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In this work we prove for all simple real groups of classical type, except for $SO^*(n)$ and $Sp(p,q)$, that if $\nu \in \mathfrak{a}^*$ does not belong to C_ρ, the convex hull of the Weyl group orbit of ρ, then the signature of the Hermitian form attached to the irreducible subquotient of the principal spherical series corresponding to ν, with integral infinitesimal character, is indefinite on K-types in $\mathfrak{p}$.

0. Introduction.

Let G be a real semisimple Lie group. Let $\mathfrak{g}_0$ be the Lie algebra of G. We will denote the complexification of any vector space V_0 by V and its dual by V^*. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a fixed Cartan decomposition corresponding to the Cartan involution θ of $\mathfrak{g}_0$. Let K be the corresponding maximal compact subgroup of G. Let us fix $T \subset K$ a maximal torus. Define $S \subset \mathfrak{t}^*$ as the set of weights of T that are sums of distinct non-compact roots.

We say that (μ, V_μ) an irreducible representation of K is *unitarily small* if the weights of μ lie in the convex hull of S. Let us state the following:

Salamanca-Vogan Conjecture. *Suppose X is an irreducible Hermitian $(\mathfrak{g}, K)$-module containing a K type in S. Then:*

(1) *If X is unitary, then the real part of the infinitesimal character belongs to the convex hull of the Weyl group orbit $W \cdot \rho$, where ρ is the semi-sum of positive roots.*
(2) *If X is not unitarizable, then the invariant hermitian form must be indefinite on unitarily small K-types.*

Using this conjecture the classification of unitary representation can be reduced to the unitarily small case.

If X is a spherical representation, the statement (1) is true by [**HJ**]. In order to move towards (2) we may assume that real part of the infinitesimal character does not belongs to $W \cdot \rho$, and the hope is that if this holds, the invariant hermitian form is negative definite in the K-types in $\mathfrak{p}$. In this work we prove for all simple real groups of classical type, except for $SO^*(n)$ and $Sp(p,q)$, that if $\nu \in \mathfrak{a}^*$ does not belong to C_ρ, then the signature of the Hermitian form attached to the irreducible subquotient of the principal spherical series corresponding to ν, with integral infinitesimal character, is indefinite on K-types in $\mathfrak{p}$.

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1. Notation and General Results.

Recall that the complexification of a Lie algebra $\mathfrak{g}_0$ will be denoted by $\mathfrak{g}$. Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$ be a maximal Abelian subspace. Let $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0) = \Sigma$ be the corresponding set of restricted roots and let $\overline{\Sigma}(\mathfrak{g}_0, \mathfrak{a}_0) = \overline{\Sigma}$ be the reduced restricted roots. If $\alpha \in \mathfrak{a}^*$, then we will denote the corresponding weight space in $\mathfrak{g}_0$ by $\mathfrak{g}_0^\alpha$ and let m_α the dimension of this subspace. For a choice of positive roots $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0) = \Sigma^+$ we have $\Pi(\mathfrak{g}_0, \mathfrak{a}_0) = \Pi$ the set of simple roots. Let $W \simeq N_K(\mathfrak{a}_0)\backslash Z_K(\mathfrak{a}_0)$ be the corresponding Weyl group. If $w \in W$, then we define $\Sigma^+(w) = \{\alpha \in \Sigma^+ \ : \ w\alpha \notin \Sigma^+\}$. We also denote the longest element in the Weyl group by w_0. If $\Omega \subseteq \Sigma^+$ then we will say that $\nu \in \mathfrak{a}^*$ is it positive (resp. *negative*) with respect to Ω, if $\mathrm{Re}\langle \alpha, \nu \rangle$ is positive (resp. negative) or zero for all $\alpha \in \Omega$.

If X is an admissible representation of G, we will also denote the corresponding $(\mathfrak{g}, K)-$module by X.

Recall that a $(\mathfrak{g}, K)-$module (π, H_π) is called *spherical* if the trivial K-type occurs in the restriction of (π, H_π) to K, i.e., H_π contains a non-trivial K-fixed vector. Then, we have the following:

Theorem 1.1. *The irreducible spherical $(\mathfrak{g}, K)-$modules (π, H_π) are in one-to-one correspondence with the W-orbits in $\mathfrak{a}^*$.*

A proof of this Theorem appears in [**K**, **BJ2**].

