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ON THE ASYMPTOTICS OF THE TRACE OF THE HEAT
KERNEL FOR THE MAGNETIC SCHRÖDINGER
OPERATOR

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**ON THE ASYMPTOTICS OF THE TRACE OF THE HEAT
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We consider the effect of a magnetic field for the asymptotic behavior of the trace of the heat kernel for the Schrödinger operator. We discuss the case where the operator has compact resolvents in spite of the fact that the electric potential is degenerate on some submanifold. According to the degree of the degeneracy, we obtain the classical and non-classical asymptotics.

1. Introduction.

In this paper, we consider the Schrödinger operator on \mathbb{R}^d with the magnetic vector potential $A(z)$ and the electric scalar potential $V(z)$:

$$(1.1) \quad H(A, V) = \frac{1}{2}(i\nabla + A(z))^2 + V(z).$$

We will assume that $H(A, V)$ and $H(0, V)$ are essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ and we denote the self-adjoint extensions by H and H_0 , respectively. It is well known that if

$$(1.2) \quad \lim_{|z| \rightarrow \infty} V(z) = +\infty,$$

H_0 has compact resolvents (c.f. for example, Reed and Simon [8]). Odenrantz [7] studied the asymptotic behavior of $\text{Tr}[\exp(-tH) - \exp(-tH_0)]$ as $t \downarrow 0$ in the case where $V(z) \approx |z|^{2p}$ ($p > 0$) and the curl of $A(z)$ is a uniform magnetic field. Matsumoto [5] extended the result to the case with more general magnetic field.

However, (1.2) is not a necessary condition in order that H_0 has compact resolvents. In spite of the lack of (1.2), there are some cases where H_0 has compact resolvents.

The motivation of this paper originates in the works of Simon [11], Robert [9] and Aramaki [3]. In order to explain this, we put

$$Z_{\text{cl}}(t) = (2\pi)^{-d} \iint e^{-t(|\zeta|^2/2 + V(z))} dz d\zeta.$$

It is well known that $\text{Tr}[\exp(-tH_0)] \leq Z_{\text{cl}}(t)$ for $t > 0$. In [11], he considered the degenerate potential $V(z)$ of the form $V(x, y) = |x|^{2p}|y|^{2q}$ ($p, q > 0$) on

\mathbb{R}^2 . Then it holds that H_0 has compact resolvents. In this case, $Z_{\text{cl}}(t) \equiv \infty$ while $\text{Tr}[\exp(-tH_0)] < \infty$ for all $t > 0$. He succeeded to get the asymptotics of $\text{Tr}[\exp(-tH_0)]$ as $t \downarrow 0$ by using the sliced Golden-Thompson inequality and the sliced bread inequality according to the cases $p \neq q$ and $p = q$. [9] and [3] considered a slightly modified potential $V(z)$ of the form

(1.3)

$$V(z) = (1 + |x|^2)^p |y|^{2q} \quad (p, q > 0 \text{ integers}), z = (x, y) \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m.$$

Then H_0 also has compact resolvents (cf. [9]). In this case, it is easy to see that $Z_{\text{cl}}(t)$ is finite for $t > 0$ in the case $pm > qn$, however, $Z_{\text{cl}}(t)$ is infinite for $t > 0$ in the case $pm \leq qn$. In the present paper, we consider the magnetic Schrödinger operator H with an electric potential $V(z)$ of the type (1.3). We want to show that if the magnetic potential $A(z)$ is relatively benign, the difference between $\text{Tr}[\exp(-tH)]$ and $\text{Tr}[\exp(-tH_0)]$ can be controlled by using the representation of the heat kernels in terms of Wiener integrals. From this, we can see the asymptotics of $\text{Tr}[\exp(-tH)]$ as $t \downarrow 0$, if we combined the estimate with our previous results as follows.

We denote the numbers of the eigenvalues counting multiplicities of H and H_0 equal to or less than λ by $N(\lambda)$ and $N_0(\lambda)$, respectively. Then Aramaki [3] obtained the following.

Proposition 1.1. *Under (1.3), there exists $\delta > 0$ such that:*

- (i) *If $pm > qn$, $N_0(\lambda) = c_1 \lambda^{(m+mq+nq)/(2q)} (1 + O(\lambda^{-\delta}))$ as $\lambda \rightarrow \infty$.*
- (ii) *If $pm = qn$, $N_0(\lambda) = c_2 \lambda^{(m+mq+nq)/(2q)} \log \lambda + c_3 \lambda^{(m+mq+nq)/(2q)} (1 + O(\lambda^{-\delta}))$ as $\lambda \rightarrow \infty$.*

(iii) *If $pm < qn$, $N_0(\lambda) = c_4 \lambda^{n(1+p+q)/(2p)} (1 + O(\lambda^{-\delta}))$ as $\lambda \rightarrow \infty$ where c_i ($i = 1, 2, 3, 4$) are some positive constants which can be calculated concretely.*

Here we note that in only the case where $pm > qn$, the problem is classical in the sense of

$$(1.4) \quad \text{vol} \left\{ (z, \zeta) \in \mathbb{R}^d \times \mathbb{R}^d; \frac{1}{2} |\zeta|^2 + V(z) \leq \lambda \right\} < \infty.$$

On the other hand, in the case where $pm \leq qn$, the problem is non-classical in the sense that the left hand side of (1.4) is the infinity. It easily follows from the well known equation

$$\text{Tr}[\exp(-tH_0)] = \int \exp(-t\lambda) dN_0(\lambda)$$

that we also have the asymptotic behavior of $\text{Tr}[\exp(-tH_0)]$ as $t \downarrow 0$.

