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An embedded surface in $\mathbf{R}^4$ is projected into $\mathbf{R}^3$ with the double point set which includes a finite number of triple points. We consider the minimal number of such triple points among all projections of embedded surfaces which are ambient isotopic to a given surface and show that for any non-negative integer N there exists a 2-component non-orientable surface in $\mathbf{R}^4$ whose minimal triple point number is equal to $2N$.

1. Introduction.

In this paper we denote the 4-dimensional Euclidian space by

$$\mathbf{R}^4 = \{(x, y, z, w) | x, y, z, w \in \mathbf{R}\}.$$

A *surface-link* is a 2-dimensional manifold F embedded in $\mathbf{R}^4$ locally flatly, each component of which is homeomorphic to a closed surface. In particular, it is called a *surface-knot* when F is connected, and it is called a *2-knot* (resp. a $\mathbf{P}^2$*-knot*) when F is homeomorphic to a 2-sphere (resp. a projective plane). Two surface-links F and F' are *equivalent* if there exists an orientation preserving homeomorphism of $\mathbf{R}^4$ which maps F onto F'. If F and F' are equivalent, we use the notation $F \cong F'$. For a surface-link F, the *type* of F is the collection of all surface-links each member in which is equivalent to F.

To describe a surface-link, we use the projection image in $\mathbf{R}^3$. For convenience, we may assume that the projection $\pi : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ determined by the w-axis is a *generic* projection for a surface-link F; that is, its double point set consists of isolated branch points, double point curves, and isolated triple points. The *broken surface diagram* or simply the *diagram* of a surface-link F is the generic projection image $\pi(F)$ such that the upper sheet and the lower sheet along each double point curve are distinguished. (To distinguish upper and lower, we often depict the diagram by erasing a small neighborhood of the curve in the lower sheet.)

Let D_F be the diagram of a surface-link F. We denote the number of the triple points on D_F by $t(D_F)$. Then the *minimal triple point number* of a surface-link F, denoted by $t(F)$, is the smallest number of the triple points among all the diagrams of surface-links with the same type as F;

$$t(F) = \min\{t(D_{F'}) | F' \cong F\}.$$

This definition is an analogy to that of the 'minimal crossing number' in classical knot theory. It is shown by Kamada that there exists a 2-knot K with $t(K) > N$ for any non-negative integer N (cf. [**6**]). And also we have $t(F) \neq 1$ for any surface-link F (cf. [**7**]). However, the minimal triple point number in 2-knot theory differs from the minimal crossing number in classical knot theory: For instance, $t(F) = 0$ does not imply that F is trivial. For example, a 2-knot K is ribbon if and only if $t(K) = 0$ (cf. [**10**]). The purpose of this paper is to prove:

Theorem 1.1. *For any positive integer N, there exists a 2-component surface-link $F = F_1 \cup F_2$ such that*

(i) *each F_i is a non-orientable surface-knot,*
(ii) $\chi(F_i) = 2 - N$ $(i = 1, 2)$,
(iii) $e(F_1) = 2N$ *and* $e(F_2) = -2N$,
(iv) $\pi_1(\mathbf{R}^4 - F) \cong \langle a, b | aba = b, bab = a \rangle$, *and*
(v) $t(F) = 2N$,

where χ denotes the Euler characteristic, and e denotes the normal Euler number.

2. Preliminaries.

We review some definitions and results on diagrams of surface-links. Refer to [**3**] for more details.

Let F be a surface-link and D_F the (broken surface) diagram of F. A *sign* of a branch point on D_F is defined as follows: There are two types of crossing information near a branch point — one is *positive* (with $+1$) and the other is *negative* (with -1) — depicted in Figure 1.

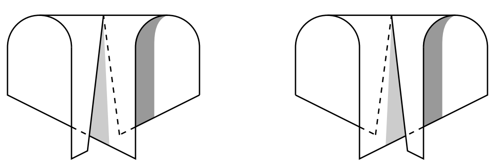

Figure 1.