This correspondence can be realized as follows: Set $\mathfrak{n}_0 = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_0^\alpha$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$. Let P=MAN the corresponding minimal parabolic subgroup of G. If $\nu \in \mathfrak{a}^*$, define

$$I_P^G(\nu) = \mathrm{Ind}_{MAN}^G(1 \otimes \nu \otimes 1)$$

where the right-hand-side is the space

$$\{f \in C^\infty(G, \mathbb{C}) \ : \ \forall man \in MAN \ \text{ s.t.} \\ f(gman) = a^{-\nu-\rho} f(g), f \ \text{ is } \ K - \text{finite}\}$$

and the $\mathfrak{g}$- action on this induced module is the left regular action, i.e.,:

$$(X \cdot f)(g) = \frac{d}{dt} f(\exp(-t \cdot X)g)_{|t=0}, \quad g \in G, \ X \in \mathfrak{g}, \ f \in I_p^G(\nu)$$

and

$$(k \cdot f)(g) = f(k^{-1}g), \quad k \in K \text{ and } f \in I_P^G(\nu)$$

It is easy to see that $I_P^G(\nu)$ is a Harish-Chandra module of finite length. Observe that by the Iwasawa decomposition $G = KAN$ of G we have

$$I_P^G(\nu)_{|K} = K - \text{finite part of } \mathrm{Ind}_M^K(1)$$

and hence by Frobenius reciprocity

$$\dim I_P^G(\nu)^K = \dim \mathrm{Ind}_M^K(1)^K = \dim \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = 1. \tag{1.2}$$

So $I_P^G(\nu)$ is a spherical $(\mathfrak{g}, K)$-module, and by (1.1) there exists a unique irreducible composition factor $J^G(\nu) = J_P^G(\nu)$ containing the trivial K-type in $I_P^G(\nu)$. It is well-known that $J^G(\nu) \simeq J^G(\nu')$ if and only if there exists $w \in W$ such that $\nu = w \cdot \nu'$, in particular $J^G(\nu)$ does not depend on the choice of the minimal parabolic subgroup containing MA. The $(\mathfrak{g}, K)$-modules $I_P^G(\nu)$ are called *spherical principal series representations.*

We denote the set of $\nu \in \mathfrak{a}^*$ such that $J^G(\nu)$ has integral infinitesimal character by $\mathfrak{a}^*_{\text{int}}$.

Let X be a $(\mathfrak{g}, K)$-module. Then we say that X admits an invariant Hermitian form if there exists a non-zero map $\omega = \omega_G : X \times X \longrightarrow \mathbb{C}$ such that:

(1) ω is linear in the first factor and conjugate linear in the second factor.
(2) $\omega(x, y) = \overline{\omega(y, x)}, \quad x, y \in X.$
(3) $\omega(k \cdot x, k \cdot y) = \omega(x, y), \quad x, y \in X, \quad k \in K.$
(4) $\omega(H \cdot x, y) = \omega(x, -\overline{H} \cdot y), \quad x, y \in X, \quad H \in \mathfrak{g}$, where $\overline{H}$ stand for complex conjugation of $\mathfrak{g}$ respect to $\mathfrak{g}_0$.

If μ, μ' are two different K-types, then, (3) implies that $\omega(X^\mu, X^{\mu'}) = 0$. Hence ω is completely described by its restriction to the K-isotopic spaces on X.

Let $\mu \in \hat{K}$. Then $X^\mu \simeq \mathrm{Hom}_K(V_\mu, X) \otimes V_\mu$ and ω induces a Hermitian form, ω^μ, on the first factor.

Definition 1.3. Let $(p(\mu), q(\mu)) := (p_X(\mu), q_X(\mu)) := (p_X^G(\mu), q_X^G(\mu))$ be the *signature* of ω^μ, i.e., $p(\mu)$ (respective $q(\mu)$) is the sum of the strictly positive (respective negative) eigenspaces of ω^μ.

Let us see when there exists this Hermitian form in $J(\nu)$. This is the one being used in this article. Choose a minimal parabolic subgroup $P = MAN$ in G. Consider $w \in W$ and ν positive with respect to $\Sigma^+(w)$. Then there exist an intertwining operator $\Psi(w) : \mathrm{I}_P^G(\nu) \longrightarrow \mathrm{I}_P^G(w\nu)$ so that $\Psi(w)$ is an isomorphism on the trivial K-type. If $w = w_0$ then the image is isomorphic to $J(\nu)$.

Recall that there is a natural non-zero Hermitian paring

$$\mathrm{I}_P^G(\nu) \times \mathrm{I}_P^G(-\overline{\nu}) \xrightarrow{\langle\,,\,\rangle} \mathbb{C}.$$

Here $\mathrm{I}^G_{MAN}(\nu)$ is naturally identified with the K-finite part of $L^2(K/M)$ and with this identification, $\langle\ ,\ \rangle$ is the inner product on $L^2(K/M)$. Now suppose there exists $w \in W$ such that $-\overline{\nu} = w\nu$ and ν is positive with respect to MAN. Then we get the Hermitian pairing

$$\begin{aligned} \mathrm{I}^G_P(\nu) \times \mathrm{I}^G_P(\nu) &\xrightarrow{\omega_\nu} \mathbb{C} \\ (v_1\ ,\ v_2) \qquad &\longrightarrow \langle v_1, \Psi(w)v_2\rangle. \end{aligned}$$

Since $\Psi(w\nu, w_0w^{-1}) : \mathrm{I}^G_P(w\nu) \longrightarrow \mathrm{I}^G_P(w_0\nu)$ is an isomorphism, we get that $\mathrm{I}^G_{MAN}(\nu)/\mathrm{Rad}\omega \simeq J^G(\nu)$. Hence ω induces a Hermitian form on $J(\nu)$. The same argument shows that $J(\nu)$ admits an invariant Hermitian form if and only if $-\overline{\nu}$ is W-conjugate to ν.