Proposition 1.2. *Under (1.3), we have:*

- (i) If $pm > qn$, $\text{Tr}[e^{-tH_0}] = d_1 t^{-(m+mq+nq)/(2q)}(1 + O(t^\delta))$ as $t \rightarrow 0$.
(ii) If $pm = qn$,
 $\text{Tr}[e^{-tH_0}] = d_2 t^{-(m+mq+nq)/(2q)} \log t^{-1} + d_3 t^{-(m+mq+nq)/(2q)}(1 + O(t^\delta))$
as $t \rightarrow 0$.
(iii) If $pm < qn$, $\text{Tr}[e^{-tH_0}] = d_4 t^{-n(1+p+q)/(2p)}(1 + O(t^\delta))$ as $t \rightarrow 0$
where d_i ($i = 1, 2, 3, 4$) are positive constants which can easily be given by c_i
(also given in Section 4) and δ is as in Proposition 1.1.

For the precise, see Aramaki [1], [2], [3], [9] and [11].

In this paper, we shall treat the case where $V(z)$ is of the type of (1.3). Under some hypotheses on the magnetic vector potential $A(z)$, we shall obtain the asymptotic behavior of $\text{Tr}[\exp(-tH) - \exp(-tH_0)]$ as $t \downarrow 0$ is better than that in Proposition 1.2 in each case. Therefore, we see that the magnetic field does not give any influence to the leading terms of the asymptotics. One of the features of this paper is that the argument is based on the probabilistic representations of the heat kernels. As we do not use any pseudodifferential calculus in the proof of the main theorem, it suffices to assume less smoothness of V and A .

The plan of this paper is as follows. In § 2, we give the main theorem. Section 3 is devoted to the proof of the main theorem. In § 4, we give an example so that the electric potential $V(x, y)$ is of the form (1.3).

2. Hypotheses and Statements.

Let $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$ and we write a variable z in \mathbb{R}^d by $z = (x, y) \in \mathbb{R}_x^n \times \mathbb{R}_y^m$. We consider the operator:

$$(2.1) \quad H(A, V) = \frac{1}{2}(i\nabla_{(x,y)} + A(x, y))^2 + V(x, y)$$

where $i = \sqrt{-1}$ and $\nabla_{(x,y)}$ denotes the gradient operator. First of all, we state the assumptions on the scalar potential $V(x, y)$.

(V.1) $V(x, y) \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ is a real valued function.

(V.2) There exist positive constants p, q and $C > 0$ such that

$$V(x, y) \geq C(1 + |x|^2)^p |y|^{2q} \quad \text{for all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Moreover, we give the assumptions for the vector potential $A(x, y)$:

(A.1) $A(x, y) = (a_1(x, y), \dots, a_d(x, y)) \in C^2(\mathbb{R}^d; \mathbb{R}^d)$.

(A.2) There exist constants a, b satisfying $0 \leq a < p$, $0 \leq b < q$, $(q+1)a < p(b+1)$ and $C_1 > 0$ such that for every $j = 1, 2, \dots, d$ and $|\alpha| \leq 2$,

$$(2.2) \quad |\partial_{x,y}^\alpha a_j(x, y)| \leq C_1(1 + |x|^2)^a |y|^{2b}.$$

By the assumptions (V.1) and (A.1), $H(A, V)$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ (c.f. Schechter [10]) and we denote the unique self-adjoint extensions of $H(A, V)$ and $H(0, V)$ by H and H_0 , respectively

as in introduction. Under (V.1), (V.2), H_0 has compact resolvents and $\exp(-tH_0)$ is of trace class, i.e., $\text{Tr}[\exp(-tH_0)]$ is finite (c.f. [1]). Since V is bounded from below, we have the diamagnetic inequality:

$$e^{-tH(A,V)} \preceq e^{-tH(0,V)} \quad \text{for every } t > 0,$$

that is to say, for all $u \in L^2(\mathbb{R}^d)$,

$$\left| (e^{-tH(A,V)}u)(z) \right| \leq (e^{-tH(0,V)}|u|)(z) \quad \text{a.e. } z \text{ for } t > 0.$$

Then it follows from Simon [13, p. 164] that

$$\text{Tr}[e^{-tH}] \leq \text{Tr}[e^{-tH_0}] \quad \text{for } t > 0,$$

so we see that $\exp(-tH)$ is also of trace class.

Then we have the main theorem.

