MINIMAL TRIPLE POINT NUMBERS
OF SOME NON-ORIENTABLE SURFACE-LINKS

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An embedded surface in $\mathbb{R}^4$ is projected into $\mathbb{R}^3$ with the double point set which includes a finite number of triple points. We consider the minimal number of such triple points among all projections of embedded surfaces which are ambient isotopic to a given surface and show that for any non-negative integer $N$ there exists a 2-component non-orientable surface in $\mathbb{R}^4$ whose minimal triple point number is equal to $2N$.

1. Introduction.

In this paper we denote the 4-dimensional Euclidian space by

$$\mathbb{R}^4 = \{(x, y, z, w) | x, y, z, w \in \mathbb{R}\}.$$ 

A surface-link is a 2-dimensional manifold $F$ embedded in $\mathbb{R}^4$ locally flatly, each component of which is homeomorphic to a closed surface. In particular, it is called a surface-knot when $F$ is connected, and it is called a 2-knot (resp. a $\mathbb{P}^2$-knot) when $F$ is homeomorphic to a 2-sphere (resp. a projective plane). Two surface-links $F$ and $F'$ are equivalent if there exists an orientation preserving homeomorphism of $\mathbb{R}^4$ which maps $F$ onto $F'$. If $F$ and $F'$ are equivalent, we use the notation $F \cong F'$. For a surface-link $F$, the type of $F$ is the collection of all surface-links each member in which is equivalent to $F$.

To describe a surface-link, we use the projection image in $\mathbb{R}^3$. For convenience, we may assume that the projection $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ determined by the $w$-axis is a generic projection for a surface-link $F$; that is, its double point set consists of isolated branch points, double point curves, and isolated triple points. The broken surface diagram or simply the diagram of a surface-link $F$ is the generic projection image $\pi(F)$ such that the upper sheet and the lower sheet along each double point curve are distinguished. (To distinguish upper and lower, we often depict the diagram by erasing a small neighborhood of the curve in the lower sheet.)

Let $D_F$ be the diagram of a surface-link $F$. We denote the number of the triple points on $D_F$ by $t(D_F)$. Then the minimal triple point number of a
surface-link $F$, denoted by $t(F)$, is the smallest number of the triple points among all the diagrams of surface-links with the same type as $F$;

$$t(F) = \min \{ t(D_{F'}) | F' \cong F \}.$$

This definition is an analogy to that of the ‘minimal crossing number’ in classical knot theory. It is shown by Kamada that there exists a 2-knot $K$ with $t(K) > N$ for any non-negative integer $N$ (cf. [6]). And also we have $t(F) \neq 1$ for any surface-link $F$ (cf. [7]). However, the minimal triple point number in 2-knot theory differs from the minimal crossing number in classical knot theory: For instance, $t(F) = 0$ does not imply that $F$ is trivial. For example, a 2-knot $K$ is ribbon if and only if $t(K) = 0$ (cf. [10]).

The purpose of this paper is to prove:

**Theorem 1.1.** For any positive integer $N$, there exists a 2-component surface-link $F = F_1 \cup F_2$ such that

1. each $F_i$ is a non-orientable surface-knot,
2. $\chi(F_i) = 2 - N$ ($i = 1, 2$),
3. $e(F_1) = 2N$ and $e(F_2) = -2N$,
4. $\pi_1(\mathbb{R}^4 - F) \cong \langle a, b | aba = b, bab = a \rangle$, and
5. $t(F) = 2N$.

where $\chi$ denotes the Euler characteristic, and $e$ denotes the normal Euler number.

### 2. Preliminaries.

We review some definitions and results on diagrams of surface-links. Refer to [3] for more details.

Let $F$ be a surface-link and $D_F$ the (broken surface) diagram of $F$. A **sign** of a branch point on $D_F$ is defined as follows: There are two types of crossing information near a branch point — one is **positive** (with +1) and the other is **negative** (with −1) — depicted in Figure 1.

![Figure 1](image-url)

**Proposition 2.1 ([1]).** For a surface-knot $F$, the sum of signs taken over all the branch points on $D_F$ is equal to the normal Euler number $e(F)$. 


Similarly to the minimal triple point number, we can also consider the minimal branch point number $b(F)$ of a surface-link $F$ as follows:

$$b(F) = \min \{ b(D_{F'}) | F' \cong F \},$$

where $b(D_{F'})$ is the number of the branch points on the diagram $D_{F'}$. In [2], Carter and Saito determined the number $b(F)$ completely as follows (they prove only the case of surface-knots, but their technique used in their paper is also applied for any surface-links).

**Proposition 2.2 ([2]).** For a surface-link $F = F_1 \cup \cdots \cup F_n$, we have

$$b(F) = |e(F_1)| + \cdots + |e(F_n)|.$$ 

Let $\Gamma_F$ be the double point set of the diagram $D_F$, which is regarded as a union of immersed loops and immersed arcs in $\mathbb{R}^3$ such that the endpoints of the immersed arcs are branch points.