Proposition 2.1 ([**1**]). *For a surface-knot F, the sum of signs taken over all the branch points on D_F is equal to the normal Euler number $e(F)$.*

Similarly to the minimal triple point number, we can also consider the *minimal branch point number* $b(F)$ of a surface-link F as follows;

$$b(F) = \min\{b(D_{F'}) | F' \cong F\},$$

where $b(D_{F'})$ is the number of the branch points on the diagram $D_{F'}$. In [**2**], Carter and Saito determined the number $b(F)$ completely as follows (they proves only the case of surface-*knots*, but their technique used in their paper is also applied for any surface-*links*).

Proposition 2.2 ([**2**]). *For a surface-link* $F = F_1 \cup \cdots \cup F_n$*, we have*

$$b(F) = |e(F_1)| + \cdots + |e(F_n)|.$$

Let Γ_F be the double point set of the diagram D_F, which is regarded as a union of immersed loops and immersed arcs in $\mathbf{R}^3$ such that the endpoints of the immersed arcs are branch points.

Suppose that Γ_F contains a simple arc (that is, an embedded arc with no triple point on it). Such a simple arc is called an *a-arc* (resp. an *m-arc*) if the two branch points of its ends have the opposite signs (resp. the same sign). We notice that the neighborhood of an a-arc (resp. an m-arc) is homeomorphic to an annulus (resp. a Möbius band). By canceling the branch points on an a-arc as illustrated in Figure 2, we have the following.

Lemma 2.3 ([**9**]). *If* Γ_F *contains an a-arc, then* F *is equivalent to a surface-link* F' *with* $t(D_{F'}) = t(D_F)$ *and* $b(D_{F'}) = b(D_F) - 2$.

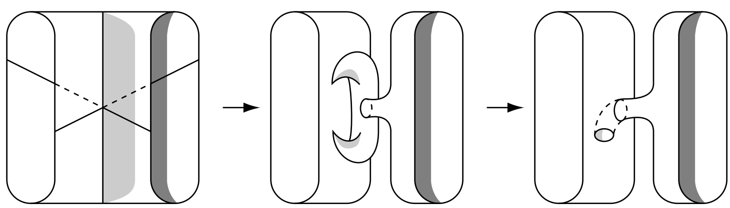

Figure 2.

A surface-link F is $\mathbf{P}^2$*-reducible* if F is equivalent to a connected sum of a standard $\mathbf{P}^2$-knot and some surface-link (refer to [**5**] for a *standard* $\mathbf{P}^2$-knot). F is $\mathbf{P}^2$*-irreducible* if F is not $\mathbf{P}^2$-reducible. Since the neighborhood of an m-arc is a punctured projective plane properly embedded in a 4-ball as depicted in Figure 3, we have the following.

Lemma 2.4 ([**8**]). *If* Γ_F *contains an m-arc, then* F *is* $\mathbf{P}^2$*-reducible.*

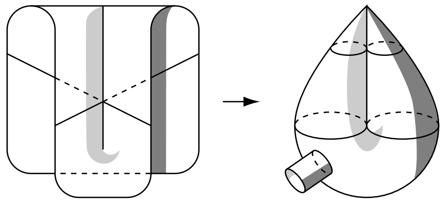

Figure 3.

The neighborhood of a triple point on D_F consists of three sheets. These sheets are labeled *top*, *middle* and *bottom*, and these indicate the relative position of the sheets with respect to the w-coordinate.

A branch point b and a triple point t on D_F are *connected* by a double point curve c if there exists a simple sub-arc c of Γ_F whose endpoints are b and t. By the deformation of F into F' as illustrated in Figure 4, we have the following.

Lemma 2.5 ([**8**], [**11**]). *Suppose that a branch point b and a triple point t on D_F are connected by a double point curve c. If the arc c is transverse to the top sheet or the bottom sheet at t, then F is equivalent to a surface-link F' with $t(D_{F'}) = t(D_F) - 1$ and $b(D_{F'}) = b(D_F)$.*

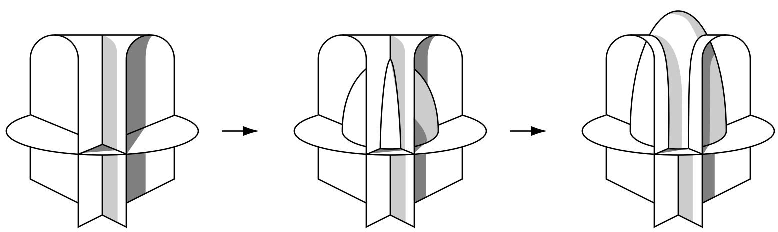

Figure 4.