Theorem 2.1. *Under the conditions (V.1), (V.2), (A.1) and (A.2), we have the following.*

(i) *The case where $pm > qn$. If $q(4a + n) - p(4b + m) \neq 0$, we have*

$$\text{Tr}[e^{-tH} - e^{-tH_0}] = O(t^{-(m+mq+nq)/(2q)+\gamma_1}) \quad \text{as } t \downarrow 0$$

and if $q(4a + n) - p(4b + m) = 0$, we have

$$\text{Tr}[e^{-tH} - e^{-tH_0}] = O(t^{-(m+mq+nq)/(2q)+\gamma_1} \log t^{-1}) \quad \text{as } t \downarrow 0$$

where

$$\gamma_1 = \min \left\{ \frac{2(q-b)}{q}, \frac{(1+q)(pm-qn)}{2pq} + \frac{2(p(1+b) - a(q+1))}{p} \right\}.$$

(ii) *The case where $pm \leq qn$. If $q(4a + n) - p(4b + m) \neq 0$, we have*

$$\text{Tr}[e^{-tH} - e^{-tH_0}] = O(t^{-n(1+p+q)/(2p)+\gamma_2}) \quad \text{as } t \downarrow 0$$

and if $q(4a + n) - p(4b + m) = 0$, we have

$$\text{Tr}[e^{-tH} - e^{-tH_0}] = O(t^{-n(1+p+q)/(2p)+\gamma_2} \log t^{-1}) \quad \text{as } t \downarrow 0$$

where

$$\gamma_2 = \min \left\{ \frac{2(q-b)}{q} + \frac{(1+q)(qn-pm)}{2pq}, \frac{2(p(1+b) - a(q+1))}{p} \right\}.$$

Remark 2.2. Since γ_1 and γ_2 are positive numbers in any cases according to (A.2), the hypothesis (A.2) on the magnetic potential certainly gives an effect to the asymptotics in each case.

Using the Karamata Tauberian theorem and [3], we have also asymptotics of distribution function $N(\lambda)$ of eigenvalues of H .

Corollary 2.3. *Addition to the hypotheses of Theorem 2.1, we assume that there exists $C_2 > 0$ such that*

$$V(x, y) \leq C_2(1 + |x|^2)^p |y|^{2q}.$$

Then we have

$$N(\lambda) = N_0(\lambda)(1 + o(1)) \quad \text{as } \lambda \rightarrow \infty.$$

Here $N_0(\lambda)$ is of the form:

(i) *If $pm > qn$, we have*

$$N_0(\lambda) = c_1 \lambda^{(m+mq+nq)/(2q)} (1 + o(1)) \quad \text{as } \lambda \rightarrow \infty.$$

(ii) *If $pm = qn$, we have*

$$N_0(\lambda) = c_2 \lambda^{(m+mq+nq)/(2q)} \log \lambda (1 + o(1)) \quad \text{as } \lambda \rightarrow \infty.$$

(iii) *If $pm < qn$, we have*

$$N_0(\lambda) = c_4 \lambda^{n(1+q+p)/(2p)} (1 + o(1)) \quad \text{as } \lambda \rightarrow \infty.$$

Here c_1, c_2, c_4 are positive constants as in Proposition 1.1. For the precise values of the constants, see [3].

3. Proof of the main theorem.

In this section, we give the proof of Theorem 2.1. Let $p_0(t; z, z')$ and $p(t; z, z')$ be the distribution kernels of $\exp(-tH_0)$ and $\exp(-tH)$, respectively. Then, by the Feynman-Kac-Itô formula, we can write these heat kernels using probabilistic representations as follows.

$$\begin{aligned} p_0(t; z, z') &= (2\pi t)^{-d/2} e^{-|z-z'|^2/(2t)} E_{0,0}^{0,0} \\ &\quad \cdot \left[\exp \left(-t \int_0^1 V(z + s(z-z') + \sqrt{t}Z_s) ds \right) \right], \\ p(t; z, z') &= (2\pi t)^{-d/2} e^{-|z-z'|^2/(2t)} E_{0,0}^{0,0} \\ &\quad \cdot \left[\exp \left(iF^t(z, z') - t \int_0^1 V(z + s(z-z') + \sqrt{t}Z_s) ds \right) \right] \end{aligned}$$

where $F^t(z, z') = \sqrt{t} \int_0^1 A(z + s(z-z') + \sqrt{t}Z_s) \circ dZ_s$. Here $E_{0,0}^{0,0}$ is the expectation with respect to the $d (= n + m)$ -dimensional pinned Brownian motion $\{Z_s\}_{0 \leq s \leq 1} = \{X_s, Y_s\}_{0 \leq s \leq 1} = \{X_s^1, \dots, X_s^n, Y_s^1, \dots, Y_s^m\}_{0 \leq s \leq 1}$ such that $Z_0 = 0 = (0, 0)$ and $Z_1 = 0 = (0, 0)$ and $\circ dZ_s$ denotes the Stratonovich integral. For the theory of these probabilistic facts, see [13] and [5]. Throughout this paper, we denote $E_{0,0}^{0,0}$ simply by E . We note that under (V.1) and (A.1), $p_0(t; z, z')$ and $p(t; z, z')$ are continuous with respect to $t > 0$ and $z, z' \in \mathbb{R}^d$. Since we study the traces of $\exp(-tH_0)$ and

