstrap argument one proves that \( \bar{\Lambda} \in C^1 ((\lambda^*, +\infty), C^2 (\bar{\Omega}) \times C^2 (\bar{\Omega})). \)

Finally we define \( \Lambda (\lambda) := (u_\lambda, \beta (u_\lambda - w_\lambda)). \) Then \( \Lambda (\lambda) \) is a solution to \((P\lambda)\) for all \( \lambda > \lambda^* \) and \( \Lambda \in C^1 ((\lambda^*, \infty), C^2 (\bar{\Omega}) \times C^2 (\bar{\Omega})). \)

### 4.2. Proof of Theorems 2.2 and 2.3

In this section we assume that the conditions of Theorem 2.1 hold. We define the operator \( B_\lambda : D(B_\lambda) \to X \) by

\[
D(B_\lambda) := \{(u, w) \in X; (\Delta u, \Delta w) \in X\},
\]

with \( \Delta u \) and \( \Delta w \) in distributional sense,

\[
B_\lambda := L_\lambda - M_{u_\lambda},
\]

with

\[
L_\lambda := \begin{pmatrix}
-\lambda^{-1} \Delta & 0 \\
0 & -\lambda^{-1} \Delta
\end{pmatrix}
\]

and \( M_{u_\lambda} \) as defined in \((34)\).

**Lemma 4.3.** For all \( \lambda > \lambda^*, \) with \( \lambda^* \) as in Lemma 4.1, the operator \( B_\lambda \) is invertible and \( B^{-1}_\lambda \in \mathcal{L}(X) \) is a positive compact operator with a positive spectral radius \( r_\lambda = r(B^{-1}_\lambda). \) Moreover \( r_\lambda \) is an eigenvalue of \( B^{-1}_\lambda \) with a corresponding positive eigenfunction.

**Proof.** Denote by \( T_\lambda \) and \( M_\lambda \) the operators \( T_{u_\lambda} \) and \( M_{u_\lambda} \) respectively. Since \( \lambda > \lambda^* \) the spectral radius \( r(T_\lambda) \) of \( T_\lambda \) satisfies

\[
0 < r(T_\lambda) < 1.
\]

Hence \( B_\lambda = L_\lambda + \omega I - (M_\lambda + \omega I) \) \((I - T_\lambda) \) is invertible with \( B^{-1}_\lambda = (I - T_\lambda)^{-1} (M_\lambda + \omega I) \) since \( (I - T_\lambda)^{-1} \) is a positive bounded operator and \( K_{\omega, \lambda} \) is a positive compact operator. Moreover, since \( r(T_\lambda) < 1 \) it follows from \([23, \text{Lemma 1.4}]\) that \( B^{-1}_\lambda \) is irreducible, and hence the spectral radius \( r_\lambda = r(B^{-1}_\lambda) \) is positive. By the Krein-Rutman Theorem \( r_\lambda \) corresponds to a positive eigenfunction. \( \square \)
Lemma 4.4. For all \( \nu > 0 \) the operator \( B_\lambda + \nu I \) is invertible. The inverse \( (B_\lambda + \nu I)^{-1} \in \mathcal{L}(X) \) is positive and compact and its spectral radius is given by \( r((B_\lambda + \nu I)^{-1}) = (r_\lambda^{-1} + \nu)^{-1} \).

