ON THE ASYMPTOTICS OF THE TRACE OF THE HEAT KERNEL FOR THE MAGNETIC SCHröDINGER OPERATOR

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We consider the effect of a magnetic field for the asymptotic behavior of the trace of the heat kernel for the Schrödinger operator. We discuss the case where the operator has compact resolvents in spite of the fact that the electric potential is degenerate on some submanifold. According to the degree of the degeneracy, we obtain the classical and non-classical asymptotics.

1. Introduction.

In this paper, we consider the Schrödinger operator on $\mathbb{R}^d$ with the magnetic vector potential $A(z)$ and the electric scalar potential $V(z)$:

(1.1) \[ H(A,V) = \frac{1}{2}(i\nabla + A(z))^2 + V(z). \]

We will assume that $H(A,V)$ and $H(0,V)$ are essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ and we denote the self-adjoint extensions by $H$ and $H_0$, respectively. It is well known that if

(1.2) \[ \lim_{|z| \to \infty} V(z) = +\infty, \]

$H_0$ has compact resolvents (c.f. for example, Reed and Simon [8]). Oden- cranez [7] studied the asymptotic behavior of $\text{Tr}[\exp(-tH) - \exp(-tH_0)]$ as $t \downarrow 0$ in the case where $V(z) \approx |z|^{2p}$ ($p > 0$) and the curl of $A(z)$ is a uniform magnetic field. Matsumoto [5] extended the result to the case with more general magnetic field.

However, (1.2) is not a necessary condition in order that $H_0$ has compact resolvents. In spite of the lack of (1.2), there are some cases where $H_0$ has compact resolvents.

The motivation of this paper originates in the works of Simon [11], Robert [9] and Aramaki [3]. In order to explain this, we put

\[ Z_{cl}(t) = (2\pi)^{-d} \int \int e^{-t(|\zeta|^2/2 + V(z))} \, dzd\zeta. \]

It is well known that $\text{Tr}[\exp(-tH_0)] \leq Z_{cl}(t)$ for $t > 0$. In [11], he considered the degenerate potential $V(z)$ of the form $V(x,y) = |x|^{2p}|y|^{2q}$ ($p, q > 0$) on...
$\mathbb{R}^2$. Then it holds that $H_0$ has compact resolvents. In this case, $Z_{\text{cl}}(t) \equiv \infty$ while $\text{Tr}[\exp(-tH_0)] < \infty$ for all $t > 0$. He succeeded to get the asymptotics of $\text{Tr}[\exp(-tH_0)]$ as $t \downarrow 0$ by using the sliced Golden-Thompson inequality and the sliced bread inequality according to the cases $p \neq q$ and $p = q$. \cite{9} and \cite{3} considered a slightly modified potential $V(z)$ of the form

$$V(z) = (1 + |x|^p)|y|^{2q} \quad (p, q > 0 \text{ integers}), \quad z = (x, y) \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m.$$ 

Then $H_0$ also has compact resolvents (cf. \cite{9}). In this case, it is easy to see that $Z_{\text{cl}}(t)$ is finite for $t > 0$ in the case $pm > qn$, however, $Z_{\text{cl}}(t)$ is infinite for $t > 0$ in the case $pm \leq qn$. In the present paper, we consider the magnetic Schrödinger operator $H$ with an electric potential $V(z)$ of the type (1.3). We want to show that if the magnetic potential $A(z)$ is relatively benign, the difference between $\text{Tr}[\exp(-tH)]$ and $\text{Tr}[\exp(-tH_0)]$ can be controlled by using the representation of the heat kernels in terms of Wiener integrals. From this, we can see the asymptotics of $\text{Tr}[\exp(-tH)]$ as $t \downarrow 0$, if we combined the estimate with our previous results as follows.

We denote the numbers of the eigenvalues counting multiplicities of $H$ and $H_0$ equal to or less than $\lambda$ by $N(\lambda)$ and $N_0(\lambda)$, respectively. Then Aramaki \cite{3} obtained the following.

**Proposition 1.1.** Under (1.3), there exists $\delta > 0$ such that:

(i) If $pm > qn$, $N_0(\lambda) = c_1 \lambda^{(m+mq+nq)/(2q)}(1 + O(\lambda^{-\delta}))$ as $\lambda \to \infty$.

(ii) If $pm = qn$, $N_0(\lambda) = c_2 \lambda^{(m+mq+nq)/(2q)} \log \lambda + c_3 \lambda^{(m+mq+nq)/(2q)}(1 + O(\lambda^{-\delta}))$ as $\lambda \to \infty$.

(iii) If $pm < qn$, $N_0(\lambda) = c_4 \lambda^{(1+p+q)/(2p)}(1 + O(\lambda^{-\delta}))$ as $\lambda \to \infty$

where $c_i$ ($i = 1, 2, 3, 4$) are some positive constants which can be calculated concretely.