Let $\{m_1, \cdots, m_n\}$ be a meridian system of a surface-link $F = F_1 \cup \cdots \cup F_n$, where m_k is a meridian of F_k $(k = 1, \cdots, n)$. Each m_k is regarded as an element of the knot group $\pi_1(\mathbf{R}^4 - F)$. Then the following is clear from the property of standard $\mathbf{P}^2$-knots.

Lemma 2.6. *If the order of each m_k is not equal to 2 in $\pi_1(\mathbf{R}^4 - F)$, then F is $\mathbf{P}^2$-irreducible.*

3. Projections and movie pictures.

For a $\mathbf{P}^2$-irreducible surface-link F we give an estimate for a lower bound of $t(F)$. However, the following lemma has no sense for an orientable surface-link; for the normal Euler number of any constituent orientable surface-knot vanishes.

Lemma 3.1. *For a $\mathbf{P}^2$-irreducible surface-link $F = F_1 \cup \cdots \cup F_n$, we have*

$$t(F) \geq (|e(F_1)| + \cdots + |e(F_n)|)/2,$$

where $e(F_i)$ denotes the normal Euler number of a surface-knot F_i ($i = 1, \cdots, n$).

Proof. By Proposition 2.2, it is sufficient to prove that

$$t(F) \geq b(F)/2$$

for any $\mathbf{P}^2$-irreducible surface-link F. Let M be the set of all diagrams of the surface-links with the same type as F whose triple point number is realizing $t(F)$;

$$M = \{D_{F'} | F' \cong F,\ t(D_{F'}) = t(F)\}.$$

Among the diagrams in M, we take a diagram, say D, whose branch point number is minimal in M. Let Γ be the double point set of D.

Since F is $\mathbf{P}^2$-irreducible, Γ contains no m-arc by Lemma 2.4. Moreover Γ contains no a-arc by Lemma 2.3; for if Γ contains an a-arc, then there exists a diagram D' in M with $b(D) > b(D')$. Hence any branch point in Γ is connected with some triple point.

On the other hand, the number of branch points connecting with each triple point in Γ is at most two; for if at least three branch points connect with a triple point, then we have a cancelling pair of a branch point and the triple point which satisfies the condition of Lemma 2.5, and so there exists a diagram $D_{F''}$ with $F'' \cong F$ and $t(D_{F''}) < t(F)$. Therefore we have

$$t(F) = t(D) \geq b(D)/2 \geq b(F)/2.$$

□

Corollary 3.2. *For a $\mathbf{P}^2$-irreducible surface-link $F = F_1 \cup \cdots \cup F_n$, if*

$$t(F) = (|e(F_1)| + \cdots + |e(F_n)|)/2,$$

then the minimal triple point number $t(F)$ is even.

Proof. From the proof of Lemma 3.1, there exists a surface-link $F' \cong F$ whose diagram $D_{F'}$ satisfies $t(D_{F'}) = t(F)$ and $b(D_{F'}) = b(F)$. Let $\Gamma_{F'}$ be the double point set of $D_{F'}$. Then the neighborhood of each triple point in $\Gamma_{F'}$ is as shown in Figure 5(A) or (B). Here the arrows along the double point curves mean a BW orientation of $\Gamma_{F'}$ (refer to [**7**] for a *BW orientation* of a double point set). Since the number of the triple points depicted in Figure 5(A) is equal to that depicted in Figure 5(B), the sum $t(D_{F'})$ is even. □

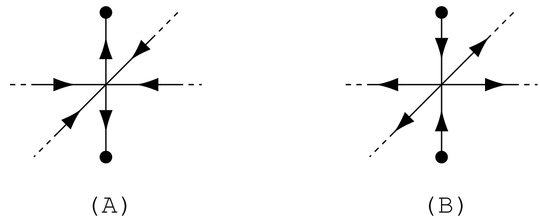

Figure 5.