Suppose that $\Gamma_F$ contains a simple arc (that is, an embedded arc with no triple point on it). Such a simple arc is called an $a$-arc (resp. an $m$-arc) if the two branch points of its ends have the opposite signs (resp. the same sign). We notice that the neighborhood of an $a$-arc (resp. an $m$-arc) is homeomorphic to an annulus (resp. a Möbius band). By canceling the branch points on an $a$-arc as illustrated in Figure 2, we have the following.

**Lemma 2.3 ([9]).** If $\Gamma_F$ contains an $a$-arc, then $F$ is equivalent to a surface-link $F'$ with $t(D_{F'}) = t(D_F)$ and $b(D_{F'}) = b(D_F) - 2$.

![Figure 2](image)

A surface-link $F$ is $\mathbb{P}^2$-reducible if $F$ is equivalent to a connected sum of a standard $\mathbb{P}^2$-knot and some surface-link (refer to [5] for a standard $\mathbb{P}^2$-knot). $F$ is $\mathbb{P}^2$-irreducible if $F$ is not $\mathbb{P}^2$-reducible. Since the neighborhood of an $m$-arc is a punctured projective plane properly embedded in a 4-ball as depicted in Figure 3, we have the following.

**Lemma 2.4 ([8]).** If $\Gamma_F$ contains an $m$-arc, then $F$ is $\mathbb{P}^2$-reducible.
The neighborhood of a triple point on $D_F$ consists of three sheets. These sheets are labeled top, middle and bottom, and these indicate the relative position of the sheets with respect to the $w$-coordinate.

A branch point $b$ and a triple point $t$ on $D_F$ are connected by a double point curve $c$ if there exists a simple sub-arc $c$ of $\Gamma_F$ whose endpoints are $b$ and $t$. By the deformation of $F$ into $F'$ as illustrated in Figure 4, we have the following.

**Lemma 2.5** ([8], [11]). Suppose that a branch point $b$ and a triple point $t$ on $D_F$ are connected by a double point curve $c$. If the arc $c$ is transverse to the top sheet or the bottom sheet at $t$, then $F$ is equivalent to a surface-link $F'$ with $t(D_{F'}) = t(D_F) - 1$ and $b(D_{F'}) = b(D_F)$.

Let $\{m_1, \cdots, m_n\}$ be a meridian system of a surface-link $F = F_1 \cup \cdots \cup F_n$, where $m_k$ is a meridian of $F_k$ ($k = 1, \cdots, n$). Each $m_k$ is regarded as an element of the knot group $\pi_1(\mathbb{R}^4 - F)$. Then the following is clear from the property of standard $P^2$-knots.

**Lemma 2.6.** If the order of each $m_k$ is not equal to $2$ in $\pi_1(\mathbb{R}^4 - F)$, then $F$ is $P^2$-irreducible.

### 3. Projections and movie pictures.

For a $P^2$-irreducible surface-link $F$ we give an estimate for a lower bound of $t(F)$. However, the following lemma has no sense for an orientable surface-link; for the normal Euler number of any constituent orientable surface-knot vanishes.
Lemma 3.1. For a $\mathbb{P}^2$-irreducible surface-link $F = F_1 \cup \cdots \cup F_n$, we have

$$t(F) \geq (|e(F_1)| + \cdots + |e(F_n)|)/2,$$

where $e(F_i)$ denotes the normal Euler number of a surface-knot $F_i$ ($i = 1, \cdots, n$).

Proof. By Proposition 2.2, it is sufficient to prove that

$$t(F) \geq b(F)/2$$

for any $\mathbb{P}^2$-irreducible surface-link $F$. Let $M$ be the set of all diagrams of the surface-links with the same type as $F$ whose triple point number is realizing $t(F)$;

$$M = \{D_{F'}|F' \cong F, t(D_{F'}) = t(F)\}.$$

Among the diagrams in $M$, we take a diagram, say $D$, whose branch point number is minimal in $M$. Let $\Gamma$ be the double point set of $D$.

Since $F$ is $\mathbb{P}^2$-irreducible, $\Gamma$ contains no $m$-arc by Lemma 2.4. Moreover $\Gamma$ contains no $a$-arc by Lemma 2.3; for if $\Gamma$ contains an $a$-arc, then there exists a diagram $D'$ in $M$ with $b(D) > b(D')$. Hence any branch point in $\Gamma$ is connected with some triple point.

On the other hand, the number of branch points connecting with each triple point in $\Gamma$ is at most two; for if at least three branch points connect with a triple point, then we have a cancelling pair of a branch point and the triple point which satisfies the condition of Lemma 2.5, and so there exists a diagram $D_{F''}$ with $F'' \cong F$ and $t(D_{F''}) < t(F)$. Therefore we have

$$t(F) = t(D) \geq b(D)/2 \geq b(F)/2.$$

□

Corollary 3.2. For a $\mathbb{P}^2$-irreducible surface-link $F = F_1 \cup \cdots \cup F_n$, if

$$t(F) = (|e(F_1)| + \cdots + |e(F_n)|)/2,$$

then the minimal triple point number $t(F)$ is even.