Proof. Let \( k = \omega + \nu \). Then
\[
B_\lambda + \nu I = L_\lambda + kI - (M_\lambda + \omega I)
= (L_\lambda + kI)(I - K_{k,\lambda}(M_\lambda + \omega I)).
\]
The operator \( K_{k,\lambda}(M_\lambda + \omega I) \) is also positive, compact and irreducible. Again using the Krein-Rutman Theorem we find that \( r(K_{k,\lambda}(M_\lambda + \omega I)) \) is an eigenvalue of the adjoint operator \( (K_{k,\lambda}(M_\lambda + \omega I))^* \) pertaining to a positive functional, say \( \Upsilon_{k,\lambda} \). Let \( h_\lambda \) be the positive eigenfunction of \( B_\lambda^{-1} \) corresponding to the eigenvalue \( r_\lambda \) as in Lemma 4.3. It holds that
\[
(42) \quad h_\lambda = (r_\lambda^{-1} + k - \omega) K_{k,\lambda} h_\lambda + K_{k,\lambda}(M_\lambda + \omega I) h_\lambda.
\]
Using (42) it follows that
\[
\langle h_\lambda, \Upsilon_{k,\lambda} \rangle = (r_\lambda^{-1} + k - \omega) \langle K_{k,\lambda} h_\lambda, \Upsilon_{k,\lambda} \rangle + \langle K_{k,\lambda}(M_\lambda + \omega I) h_\lambda, \Upsilon_{k,\lambda} \rangle
\geq (r_\lambda^{-1} + k - \omega) \langle h_\lambda, \Upsilon_{k,\lambda} \rangle
= r(K_{k,\lambda}(M_\lambda + \omega I)) \langle h_\lambda, \Upsilon_{k,\lambda} \rangle.
\]
Hence \( r(K_{k,\lambda}(M_\lambda + \omega I)) < 1 \) and the operator \( B_\lambda + \nu I \) is invertible with
\[
(B_\lambda + \nu I)^{-1} = (I - K_{k,\lambda}(M_\lambda + \omega I))^{-1} (L_\lambda + kI)^{-1}.
\]
Moreover \( (B_\lambda + \nu I)^{-1} \) is compact and positive and irreducible. Since
\[
(B_\lambda + \nu I)^{-1} h_\lambda = (r_\lambda^{-1} + \nu)^{-1} h_\lambda
\]
we see that \( (r_\lambda^{-1} + \nu)^{-1} \) is an eigenvalue to which a positive eigenfunction pertains. It then follows from the irreducibility of \( (B_\lambda + \nu I)^{-1} \) that the spectral radius of this operator must be \( (r_\lambda^{-1} + \nu)^{-1} \).

\[\square\]

Lemma 4.5. If \( \mu \in \mathbb{C} \) is such that \( \text{Re} \ \mu < r_\lambda^{-1} \) then \( \mu \) is in the resolvent set of \( B_\lambda \).

Proof. Let \( h \in X \) be arbitrary and consider the equation
\[
(43) \quad B_\lambda g - \mu g = h,
\]
where \( \text{Re} \ \mu < r_\lambda^{-1} \). Choose \( \nu \in \mathbb{R} \) large enough such that
\[
(\text{Re} \ \mu)^2 + 2\nu (\text{Re} \ \mu - r_\lambda^{-1}) + (\text{Im} \ \mu)^2 < r_\lambda^{-2}
\]
and \( \mu + \nu \neq 0 \). Then
\[
|\mu + \nu|^2 = (\text{Re} \ \mu)^2 + 2\nu \text{Re} \ \mu + \nu^2 + (\text{Im} \ \mu)^2
< r_\lambda^{-2} + 2\nu r_\lambda^{-1} + \nu^2,
\]

and hence $0 < |\mu + \nu| < r^{-1}_\lambda + \nu$. Equation (43) is equivalent with
\[(B_\lambda + \nu I)g - (\mu + \nu)g = h.\]
Using Lemma 4.4 we can rewrite this as
\[\left( (B_\lambda + \nu I)^{-1} - (\mu + \nu)^{-1} I \right) g = -(\mu + \nu)^{-1}(B_\lambda + \nu I)^{-1}h.\]
Since $\left| (\mu + \nu)^{-1} \right| > (r^{-1}_\lambda + \nu)^{-1} = r((B_\lambda + \nu I)^{-1})$ we have that $(\mu + \nu)^{-1}$ is in the resolvent set $(B_\lambda + \nu I)^{-1}$ and hence (44) has a unique solution. It follows from the closed graph theorem that $\mu$ is in the resolvent set of $B_\lambda$. \hfill \square