$\exp(-tH)$, it suffices to consider the heat kernels on the diagonal set only. I.e.,

$$(3.1) \quad p_0(t; z, z) = (2\pi t)^{-d/2} E \left[\exp \left(-t \int_0^1 V(z + \sqrt{t}Z_s) ds \right) \right],$$

$$(3.2) \quad p(t; z, z) = (2\pi t)^{-d/2} E \left[\exp \left(iF^t(z) - t \int_0^1 V(z + \sqrt{t}Z_s) ds \right) \right]$$

where $F^t(z) = \sqrt{t} \int_0^1 A(z + \sqrt{t}Z_s) \circ dZ_s$

Let $\{Z_s\}_{0 \leq s \leq 1} = \{X_s, Y_s\}_{0 \leq s \leq 1}$ be defined on a probability space (Ω, \mathcal{F}, P) and define

$$(3.3) \quad \xi = \sup_{0 \leq s \leq 1} |X_s|.$$

Then it follows from Lévy's work that:

Lemma 3.1. *For every $R > 0$,*

$$P(\xi \geq R) \leq 2ne^{-2R^2/n}.$$

For the proof, see Simon [12], Itô and McKean [4] and [5, Lemma 1].

From now, we denote various constants independent of $t > 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ by the same notations C, C_j ($j = 1, 2, \dots$) etc..

Lemma 3.2. *Under the assumptions (A.1) and (A.2), there exists a constant $C > 0$ such that*

$$(3.4) \quad E[|F^t(x, y)|^4] \leq Ct^4(1 + |x|^2)^{4a}|y|^{8b}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $t \in (0, 1)$.

Proof. Since the proof is essentially the same as [5, Lemma 2], we give only an outline of the proof.

Let $\{w_s\}_{0 \leq s \leq 1}$ be the standard d -dimensional Brownian motion defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Then, the pinned Brownian motion $\{Z_s\}_{0 \leq s \leq 1}$ such that $Z_0 = Z_1 = 0$ is the solution of the stochastic differential equation:

$$dZ_s^i = dw_s^i - \frac{Z_s^i}{1-s} ds \quad (0 < s < 1), \quad Z_0^i = 0 \quad (i = 1, 2, \dots, d).$$

Thus, by the Itô formula, we have

$$\begin{aligned}
 F^t(z) &= t \sum_{i,j=1}^d \int_0^1 dw_s^i \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) dw_u^j \\
 &\quad - t \sum_{i,j=1}^d \int_0^1 dw_s^i \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) \frac{Z_u^j}{1-u} du \\
 &\quad + \frac{1}{2} t^{3/2} \sum_{i,j=1}^d \int_0^1 dw_s^i \int_0^s \partial_j^2 a_i(z + \sqrt{t}Z_u) du \\
 &\quad - t \sum_{i,j=1}^d \int_0^1 \frac{Z_s^i}{1-s} ds \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) dw_u^j \\
 &\quad + t \sum_{i,j=1}^d \int_0^1 \frac{Z_s^i}{1-s} ds \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) \frac{Z_u^j}{1-u} du \\
 &\quad - \frac{1}{2} t^{3/2} \sum_{i,j=1}^d \int_0^1 \frac{Z_s^i}{1-s} ds \int_0^s \partial_j^2 a_i(z + \sqrt{t}Z_u) du \\
 &\quad + \frac{1}{2} t \sum_{i=1}^d \int_0^1 \partial_i a_i(z + \sqrt{t}Z_u) ds.
 \end{aligned}$$

Since Z_s^i is the Gaussian random variable of mean 0 and variance $s(1-s)$, we have

$$E[|Z_s^i|^{2m}] = (2m-1)!!(s(1-s))^m \quad \text{for } m = 1, 2, \dots$$

Using this equality, the Hölder inequality and (A.2), we can prove the lemma. \square

Lemma 3.3. *For every $x \in \mathbb{R}^n$, put $A_{|x|} = -\frac{1}{2}\Delta_y + c|x|^{2p}|y|^{2q}$ on $L^2(\mathbb{R}^m)$ where c is a positive constant and let $e^{-tA_{|x|}}(y, y')$ be the kernel of $e^{-tA_{|x|}}$ and $J(t; |x|, y) = e^{-tA_{|x|}}(y, y)$. Then, we have following:*

(i) *There exist constants C_j ($j = 1, 2, 3$) such that*

$$(3.5) \quad |J(t; 1, y)| \leq C_1 t^{-m/2} (e^{-C_2 t|y|^{2q}} + e^{-C_3 |y|^2/t}).$$

(ii) *For every $\lambda > 0$,*

$$(3.6)$$

$$J(t; |x|, y) = \lambda^{-m} J(\lambda^{-2}t; |\lambda^{(1+q)/p}x|, \lambda^{-1}y) \quad \text{for all } t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Proof. For (i), see Matsumoto [6, Lemma 3.1]. For (ii), it follows from the Feynman-Kac formula that

$$J(t; |x|, y) = (2\pi t)^{-m/2} E[e^{-t \int_0^1 c|x|^{2p}|y + \sqrt{t}Y_s|^{2q} ds}].$$

Thus we have

$$\begin{aligned} J(t; |x|, \lambda y) &= \lambda^{-m} (2\pi\lambda^{-2}t)^{-m/2} E \left[e^{-\lambda^{-2}t} \int_0^1 c |\lambda^{(1+q)/p} x|^{2p} |y + \sqrt{\lambda^{-2}t} Y_s|^{2q} ds \right] \\ &= \lambda^{-m} J(\lambda^{-2}t; |\lambda^{(1+q)/p} x|, y). \end{aligned}$$

□

Now, we give the proof of Theorem 2.1.