Here we note that in only the case where $pm > qn$, the problem is classical in the sense of

$$\text{vol} \left\{ (z, \zeta) \in \mathbb{R}^d \times \mathbb{R}^d; \frac{1}{2} |\zeta|^2 + V(z) \leq \lambda \right\} < \infty. \quad (1.4)$$

On the other hand, in the case where $pm \leq qn$, the problem is non-classical in the sense that the left hand side of (1.4) is the infinity. It easily follows from the well known equation

$$\text{Tr}[\exp(-tH_0)] = \int \exp(-t\lambda) dN_0(\lambda)$$

that we also have the asymptotic behavior of $\text{Tr}[\exp(-tH_0)]$ as $t \downarrow 0$.

**Proposition 1.2.** Under (1.3), we have:
1. Hypotheses and Statements.

Let $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$ and we write a variable $z$ in $\mathbb{R}^d$ by $z = (x, y) \in \mathbb{R}_x^n \times \mathbb{R}_y^m$. We consider the operator:

$$H(A, V) = \frac{1}{2} (i \nabla_{(x, y)} + A(x, y))^2 + V(x, y)$$

where $i = \sqrt{-1}$ and $\nabla_{(x, y)}$ denotes the gradient operator. First of all, we state the assumptions on the scalar potential $V(x, y)$.

(V.1) $V(x, y) \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ is a real valued function.

(V.2) There exist positive constants $p, q$ and $C > 0$ such that

$$V(x, y) \geq C(1 + |x|^2)^p|y|^{2q} \text{ for all } (x, y) \in \mathbb{R}_x^n \times \mathbb{R}_y^m.$$ 

Moreover, we give the assumptions for the vector potential $A(x, y)$:

(A.1) $A(x, y) = (a_1(x, y), \ldots, a_d(x, y)) \in C^2(\mathbb{R}^d; \mathbb{R}^d)$.

(A.2) There exist constants $a, b$ satisfying $0 \leq a < p$, $0 \leq b < q$, $(q + 1)a < p(b + 1)$ and $C_1 > 0$ such that for every $j = 1, 2, \ldots, d$ and $|\alpha| \leq 2$,

$$|\partial_{x,y}^\alpha a_j(x, y)| \leq C_1(1 + |x|^2)^a|y|^{2b}.$$

By the assumptions (V.1) and (A.1), $H(A, V)$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ (c.f. Schechter [10]) and we denote the unique self-adjoint extensions of $H(A, V)$ and $H(0, V)$ by $H$ and $H_0$, respectively.
as in introduction. Under (V.1), (V.2), $H_0$ has compact resolvents and $\exp(-tH_0)$ is of trace class, i.e., $\text{Tr}[\exp(-tH_0)]$ is finite (c.f. [1]). Since $V$ is bounded from below, we have the diamagnetic inequality:

$$e^{-tH(A,V)} \preceq e^{-tH(0,V)}$$

for every $t > 0$,

that is to say, for all $u \in L^2(\mathbb{R}^d)$,

$$\left|(e^{-tH(A,V)}u)(z)\right| \leq (e^{-tH(0,V)}|u|)(z) \quad \text{a.e.} \quad z \quad \text{for} \quad t > 0.$$

Then it follows from Simon [13, p. 164] that

$$\text{Tr}\left[e^{-tH}\right] \leq \text{Tr}\left[e^{-tH_0}\right] \quad \text{for} \quad t > 0,$$

so we see that $\exp(-tH)$ is also of trace class.

Then we have the main theorem.

**Theorem 2.1.** Under the conditions (V.1), (V.2), (A.1) and (A.2), we have the following.

(i) The case where $pm > qn$. If $q(4a + n) - p(4b + m) \neq 0$, we have

$$\text{Tr}\left[e^{-tH} - e^{-tH_0}\right] = O(t^{-m+mq+nq/(2q)+\gamma_1}) \quad \text{as} \quad t \downarrow 0$$

and if $q(4a + n) - p(4b + m) = 0$, we have

$$\text{Tr}\left[e^{-tH} - e^{-tH_0}\right] = O(t^{-m+mq+nq/(2q)+\gamma_1 \log t^{-1}}) \quad \text{as} \quad t \downarrow 0$$

where

$$\gamma_1 = \min \left\{ \frac{2(q - b)}{q}, \frac{(1 + q)(pm - qn)}{2pq} + \frac{2(p(1 + b) - a(q + 1))}{p} \right\}.$$

(ii) The case where $pm \leq qn$. If $q(4a + n) - p(4b + m) \neq 0$, we have

$$\text{Tr}\left[e^{-tH} - e^{-tH_0}\right] = O(t^{-n(1+p+q)/(2p)+\gamma_2}) \quad \text{as} \quad t \downarrow 0$$

and if $q(4a + n) - p(4b + m) = 0$, we have

$$\text{Tr}\left[e^{-tH} - e^{-tH_0}\right] = O(t^{-n(1+p+q)/(2p)+\gamma_2 \log t^{-1}}) \quad \text{as} \quad t \downarrow 0$$

where

$$\gamma_2 = \min \left\{ \frac{2(q - b)}{q} + \frac{(1 + q)(qn - pm)}{2pq}, \frac{2(p(1 + b) - a(q + 1))}{p} \right\}.$$

**Remark 2.2.** Since $\gamma_1$ and $\gamma_2$ are positive numbers in any cases according to (A.2), the hypothesis (A.2) on the magnetic potential certainly gives an effect to the asymptotics in each case.

