Pacific Journal of Mathematics

INJECTIVITY RADII OF HYPERBOLIC POLYHEDRA

Joseph D. Masters

Volume 197 No. 2 February 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 197, No. 2, 2001

INJECTIVITY RADII OF HYPERBOLIC POLYHEDRA

Joseph D. Masters

We define the *injectivity radius* of a Coxeter polyhedron in $\mathbf{H}^3$ to be half the shortest translation length among hyperbolic/loxodromic elements in the orientation-preserving reflection group. We show that, for finite-volume polyhedra, this number is always less than 2.6339..., and for compact polyhedra it is always less than 2.1225... .

1. Introduction.

Hyperbolic reflection groups are discrete groups of isometries of hyperbolic space generated by reflections in the faces of a polyhedron. They provided some of the earliest known examples of Kleinian groups, and have been well-studied (see [**V**]). In this paper, we prove that 3-dimensional hyperbolic reflection groups always contain short elements.

To be more precise, let $\Gamma^+(P)$ be the group of orientation-preserving isometries generated by reflections in the faces of the polyhedron P. We define $\mathrm{injrad}\,(\Gamma^+(P))$ to be half the shortest translation length among hyperbolic/loxodromic elements of $\Gamma^+(P)$. Then we have:

Theorem 4.1 (Main Theorem). *Let P be a finite-volume Coxeter polyhedron in $\mathbf{H}^3$. Then* $\mathrm{injrad}(\Gamma^+(P)) < \cosh^{-1}(7) = 2.6639....$ *If P is compact, then* $\mathrm{injrad}(\Gamma^+(P)) < \cosh^{-1}(3 + 4\cos(2\pi/5)) = 2.1225....$

Remarks.

1. It is known (see [**W**]) that if $\{\mathbf{H}^3/\Gamma_i\}$ is a family of closed hyperbolic 3-manifolds, and if $\{\mathrm{rank}\,(\Gamma_i)\}$ is bounded, then $\{\mathrm{injrad}\,(\mathbf{H}^3/\Gamma_i)\}$ is also bounded (recall the *rank* of a finitely generated group is the cardinality of a minimal generating set). Observe that by [**B**], $\mathrm{Rank}\,(\Gamma(P))$ increases with the number of sides of P, so Theorem 4.1 is not covered by [**W**].

2. For more about short geodesics in hyperbolic 3-manifolds, see [**AR**].

3. We speculate that the bounds may be sharp, but we do not know a proof.

Idea of Proof. To prove the Main Theorem, we must show that every three-dimensional hyperbolic reflection group contains a hyperbolic element with suitably short translation length. This can usually be done by finding two non-adjacent faces of the polyhedron which are suitably close; the short element is obtained by composing the reflections in the corresponding hyperplanes.

A result of Nikulin's guarantees that every Coxeter polygon which is not a triangle has two non-adjacent sides which are close, and a two-dimensional version of the Main Theorem follows easily. For most polyhedra, we can use Nikulin's result to show that two non-adjacent faces are close; the exceptions are those which contain "non-prismatic" faces (see Section 2 for a definition). We show that these exceptional cases always contain triangular faces. Then, after extending to the sphere at infinity, we use combinatorics and Euclidean geometry to deduce the existence of a short element. The results on hyperbolic polyhedra contained in Section 3 allow us to sharpen the bound.

Organization. Section 2 contains basic definitions; Section 3 contains some general, technical results about hyperbolic polyhedra; Section 4 contains the proof of the Main Theorem.

Acknowledgments. I would like to thank Alan Reid for his help and encouragement. Thanks also to Gary Hamrick and David Bachman for valuable conversations, Daryl Cooper for helpful correspondence, and the referee for valuable comments.

2. Definitions.

A *convex polyhedron*, P, in $\mathbf{H}^n$ is a countable intersection of closed half-spaces: $P = \bigcap_i H_i^+$, where H_i denotes a hyperplane and H_i^+ the corresponding closed half-space. When n=2, we use the term *polygon* instead. If P is a convex polyhedron in $\mathbf{H}^n$, we let $\Gamma(P)$ denote the group of isometries generated by reflections in the bounding hyperplanes of P. $\Gamma^+(P)$ denotes its orientation-preserving subgroup of index 2. We say that a finite-volume, convex polyhedron P is a *Coxeter polyhedron* if its dihedral angles are all integer submultiples of π; if P is a Coxeter polyhedron, then $\Gamma^+(P)$ is discrete. Two faces of P are *adjacent* if they share an edge. Given a hyperplane H in $\mathbf{H}^n$, ρ_H will denote the isometry obtained by reflection in H. We will denote the hyperbolic distance between two sets X and Y in $\mathbf{H}^n$ by $d(X, Y)$.

Given an n-sided face F of P ($n > 3$), label the edges of F by $E_1, \dots, E_n$, where E_i shares a vertex with E_{i+1} for $i = 1, 2, \dots, n-1$, and label the adjacent faces by $F_1, \dots, F_n$, where F and F_i share edge E_i (we say the faces adjacent to F are labeled "cyclicly"). We say that F is *prismatic* if, for $i, j = 1, \dots, n$, $|i - j| > 1 \pmod n$ implies F_i is non-adjacent to F_j.