Proof of Theorem 2.2. Consider the operator $A_\lambda$ defined in (7). It holds that $\mu$ is in the resolvent set of $A_\lambda$ if and only if $\mu/\lambda$ is in the resolvent set of $B_\lambda$. Indeed if $\mu/\lambda$ is in the resolvent set of $B_\lambda$ the operator defined by
\[(u,v) \mapsto (\varphi, \beta \varphi - \beta \psi)\]
where $(\varphi, \psi) := \lambda^{-1}(B_\lambda - \mu/\lambda)^{-1}(u, u - \frac{1}{\beta} v)$ is directly seen to be $(A_\lambda - \mu)^{-1}$. Conversely, for $\mu$ in the resolvent set of $A_\lambda$ the operator defined by
\[(u, w) \mapsto (\varphi, \varphi - (1/\beta) \theta)\]
where $(\varphi, \theta) := \lambda(A_\lambda - \mu I)^{-1}(u, \beta u - \beta w)$ is $(B_\lambda - \mu/\lambda)^{-1}$. Hence, using the last lemma we have that all $\mu \in \mathbb{C}$ with $\text{Re}\mu \leq \nu \lambda := \lambda r^{-1}_\lambda$ that $\mu$ is in the resolvent set of $A_\lambda$. \hfill \square

Proof of Theorem 2.3. The theorem follows directly from Lemma 3.5 and the fact that $\Lambda(\lambda)$ is the unique solution in $[Z_\lambda, Y]$. \hfill \square

A. Appendix.

We recall some facts about quasimonotone systems. We remark that in this section $\Omega$ may be an unbounded domain.

Definition A.1. A system of elliptic equations
\[(45) \begin{cases} -\Delta u &= F_1(x, u, w) \quad \text{in } \Omega, \\ -\Delta w &= F_2(x, u, w) \quad \text{in } \Omega, \end{cases}\]
with $F_i \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R})$ is called quasimonotone if
\[\frac{\partial F_1}{\partial u}(x, u, w) \geq 0 \quad \text{and} \quad \frac{\partial F_2}{\partial w}(x, u, w) \geq 0 \quad \text{for all } (x, u, w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}.\]

This definition suffices for our purposes. For a more general definition we refer to [16].
Definition A.2. A pair \((u, w) \in C(\bar{\Omega}) \times C(\bar{\Omega})\) is called a subsolution to the problem

\[
\begin{aligned}
-\Delta u &= F_1(x, u, w) \quad \text{in } \Omega, \\
-\Delta w &= F_2(x, u, w) \quad \text{in } \Omega, \\
(u, w) &= (\varphi, \psi) \quad \text{on } \Gamma = \partial \Omega,
\end{aligned}
\]

with \(\varphi, \psi \in C(\Gamma)\) if

1. it holds in \(D'(\Omega)\)-sense that
   \[-\Delta u \leq F_1(x, u, w),
   -\Delta w \leq F_2(x, u, w);\]

Supersolutions are defined by reversing the inequality signs. If \((u, w)\) is both a subsolution and a supersolution then it is called a \(C\)-solution.

We note that if \(\Omega\) is a bounded smooth domain and \(F_1, F_2\) are \(C^1\) then a \(C\)-solution \((u, w)\) is in \(C^2(\bar{\Omega}) \times C^2(\bar{\Omega})\). We often use the following results from [16, Theorem 1.3].

Proposition A.3. Let \(\Omega\) be a bounded smooth domain and assume that (46) is quasimonotone.

1. If \((u_i, w_i), i = 1, 2,\) are subsolutions to this system then \((u, w)\) defined by
   \[
   (u (x), w (x)) := \left(\max_{1,2} \{u_i (x)\}, \max_{1,2} \{w_i (x)\}\right)
   \]
   is again a subsolution to (46).