Using the Karamata Tauberian theorem and [3], we have also asymptotics of distribution function $N(\lambda)$ of eigenvalues of $H$. 
Corollary 2.3. Addition to the hypotheses of Theorem 2.1, we assume that there exists \( C_2 > 0 \) such that
\[
V(x, y) \leq C_2(1 + |x|^2)^p|y|^{2q}.
\]
Then we have
\[
N(\lambda) = N_0(\lambda)(1 + o(1)) \quad \text{as} \quad \lambda \to \infty.
\]
Here \( N_0(\lambda) \) is of the form:
(i) If \( pm > qn \), we have
\[
N_0(\lambda) = c_1\lambda^{(m+mq+qn)/(2q)}(1 + o(1)) \quad \text{as} \quad \lambda \to \infty.
\]
(ii) If \( pm = qn \), we have
\[
N_0(\lambda) = c_2\lambda^{(m+mq+qn)/(2q)} \log \lambda(1 + o(1)) \quad \text{as} \quad \lambda \to \infty.
\]
(iii) If \( pm < qn \), we have
\[
N_0(\lambda) = c_4\lambda^{n(1+q+p)/(2p)}(1 + o(1)) \quad \text{as} \quad \lambda \to \infty.
\]
Here \( c_1, c_2, c_4 \) are positive constants as in Proposition 1.1. For the precise values of the constants, see [3].

3. Proof of the main theorem.

In this section, we give the proof of Theorem 2.1. Let \( p_0(t; z, z') \) and \( p(t; z, z') \) be the distribution kernels of \( \exp(-tH_0) \) and \( \exp(-tH) \), respectively. Then, by the Feynman-Kac-Itô formula, we can write these heat kernels using probabilistic representations as follows.
\[
p_0(t; z, z') = (2\pi t)^{-d/2}e^{-|z-z'|^2/(2t)} E_{0,0}^{0,0} \\
\cdot \left[ \exp \left( -t \int_0^1 V(z + s(z - z') + \sqrt{t}Z_s)ds \right) \right],
\]
\[
p(t; z, z') = (2\pi t)^{-d/2}e^{-|z-z'|^2/(2t)} E_{0,0}^{0,0} \\
\cdot \left[ \exp \left( iF^t(z, z') - t \int_0^1 V(z + s(z - z') + \sqrt{t}Z_s)ds \right) \right]
\]
where \( F^t(z, z') = \sqrt{t} \int_0^1 A(z + s(z - z') + \sqrt{t}Z_s) \circ dZ_s \). Here \( E_{0,0}^{0,0} \) is the expectation with respect to the \( d (= n + m) \)-dimensional pinned Brownian motion \( \{Z_s\}_{0 \leq s \leq 1} = \{X_s, Y_s\}_{0 \leq s \leq 1} = \{X^1_s, \ldots, X^n_s, Y^1_s, \ldots, Y^m_s\}_{0 \leq s \leq 1} \) such that \( Z_0 = 0 = (0, 0) \) and \( Z_1 = 0 = (0, 0) \) and \( dZ_s \) denotes the Stratonovich integral. For the theory of these probabilistic facts, see [13] and [5]. Throughout this paper, we denote \( E_{0,0}^{0,0} \) simply by \( E \). We note that under (V.1) and (A.1), \( p_0(t; z, z') \) and \( p(t; z, z') \) are continuous with respect to \( t > 0 \) and \( z, z' \in \mathbb{R}^d \). Since we study the traces of \( \exp(-tH_0) \) and
exp(−tH), it suffices to consider the heat kernels on the diagonal set only. I.e.,

\begin{align}
\tag{3.1} p_0(t; z, z) &= (2\pi t)^{-d/2}E \left[ \exp \left( -t \int_0^1 V(z + \sqrt{t}Z_s)ds \right) \right], \\
\tag{3.2} p(t; z, z) &= (2\pi t)^{-d/2}E \left[ \exp \left( iF^t(z) - t \int_0^1 V(z + \sqrt{t}Z_s)ds \right) \right]
\end{align}

where $F^t(z) = \sqrt{t} \int_0^1 A(z + \sqrt{t}Z_s) \circ dZ_s$.

Let $\{Z_s\}_{0 \leq s \leq 1} = \{X_s, Y_s\}_{0 \leq s \leq 1}$ be defined on a probability space $(\Omega, \mathcal{F}, P)$ and define

\begin{equation}
\tag{3.3} \xi = \sup_{0 \leq s \leq 1} |X_s|.
\end{equation}

Then it follows from Lévy’s work that:

**Lemma 3.1.** For every $R > 0$,

\[ P(\xi \geq R) \leq 2ne^{-2R^2/n}. \]

For the proof, see Simon [12], Itô and McKean [4] and [5, Lemma 1].