By a *hyperbolic n-manifold*, $\mathbf{H}^n/\Gamma$, we mean the quotient of hyperbolic n-space by a discrete group of isometries acting freely. If Γ has torsion, the quotient space H^n/Γ is a *hyperbolic n-orbifold*. The *injectivity radius* of a hyperbolic manifold $M = \mathbf{H}^n/\Gamma$ is equal to $sup\{\alpha \in \mathbb{R}^+ |$ *every point* $x \in M$ *is the center of an embedded ball of radius* $\alpha\}$. We shall generalize this definition to the case where Γ has torsion: The *injectivity radius of a Kleinian group* Γ, denoted injrad (Γ), is equal to half the shortest translation length among hyperbolic/loxodromic elements of Γ. Note that this agrees with the usual notion when Γ is torsion-free. The *injectivity radius of a Coxeter polyhedron* P is equal to injrad $(\Gamma^+(P))$. Given an element g of Γ, we will denote its translation length by $\ell(g)$.

3. Hyperbolic Polyhedra.

The following technical result shall be used in our proof of the Main Theorem.

Theorem 3.1. *Let* P *be a finite-volume, convex polyhedron in* $\boldsymbol{H}^3$ *with acute dihedral angles and no triangular faces. Then* P *has a prismatic face. If, furthermore,* P *is compact, then* P *has a prismatic quadrilateral or pentagonal face.*

Proof. We first consider the finite-volume case.

Lemma 3.2. *Let* P *be a finite-volume, convex polyhedron in* $\boldsymbol{H}^3$ *with acute dihedral angles and no triangular faces, and suppose that* P *contains at least one non-prismatic face. Let* G *be a planar graph representing the* 1*-skeleton of* P *(see Fig.* 1*). Then there are three non-prismatic faces of* P *which bound a region in* G *consisting entirely of prismatic faces.*

Proof. Let F be a non-prismatic face of P, and label its adjacent faces cyclicly by $F_1, F_2, \ldots$. We have that F_i and F_j are adjacent for some i, j with $|i-j| > 1$ (see Fig. 1). Note that F, F_i, and F_j bound a region R, and that F_i and F_j are also non-prismatic. Since P contains no triangular faces, R cannot be a face. And if the lemma is false, R must contain a non-prismatic face F'. Label the faces adjacent to F' cyclicly by $F'_1, F'_2, \ldots$. Then for some k, ℓ with $|k-\ell| > 1$, F'_k and F'_ℓ are adjacent. Then F', F'_k, and F'_ℓ bound a triangular region $R' \subsetneq R$. Again, if the lemma is false, R' must contain a non-prismatic face F'', creating a triangular region $R'' \subsetneq R'$. Since P has a finite number of sides, this process must eventually terminate with three non-prismatic faces bounding a region containing only prismatic faces. This proves the lemma, and the non-compact case of Theorem 3.1 follows immediately. $\square$

To prove the stronger statement for the compact case, we shall need the following combinatorial lemma:

Lemma 3.3. *Let* P *be a compact, convex polyhedron in* $\boldsymbol{H}^3$ *with acute dihedral angles. Then* P *contains a face with* 5 *or fewer edges.*

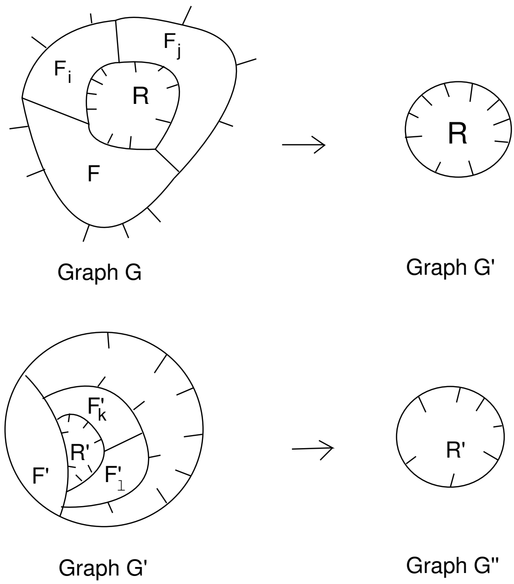

Figure 1. Reducing the graph around a non-prismatic face.

Proof. This follows from an Euler characteristic count. Let $|V|$ = number of vertices of P, $|E|$ = number of edges of P, and $|F|$ = number of faces of P. Since P is a compact, convex hyperbolic polyhedron, P is *simple*– i.e., each vertex is shared by exactly 3 different edges (see [**A1**]). So $|V| = 2|E|/3$.

We have $|V| - |E| + |F| = 2$.

$|F| - |E|/3 = 2$.

$|F|(1 - |E|/3|F|) = 2$.

So $|E|/|F| < 3$.

So the average number of edges per face < 6. So P must contain a face with 5 or fewer edges. □

Now suppose P is compact. If all faces of P are prismatic, we are done by Lemma 3.3. So suppose P contains a non-prismatic face. Then by Lemma 3.2, there are three non-prismatic faces of P which bound a region R in G consisting entirely of prismatic faces. We need to show that R contains a face with 5 or fewer edges. We form $\hat{R}$ from R by subtracting the three vertices, v_1, v_2, v_3, of non-prismatic faces on the boundary of R, and then adding a face to the exterior of R, so that $\hat{R}$ is topologically a sphere (see Fig. 2).