Now, we will define the set of K-types where we will work. So, let's define as the $\mathfrak{p}_0$-*representation of* K the homomorphism

$$\begin{aligned} K &\longrightarrow GL(\mathfrak{p}_0) \\ k &\longmapsto Ad(k)_{|\mathfrak{p}_0}, \end{aligned}$$

and the $\mathfrak{p}$-*representation of* K the complexification of $\mathfrak{p}_0$-representation.

Recall that if $\mathfrak{g}_0$ is simple, then $\mathfrak{p}$ is either irreducible or it is a direct sum of two inequivalent irreducible representations. Consider the following set of K-types:

$$\mathfrak{p} = \{\mu \in \hat{K}\ :\ \mathrm{Hom}_K(\mu, \mathfrak{p}) \neq 0\}. \tag{1.4}$$

Through this work, we will consider the signature over this set of K-types.

Finally we will define C_ρ as the convex hull of points $w\rho$ with $w \in W$. This set is also characterized in terms of positive roots by the following:

Proposition 1.5. *The set C_ρ coincides with the set of all weights of the form*

$$r = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha \qquad \textit{where } -1/2 \leq c_\alpha \leq 1/2,$$

and each α is counted with multiplicity.

For a proof of this result we refer to [**SV**].

2. Problem.

Take $\nu \in \mathfrak{a}^*_{int}$ such that $J(\nu)$ has integral infinitesimal character and admits an invariant Hermitian form. Then, we will prove the following

Theorem 2.1. *If $\nu \notin C_\rho$ then $q_\mu(\nu) > 0$, where μ is a K-type in $\mathfrak{p}$ and q as in Definition* 1.3.

The main problem here is that this Hermitian form has no known general expression. However, Bang-Jensen [**BJ**], proves some useful Theorems, that give conditions for an invariant Hermitian form on an irreducible spherical representation, with integral infinitesimal character, to be positive definite on the K-types $\mu \in \mathfrak{p}$, in terms of Langlands data, for the simple groups of classical type except $SO^*(n)$ and $Sp(p,q)$.

These Theorems will be the main tool we will use in order to prove Theorem 2.1. This will be done case by case. Bang-Jensen's results are used to get an explicit characterization of $\nu \in \mathfrak{a}^*$, that is crucial for the proof of Theorem 2.1.

3. Case $SL(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Let μ be a K-type in $\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$. We identify $\mathfrak{a}^*$ with $\mathbb{C}^n$ such that $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0) = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$. Then $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{(e_i - e_j) \mid 1 \leq i < j \leq n\}$. Put $r = \dim_{\mathbb{R}} \mathbb{F}$. Then, $\dim \mathfrak{g}_0^\alpha = r$. Then $\mathfrak{a}^*_{\text{int}} = \{\nu \in \mathbb{C}^n \mid \nu_i \equiv 0 \mod r\mathbb{Z},\ i = 1, \ldots, n\}$. For $\nu \in \mathfrak{a}^*_{\text{int}}$ and $i \in \mathbb{Z}$ we define

$$R(i) := R_\nu(i) := \#\{\nu_j \ : \ \nu_j = r \cdot i\}.$$

Now if $\nu \in \mathfrak{a}^*_{\text{int}}$, then $J^G(\nu)$ admits an invariant Hermitian form if and only if $R_\nu(i) = R_\nu(-i)$ for all i.

Then we have the following:

Theorem 3.1. *Assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Suppose $\nu \in \mathfrak{a}^*_{\text{int}}$ and $J^G(\nu)$ admits an invariant hermitian form, ω. Then ω is positive definite on $J^G(\nu)^\mu$ if and only if $R_\nu(i+1) \leq R_\nu(i)$ for all $i \geq 0$.*

Proof. See Theorem 5.2 in [**BJ**].

We will now prove the following proposition.

Proposition 3.2. *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If $R_\nu(i+1) \leq R_\nu(i)$ for all $i \geq 0$ then $\nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

Proof. We will give the proof only for $Sl(n, \mathbb{R})$. The case $Sl(n, \mathbb{C})$ follows immediately from the definition of $\mathfrak{a}^*_{\text{int}}$ and the fact that for this group, each root has multiplicity 2. Take $\nu \in \mathfrak{a}^*_{\text{int}}$ such that:

(1) $R_\nu(i+1) \leq R_\nu(i)$ for all $i \geq 0$,
(2) $R_\nu(i) = R_\nu(-i)$ for all i;

thus, we are assuming that ω is positive definite on $J^G(\nu)^\mu$. By definition, C_ρ is stable by Weyl group action, hence by (1) and (2) we can consider,

(3.3)
$$\nu = (k, \ldots, k, k-1, \cdots, k-1, \cdots\cdots, 1, \cdots, 1, 0, \cdots, 0, -1, \cdots, -1, \cdots$$
$$\cdots\cdots, -(k-1), \cdots, -(k-1), -k, \cdots, -k)$$

where $k \leq [\frac{n}{2}] - 1$.