From now, we denote various constants independent of $t > 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ by the same notations $C, C_j \ (j = 1, 2, \ldots)$ etc.

**Lemma 3.2.** Under the assumptions (A.1) and (A.2), there exists a constant $C > 0$ such that

\begin{equation}
\tag{3.4} E \left[ |F^t(x, y)|^4 \right] \leq Ct^4(1 + |x|^2)^{4a}|y|^{8b}
\end{equation}

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $t \in (0, 1)$.

**Proof.** Since the proof is essentially the same as [5, Lemma 2], we give only an outline of the proof.

Let $\{w_s\}_{0 \leq s \leq 1}$ be the standard $d$-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Then, the pinned Brownian motion $\{Z_s\}_{0 \leq s \leq 1}$ such that $Z_0 = Z_1 = 0$ is the solution of the stochastic differential equation:

\[ dZ^i_s = dw^i_s - \frac{Z^i_s}{1 - s}ds \quad (0 < s < 1), \quad Z^i_0 = 0 \quad (i = 1, 2, \ldots, d). \]
Thus, by the Itô formula, we have
\[
F^t(z) = t \sum_{i,j=1}^d \int_0^1 dw_i \int_0^s \partial_j a_i(z + \sqrt{t}Z_u)dw^j_u
\]
\[
- t \sum_{i,j=1}^d \int_0^1 dw_i \int_0^s \partial_j a_i(z + \sqrt{t}Z_u)\frac{Z^j_u}{1-u} du
\]
\[
+ \frac{1}{2} t^{3/2} \sum_{i,j=1}^d \int_0^1 dw_i \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) \frac{Z^j_u}{1-u} du
\]
\[
- t \sum_{i,j=1}^d \int_0^1 \frac{Z^j_i}{1-s} ds \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) \frac{Z^j_u}{1-u} du
\]
\[
+ t \sum_{i,j=1}^d \int_0^1 \frac{Z^j_i}{1-s} ds \int_0^s \partial_j a_i(z + \sqrt{t}Z_u) \frac{Z^j_u}{1-u} du
\]
\[
- \frac{1}{2} t^{3/2} \sum_{i,j=1}^d \int_0^1 \frac{Z^j_i}{1-s} ds \int_0^s \partial^2_j a_i(z + \sqrt{t}Z_u) du
\]
\[
+ \frac{1}{2} t^1 \sum_{i=1}^d \int_0^1 \partial_i a_i(z + \sqrt{t}Z_u) ds.
\]

Since \(Z^i_s\) is the Gaussian random variable of mean 0 and variance \(s(1-s)\), we have
\[
E[|Z^i_s|^{2m}] = (2m-1)!!(s(1-s))^m \quad \text{for} \quad m = 1, 2, \ldots.
\]
Using this equality, the Hölder inequality and (A.2), we can prove the lemma.

\[\square\]

**Lemma 3.3.** For every \(x \in \mathbb{R}^n\), put \(A_{|x|} = -\frac{1}{2} \Delta_y + c|x|^{2p}|y|^{2q}\) on \(L^2(\mathbb{R}^m)\) where \(c\) is a positive constant and let \(e^{-tA_{|x|}}(y, y')\) be the kernel of \(e^{-tA_{|x|}}\) and \(J(t; |x|, y) = e^{-tA_{|x|}}(y, y)\). Then, we have following:

(i) There exist constants \(C_j\) \((j = 1, 2, 3)\) such that

\[
|J(t; 1, y)| \leq C_1 t^{-m/2} \left(e^{C_2 t|y|^{2q}} + e^{C_3 |y|^{2/t}}\right).
\]

(ii) For every \(\lambda > 0\),

\[
J(t; |x|, y) = \lambda^{-m} J(\lambda^{-2} t; |\lambda^{1+q/p} x|, \lambda^{-1} y) \quad \text{for all} \ t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.
\]

**Proof.** For (i), see Matsumoto [6, Lemma 3.1]. For (ii), it follows from the Feynman-Kac formula that

\[
J(t; |x|, y) = (2\pi t)^{-m/2} E[e^{-t \int_0^1 c|x|^{2p}|y_{s+y}|^{2q} ds}].
\]
Thus we have
\[ J(t; |x|, \lambda y) = \lambda^{-m}(2\pi\lambda^{-2}t)^{-m/2}e^{\lambda t}\int_0^1 c(\lambda^{1+q}/p|x|^2p)y + \sqrt\lambda^{-1}tY_s|^{2}\omega ds \]
\[ = \lambda^{-m}J(\lambda^{-2}t; |\lambda^{1+q}/p|x|, y). \]

Now, we give the proof of Theorem 2.1.