Then, arguing as in Lemma 3.3, $\hat{R}$ must contain a face F with 5 or fewer edges. We claim that R must also contain such a face. This will certainly be true unless

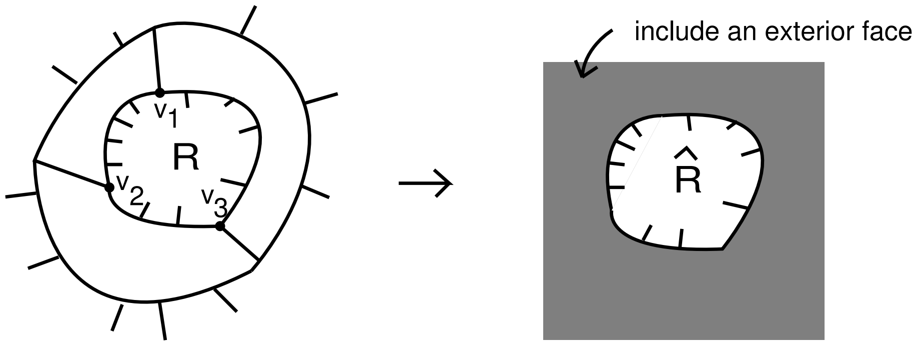

Figure 2. To form $\hat{R}$ from R, we remove three vertices and include an exterior face.

1. F is the face on the exterior of R.

or

2. F is a pentagon, and F contains one of the edges from which a vertex was deleted (note that F can contain at most one of the v_i's, since it is prismatic).

If F is the face on the exterior and F is a triangle, then we claim that R must consist of three quadrilaterals, as in Fig. 3b. For otherwise, R must contain three non-prismatic faces, contrary to assumption (see Fig. 3b).

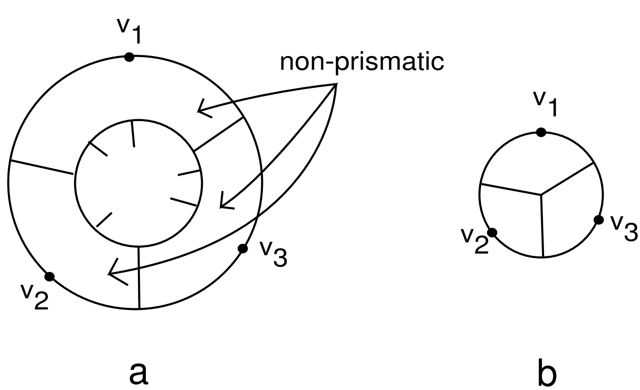

Figure 3. If the exterior is a triangle, then R will contain either a) non-prismatic faces, or b) three quadrilaterals.

So if the theorem is false, then $\hat{R}$ contains at most four faces with fewer than 6 sides; at least three of them must be pentagons, and none are triangles. We shall show that this is impossible.

Number the faces of $\hat{R}$ by $F_1, F_2, \dots, F_n$. Let $|E_j|$ be the number of sides of F_j. Then $|E_1| + \cdots + |E_n| = 2|F|$, where $|F|$ is the total number of faces of $\hat{R}$. Since P is simple, the graph is trivalent, so $|F|$ is divisible by 3; hence $|E_1| + \cdots + |E_n|$ is divisible by 6. Since also the average number of sides per face is < 6, we have:

$$6n - 5 \leq \sum_{i=1,\dots,n} |E_i| < 6n.$$

Thus the sum is not divisible by 6, for a contradiction. □

4. Injectivity radius of hyperbolic polyhedra.

In this section we prove the Main Theorem:

Theorem 4.1. *Let P be a finite-volume Coxeter polyhedron in $\boldsymbol{H}^3$. Then* $\operatorname{injrad}(\Gamma^+(P)) < \cosh^{-1}(7) = 2.6339\dots$ *If P is compact, then* $\operatorname{injrad}(\Gamma^+(P)) < \cosh^{-1}(3 + 4\cos(2\pi/5)) = 2.1226\dots$.

An important part of the proof of Theorem 4.1 is played by the following 2-dimensional result.

Theorem 4.2. *Let P be a finite-area Coxeter n-gon in $\boldsymbol{H}^2$, $n > 3$. Then* $\operatorname{injrad}(\Gamma^+(P)) \leq \cosh^{-1}(3 + 4\cos(2\pi/n)) < \cosh^{-1}(7) = 2.6339\dots$ *If $n = 3$, then* $\operatorname{injrad}(\Gamma^+(P)) \leq \cosh^{-1}(3 + 4\cos(2\pi/4))$. *If P is compact, then all the inequalities are strict.*

Proof. The theorem is a consequence of the following lemma, which is a re-phrasing of ([**N**], Theorem 3.2.1]).