In order to complete this proof, we will need the following:

Lemma 3.4. *Take ν as above and consider $R(0) \geq 1$ (otherwise, $\nu \equiv 0$ by (1)). Then*

$$2(k-i)-2 < n - \left(2\sum_{j=0}^{i} R(k-j)\right) + 1 \qquad \textit{where } i = 0, \ldots, k-1. \tag{3.5}$$

Proof. Since $n = 2\sum_{j=0}^{k-1} R(k-j) + R(0)$ we have

$$\begin{aligned} 2(k-i)-2 \leq 2\sum_{j=i+1}^{k-1} R(k-j) + R(0) &= n - 2\sum_{j=0}^{i} R(k-j) \\ &< n - 2\sum_{j=0}^{i} R(k-j) + 1. \end{aligned}$$

Then, (3.5) has been proved.

Proof of Proposition 3.2 (Continuation). Let define

$$\mathcal{T}(m) = \begin{cases} \sum_{t=0}^{m-1} R(k-t), & \text{if } m > 0 \\ 0, & \text{if } m = 0 \end{cases}. \tag{3.6}$$

Now, it is easy to check, that Lemma 3.4 allows us to rewrite ν as follows:

$$\begin{aligned} \nu = & \frac{1}{2}\sum_{m=0}^{k-1}\sum_{j=1}^{R(k-m)} \Bigg\{ (e_{\mathcal{T}(m)+j} - e_{n-\mathcal{T}(m)-j+1}) \\ & + \sum_{s=\mathcal{T}(m)+j}^{2(k-m)-2+\mathcal{T}(m)+j} [(e_{\mathcal{T}(m)+j} - e_{s+1}) + (e_{s+1} - e_{n-\mathcal{T}(m)-j+1})] \Bigg\}. \end{aligned}$$

And so, Proposition 3.2 is proved. □

Proof of Theorem 2.1 for $Sl(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$. This is immediate from Proposition 3.2, Theorem 3.1 and Proposition 1.5. □

4. Case $Sp(2n, \mathbb{F})$, $n > 2$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Now, assume that $G = Sp(2n, \mathbb{R})$ or $G = Sp(2n, \mathbb{C})$. In this case, $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ is of type C_n, and identifying $\mathfrak{a} \simeq \mathbb{C}^n$ we have that the restricted roots are $\{\pm e_i \pm e_j, \pm 2e_l\}$. Put $r = \dim_{\mathbb{R}} \mathbb{F}$.

Here, $J^G(\nu)$ has integral infinitesimal character if and only if $\nu_i \in r\mathbb{Z}$. Take $\nu \in \mathfrak{a}^*$, and replace it by a Weyl group conjugate, then we may assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0 \tag{4.1}$$

and with this assumption define

$$R(i) = R_\nu(i) = \#\{j \ : \ \nu_j = ri\}, \quad i \geq 0.$$

Then we can state the following:

Theorem 4.2. *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

(1) $R(i+1) \leq R(i) + 1$, *for* $i \geq 1$;
(2) *If* $R(i+1) = R(i) + 1$, *for* $i \geq 1$, *then* $R(i)$ *is odd;*
(3) $R(1) \leq 2R(0) + 2$.

Proof. See Theorem 8.4 in [**BJ**].

Now, we can prove the following:

Proposition 4.3. *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(3) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

Proof. Let us assume that $G = Sp(2n, \mathbb{R})$ and so $r = 1$. It follows by Condition (2) in Theorem 4.2, that if $R(j) = 0$ for any $j \geq 1$ then $R(j+1) = 0$. This implies that if $R(1) \neq 0$ then

$$\nu = (k, \dots, k, k-1, \dots, k-1, \dots\dots, 1, \dots, 1, 0, \dots, 0)$$

or

$$\nu = (k, \dots, k, k-1, \dots, k-1, \dots\dots, 1)$$

where $k \leq n$. Now, since $n = \sum_{j=0}^{k} R(k-j)$, we have that

$$k - j \leq n - \sum_{i=0}^{j} R(k-i) \tag{4.4}$$

with $j = 0, \dots, k-2$. Let us define $\mathcal{T}(m)$ as in (3.6), and Inequality 4.4 allows us to write ν as follows

$$\begin{aligned}\nu = \frac{1}{2} \sum_{m=0}^{k-2} \sum_{j=1}^{R(k-m)} \Bigg\{ & 2e_{\mathcal{T}(m)+j} \\ & + \sum_{s=\mathcal{T}(m)+j+1}^{(k-m)-1+\mathcal{T}(m)+j} \left[(e_{\mathcal{T}(m)+j} - e_s) + (e_{\mathcal{T}(m)+j} + e_s) \right] \Bigg\} \\ + \frac{1}{2} \sum_{j=1}^{R(1)} 2e_{\mathcal{T}(k-1)+j}. &\end{aligned}$$