**Proof of Theorem 2.1.**

Since \( p(t; x, y, x, y) \) is a real valued function, using (3.1) and (3.2), we can write
\[
I(t; x, y) = |p(t; x, y, x, y) - p_0(t; x, y, x, y)|
\]
\[
= (2\pi t)^{-d/2}E[(\cos F^t(x, y) - 1)e^{-t}\int_0^1 V(x + \sqrt{t}X_s, y + \sqrt{t}Y_s)ds].
\]

Since \( 0 \leq \cos \theta \leq \theta^2/2 \) for \( \theta \in \mathbb{R} \), \( I(t; x, y) \) is estimated by
\[
\frac{1}{2}(2\pi t)^{-d/2}E[F^t(x, y)^2e^{-t}\int_0^1 V(x + \sqrt{t}X_s, y + \sqrt{t}Y_s)ds].
\]

By the Schwartz inequality, Lemma 3.2 and hypothesis (A.2) and (V.2), we have
\[
I(t; x, y) \leq C_1 t^{-d/2}E[F^t(x, y)^4]\frac{1}{2}E[e^{-2t}\int_0^1 V(x + \sqrt{t}X_s, y + \sqrt{t}Y_s)ds]^{1/2}
\]
\[
\leq C_2 t^{-d/2}(1 + |x|^2)^{2\alpha}E[e^{-C_3 t}\int_0^1 (1 + |x + \sqrt{t}X_s|^2)^p|y + \sqrt{t}Y_s|^{2\alpha}ds]^{1/2}.
\]

Define the function \( \xi \) by (3.3) and let \( \chi \) be the characteristic function of the sets \( \{ \xi \geq |x|/2\sqrt{t} \} \).

Now we decompose
\[
(3.7) \quad K(t; x, y) \equiv E[e^{-C_3 t}\int_0^1 (1 + |x + \sqrt{t}X_s|^2)^p|y + \sqrt{t}Y_s|^{2\alpha}ds]
\]
into the form \( K(t; x, y) = \sum_{j=1}^2 K_j(t; x, y) \) where
\[
K_1(t; x, y) = E[e^{-C_3 t}\int_0^1 (1 + |x + \sqrt{t}X_s|^2)^p|y + \sqrt{t}Y_s|^{2\alpha}ds \chi]
\]
\[
K_2(t; x, y) = E[e^{-C_3 t}\int_0^1 (1 + |x + \sqrt{t}X_s|^2)^p|y + \sqrt{t}Y_s|^{2\alpha}ds (1 - \chi)].
\]

Then we note that \( K(t; x, y)^{1/2} \leq \sum_{j=1}^2 K_j(t; x, y)^{1/2} \). At first, we consider \( K_1(t; x, y) \). Since \( (1 + |x + \sqrt{t}X_s|^2)^p \geq 1 \), we see that
\[
K_1(t; x, y) \leq E[e^{-C_3 t}\int_0^1 |y + \sqrt{t}Y_s|^{2\alpha}ds]E[\chi]
\]
\[
= E[e^{-C_3 t}\int_0^1 |y + \sqrt{t}Y_s|^{2\alpha}ds]P(\{ \xi \geq |x|/2\sqrt{t} \}).
\]

By Lemma 3.1 and Lemma 3.3 (i), we have
\[
K_1(t; x, y) \leq C_4(e^{-C_5 t}|y|^{2\alpha} + e^{-C_5 |y|^2/t})e^{-C_7 |x|^2/t}.
\]
Therefore,
\[ K_1(t) = t^{2-d/2} \int \int (1 + |x|^2)^{2a} |y|^{4b} K_1(t; x, y)^{1/2} dx dy \]
\[ \leq C_4 t^{2-d/2} \int (1 + |x|^2)^{2a} e^{-C_7|x|^2/t} dx \int |y|^{4b} (e^{-C_5|y|^2/t} + e^{-C_6|y|^2/t}) dy. \]

Since a simple computation leads to
\[ \int (1 + |x|^2)^{2a} e^{-C_1|x|^2/t} dx = O(t^{n/2}) \quad \text{as} \quad t \to 0 \]
and
\[ \int |y|^{4b} (e^{-C_5|y|^2/t} + e^{-C_6|y|^2/t}) dy = O(t^{-(4b+m)/(2q)}), \]
we have an estimate of \( K_1(t) \):
\[ K_1(t) \leq C_8 t^{2-m/2-(4b+m)/(2q)} = O(t^{-(m+mq+nq)/(2q)+2(1-b/q)+n/2}). \quad (3.8) \]

Secondly, we consider \( K_2(t; x, y) \). Since we have \(|x + \sqrt{t} X_s| \geq |x|/2\) on \( \text{supp}(1 - \chi) \),
\[ K_2(t; x, y) \leq E \left[ e^{-C_1 t L_1^{1+|x|^2}|y+\sqrt{t} Y_s|^{2q}} ds \right]. \quad (3.9) \]