2. If \((u, w)\) is a subsolution and \((\overline{u}, \overline{w})\) a supersolution to (46) then there exists a \(C\)-solution \((u, w)\) to (46) with
   \[
   (u, w) \leq (u, w) \leq (\overline{u}, \overline{w}).
   \]

We give some results for \(\Omega = \mathbb{R}^N_+ := \{(x_1, x') \in \mathbb{R}, x' \in \mathbb{R}^{N-1}\}\). The first is that one has also for quasimonotone systems the existence of a minimal and maximal solutions between an ordered pair of sub- and supersolutions.

Lemma A.4. Consider the following halfspace problem:

\[
\begin{aligned}
-\Delta u &= F_1(u, w) \quad \text{in } \mathbb{R}^N_+,
-\Delta w &= F_2(u, w) \quad \text{in } \mathbb{R}^N_+,
(u, w) &= (0, 0) \quad \text{on } \partial \mathbb{R}^N_+,
\end{aligned}
\]

with \(F \in C^{1, \alpha} (\mathbb{R} \times \mathbb{R}), 0 < \alpha < 1,\) and suppose this system is quasimonotone. If there exists a bounded subsolution \((\underline{u}, \underline{w})\) and bounded supersolution \((\overline{u}, \overline{w})\) to this system with \((u, w) \leq (\underline{u}, \underline{w})\), then there exist a maximal and a minimal \(C^{2, \alpha}\)-solution in the order interval \([u, w], (\overline{u}, \overline{w})\) to this problem.
The proof of this lemma is almost the same as for bounded domains. We only observe that if \( \omega > 0 \) is such that \( \frac{\partial}{\partial u} F_1 (u, w) + \omega \geq 0 \) and \( \frac{\partial}{\partial w} F_2 (u, w) + \omega \geq 0 \) for \((u, w) \leq (u, w) \leq (\bar{u}, \bar{w})\) then one can define inductively
\[
(u_0, w_0) = (u, w), \quad (u_{n+1}, w_{n+1}) = T (u_n, w_n), \quad n = 0, 1, 2, \ldots
\]
with \((u, w) = T (u_n, w_n)\) the unique solution to the linear problem
\[
\begin{cases}
(-\Delta + \omega_1) u &= F_1 (u_n, w_n) + \omega u_n \quad \text{in } \mathbb{R}^N_+,
(-\Delta + \omega_2) w &= F_2 (u_n, w_n) + \omega w_n \quad \text{in } \mathbb{R}^N_+,
 u &= w = 0 \quad \text{on } \partial \mathbb{R}^N_+.
\end{cases}
\]
That this system has a unique solution follows from the fact that if \( k > 0 \) and \( g \in L^\infty (\mathbb{R}^N) \) then there exists a unique \( u \in L^\infty (\mathbb{R}^N) \cap C (\mathbb{R}^N) \) such that \(-\Delta u + ku = f \) in \( D' (\mathbb{R}^N) \)-sense and \( u = 0 \) on \( \partial \mathbb{R}^N_+ \), see see e.g., [6, Proposition 27, p. 635]. Since the system is quasimonotone we have, see also [16], that
\[
(u, w) \leq (u_n, w_n) \leq (u_{n+1}, w_{n+1}) \leq (\bar{u}, \bar{w}) \quad \text{for } n = 0, 1, 2, \ldots.
\]
Letting \( n \to \infty \) one obtains a solution.

The next lemma is used to reduce the study of equations on \( \mathbb{R}^N_+ \) to the study of inequalities on \( \mathbb{R}^+ \).