The case $G = Sp(2n, \mathbb{C})$ follows from this using that $r = 2$ and the multipilcities of the positive roots. Hence, in this way we have proved Proposition 4.3. □

Now, we can give the:

Proof of Theorem 2.1 for $Sp(2n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$. It follows from Proposition 4.3, Theorem 4.2 and Proposition 1.5. □

5. Case $SO(p, q)$, $p > q + 1$ and $SU(p, q)$, $p > q$.

Let us assume that $G = SO(p, q)$, $p > q + 1$ or $G = SU(p, q)$, $p > q$; and as usual identify $\mathfrak{a} \simeq \mathbb{C}^q$. Then the restricted roots become $\{\pm e_i \pm e_j, \pm e_l\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(p, q) \\ 2, & \text{if } G = SU(p, q). \end{cases}$$

Define, $\epsilon = 0, 1$ by $\epsilon \equiv p - q - 1 + r \bmod 2\mathbb{Z}$. In these cases, $J(\nu)$ has integral infinitesimal character if and only if

$$2\nu_i \equiv r\epsilon \bmod 2r\mathbb{Z}, \qquad i = 1, \ldots, q.$$

We can replace ν by a Weyl group conjugate, and assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0. \tag{5.1}$$

With this assumption we define

$$R(i) = R_\nu(i) = \#\left\{j \ : \ \nu_j = r\left(i + \frac{\epsilon}{2}\right)\right\}, \quad i \geq 1 \tag{5.2}$$

and

$$R(0) = R_\nu(0) = (2 - \epsilon)\#\left\{j \ : \ \nu_j = r\left(\frac{\epsilon}{2}\right)\right\}. \tag{5.3}$$

Take $s = \frac{p-q-1+r-\epsilon}{2}$. Now, we can state the following:

Theorem 5.4. *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

(1) *$R(i + 1) \leq R(i) + 1$, for $i \geq 0$, $i \neq s - 1$;*
(2)

$$R(i+1) = R(i) + 1, \ \textit{for } i \geq 0, \, i \neq s - 1 \ \textit{then} \ \begin{cases} R(i) \ \textit{is even}, & \textit{if } i < s \\ R(i) \ \textit{is odd}, & \textit{if } i > s; \end{cases}$$

(3) *$R(s) \leq R(s - 1) + 2$;*
(4) *$R(s) = R(s - 1) + 2$, then $R(s - 1)$ is even.*

Proof. Cf. Theorem 6.2 in [**BJ**].

Now, we can prove the following:

Proposition 5.5. *Consider $\nu \in \mathfrak{a}^*_{int}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(4) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

Proof. Assume that $r = 1$ and $\epsilon = 0$, in other words that $G = SO(p,q)$ with $p - q$ even. So $s = \frac{p-q}{2} \in \mathbb{Z}$. In this setting $J(\nu)$ has integral infinitesimal character if and only if $\nu_i \in \mathbb{Z}$ and

$$R(i) = \#\{n \ : \ \nu_n = i\} \quad i \geq 1$$

and

$$R(0) = 2\#\{n \ : \ \nu_n = 0\}.$$

Then by conditions (1)-(4), Formula (5.1) and assuming $R(s) \neq 0$ and there exists $j < s$ such that $R(j) \neq 0$ we can consider ν as follows

$$\nu = (s+k,\dots,s+k,s+(k-1),\dots,s+(k-1),\dots\dots,s,\dots,s, \nu_j,\dots,\nu_j,\dots\dots,\nu_1,\dots\nu_1) \tag{5.6}$$

with $0 \leq \nu_i < s$, $i = 1,\dots,j$.

In order to complete the proof of this proposition, we will need the following:

Lemma 5.7. *Take ν as above. Then*

$$k - i + 1 \leq q - \sum_{r=0}^{i} R(s+k-r).$$

Proof. Since $R(s) \neq 0$ we have, by condition (2) in Theorem 5.4 $R(s+m) \neq 0$, for $m = 1,\dots,k$. Hence, using that $q = \sum_{r=0}^{k} R(s+k-r) + \sum_{p=0}^{j-1} R(\nu_{j-p})$ we have

$$\begin{aligned} k - i + 1 &\leq \sum_{r=i+1}^{k} R(s+k-r) + \sum_{p=0}^{j-1} R(\nu_{j-p}) \\ &= q - \sum_{r=0}^{i} R(s+k-r). \end{aligned}$$