Now, we decompose
\[ K_2(t) = t^{2-d/2} \int \int (1 + |x|^2)^{2a} |y|^{4b} K_2(t; x, y)^{1/2} dx dy \]
\[ = K_{2,1}(t) + K_{2,2}(t) \]
where
\[ K_{2,1}(t) = t^{2-d/2} \int \int _{|x| \leq 1} (1 + |x|^2)^{2a} |y|^{4b} K_2(t; x, y)^{1/2} dx dy, \]
\[ K_{2,2}(t) = t^{2-d/2} \int \int _{|x| \geq 1} (1 + |x|^2)^{2a} |y|^{4b} K_2(t; x, y)^{1/2} dx dy. \]

For the estimate of \( K_{2,1}(t) \), we use \((1 + |x|^2)^p \geq 1\) in (3.9). Thus we have
\[ K_{2,1}(t) \leq t^{2-d/2} \int _{|x| \leq 1} (1 + |x|^2)^{2a} |y|^{4b} E \left[ e^{-C_2 t L_1^{1+|y+\sqrt{t} Y_s|^{2q}} ds} \right]^{1/2} dx dy. \]

Here, by Lemma 3.3 (i),
\[ E \left[ e^{-C_2 t L_1^{1+|y+\sqrt{t} Y_s|^{2q}} ds} \right]^{1/2} = \left\{ \left( 2\pi t \right)^{m/2} e^{-t\left( -\frac{1}{2} \Delta + C_4 |y|^{2q} \right)} (y, y) \right\}^{1/2} \leq C_3 t^{m/4} \left\{ e^{-C_4 |y|^{2q}} + e^{-C_5 |y|^2/t} \right\}. \]
Therefore, we have
\[ K_{2,1}(t) \leq C_6 t^{-(m+q)/(2q)+2(1-b/q)+m/4}. \quad (3.10) \]
For the estimate of $K_{2,2}(t)$, we use $(1 + |x|^2)^p \geq |x|^{2p}$ in (3.9). Since
\begin{equation}
K_2(t; x, y) \leq E\left[e^{-C_2 t \int_0^1 |x|^{2p} y + \sqrt{Y_s} |x|^{2q} ds}\right],
\end{equation}
we have
\begin{align*}
K_{2,2}(t) & \leq t^{2-d/2} \int_{|x| \geq 1} (1 + |x|^2)^{2a} |y|^{4b} E\left[e^{-C_2 t \int_0^1 |x|^{2p} y + \sqrt{Y_s} |x|^{2q} ds}\right]^{1/2} dx dy \\
& = t^{2-d/2} \int_{|x| \geq 1} (1 + |x|^2)^{2a} dx \int |y|^{4b} (2\pi t)^{m/4} J(t; |x|, y)^{1/2} dy
\end{align*}
where $J(t; |x|, y)$ is as in Lemma 3.3 (ii) with $c = C_2$. If we define
\[ F(t; x) = (2\pi t)^{m/4} \int |y|^{4b} J(t; |x|, y)^{1/2} dy, \]
it follows from Lemma 3.3 (ii) with $\lambda = t^{1/2}$ and the change of variable $t^{-1/2}y \to y$ that
\[ F(t; x) = t^{2b+3m/4} F(1; t^{(1+q)/(2p)} x). \]
Thus we have
\begin{align*}
K_{2,2}(t) & = t^{2-d/2} \int_{|x| \geq 1} (1 + |x|^2)^{2a} F(t; x) dx \\
& \leq C_7 t^{2-d/2+2b+m/2} \int_{|x| \geq 1} |x|^{4a} F(1; t^{(1+q)/(2p)} x) dx.
\end{align*}
A change of variable $t^{(1+q)/(2p)} x \to x$ in the last integral leads to
\[ K_{2,2}(t) \leq C_7 t^{2-d/2+2b+m/2-(1+q)(4a-n)/(2p)} \int_{|x| \geq t^{(1+q)/(2p)}} |x|^{4a} F(1; x) dx. \]
Here we need the following lemma:

**Lemma 3.4.** Under the above notations, there exist constants $C_j$ ($j = 1, 2, 3$) such that
\[ F(1; x) \leq \begin{cases} 
C_1 |x|^{-(4b+m)p/q}, & \text{for } |x| \leq 1 \\
C_2 e^{-C_3 |x|^{2p/(q+1)}}, & \text{for } |x| \geq 1. 
\end{cases} \]