Proof of Theorem 2.1.

Since $p(t; x, y, x, y)$ is a real valued function, using (3.1) and (3.2), we can write

$$\begin{aligned} I(t; x, y) &\equiv |p(t; x, y, x, y) - p_0(t; x, y, x, y)| \\ &= (2\pi t)^{-d/2} \left| E \left[(\cos F^t(x, y) - 1) e^{-t \int_0^1 V(x + \sqrt{t} X_s, y + \sqrt{t} Y_s) ds} \right] \right|. \end{aligned}$$

Since $0 \leq 1 - \cos \theta \leq \theta^2/2$ for $\theta \in \mathbb{R}$, $I(t; x, y)$ is estimated by

$$\frac{1}{2} (2\pi t)^{-d/2} E \left[F^t(x, y)^2 e^{-t \int_0^1 V(x + \sqrt{t} X_s, y + \sqrt{t} Y_s) ds} \right].$$

By the Schwartz inequality, Lemma 3.2 and hypothesis (A.2) and (V.2), we have

$$\begin{aligned} I(t; x, y) &\leq C_1 t^{-d/2} E \left[F^t(x, y)^4 \right]^{1/2} E \left[e^{-2t \int_0^1 V(x + \sqrt{t} X_s, y + \sqrt{t} Y_s) ds} \right]^{1/2} \\ &\leq C_2 t^{2-d/2} (1 + |x|^2)^{2a} |y|^{4b} E \left[e^{-C_3 t \int_0^1 (1 + |x + \sqrt{t} X_s|^2)^p |y + \sqrt{t} Y_s|^{2q} ds} \right]^{1/2}. \end{aligned}$$

Define the function ξ by (3.3) and let χ be the characteristic function of the sets $\{\xi \geq |x|/2\sqrt{t}\}$.

Now we decompose

$$(3.7) \quad K(t; x, y) \equiv E \left[e^{-C_3 t \int_0^1 (1 + |x + \sqrt{t} X_s|^2)^p |y + \sqrt{t} Y_s|^{2q} ds} \right]$$

into the form $K(t; x, y) = \sum_{j=1}^2 K_j(t; x, y)$ where

$$\begin{aligned} K_1(t; x, y) &= E \left[e^{-C_3 t \int_0^1 (1 + |x + \sqrt{t} X_s|^2)^p |y + \sqrt{t} Y_s|^{2q} ds} \chi \right] \\ K_2(t; x, y) &= E \left[e^{-C_3 t \int_0^1 (1 + |x + \sqrt{t} X_s|^2)^p |y + \sqrt{t} Y_s|^{2q} ds} (1 - \chi) \right]. \end{aligned}$$

Then we note that $K(t; x, y)^{1/2} \leq \sum_{j=1}^2 K_j(t; x, y)^{1/2}$. At first, we consider $K_1(t; x, y)$. Since $(1 + |x + \sqrt{t} X_s|^2)^p \geq 1$, we see that

$$\begin{aligned} K_1(t; x, y) &\leq E \left[e^{-C_3 t \int_0^1 |y + \sqrt{t} Y_s|^{2q} ds} \right] E[\chi] \\ &= E \left[e^{-C_3 t \int_0^1 |y + \sqrt{t} Y_s|^{2q} ds} \right] P(\{\xi \geq |x|/2\sqrt{t}\}). \end{aligned}$$

By Lemma 3.1 and Lemma 3.3 (i), we have

$$K_1(t; x, y) \leq C_4 (e^{-C_5 t |y|^{2q}} + e^{-C_6 |y|^2/t}) e^{-C_7 |x|^2/t}.$$

Therefore,

$$\begin{aligned} K_1(t) &\equiv t^{2-d/2} \iint (1 + |x|^2)^{2a} |y|^{4b} K_1(t; x, y)^{1/2} dx dy \\ &\leq C_4 t^{2-d/2} \int (1 + |x|^2)^{2a} e^{-C_7|x|^2/t} dx \int |y|^{4b} (e^{-C_5 t|y|^{2q}} + e^{-C_6|y|^2/t}) dy. \end{aligned}$$

Since a simple computation leads to

$$\int (1 + |x|^2)^{2a} e^{-C_1|x|^2/t} dx = O(t^{n/2}) \quad \text{as } t \rightarrow 0$$

and

$$\int |y|^{4b} (e^{-C_5 t|y|^{2q}} + e^{-C_6|y|^2/t}) dy = O(t^{-(4b+m)/(2q)}),$$

we have an estimate of $K_1(t)$:

$$(3.8) \quad K_1(t) \leq C_8 t^{2-m/2-(4b+m)/(2q)} = O(t^{-(m+mq+nq)/(2q)+2(1-b/q)+n/2}).$$

Secondly, we consider $K_2(t; x, y)$. Since we have $|x + \sqrt{t}X_s| \geq |x|/2$ on $\text{supp}(1 - \chi)$,

$$(3.9) \quad K_2(t; x, y) \leq E[e^{-C_1 t \int_0^1 (1+|x|^2)^p |y + \sqrt{t}Y_s|^{2q} ds}].$$