Lemma 4.3. *Let P be a finite area, convex n-gon in $\boldsymbol{H}^2$. Label the bounding geodesics of P cyclicly by $H_1, H_2, \dots, H_n$. Then for some i, $d(H_i, H_{i+2}) \leq$* $\cosh^{-1}(3 + 4\cos(2\pi/n))$ *(subscripts taken* mod n*). If P is compact, the inequality is strict.*

Proof. We will reproduce Nikulin's proof that $d(H_i, H_{i+2}) < \cosh^{-1}(7)$. The proof of the finer estimate is a bit more complicated, and we omit it (see [**N**]).

We use the Lobachevsky model for $\mathbf{H}^2$. First, pick a point p on the interior of P. Project the vertices of P onto the circle at infinity along rays emanating from p. This determines an ideal polygon P' with bounding geodesics $H'_1, H'_2, \dots$. We claim that $d(H_i, H_{i+2}) \leq d(H'_i, H'_{i+2})$: The distance between H'_i and H'_{i+2} is measured along a mutually orthogonal geodesic segment α (see Fig. 4a); since α must intersect H_i and H_{i+2}, $d(H_i, H_{i+2}) < \operatorname{length}(\alpha) = d(H'_i, H'_{i+2})$.

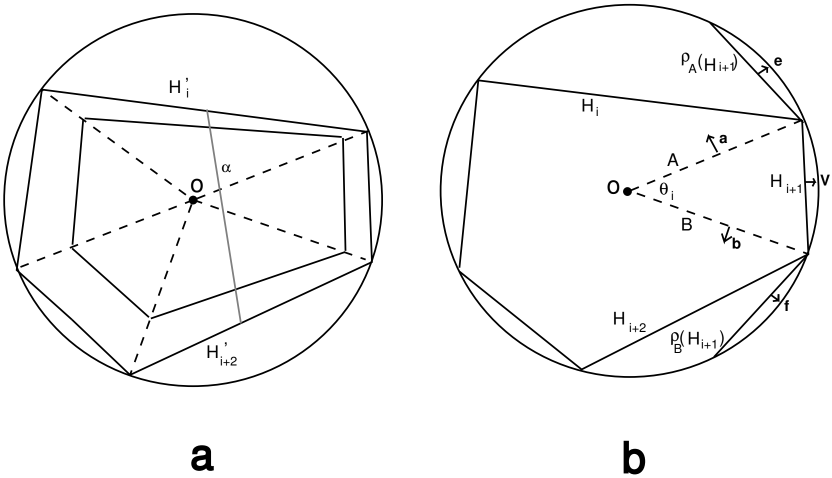

Figure 4. a) The faces of the ideal polygon are farther apart. b) Notation for Lemma 3.2.

So it is enough to consider the case where P is an ideal polygon. Pick i such that the Euclidean angle $\theta_i = v_i O v_{i+1}$ is minimal, where O is the origin. Let A be the diameter through v_i and let B be the diameter through v_{i+1}. Then $d(H_i, H_{i+2}) \leq d(\rho_A(H_{i+1}), \rho_B(H_{i+1})) = \cosh^{-1}(-\langle \mathbf{e}, \mathbf{f} \rangle)$, where $\mathbf{e}$ is the unit normal to $\rho_A(H_{i+1})$, $\mathbf{f}$ is the unit normal to $\rho_B(H_{i+1})$, and $\langle .|. \rangle$ is the inner product $\langle (x_1, y_1, z_1) | (x_2, y_2, z_2) \rangle = x_1 x_2 + y_1 y_2 - z_1 z_2$ (see Fig. 4b).

Now, let $\mathbf{v}, \mathbf{a}$ and $\mathbf{b}$ be outward unit normals to H_{i+1}, A and B, respectively. Then we have:

$$\begin{aligned}\langle \mathbf{e}, \mathbf{f} \rangle &= \langle \mathbf{v} - \mathbf{2}\langle \mathbf{v}, \mathbf{a} \rangle \mathbf{a}, \mathbf{v} - \mathbf{2}\langle \mathbf{v}, \mathbf{b} \rangle \mathbf{b} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} \rangle - \mathbf{2}\langle \mathbf{v}, \mathbf{a} \rangle^{\mathbf{2}} - \mathbf{2}\langle \mathbf{v}, \mathbf{b} \rangle^{\mathbf{2}} + \mathbf{4}\langle \mathbf{v}, \mathbf{a} \rangle \langle \mathbf{v}, \mathbf{b} \rangle \langle \mathbf{a}, \mathbf{b} \rangle \\ &= 1 - 2 - 2 - 4\cos(\theta_i).\end{aligned}$$

So $d(H_i, H_{i+2}) < \cosh^{-1}(7)$. □

We now resume the proof of Theorem 4.2. Let P be a Coxeter n-gon, and suppose first $n > 3$. Pick two non-adjacent edges such that the corresponding geodesics H and H' are less than $\cosh^{-1}(3 + 4\cos(2\pi/n))$ apart. Let $g = \rho_H \rho_{H'}$ in $\Gamma^+(P)$. Since P has acute angles, H is disjoint from H', so g is hyperbolic. Let α denote the axis of g, and note that it is perpendicular to both H_i and H_{i+2} (see Fig. 5). $\ell(g)$ is given by $d(p, g(p))$, where p is any point on α. Taking p to be $\alpha \cap H_{i+2}$, it is easy to see that $\ell(g) = 2d(H_i, H_{i+2})$. Therefore $\operatorname{injrad}(\Gamma^+(P)) \leq \ell(g)/2 = d(H, H') < \cosh^{-1}(3 + 4\cos(2\pi/n))$.