To describe a surface-link F, we also use a movie picture method [**4**]; for any subset S of $\mathbf{R}$, we denote $S \times \mathbf{R}^3 \subset \mathbf{R} \times \mathbf{R}^3 \cong \mathbf{R}^4$ by $\mathbf{R}^3 S$. If $S = \{x_0\}$, we write $\mathbf{R}^3[x_0]$. Taking the x-coordinate as a height function, we consider a surface-link F to be a one-parameter family of subsets in $\mathbf{R}^3$ that are the intersections $F_x = F \cap \mathbf{R}^3[x]$ $(-\infty < x < \infty)$. If F_x is a classical link, it is called a *cross-sectional link*.

We consider the relationship between a surface-link described by the projection method and that described by the movie pictured method. Let π' be the projection $\pi' : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ with $(y, z, w) \longmapsto (y, z)$. Then the projection $\pi : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ determined by the w-axis is considered to be

$$\mathrm{id} \times \pi' : \mathbf{R}^4 \cong \mathbf{R} \times \mathbf{R}^3 \longrightarrow \mathbf{R} \times \mathbf{R}^2 \cong \mathbf{R}^3.$$

Hence, the projection image $\pi(F)$ of a surface-link F is also considered to be a family of the projection images $\pi'(F_x)$ $(-\infty < x < \infty)$.

We notice that a crossing in the (classical link) diagram of each cross-sectional link F_x corresponds to a double point in the diagram D_F of F. If consecutive cross-sectional links $\{F_x\}$ $(x_0 \leq x \leq x_1)$ represent a deformation of a Reidemeister move I (resp. a Reidemeister move III), it produces a branch point (resp. a triple point) in D_F.

Example 3.3. In Figure 6(A), we depict a 2-component surface A properly embedded in $\mathbf{R}^3[0, 1]$, each component of which is homeomorphic to a 2-punctured projective plane. Since the projection image determined by the w-axis is shown in Figure 6(B), its double point set contains four branch points and two triple points (see Figure 6(C)).

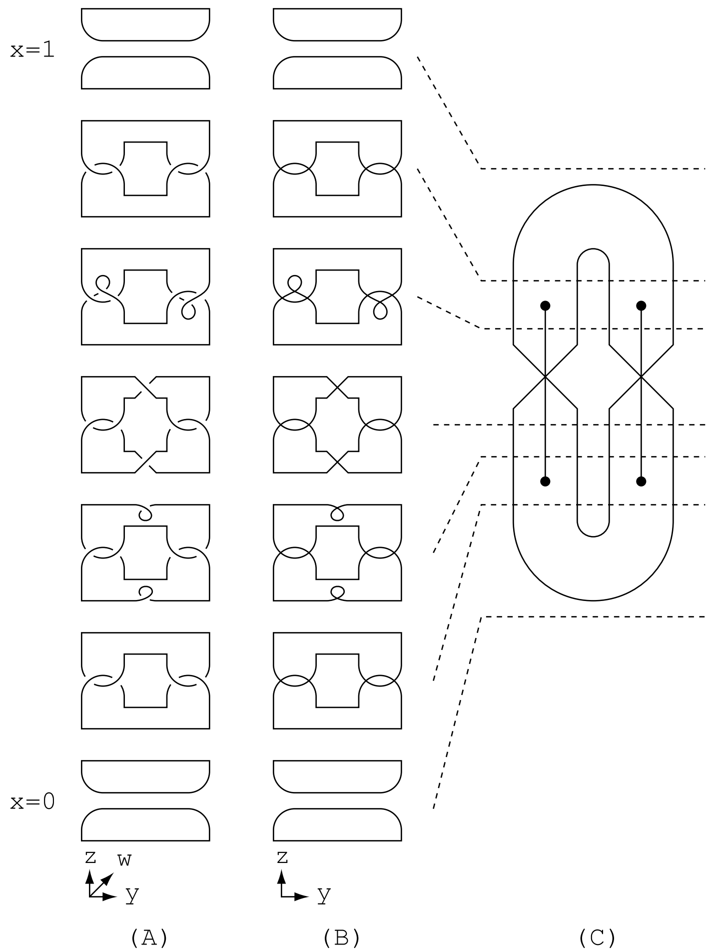

Figure 6.