So, the proof of the lemma is complete. □

Proof Proposition 5.5. (Cont.) Recalling that each root $e_i \pm e_j$ has multiplicity 1 and e_l has multiplicity $p - q = 2s$, and defining

$$\mathcal{T}(m) = \begin{cases} \sum_{i=0}^{m-1} R(s+k-i), & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \end{cases}$$

and

$$\mathcal{S}(n) = \begin{cases} \sum_{i=0}^{n-1} R(\nu_{j-i}), & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases}$$

we can, since is easy to check that $k-m \leq q-\sum_{j=0}^{m} R(s+k-j)$ for $m=0,\dots,k-1$ and $\nu_i < s$ for $i=1,\dots,j$, rewrite ν as follows

$$\begin{aligned}\nu = &\left(\sum_{m=0}^{k-1}\sum_{r=1}^{R(s+k-m)}\frac{1}{2}\left\{\sum_{t=1}^{k-m}[(e_{\mathcal{T}(m)+r}-e_{\mathcal{T}(m)+r+t})\right.\right.\\ &\left.\left.+(e_{\mathcal{T}(m)+r}+e_{\mathcal{T}(m)+r+t})]\right\}+s\,e_{\mathcal{T}(m)+r}\right)+\sum_{r=1}^{R(s)} se_{\mathcal{T}(s-1)+r}\\ &+\sum_{n=0}^{j-1}\sum_{p=1}^{R(\nu_{j-n})}\nu_{j-n}e_{\mathcal{T}(k+1)+\mathcal{S}(n)+p}.\end{aligned}$$

Now let us keep $r=1$ and consider $\epsilon=1$, i.e., $G=SO(p,q)$, but now, $p-q$ is odd. Here $s=\frac{p-q-1}{2}$ and $J(\nu)$ has integral infinitesimal character if and only if $\nu_n=\frac{2l_n+1}{2}$ with $l_n\in\mathbb{Z}_+$ and $n=1,\dots,q$. Then, we have

$$R(i)=\#\left\{n\ :\ \nu_n=\frac{2i+1}{2}\right\}\qquad i\geq 1$$

and

$$R(0)=\#\left\{n\ :\ \nu_n=\frac{1}{2}\right\}.$$

Again, as before, we can assume that

$$\begin{aligned}\nu=&\left(\frac{2(s+k)+1}{2},\dots,\frac{2(s+k)+1}{2},\frac{2(s+(k-1))+1}{2},\right.\\ &\dots,\frac{2(s+(k-1))+1}{2},\dots,\frac{2s+1}{2},\dots,\frac{2s+1}{2},\frac{2l_j+1}{2},\\ &\left.\dots,\frac{2l_j+1}{2},\dots\dots,\frac{2l_1+1}{2},\dots,\frac{2l_1+1}{2}\right)\end{aligned}\tag{5.8}$$

with $0\leq l_i<s$, $i=1,\dots,j$. By Lemma 5.7 and since $l_i+\frac{1}{2}<s$, we can rewrite ν as follows

$$\begin{aligned}\nu=&\left(\sum_{m=0}^{k}\sum_{r=1}^{R(s+k-m)}\frac{1}{2}\sum_{t=0}^{k-m}[(e_{\mathcal{T}(m)+r}-e_{\mathcal{T}(m)+r+t+1})\right.\\ &\left.+(e_{\mathcal{T}(m)+r}+e_{\mathcal{T}(m)+r+t+1})]+\left(s-\frac{1}{2}\right)e_{\mathcal{T}(m)+r}\right)\\ &+\sum_{n=0}^{j}\sum_{u=1}^{R(l_{j-n})}\left(l_{j-n}+\frac{1}{2}\right)e_{\mathcal{T}(k+1)+\mathcal{S}(n)+u}.\end{aligned}$$

When $R(s)=0$, condition (2) in Theorem 5.4 implies that $R(s+j)=0$ for all j. Then this case or when $R(\nu_i)=0$ for all i, can be easily deduced from the cases above, and hence, we have completed this proof for $SO(p,q)$

and $r = 1$ The cases corresponding to $SU(p,q)$ follows almost immediately from this cases above using, as before, the multiplicities of the positive roots for this group. $\square$

Then we can give the:

Proof of Theorem 2.1 for $SO(p,q)$, $p > q+1$ and $SU(p,q)$, $p > q$. Is immediate from Proposition 5.5, Theorem 5.4 and Proposition 1.5. $\square$

6. Case $SU(n,n)$, $n \geq 2$.

In this case, identifying $\mathfrak{a} \simeq \mathbb{C}^n$ we have that the restricted roots are $\{\pm e_i \pm e_j,\ \pm 2e_l\}$. $J^G(\nu)$ has integral infinitesimal character if and only if $\nu_i \equiv \epsilon \mod 2\mathbb{Z}$, $\epsilon = 0, 1$. Take $\nu \in \mathfrak{a}^*$. Again, we may assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0. \tag{6.1}$$

Hence we define

$$R(i) = R_\nu(i) = \#\{j \ : \ \nu_j = 2i + \epsilon\}, \quad i \geq 1 \tag{6.2}$$

and

$$R(0) = (2\epsilon)\#\{j \ : \ \nu_j = \epsilon\}. \tag{6.3}$$

Now, we can state the following:

Theorem 6.4. *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

1. *$R(i+1) \leq R(i) + 1$, for $i \geq 0$;*
2. *$R(i+1) = R(i) + 1$, then $R(i)$ is odd.*

Proof. See Theorem 7.1 in [**BJ**].