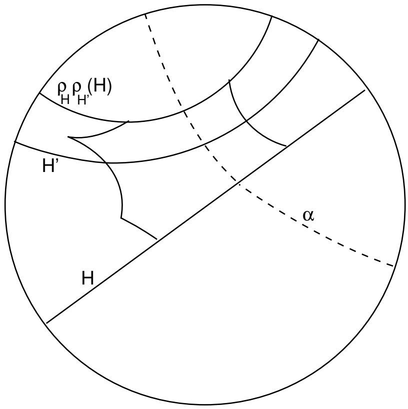

Figure 5. The distance from H to $H\prime$ is half the translation length of $\rho_H \rho_{H\prime}$.

If P is a triangle, suppose first that P has no right angles, and is compact. Label the vertices of P by v_0, v_1, v_2, with corresponding angles $\theta_0, \theta_1, \theta_2$ (see Fig. 6a). One of the angles, say θ_0, must be $\leq \pi/4$. Consider the quadrilateral Q, with angles $(\theta_0, 2\theta_1, \theta_0, 2\theta_2)$, obtained by reflecting P along $v_1 v_2$. Let H and H' be non-adjacent bounding geodesics of Q, with $d(H, H') < \cosh^{-1}(3 + 4\cos(2\pi/4))$. H and H' are disjoint: For if they intersected, they would create a triangle T with angles $(\pi - \theta_0, \pi - 2\theta_i, x)$, for i = 1 or 2; however $\theta_0 + 2\theta_i < \pi$, so T would have angle sum $> \pi$, which is impossible. Therefore, as above, $g = \rho_H \rho_{H'}$ is a hyperbolic element of $\Gamma^+(P)$ with $\ell(g)/2 < \cosh^{-1}(3 + 4\cos(2\pi/4))$.

If P has a right angle at a vertex v_0, and is compact, reflect P along $v_0 v_1$ to obtain a new triangle, P', with angles $(2\theta_1, \theta_2, \theta_2)$ (see Fig. 6b). Reflect P' along $v_0 v_2$ to obtain a quadrilateral Q with angles $(2\theta_1, 2\theta_2, 2\theta_1, 2\theta_2)$. Then, again, the opposite geodesics bounding Q must be disjoint, or else they would create a triangle with angles $(\pi - 2\theta_1, \pi - 2\theta_2, x)$, which is impossible since $\theta_1 + \theta_2 < \pi/2$. So again we obtain a hyperbolic element g with $\ell(g)/2 < \cosh^{-1}(3 + 4\cos(2\pi/4))$.

If P has an ideal vertex v_0, then reflecting P along $v_1 v_2$ creates a quadrilateral Q whose bounding geodesics clearly cannot intersect, so that again we obtain the required hyperbolic element. □

As a corollary of Theorem 4.2, we have:

Corollary 4.4. *Let P be a finite-volume Coxeter polyhedron in $\boldsymbol{H}^3$ with an n-sided prismatic face. Then* $\operatorname{injrad}(\Gamma^+(P)) \leq \cosh^{-1}(3 + 4\cos(2\pi/n))$.

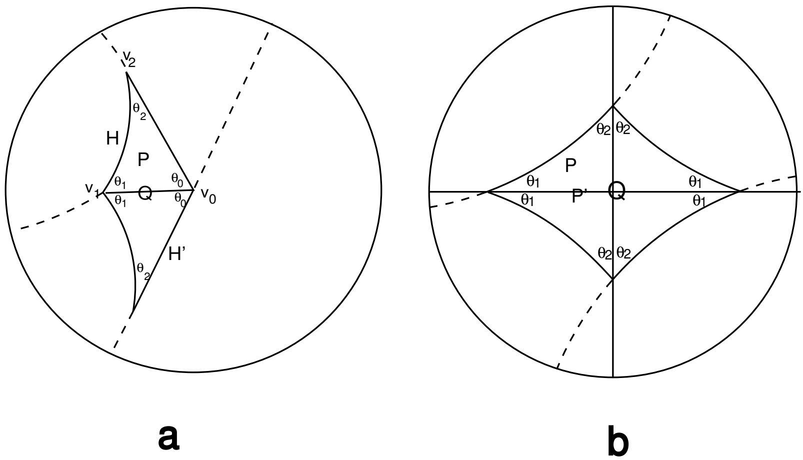

Figure 6. We can reflect to obtain a quadrilateral.