For any positive integer N, we construct a 2-component link $F(N)$ as follows;

$$F(N) \cap \mathbf{R}^3[x] = \begin{cases} B \cup B' & \text{for } x = 0, \\ A \cap \mathbf{R}^3[x] & \text{for } 0 < x \leq 1, \\ A \cap \mathbf{R}^3[x-1] & \text{for } 1 \leq x \leq 2, \\ \cdots & \\ A \cap \mathbf{R}^3[x-(N-2)] & \text{for } N-2 \leq x \leq N-1, \\ A \cap \mathbf{R}^3[x-(N-1)] & \text{for } N-1 \leq x < N, \\ B \cup B' & \text{for } x = N, \\ \phi & \text{otherwise,} \end{cases}$$

where $B \cup B'$ is a union of two standard 2-disks which bounds the trivial link $A \cap \mathbf{R}^3[0]$ $(= A \cap \mathbf{R}^3[1])$. Then the double point set of the diagram $D_{F(N)}$ consists of a union of N copies of the set in Figure 6(C). We notice that the surface-link $F(1)$ is $8_1^{-1,-1}$ in the list of [**12**].

Proof of Theorem 1.1. We prove that $F(N)$ in Example 3.3 satisfies (i) to (v) in Theorem 1.1. It is easy to verify that each component of $F(N)$ is a (trivial) non-orientable surface-knot with the Euler characteristic $2-N$. The property (iii) is followed by Proposition 2.1 (we recall that a Reidemeister move I corresponds to a branch point). For the calculation of $\pi_1(\mathbf{R}^4 - F)$, it is useful to refer to [**4**].

We will only prove that the property (v); $t(F(N)) = 2N$. Since the knot group of $F(N)$,

$$\langle a, b | aba = b, bab = a \rangle,$$

is the quaternion group, and since $\{a, b\}$ is a meridian system of $F(N)$, the order of each meridian is 4. By Lemma 2.6, $F(N)$ is $\mathbf{P}^2$-irreducible. Hence by the property (iii) and Lemma 3.1, we have $t(F(N)) \geq (|2N| + |-2N|)/2 = 2N$. On the other hand, $F(N)$ has the diagram whose double point set contains $2N$ triple points as shown in Example 3.3. So we have $t(F(N)) \leq 2N$. $\square$

References

[1] T.F. Banchoff, *Double tangency theorems for pairs of submanifolds*, in Geometry Symposium Utrecht 1980, ed. Looijenga, Seirsma, and Takens, LNM, Vol. 894, Springer-Verlag, (1981), 26-48.

[2] J.S. Carter and M. Saito, *Canceling branch points on projections of surfaces in* 4-*space*, Proc. of the A.M.S., **116**(**1**) (Sept. 1992), 229-237.

[3] ______, *Knotted surfaces and their diagrams*, Surveys and monographs, Vol. 55, A.M.S., Jan. 1998.

[4] R.H. Fox, *A quick trip through knot theory*, in Topology of 3-manifolds and related topics (Georgia, 1961), 120-167, Prentice-Hall.

[5] S. Kamada, *Non-orientable surfaces in* 4-*space*, Osaka J. Math., **26** (1989), 367-385.

[6] ______, 2-*dimensional braids and chart descriptions*, in topics in knot theory (Erzurum/Turkey, 1992), 277-287, NATO ASI Ser. C, **399**, Kluwer Academic Publ.

[7] S. Satoh, *Lifting a generic surface in* 3-*space to an embedded surface in* 4-*space*, to appear in Topology Appl.

[8] ______, *On non-orientable surfaces in* 4-*space which are projected with at most one triple point*, to appear in Proc. of the A.M.S.

[9] ______, *On simply knotted tori in* S^4 II, in Proceedings of Knots 96 (1997), World Scientific Publishing Co., 551-568.

[10] T. Yajima, *On simply knotted spheres in* $\mathbf{R}^4$, Osaka J. Math., **1** (1969), 133-152.

[11] S. Yamagata, *Some conditions and some examples for maps which are projections of 2-knots*, Master thesis, Hiroshima Univ., 1996 (in Japanese).

[12] K. Yoshikawa, *An enumeration of surfaces in four space*, Osaka J. Math., **31** (1994), 497-522.

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