**Lemma A.5.** Suppose that \((U, W) \in C^2 (\mathbb{R}^N_+) \cap C (\overline{\mathbb{R}^N_+})\) is a bounded solution of
\[
\begin{cases}
-\Delta U &= F_1 (x_1, U, W) \quad \text{in } \mathbb{R}^N_+,
-\Delta W &= F_2 (x_1, U, W) \quad \text{in } \mathbb{R}^N_+,
 U &= W = 0 \quad \text{on } \partial \mathbb{R}^N_+,
\end{cases}
\]
(48)
with \( F_i (x_1, s, t) \in C^{1, \alpha} (\overline{\mathbb{R}^N_+}) \) and \( 0 < \alpha < 1 \). Assume (48) is quasimonotone and that \( |F_i (x_1, s, t)| \leq h (s, t) \) with \( h \) a continuous function on \( \mathbb{R}^2 \). Define \((u, w)\) by
\[
(u, w) (x_1) := (\sup_{x_1' \in \mathbb{R}^{N-1}} U (x_1, x), \sup_{x_1' \in \mathbb{R}^{N-1}} W (x_1, x)).
\]
It holds that \( u, w \in C [0, \infty) \) with \( u (0) = w (0) = 0 \) and in \( D' (\mathbb{R}^+) \)-sense that
\[
\begin{align}
-u'' &\leq F_1 (x_1, u, w) \quad \text{in } \mathbb{R}^+, \\
-w'' &\leq F_2 (x_1, u, w).
\end{align}
\]
(49)\hspace{1cm} (50)

**Proof.** Since \( U \) and \( W \) are bounded, \( \Delta U \) and \( \Delta W \) are also bounded. From this and the fact that \( U = W = 0 \) on \( \partial \mathbb{R}^N_+ \) one obtains by standard regularity results that \( U, W \in C^{2, \alpha} (\overline{\mathbb{R}^N_+}) \). In particular we have uniform bounds on the first order derivatives of \( U \) and \( W \).
Let \( \{ q_j : j = 1, 2, \ldots \} \) be a numbering of \( \mathbb{Q}^{N-1} \) and define the functions \( U_j \) and \( W_j \) on \( \mathbb{R}^N_+ \) by

\[
(U_j, W_j)(x) = (U, W)(x + (0, q_j)).
\]

For \( k = 1, 2, \ldots \), we define \( (S_k, T_k) \) on \( \mathbb{R}^N_+ \) by

\[
(S_k, T_k)(x) = (\sup_{1 \leq j \leq k} U_j(x), \sup_{1 \leq j \leq k} W_j(x)),
\]

and let \( (S, T)(x) := \lim_{k \to \infty} (S_k, T_k)(x) \). It follows from the uniform continuity of \( U \) and \( W \) that \( (S(x), T(x)) = (u(x_1), w(x_1)) \).

Since the system is quasimonotone it follows from Proposition A.3 and induction that in \( \mathcal{D}'(\mathbb{R}^N_+)-\)sense \( -\Delta S_k \leq F_1(x_1, S_k, T_k)z \) for every \( k = 1, 2, \ldots \). By dominated convergence it then follows that \( -\Delta u \leq F_1(x_1, u, w) \) in \( \mathcal{D}'(\mathbb{R}^N_+)-\)sense. In particular if \( z_1 \in D^+(\mathbb{R}^+) \) we set \( z := z_1 z_2 \) with \( z_2 \in D^+(\mathbb{R}^{N-1}) \), \( z_2 \neq 0 \) one sees that this implies (49).

Since \( F_1(x_1, u, w) \) is bounded there exists \( M > 0 \) such that \( F_1(x_1, u, w) - 2M \leq 0 \) on \( \mathbb{R}^+ \). Then \( -(u + Mx_1^2)'' \leq 0 \) in \( \mathcal{D}'(\mathbb{R}^+)-\)sense. Hence \( x_1 \mapsto u(x_1) + Mx_1^2 \) is convex and consequently continuous on \( (0, \infty) \). Since \( \frac{\partial}{\partial x_1} U \) is uniformly bounded and \( U(0, x') = 0 \) for all \( x' \in \mathbb{R}^{N-1} \), it follows that \( u \) is continuous in \( 0 \) with \( u(0) = 0 \). The result for \( w \) is obtained mutatis mutandis.