Proof. Consider the n-sided prismatic face F_0, and label the faces adjacent to F_0 cyclicly by $F_1, \ldots, F_n$. Let H_j denote the hyperplane spanned by F_j, and let $H_{i,j} = H_i \cap H_j$. By Lemma 4.3, there is some i such that $d(H_{0,i}, H_{0,i+2}) \leq \cosh^{-1}(3 + 4\cos(2\pi/n))$. Then $d(H_i, H_{i+2}) \leq \cosh^{-1}(3 + 4\cos(2\pi/n))$. Since F_i and F_{i+2} are non-adjacent, H_i and H_{i+2} do not intersect by [**A1**]. So $g = \rho_{H_i}\rho_{H_{i+2}}$ is a hyperbolic element of $\Gamma^+(P)$. So $\text{injrad}(\Gamma^+(P)) \leq \ell(g)/2 \leq \cosh^{-1}(3 + 4\cos(2\pi/n))$. □

We now prove the Main Theorem.

Proof of 4.1.

Compact case.

Case 1: P has no triangular faces.

By Theorem 3.1, P contains a prismatic quadrilateral or pentagonal face, and so by Corollary 4.4, we are done.

Case 2: P has a triangular face, but P is not a simplex.

Let F_0 be the triangular face, and label the faces adjacent to F_0 by F_1, F_2 and F_3. Let $P\prime = \bigcap_{i=0,1,2,3} H_i^+$, where H_i denotes the hyperplane spanned by F_i (recall Section 2 for the definition of H_i^+). Let C_i and C_i^+ denote the boundary at infinity of H_i and H_i^+, respectively. Let θ_{ij} = dihedral angle between F_i and $F_j = \pi/n_{ij}$ (here we are again using the fact that in the compact case the polyhedra are simple). We label the dihedral angles of $P\prime$ by $\mathbf{A}_{P\prime} = ((\mathrm{n}_{01}, \mathrm{n}_{02}, \mathrm{n}_{03}), (\mathrm{n}_{12}, \mathrm{n}_{23}, \mathrm{n}_{31}))$. Then,

I. $$1/n_{0i} + 1/n_{0j} + 1/n_{ij} > 1, \quad \text{and}$$

II. $$1/n_{12} + 1/n_{23} + 1/n_{31} < 1.$$

One may then easily verify that the only 5 possibilities for (n_{01}, n_{02}, n_{03}) are: (2,2,2), (2,2,3), (2,2,4), (2,2,5) and (2,3,3).

Case 2a: $\mathbf{A}_{P\prime} = ((2,2,2),(n_{12},n_{23},n_{31}))$ or $((2,2,4),(n_{12},n_{23},n_{31}))$.

Let Stab $_{F_0}$ denote the subgroup of $\Gamma^+(P)$ which leaves F_0 invariant. Then Theorem 5.4 of [**BM**] implies that Stab $_{F_0}$ contains a triangle group. So by Theorem 4.2, we are done.

Case 2b: $\mathbf{A}_{P\prime} = ((2,3,3),(n_{12},n_{23},n_{31}))$.

By I and II, the only two possibilities (modulo relabeling of edges) are $\mathbf{A}_{P\prime} = ((2,3,3),(4,2,5))$ or $((2,3,3),(5,2,5))$. Consider $P\prime\prime = P\prime \cup \rho_{H_1}(P\prime)$ (see Fig. 7a). Let Q denote the quadrilateral created in H_0. Conjugate so that, in the upper half space model, C_1 is the imaginary axis (see Fig. 7b), and $\infty \in C_i^+ - C_i$ for i = 2,3. $H_1 \cap H_2 \cap H_3 = \emptyset$, since the angles θ_{0i} are acute; therefore, $C_2^+ \cap C_3^+ \cap \rho_{H_1}(C_2^+) \cap \rho_{H_1}(C_3^+) = Q_\infty^1 \cup Q_\infty^2$, where Q_∞^1, Q_∞^2 are quadrilateral regions in $S_\infty^2 = \hat{\mathbb{C}}$, and $\infty \in Q_\infty^2$. The dihedral angles of Q_∞^1 are $(\theta_{23}, 2\theta_{12}, \theta_{23}, 2\theta_{31})$.

We claim that $C_2 \cap \rho_{H_1}(C_3) = C_3 \cap \rho_{H_1}(C_2) = \emptyset$, in which case $H_2, \rho_{H_1}(H_3)$ are disjoint, and $H_3, \rho_{H_1}(H_2)$ are disjoint. Otherwise, three circles would intersect to create a triangle T with angle sum $> \pi$ (see Fig. 7c). However, T has only one positively curved side — call its curvature κ — and it has another, longer side of curvature $-\kappa$. So, by the Gauss-Bonnet formula, T has angle sum $< \pi$, for a contradiction. Hence $g_1 = \rho_{H_2}\rho_{\rho_{H_1}(H_3)}$ and $g_2 = \rho_{H_3}\rho_{\rho_{H_1}(H_2)}$ are both hyperbolic elements of $\Gamma^+(P)$. By Lemma 4.3, two of the opposite faces of Q must be less than $\cosh^{-1}(3 + 4\cos(2\pi/4))$ apart. It follows that either g_1 or g_2 has suitably short translation length.

Case 2c: $\mathbf{A}_{P\prime} = ((2,2,3),(n_{12},n_{23},n_{31}))$ or $((2,2,5),(n_{12},n_{23},n_{31}))$.