Proof. From the proof of Lemma 3.1, there exists a surface-link $F' \cong F$ whose diagram $D_{F'}$ satisfies $t(D_{F'}) = t(F)$ and $b(D_{F'}) = b(F)$. Let $\Gamma_{F'}$ be the double point set of $D_{F'}$. Then the neighborhood of each triple point in $\Gamma_{F'}$ is as shown in Figure 5(A) or (B). Here the arrows along the double point curves mean a BW orientation of $\Gamma_{F'}$ (refer to [7] for a BW orientation of a double point set). Since the number of the triple points depicted in Figure 5(A) is equal to that depicted in Figure 5(B), the sum $t(D_{F'})$ is even. □
To describe a surface-link $F$, we also use a movie picture method \cite{4}; for any subset $S$ of $\mathbb{R}$, we denote $S \times \mathbb{R}^3 \subset \mathbb{R} \times \mathbb{R}^3 \cong \mathbb{R}^4$ by $\mathbb{R}^3 S$. If $S = \{x_0\}$, we write $\mathbb{R}^3 [x_0]$. Taking the $x$-coordinate as a height function, we consider a surface-link $F$ to be a one-parameter family of subsets in $\mathbb{R}^3$ that are the intersections $F_x = F \cap \mathbb{R}^3 [x]$ ($-\infty < x < \infty$). If $F_x$ is a classical link, it is called a cross-sectional link.

We consider the relationship between a surface-link described by the projection method and that described by the movie pictured method. Let $\pi'$ be the projection $\pi' : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with $(y, z, w) \longrightarrow (y, z)$. Then the projection $\pi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ determined by the $w$-axis is considered to be

$$
\text{id} \times \pi' : \mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R} \times \mathbb{R}^2 \cong \mathbb{R}^3.
$$

Hence, the projection image $\pi(F)$ of a surface-link $F$ is also considered to be a family of the projection images $\pi'(F_x)$ ($-\infty < x < \infty$).

We notice that a crossing in the (classical link) diagram of each cross-sectional link $F_x$ corresponds to a double point in the diagram $D_F$ of $F$. If consecutive cross-sectional links $\{F_x\}$ ($x_0 \leq x \leq x_1$) represent a deformation of a Reidemeister move I (resp. a Reidemeister move III), it produces a branch point (resp. a triple point) in $D_F$.

**Example 3.3.** In Figure 6(A), we depict a 2-component surface $A$ properly embedded in $\mathbb{R}^3[0, 1]$, each component of which is homeomorphic to a 2-punctured projective plane. Since the projection image determined by the $w$-axis is shown in Figure 6(B), its double point set contains four branch points and two triple points (see Figure 6(C)).
For any positive integer $N$, we construct a 2-component link $F(N)$ as follows:

$$F(N) \cap \mathbb{R}^3[x] = \begin{cases} 
B \cup B' & \text{for } x = 0, \\
A \cap \mathbb{R}^3[x] & \text{for } 0 < x \leq 1, \\
A \cap \mathbb{R}^3[x - 1] & \text{for } 1 \leq x \leq 2, \\
\cdot \cdot \cdot & \\
A \cap \mathbb{R}^3[x - (N - 2)] & \text{for } N - 2 \leq x \leq N - 1, \\
A \cap \mathbb{R}^3[x - (N - 1)] & \text{for } N - 1 \leq x < N, \\
B \cup B' & \text{for } x = N, \\
\phi & \text{otherwise,}
\end{cases}$$
where \( B \cup B' \) is a union of two standard 2-disks which bounds the trivial link \( A \cap \mathbb{R}^3[0] (= A \cap \mathbb{R}^3[1]) \). Then the double point set of the diagram \( D_{F(N)} \) consists of a union of \( N \) copies of the set in Figure 6(C). We notice that the surface-link \( F(1) \) is \( S_{1,-1}^1 \) in the list of [12].

**Proof of Theorem 1.1.** We prove that \( F(N) \) in Example 3.3 satisfies (i) to (v) in Theorem 1.1. It is easy to verify that each component of \( F(N) \) is a (trivial) non-orientable surface-knot with the Euler characteristic \( 2 - N \). The property (iii) is followed by Proposition 2.1 (we recall that a Reidemeister move I corresponds to a branch point). For the calculation of \( \pi_1(\mathbb{R}^4 - F) \), it is useful to refer to [4].

We will only prove that the property (v); \( t(F(N)) = 2N \). Since the knot group of \( F(N) \),
\[
\langle a, b | aba = b, bab = a \rangle,
\]
is the quaternion group, and since \( \{a, b\} \) is a meridian system of \( F(N) \), the order of each meridian is 4. By Lemma 2.6, \( F(N) \) is \( \mathbb{P}^2 \)-irreducible. Hence by the property (iii) and Lemma 3.1, we have \( t(F(N)) \geq (|2N| + | - 2N|)/2 = 2N \). On the other hand, \( F(N) \) has the diagram whose double point set contains \( 2N \) triple points as shown in Example 3.3. So we have \( t(F(N)) \leq 2N \). \( \square \)

**References**

[8] , *On non-orientable surfaces in 4-space which are projected with at most one triple point*, to appear in Proc. of the A.M.S.


Received April 20, 1999 and revised July 7, 1999.

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