**Proof.** Since $J(t; |x|, x^{-p/(1+q)} y) = |x|^{pm/(1+q)} J(|x|^{2p/(1+q)} t; 1, y)$, we have
\begin{align*}
F(1; x) & = (2\pi)^{m/4} \int |y|^{4b} J(1; |x|, y)^{1/2} dy \\
& = (2\pi)^{m/4} \int |y|^{4b} |x|^{pm/(2(q+1))} J(|x|^{2p/(q+1)}; 1, |x|^{p/(q+1)} y)^{1/2} dy.
\end{align*}
The change of variable \(|x|^{p/(q+1)}y \to y\) leads to (3.12)
\[
F(1, x) = (2\pi)^{m/4} |x|^{-(4b+m)p/(q+1) + pm/(2(q+1))} \int |y|^{4b} J(|x|^{2p/(q+1)}; 1, y)^{1/2} dy.
\]
Since by Lemma 3.3 (i),
\[
J(|x|^{2p/(q+1)}; 1, y)^{1/2} \leq C_3 |x|^{-pm/2(q+1)} \left(e^{-C_1 |x|^{2p/(q+1)}y^{2q}} + e^{-C_2 |y|^2 |x|^{-2p/(q+1)}}\right),
\]
we have
\[
F(1, x) \leq C_3 |x|^{-(4b+m)p/(q+1)} \int |y|^{4b} \left(e^{-C_1 |x|^{2p/(q+1)}y^{2q}} + e^{-C_2 |y|^2 |x|^{-2p/(q+1)}}\right) dy.
\]
By a change of variable \(|x|^{p/(q+1)}y \to y\) in the first term and \(|x|^{-p/(q+1)}y \to y\) in the second term in the last integral, we have
\[
F(1, x) \leq C_4 |x|^{-(4b+m)p/q} \int |y|^{4b} \left(e^{-C_1 |y|^2} + e^{-C_2 |y|^2} \right) dy \leq C_5 (|x|^{-(4b+m)p/q + 1}).
\]
When \(|x| \geq 1\), we write
\[
F(1, x) = (2\pi)^{m/4} |x|^{-(8b+m)p/(2(q+1))} \int |y|^{4b} J(|x|^{2p/(q+1)}; 1, y)^{1/2} dy.
\]
Since \(-\frac{1}{2} \Delta_y + 2c|y|^{2q}\) is positive definite, there exists \(c_1 > 0\) such that
\[-\frac{1}{2} \Delta_y + 2c|y|^{2q} \geq 2c_1.\]
Since \(|x| \geq 1\), using Lemma 3.3 (i),
\[
J(|x|^{2p/(q+1)}; 1, y) \leq C_6 e^{-c_1 |x|^{2p/(q+1)}} \left(e^{-C_3 |y|^2} + e^{-C_4 |y|^2} \right).
\]
Therefore, for \(|x| \geq 1\), we have
\[
\int |y|^{4b} J(|x|^{2p/(q+1)}; 1, y)^{1/2} dy \leq C_7 e^{-c_1 |x|^{2p/(q+1)}}.
\]
This completes the proof.

\(\square\)

End of the proof of Theorem 2.1.

By Lemma 3.4, we have
\[
K_{2,2}(t) \leq C_8 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \int_{|x| \geq t^{(1+q)/(2p)}} |x|^{4a} F(1, x) dx \\
\leq C_8 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \left\{ \int_{t^{(1+q)/(2p)} \leq |x| \leq 1} |x|^{4a-(4b+m)p/q} dx + \int_{|x| \geq 1} |x|^{4a} e^{-c_1 |x|^{2p/(q+1)}} dx \right\}.
\]
Therefore, we see that

\[
K_{2,2}(t) \leq \begin{cases} 
C_9 t^{2-d/2+2b+m/2} \left\{ t^{-(1+q)(4a+n)/(2p)} + t^{-(1+q)(4b+m)/(2q)} \right\}, \\
\text{if } (4a+n)q - (4b+m)p \neq 0, \\
C_9 t^{2-d/2+2b+m/2-(1+q)(4a+n)/(2p)} \log t^{-1}, \\
\text{if } (4a+n)q - (4b+m)p = 0.
\end{cases}
\]

Taking (3.8), (3.10) and (3.13) into consideration, we see that the proof follows.

4. An Example.

In this section, we treat the case where the electric potential is of the form (1.3) and the magnetic potential \( A(x,y) \) satisfies (A.1) and (A.2).

Let \( V(x,y) = (1 + |x|^2)^p |y|^{2q} \) \((p, q > 0 \text{ integers})\).

Corollary 4.1. There exists \( \delta_1 > 0 \) such that:

(i) If \( pm > qn \), \( \text{Tr}[e^{-tH}] = d_1 t^{-(m+mq+qn)/(2q)}(1 + O(t^{\delta_1})) \) as \( t \to 0 \).

(ii) If \( pm = qn \), 
\[
\text{Tr}[e^{-tH}] = d_2 t^{-(m+mq+qn)/(2q)} \log t^{-1} + d_3 t^{-(m+mq+qn)/(2q)}(1 + O(t^{\delta_1}))
\]

as \( t \to 0 \).

(iii) If \( pm < qn \), \( \text{Tr}[e^{-tH}] = d_4 t^{-(m+q)/(2p)}(1 + O(t^{\delta_1})) \) as \( t \to 0 \)

where \( d_i \ (i = 1, 2, 3, 4) \) are given as below.

For the proof, it suffices to note

\[
e^{-tH} = e^{-tH_0} + (e^{-tH} - e^{-tH_0})
\]

and apply Theorem 2.1 and Proposition 1.2.