Now, we decompose

$$\begin{aligned} K_2(t) &\equiv t^{2-d/2} \iint (1 + |x|^2)^{2a} |y|^{4b} K_2(t; x, y)^{1/2} dx dy \\ &= K_{2,1}(t) + K_{2,2}(t) \end{aligned}$$

where

$$\begin{aligned} K_{2,1}(t) &= t^{2-d/2} \iint_{|x| \leq 1} (1 + |x|^2)^{2a} |y|^{4b} K_2(t; x, y)^{1/2} dx dy, \\ K_{2,2}(t) &= t^{2-d/2} \iint_{|x| \geq 1} (1 + |x|^2)^{2a} |y|^{4b} K_2(t; x, y)^{1/2} dx dy. \end{aligned}$$

For the estimate of $K_{2,1}(t)$, we use $(1 + |x|^2)^p \geq 1$ in (3.9). Thus we have

$$K_{2,1}(t) \leq t^{2-d/2} \iint_{|x| \leq 1} (1 + |x|^2)^{2a} |y|^{4b} E[e^{-C_2 t \int_0^1 |y + \sqrt{t}Y_s|^{2q} ds}]^{1/2} dx dy.$$

Here, by Lemma 3.3 (i),

$$\begin{aligned} E[e^{-C_2 t \int_0^1 |y + \sqrt{t}Y_s|^{2q} ds}]^{1/2} &= \{(2\pi t)^{m/2} e^{-t(-\frac{1}{2}\Delta_y + C_4|y|^{2q})}(y, y)\}^{1/2} \\ &\leq C_3 t^{m/4} \{e^{-C_4 t|y|^{2q}} + e^{-C_5|y|^2/t}\}. \end{aligned}$$

Therefore, we have

$$(3.10) \quad K_{2,1}(t) \leq C_6 t^{-(m+nq+mq)/(2q)+2(1-b/q)+m/4}.$$

For the estimate of $K_{2,2}(t)$, we use $(1 + |x|^2)^p \geq |x|^{2p}$ in (3.9). Since

$$(3.11) \quad K_2(t; x, y) \leq E[e^{-C_2 t \int_0^1 |x|^{2p} |y + \sqrt{t} Y_s|^{2q} ds}],$$

we have

$$\begin{aligned} K_{2,2}(t) &\leq t^{2-d/2} \iint_{|x| \geq 1} (1 + |x|^2)^{2a} |y|^{4b} E[e^{-C_2 t \int_0^1 |x|^{2p} |y + \sqrt{t} Y_s|^{2q} ds}]^{1/2} dx dy \\ &= t^{2-d/2} \int_{|x| \geq 1} (1 + |x|^2)^{2a} dx \int |y|^{4b} (2\pi t)^{m/4} J(t; |x|, y)^{1/2} dy \end{aligned}$$

where $J(t; |x|, y)$ is as in Lemma 3.3 (ii) with $c = C_2$. If we define

$$F(t; x) = (2\pi t)^{m/4} \int |y|^{4b} J(t; |x|, y)^{1/2} dy,$$

it follows from Lemma 3.3 (ii) with $\lambda = t^{1/2}$ and the change of variable $t^{-1/2}y \rightarrow y$ that

$$F(t; x) = t^{2b+3m/4} F(1; t^{(1+q)/(2p)}x).$$

Thus we have

$$\begin{aligned} K_{2,2}(t) &= t^{2-d/2} \int_{|x| \geq 1} (1 + |x|^2)^{2a} F(t; x) dx \\ &\leq C_7 t^{2-d/2+2b+m/2} \int_{|x| \geq 1} |x|^{4a} F(1; t^{(1+q)/(2p)}x) dx. \end{aligned}$$

A change of variable $t^{(1+q)/(2p)}x \rightarrow x$ in the last integral leads to

$$K_{2,2}(t) \leq C_7 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \int_{|x| \geq t^{(1+q)/(2p)}} |x|^{4a} F(1; x) dx.$$

Here we need the following lemma:

Lemma 3.4. *Under the above notations, there exist constants C_j ($j = 1, 2, 3$) such that*

$$F(1; x) \leq \begin{cases} C_1 |x|^{-(4b+m)p/q}, & \text{for } |x| \leq 1 \\ C_2 e^{-C_3 |x|^{2p/(q+1)}}, & \text{for } |x| \geq 1. \end{cases}$$

Proof. Since $J(t; |x|, |x|^{-p/(1+q)}y) = |x|^{pm/(1+q)} J(|x|^{2p/(1+q)}t; 1, y)$, we have

$$\begin{aligned} F(1; x) &= (2\pi)^{m/4} \int |y|^{4b} J(1; |x|, y)^{1/2} dy \\ &= (2\pi)^{m/4} \int |y|^{4b} |x|^{pm/2(q+1)} J(|x|^{2p/(q+1)}; 1, |x|^{p/(q+1)}y)^{1/2} dy. \end{aligned}$$

The change of variable $|x|^{p/(q+1)}y \rightarrow y$ leads to

(3.12)

$$F(1, x) = (2\pi)^{m/4} |x|^{-(4b+m)p/(q+1)+pm/(2(q+1))} \int |y|^{4b} J(|x|^{2p/(q+1)}; 1, y)^{1/2} dy.$$