Without loss of generality, assume $n_{13} \geq n_{23}$, so by II, $n_{13} \geq 3$. First suppose $n_{12}, n_{23} \geq 3$. As in Case 2b, reflect in H_1 to create a polyhedron $P\prime\prime$ with a quadrilateral face Q. Again, on S_∞^2 we see a quadrilateral Q_∞^1 with

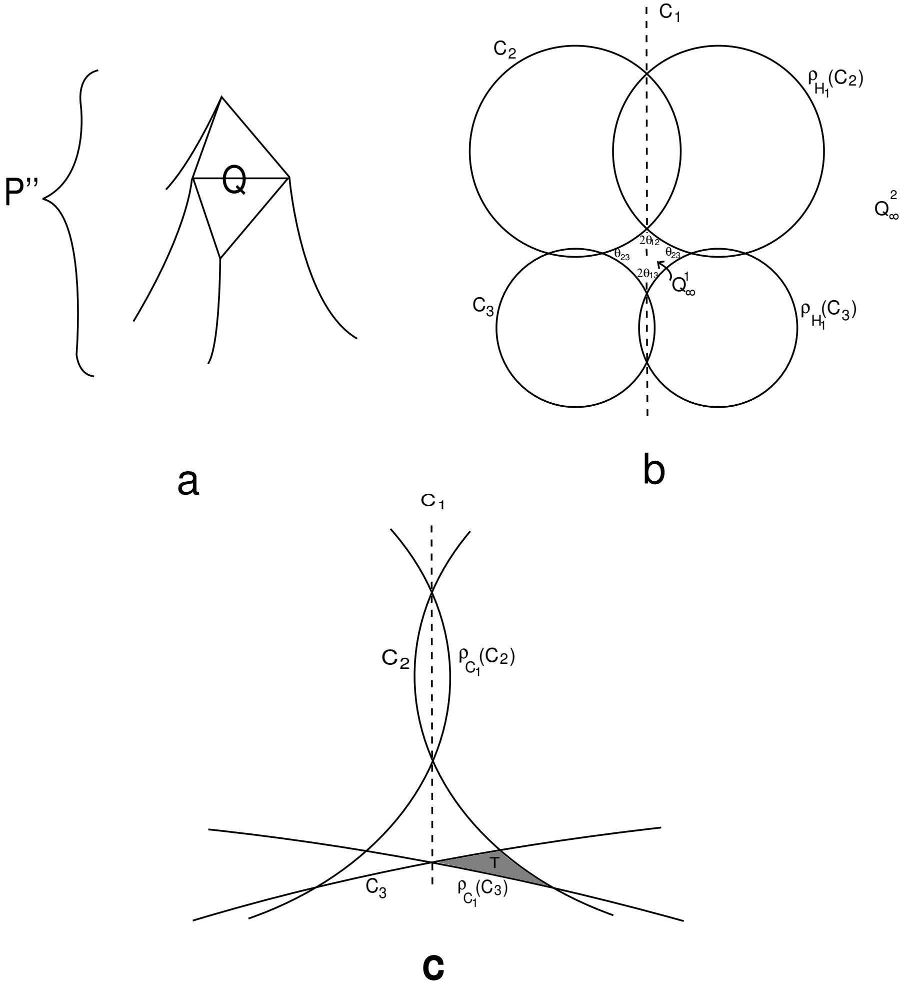

Figure 7. a) The polyhedron $P\prime\prime$. b) The view on the sphere at infinity. c) If opposite circles intersect, a triangle is formed.

angles $(\theta_{23}, 2\theta_{12}, \theta_{23}, 2\theta_{31})$. Since the sum of any two adjacent angles of Q^1_∞ is $\leq \pi$, we can argue as in Case 2b to show that opposite circles bounding $Q\prime$ must be disjoint, thus creating a hyperbolic element with suitably short translation length.

If $n_{12} = 2$, then by I and II, n_{23} and $n_{31} \geq 4$. Then after reflecting in H_1 and in H_2, we see on S^2_∞ an acute quadrilateral (see Fig. 8), and we may argue as above.

If $n_{23} = 2$, then by I and II, $n_{12} \geq 4$.

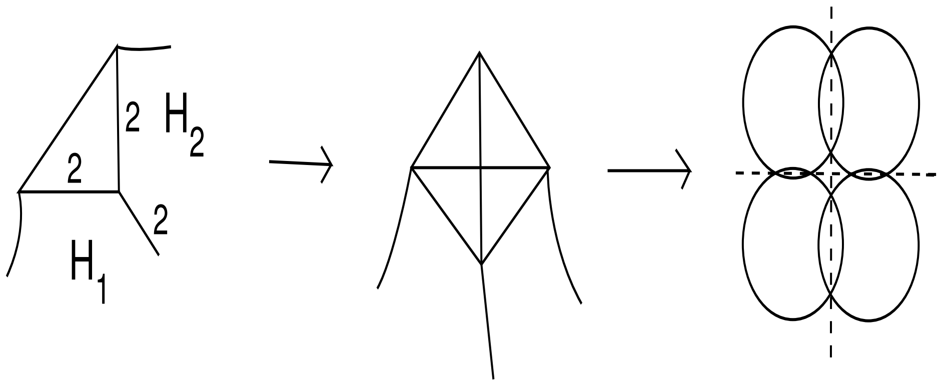

Figure 8. Reflect twice, then view at infinity.