With this, we can prove the following:

Proposition 6.5. *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(2) above are satisfied then $\nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

Proof. Let us first assume that $\epsilon = 0$. By (6.3) we have that $R(0) = 0$. Then it follows from condition (2) in Theorem 6.4 that $\nu \equiv 0$. So we can suppose that $\epsilon = 1$ and again, by condition (2), we can consider that

$$\nu = (2k+1, \ldots, 2k+1, \ldots\ldots, 1, \ldots, 1) \tag{6.6}$$

with $R(0) \geq 2$. Since $n = \sum_{i=0}^{k-1} R(k-i) + \frac{1}{2}R(0)$, we have

$$k - m < n - \sum_{i=0}^{m} R(k-i), \qquad m = 0, \ldots, k-1$$

and this formula plus the fact that each root $e_i \pm e_j$ has multiplicity 2 and each $2e_l$ has multiplicity one, allows us to say that

$$\nu = \left(\sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \sum_{t=1}^{k-m} [(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})] \right.$$
$$\left. + e_{\mathcal{T}(m)+r} \right) + \sum_{r=1}^{\frac{1}{2}R(0)} e_{\mathcal{T}(k)+r}$$

and in this way, we have completed this proposition. □

Hence, we can give the:

Proof of Theorem 2.1 for $SU(n,n)$, $n > 2$. It is immediate from Proposition 6.5, Theorem 6.4 and Proposition 1.5. □

7. Case $SO(n+1,n)$ and $SO(2n+1,\mathbb{C})$, $n \geq 2$.

Let us assume that $G = SO(n+1,n)$ or $G = SO(2n+1,\mathbb{C})$, and as usual identify $\mathfrak{a} \simeq \mathbb{C}^n$. Then the restricted roots become $\{\pm e_i \pm e_j, \pm e_l\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(n+1,n) \\ 2, & \text{if } G = SO(2n+1,\mathbb{C}). \end{cases}$$

In these cases, $J(\nu)$ integral infinitesimal character if and only if

$$\nu_i \in r\mathbb{Z} + r\frac{\epsilon}{2}, \text{ with } \epsilon = 0, 1.$$

We can replace ν by a Weyl group conjugate, and assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0 \tag{7.1}$$

and so, we define

$$R(i) = R_\nu(i) = \#\left\{ j \ : \ \nu_j = r\left(i + \frac{\epsilon}{2}\right) \right\}, \quad i \geq 1 \tag{7.2}$$

and

$$R(0) = R_\nu(0) = (2-\epsilon)\#\left\{ j \ : \ \nu_j = r\frac{\epsilon}{2} \right\}. \tag{7.3}$$

Let us see the following:

Theorem 7.4. *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

1. $R(i+1) \leq R(i) + 1$, *for* $i \geq 0$;
2. *if* $R(i+1) = R(i) + 1$, *for* $i \geq 1$, *then* $R(i)$ *is odd;*
3. $R(0) > 2 - \epsilon$.

Proof. Cf. Theorem 9.4 in [**BJ**]. The condition $R(0) > 2-\epsilon$ does not appear in [**BJ**]. However, it is easy to see by inspection of the proof that this condition is needed.

Now, we can prove the following:

Proposition 7.5. *Consider $\nu \in \mathfrak{a}^*_{\mathrm{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(3) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

Proof. Let us assume that r=1. The other case, r=2, follows from this one. And also consider $\epsilon = 0$, so here we have that $J(\nu)$ admits integral infinitesimal character if and only if $\nu_i \in \mathbb{Z}$. Hence by condition (2) in Theorem 7.4 we have that

$$\nu = (k,\dots,k,k-1,\dots,k-1,\dots\dots,1,\dots,1,0,\dots,0). \tag{7.6}$$

As before, since $R(0) \geq 2$, we can prove that $k-i \leq n - \sum_{j=0}^{i} R(k-j)$, and this inequality allows us to write down

$$\nu = \frac{1}{2}\sum_{m=0}^{k-1}\sum_{r=1}^{R(k-m)}\sum_{t=1}^{k-m}[(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})]$$

where $\mathcal{T}(m)$ is defined in (3.6). The case $\epsilon = 1$ follows from this one, using that

$$\nu = \nu_1 + \left(\frac{1}{2},\dots,\frac{1}{2}\right)$$

with ν_1 as in (7.6). Since we do not use e_i's in the case above, we can put

$$\nu = \nu_1 + \frac{1}{2}\sum_{i=1}^{n} e_i$$

and we have completed the proof of this proposition. □

Then we have:

Proof of Theorem 2.1 for $SO(n+1,n)$ and $SO(2n+1,\mathbb{C})$, $n \geq 2$. It follows from Proposition 7.5, Theorem 7.4 and Proposition 1.5. □

8. Case $SO(n,n)$ and $SO(2n,\mathbb{C})$, $n \geq 4$.

Let us assume that $G = SO(n,n)$ or $G = SO(2n,\mathbb{C})$. In this case $\Sigma(\mathfrak{g}_0,\mathfrak{a}_0)$ is of type D_n and, if as usual we identify $\mathfrak{a} \simeq \mathbb{C}^n$, then the restricted roots become $\{\pm e_i \pm e_j\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(n,n) \\ 2, & \text{if } G = SO(2n,\mathbb{C}). \end{cases}$$

Here, $J(\nu)$ integral infinitesimal character if and only if

$$\nu_i \in r\mathbb{Z} + r\frac{\epsilon}{2}, \text{ with } \epsilon = 0, 1.$$

Again, we can assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq |\nu_n| \geq 0 \tag{8.1}$$

and define

$$R(i) = R_\nu(i) = \#\left\{j \ : \ \nu_j = r\left(i + \frac{\epsilon}{2}\right)\right\}, \quad i \geq 1 \tag{8.2}$$

and

$$R(0) = R_\nu(0) = (2-\epsilon)\#\left\{j \ : \ \nu_j = r\frac{\epsilon}{2}\right\} + (1-\epsilon). \tag{8.3}$$

Take $\nu \in \mathfrak{a}^*_{\text{int}}$, then $J(\nu)$ admits an invariant Hermitian form if and only if n is even, or n is odd, $\epsilon = 0$ and $R(0) > 1$. Let us see the following:

Theorem 8.4. *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

1. $R(i+1) \leq R(i) + 1$, *for* $i \geq 0$*;*
2. $R(i+1) = R(i) + 1$, *for* $i \geq 1$, *then* $R(i)$ *is odd;*
3. $R(0)$ *is odd;*
4. $R(0) > 1$.

Proof. See Theorem 10.3 in [**BJ**]. The condition $R(0) > 1$ does not appear in the statement of this theorem in [**BJ**], but it is easy to see, checking the proof, that, otherwise, $q_\mu(\nu) > 0$.

Now, we can prove the following:

Proposition 8.5. *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(4) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

Proof. Let us assume that $r = 1$, $\epsilon = 0$. Since the Weyl group is the group of permutation and sign changes involving an even number of signs of the set of n elements, by condition (2) and $R(0) > 1$, we can suppose that $\nu_i \geq 0$ and

$$\nu = (k, \ldots, k, k-1, \ldots, k-1, \ldots\ldots, 1, \ldots, 1, 0, \ldots, 0).$$

Since $R(0) > 1$, we have $k - i \leq n - \sum_{j=0}^{i} R(k-j)$, and thus

$$\nu = \frac{1}{2}\sum_{m=0}^{k-1}\sum_{r=1}^{R(k-m)}\sum_{t=1}^{k-m}[(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})],$$

where $\mathcal{T}(m)$ was defined in (3.6). Now, let us assume that $\epsilon = 1$ and n even. So, by a conjugation by the Weyl group we can assume that $\nu_i \geq 0$ for $i = 0, \ldots, n-1$ and moreover

$$\nu = \left(k + \frac{1}{2}, \ldots, k + \frac{1}{2}, k - 1 + \frac{1}{2}, \ldots, k - 1 + \frac{1}{2}, \right.$$
$$\left. \ldots\ldots, 1 + \frac{1}{2}, \ldots, 1 + \frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2}\right).$$

Since $R(0) > 1$ and by condition (3) in Theorem 8.4 we can prove that $k - i + 1 \leq n - \sum_{j=0}^{i} R(k-j)$, and defining $\delta = \begin{cases} 1, & \text{if } \nu_n = \frac{1}{2} \\ 0, & \text{if } \nu_n = -\frac{1}{2}. \end{cases}$ we can rewrite ν as follows

$$\nu = \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \sum_{t=1}^{k-m} \frac{1}{2}\Big\{[(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})]$$
$$+ (e_{\mathcal{T}(m)+r} + (-1)^{\mathcal{T}(m)+r+\delta} e_n)\Big\} + \sum_{j=1}^{R(0)-1} \frac{1}{2}(e_{\mathcal{T}(k)+j} + (-1)^j e_n)$$

where $\mathcal{T}(m)$ was defined in (3.6). And, since n is even and $R(0)$ is odd, we have completed our proof. The case $SO(2n, \mathbb{C})$ follows as above using multiplicities of positive roots for this particular case. ☐

Then we can give the:

Proof of Theorem 2.1 for $SO(n,n)$ and $SO(2n,\mathbb{C})$, $n \geq 4$. It is immediate from Proposition 8.5, Theorem 8.4 and Proposition 1.5. ☐

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