Since by Lemma 3.3 (i),

$$\begin{aligned} & J(|x|^{2p/(q+1)}; 1, y)^{1/2} \\ & \leq C_3 |x|^{-pm/2(q+1)} (e^{-C_1 |x|^{2p/(q+1)} |y|^{2q}} + e^{-C_2 |y|^2 |x|^{-2p/(q+1)}}), \end{aligned}$$

we have

$$F(1, x) \leq C_3 |x|^{-(4b+m)p/(q+1)} \int |y|^{4b} (e^{-C_1 |x|^{2p/(q+1)} |y|^{2q}} + e^{-C_2 |y|^2 |x|^{-2p/(q+1)}}) dy.$$

By a change of variable $|x|^{p/q(q+1)}y \rightarrow y$ in the first term and $|x|^{-p/(q+1)}y \rightarrow y$ in the second term in the last integral, we have

$$\begin{aligned} F(1, x) & \leq C_4 |x|^{-(4b+m)p/q} \int |y|^{4b} (e^{-C_1 |y|^{2q}} + e^{-C_2 |y|^2}) dy \\ & \leq C_5 (|x|^{-(4b+m)p/q} + 1). \end{aligned}$$

When $|x| \geq 1$, we write

$$F(1, x) = (2\pi)^{m/4} |x|^{-(8b+m)p/(2(q+1))} \int |y|^{4b} J(|x|^{2p/(q+1)}; 1, y)^{1/2} dy.$$

Since $-\frac{1}{2}\Delta_y + 2c|y|^{2q}$ is positive definite, there exists $c_1 > 0$ such that $-\frac{1}{2}\Delta_y + 2c|y|^{2q} \geq 2c_1$. Since $|x| \geq 1$, using Lemma 3.3 (i),

$$J(|x|^{2p/(q+1)}; 1, y) \leq C_6 e^{-c_1 |x|^{2p/(q+1)}} (e^{-C_3 |y|^{2q}} + e^{-C_4 |y|^2}).$$

Therefore, for $|x| \geq 1$, we have

$$\int |y|^{4b} J(|x|^{2p/(q+1)}; 1, y)^{1/2} dy \leq C_7 e^{-c_1 |x|^{2p/(q+1)}}.$$

This completes the proof. \square

End of the proof of Theorem 2.1.

By Lemma 3.4, we have

$$\begin{aligned} K_{2,2}(t) & \leq C_8 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \int_{|x| \geq t^{(1+q)/(2p)}} |x|^{4a} F(1, x) dx \\ & \leq C_8 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \\ & \quad \cdot \left\{ \int_{t^{(1+q)/(2p)} \leq |x| \leq 1} |x|^{4a-(4b+m)p/q} dx + \int_{|x| \geq 1} |x|^{4a} e^{-c_1 |x|^{2p/(q+1)}} dx \right\}. \end{aligned}$$

Therefore, we see that

$$(3.13) \quad K_{2,2}(t) \leq \begin{cases} C_9 t^{2-d/2+2b+m/2} \{t^{-(1+q)(4a+n)/(2p)} + t^{-(1+q)(4b+m)/(2q)}\}, \\ \quad \text{if } (4a+n)q - (4b+m)p \neq 0, \\ C_9 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \log t^{-1}, \\ \quad \text{if } (4a+n)q - (4b+m)p = 0. \end{cases}$$

Taking (3.8), (3.10) and (3.13) into consideration, we see that the proof follows.

4. An Example.

In this section, we treat the case where the electric potential is of the form (1.3) and the magnetic potential $A(x, y)$ satisfies (A.1) and (A.2).

Let $V(x, y) = (1 + |x|^2)^p |y|^{2q}$ ($p, q > 0$ integers).

Corollary 4.1. *There exists $\delta_1 > 0$ such that:*

- (i) *If $pm > qn$, $\text{Tr}[e^{-tH}] = d_1 t^{-(m+mq+nq)/(2q)} (1 + O(t^{\delta_1}))$ as $t \rightarrow 0$.*
- (ii) *If $pm = qn$,
 $\text{Tr}[e^{-tH}] = d_2 t^{-(m+mq+nq)/(2q)} \log t^{-1} + d_3 t^{-(m+mq+nq)/(2q)} (1 + O(t^{\delta_1}))$
as $t \rightarrow 0$.*
- (iii) *If $pm < qn$, $\text{Tr}[e^{-tH}] = d_4 t^{-n(1+p+q)/(2p)} (1 + O(t^{\delta_1}))$ as $t \rightarrow 0$*

where d_i ($i = 1, 2, 3, 4$) are given as below.

For the proof, it suffices to note

$$e^{-tH} = e^{-tH_0} + (e^{-tH} - e^{-tH_0})$$

and apply Theorem 2.1 and Proposition 1.2.