Finally we prove a direct analogue for a quasimonotone system of the sweeping principle for scalar equations in [15]. Suppose that \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1, \Gamma_2 \in C^2 \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \). Here \( \Gamma_i \) may be empty. Let \( e \in C^1(\bar{\Omega}) \) be such that \( e(x) > 0 \) for \( x \in \Omega \cup \Gamma_1 \) and \( e(x) = 0 \), \( \frac{\partial e}{\partial n}(x) < 0 \) for \( x \in \Gamma_2 \) where \( n \) is the outward normal and let \( C_e(\bar{\Omega}) \) be as in (38), see also [1].

**Proposition A.6.** Suppose that (46) is quasimonotone. If \( (u, w) \) is a supersolution, and \( \{(u_t, w_t) : t \in [0, 1]\} \) is a family of subsolutions such that

1. \((u_t, w_t) < (g_1, g_2) \) on \( \Gamma_1 \) and \((u_t, w_t) = (g_1, g_2) \) on \( \Gamma_2 \) for all \( t \in [0, 1] \);
2. \( t \mapsto u_t - u_0 \) and \( t \mapsto w_t - w_0 \) is continuous from \([0, 1]\) into \( C_e(\bar{\Omega}) \);
3. \((u_0, w_0) \leq (u, w) \) in \( \Omega \);
4. \( u_t \neq u \) and \( w_t \neq w \) for all \( t \in [0, 1] \);

then there exists \( r > 0 \) such that \( (u, w) - (u_t, w_t) > (re, re) \) for all \( t \in [0, 1] \).

**Proof.** Let \( S = \{ t \in [0, 1] ; (u_t, w_t) \leq (u, w) \} \). By assumption \( 0 \in S \). Since convergence in \( C_e(\bar{\Omega}) \) implies pointwise convergence it follows that \( S \) is closed. Let \( t_0 \in S \). It holds with \( \omega \) large enough in \( \mathcal{D}'(\Omega)-\)sense that

\[
-\Delta (u - u_{t_0}) + \omega (u - u_{t_0}) \geq F_1(u, w) + \omega u - F_1(u_{t_0}, w_{t_0}) + \omega u_{t_0} = F_1(u, w) + \omega u - F_1(u_{t_0}, w) + \omega u_{t_0} + F_1(u_{t_0}, w) - F_1(u_{t_0}, w_{t_0}) \geq 0.
\]

Since \( u \neq u_{t_0} \) there exists \( s' > 0 \) such that \( u - u_{t_0} > s' e_0 \) with \( e_0 \) a \( C^1(\bar{\Omega}) \) function with \( e(x) > 0 \) for \( x \in \Omega \), \( e_0(x) = 0 \) and \( \frac{\partial e_0}{\partial n}(x) < 0 \) for \( x \in \Gamma \),
see [3, Corollary p. 581]. Since $u(x) - u_{t_0}(x) > 0$ for $x \in \Gamma_1$ and $\Gamma_1$ is compact, there exists $s_1 > 0$ such $u - u_{t_0} \geq s_1 e$. In the same way there exists $s_2$ such that $w - w_{t_0} \geq s_2 e$. By hypothesis 2 there exists $\delta > 0$ such that \|u_t - u_{t_0}\|_e, \|w_t - w_{t_0}\|_e < s/2$ for all $t \in [0, 1]$ for which $|t - t_0| < \delta$.

This implies that for all such $t$ we have that $u_t - u_{t_0} \leq \frac{s}{2} e$ and hence $u - u_t = u - u_{t_0} - (u_{t_0} - u_t) \geq \frac{s}{2} e$ and in the same way $w - w_t \geq \frac{s}{2} e$. Hence $S$ is open and we have that $S = [0, 1]$. By the compactness of $[0, 1]$ and by hypotheses (2) it follows that there exists $r > 0$ such that $u - u_t \geq re$ and $w - w_t \geq re$.

\[ \square \]

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References


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