— If $n_{12} = 4$, then after reflecting in H_2, we can reduce to the case where $n_{12} = 2$.
— If $n_{12} \geq 5$, then by II, $n_{23} \geq 4$, and reflecting in H_2 creates an acute quadrilateral. So we may argue as in Case 2b.

Case 3: P is a simplex.

By [**L**] there are only nine congruence classes of compact simplices in $\mathbf{H}^3$. By [**M**] (see also [**BM**]), eight of these contain triangle groups, so in these cases the result follows from Theorem 4.2. Denote the remaining tetrahedron by T_8; label its faces $F_1, \ldots, F_4$; and let π/n_{ij} be the dihedral angle between F_i and F_j. We have $n_{12} = 2$, $n_{13} = 3$, $n_{14} = 4$, $n_{23} = 5$, $n_{24} = 3$, and $n_{34} = 4$. Let H_i be the hyperplane spanned by F_i. It is not difficult to construct T_8 explicitly in Lobachevsky space and then compute the faithful discrete representation of $\Gamma^+(T_8)$ in $O(3,1)$. Then it is straightforward to compute that $\rho_{H_3}\rho_{H_4}\rho_{H_2}\rho_{H_1}\rho_{H_4}\rho_{H_2}$ is a hyperbolic element with translation length $1.66131... < 2\cosh^{-1}(3 + 4\cos(\pi/5))$.

Non-compact case:

By Theorem 3.1 and Corollary 4.4, it is enough to consider the case where P has a triangular face, F_0. As in the compact case, the faces adjacent to F_0 form a polyhedron $P\prime$.

If none of the vertices of F_0 are ideal, then the proof for the compact case carries over without change. So suppose F_0 has an ideal vertex.

Ideal vertices may be either tri-valent or 4-valent (see [**A2**]). If the vertices of F_0 are all trivalent then condition II still holds; condition I holds at regular vertices, and changes to an equality at ideal vertices. Then the techniques from the compact case are sufficient to prove the theorem– we omit the details. If one of the vertices is 4-valent, then two of the sides adjacent to F_0 are tangent on S^2_∞ (see Fig. 9). One may now argue as in the compact case, treating the tangent sides as adjacent with dihedral angle 0. The theorem follows.

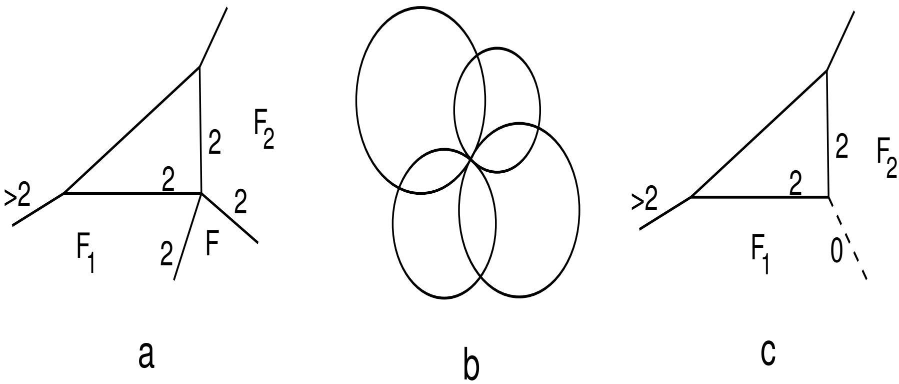

Figure 9. a) A triangular region with a 4-valent ideal vertex. b) A 4-valent ideal vertex on S^2_∞. c) We may remove F and view F_1 and F_2 as adjacent with dihedral angle 0.

□

References

[AR] C.C. Adams and A.W. Reid, *Systoles for hyperbolic knot and link complements*, Math. Proc. Camb. Phil. Soc., to appear.

[A1] E.M. Andreev, *On convex polyhedra in Lobačevskiĭ spaces*, Math. USSR-Sb., **10** (1970), 413-440.

[A2] ———, *On convex polyhedra of finite volume in Lobačevskiĭ spaces*, Math. USSR-Sb., **12** (1970), No. 2, 255-259.

[BM] T. Baskan and A.M. Macbeath, *Centralizers of reflections in crystallographic groups*, Math. Proc. Camb. Phil. Soc., **92** (1982), 79-91.

[B] K. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[L] F. Lanner, *On complexes with transitive groups of automorphisms*, Comm. Sem. Math. Univ. Lund, **11** (1950), 1-71.

[M] C. Maclachlan, *Triangle subgroups of hyperbolic tetrahedral groups*, Pacific J. Math., **176** (1996), 195-203.

[N] V.V. Nikulin, *On arithmetic groups generated by reflections in Lobačevskiĭ spaces*, Math. USSR Izv., **16** (1981), 573-601.

[V] E.B. Vinberg, *Hyperbolic reflection groups*, Russian Math. Surveys, **40** (1985), 31-75.

[W] M. White, preprint.

Received September 17, 1997 and revised March 7, 2000.

Department of Mathematics
University of Texas
Austin, TX 78712

Department of Mathematics
MS 136, Rice University
Houston TX 77005-1892
E-mail address: mastersj@rice.edu