In order to get the values of the constants d_i , let $A = -\frac{1}{2}\Delta + |y|^{2q}$ on $L^2(\mathbb{R}^m)$. According to [3], the complex powers A^{-s} ($s \in \mathbb{C}$) of A are defined for large $\text{Re } s > 0$, the trace $\text{Tr}[A^{-s}]$ has a meromorphic extension $Z_A(s)$ in \mathbb{C} whose singularities are all simple poles $\{s_j = (m(1+q) - j)/(2q)\}_{j=0,1,\dots}$. Therefore, we can write

$$Z_A(s) = \frac{\text{Res}(s_0)}{s - s_0} + C_{(m,q)} + O(|s - s_0|) \quad \text{as } s \rightarrow s_0.$$

Here $\text{Res}(s_0)$ denotes the residue of $Z_A(s)$ at s_0 and $C_{(m,q)}$ is a constant. Then we have

$$\begin{aligned}
 d_1 &= 2^{1-d/2} \frac{1}{q(1+q)} \frac{\Gamma(m/(2q))\Gamma((pm-qn)/(2q))}{\Gamma(m/2)\Gamma(pm/(2q))}, \\
 d_2 &= 2^{1-d/2} \frac{1}{pq} \frac{\Gamma(m/(2q))}{\Gamma(m/2)\Gamma(n/2)}, \\
 d_3 &= 2^{1-d/2} \frac{1}{\Gamma(n/2)} \left[\frac{\Gamma(m/(2q))\psi((m+mq+nq)/(2q))}{pq\Gamma(m/2)} \right. \\
 &\quad + C_{(m,q)} \frac{(1+q)\Gamma((m+mq)/(2q))}{p} - \frac{\gamma\Gamma(m/(2q))}{q(1+q)\Gamma(m/2)} \\
 &\quad \left. + \frac{\Gamma(m/(2q))}{pq\Gamma(m/2)} \left\{ \psi\left(\frac{m+mq}{2q}\right) - \frac{p}{1+q}\psi\left(\frac{n}{2}\right) - \psi\left(\frac{m+mq+nq}{2q}\right) \right\} \right] \\
 d_4 &= 2^{-n/2} \frac{(1+q)\Gamma((m+mq)/(2q))\Gamma(n(1+p+q)/(2p))}{p\Gamma(n/2)\Gamma((m+mq+nq)/(2q))} Z_A\left(\frac{n(1+q)}{2p}\right)
 \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and γ is the Euler constant.

In the particular case where $q = m = 1$, we have $s_0 = 1$, $\text{Res}(s_0) = 2^{-1/2}$ and $C_{(1,1)} = 2^{-1/2}(\gamma + \log 2)$. Moreover, for $\text{Res} > 1$, we see that $Z_A(s) = (2^{s/2} - 2^{-s/2})R(s)$ where $R(s)$ is the Riemann zeta function. Thus we have

$$\begin{aligned}
 d_1 &= 2^{-(1+n)/2} \frac{\Gamma((p-n)/2)}{\Gamma(p/2)}, \\
 d_2 &= 2^{(1-n)/2} \frac{1}{p\Gamma(n/2)}, \\
 d_3 &= 2^{(1-n)/2} \frac{1}{\Gamma(n/2)} \left[\frac{2-p+2^{3/2}}{2p}\gamma - \frac{p}{2}\psi\left(\frac{n}{2}\right) + \frac{2^{1/2}}{p}\log 2 \right], \\
 d_4 &= 2^{1-n/2} \frac{\Gamma(n(2+p)/(2p))}{p\Gamma(n/2)\Gamma(1+n/2)} (2^{n/(2p)} - 2^{-n/(2p)}) R\left(\frac{n}{p}\right).
 \end{aligned}$$

For a more precise argument, see [3].

References

- [1] J. Aramaki, *On an extension of the Ikehara Tauberian theorem*, Pacific J. Math., **133**(1) (1988), 13-30.
- [2] ———, *Complex powers of vector valued operators and their applications to asymptotic behavior of eigenvalues*, J. Funct. Anal., **87**(2) (1989), 294-320.
- [3] ———, *An extension of the Ikehara Tauberian theorem and its application*, Acta Math. Hungarica, **71**(4) (1996), 297-326.
- [4] K. Itô and H.P. McKeane, Jr., *Diffusion Processes and Their Sample Paths*, Springer, Berlin, Heidelberg, New York, 1974.

- [5] H. Matsumoto, *The short time asymptotics of the trace of the heat kernels for the magnetic Schrödinger operators*, J. Math. Soc. Japan, **42**(4) (1990), 677-689.
- [6] ———, *Classical and non-classical eigenvalue asymptotics for the magnetic Schrödinger operators*, J. Funct. Anal., **95** (1991), 460-482.
- [7] K. Odencrantz, *The effect of a magnetic field on asymptotics of the trace of the heat kernel*, J. Funct. Anal., **79** (1988), 398-422.
- [8] M. Reed and B. Simon, *Method of Modern Mathematical Physics Vol. IV, Analysis of Operators*, Academic Press, New York, 1977.
- [9] D. Robert, *Comportement asymptotique des valeurs propres d'opérateurs du type Schrödinger a potentiel dégénéré*, J. Math. Pures et Appl., **61** (1982), 275-300.
- [10] M. Schechter, *Spectra of Partial Differential Operators*, North-Holland, Amsterdam, 1971.
- [11] B. Simon, *Non-classical eigenvalue asymptotics*, J. Funct. Anal., **53** (1983), 84-98.
- [12] ———, *Semiclassical analysis of low lying eigenvalues II, Tunneling*, Ann. of Math., **120** (1984), 89-118.
- [13] ———, *Functional integration and quantum physics*, Academic Press, New York, San Francisco, London, 1979.

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