In order to get the values of the constants \( d_i \), let \( A = -\frac{1}{2} \Delta + |y|^{2q} \) on \( L^2(\mathbb{R}^m) \). According to [3], the complex powers \( A^{-s} \ ((s \in \mathbb{C}) \) of \( A \) are defined for large \( \text{Re} s > 0 \), the trace \( \text{Tr}[A^{-s}] \) has a meromorphic extension \( Z_A(s) \) in \( \mathbb{C} \) whose singularities are all simple poles \( \{ s_j = (m(1+q)-j)/(2q) \}_{j=0,1,...} \).

Therefore, we can write

\[
Z_A(s) = \frac{\text{Res}(s_0)}{s - s_0} + C_{(m,q)} + O(|s - s_0|) \quad \text{as } s \to s_0.
\]
Here $\text{Res}(s_0)$ denotes the residue of $Z_A(s)$ at $s_0$ and $C_{(m,q)}$ is a constant. Then we have

\[
d_1 = 2^{1-d/2} \frac{1}{q(1+q)} \Gamma(m/(2q)) \Gamma((pm - qn)/(2q)) / \Gamma(m/2) \Gamma(pm/(2q)),
\]

\[
d_2 = 2^{1-d/2} \frac{1}{pq(1+q)} \Gamma(m/(2q)) / \Gamma(m/2) \Gamma(n/(2q)),
\]

\[
d_3 = 2^{1-d/2} \frac{1}{pq(1+q)} \left( \Gamma(m/(2q)) \psi((m + mq + nq)/(2q)) / \psi(m/q) - \frac{\gamma \Gamma(m/(2q))}{q(1+q) \Gamma(m/2)} \right)
\]

\[
+ \frac{\Gamma(m/(2q))}{pq(1+q) \Gamma(m/2)} \left( \psi \left( \frac{m + mq}{2q} \right) - \frac{p}{1+q} \psi \left( \frac{n}{2q} \right) - \psi \left( \frac{m + mq + nq}{2q} \right) \right) \bigg],
\]

\[
d_4 = 2^{-n/2} \frac{1 + q}{pq(1+q) \Gamma(m/2) \Gamma(n/(2q)) \Gamma((m + mq + nq)/(2q))} \Gamma((n + 1 + q)/(2p)) Z_A \left( \frac{n(1 + q)}{2p} \right)
\]

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and $\gamma$ is the Euler constant.

In the particular case where $q = m = 1$, we have $s_0 = 1$, $\text{Res}(s_0) = 2^{-1/2}$ and $C_{(1,1)} = 2^{-1/2} (\gamma + \log 2)$. Moreover, for $\text{Res} > 1$, we see that $Z_A(s) = (2^{s/2} - 2^{-s/2}) R(s)$ where $R(s)$ is the Riemann zeta function. Thus we have

\[
d_1 = 2^{-(1+n)/2} \frac{\Gamma((p - n)/2)}{\Gamma(p/2)},
\]

\[
d_2 = 2^{(1-n)/2} \frac{1}{p \Gamma(n/2)},
\]

\[
d_3 = 2^{(1-n)/2} \frac{1}{\Gamma(n/2)} \left[ \frac{2 - p + 2^{3/2}}{2p} \gamma - \frac{p}{2} \psi \left( \frac{n}{2} \right) + \frac{21/2}{p} \log 2 \right],
\]

\[
d_4 = 2^{1-n/2} \frac{\Gamma((n + 2p)/(2p))}{p \Gamma(n/2) \Gamma(1 + n/2)} (2^{n/(2p)} - 2^{-n/(2p)}) R \left( \frac{n}{p} \right).
\]

For a more precise argument, see [3].

References


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IMAGINARY QUADRATIC FIELDS \( k \) WITH
\( \text{Cl}_2(k) \simeq (2, 2^m) \) AND RANK \( \text{Cl}_2(k^1) = 2 \)

E. BENJAMIN, F. LEMMERMEYER, AND C. SNYDER

Let \( k \) be an imaginary quadratic number field and \( k^1 \) the
Hilbert 2-class field of \( k \). We give a characterization of those
\( k \) with \( \text{Cl}_2(k) \simeq (2, 2^m) \) such that \( \text{Cl}_2(k^1) \) has 2 generators.

1. Introduction.

Let \( k \) be an algebraic number field with \( \text{Cl}_2(k) \), the Sylow 2-subgroup of its
ideal class group, \( \text{Cl}(k) \). Denote by \( k^1 \) the Hilbert 2-class field of \( k \) (in the
wide sense). Also let \( k^n \) (for \( n \) a nonnegative integer) be defined inductively
as: \( k^0 = k \) and \( k^{n+1} = (k^n)^1 \). Then

\[
k^0 \subseteq k^1 \subseteq k^2 \subseteq \cdots \subseteq k^n \subseteq \cdots
\]

is called the 2-class field tower of \( k \). If \( n \) is the minimal integer such that
\( k^n = k^{n+1} \), then \( n \) is called the length of the tower. If no such \( n \) exists, then
the tower is said to be of infinite length.

At present there is no known decision procedure to determine whether or
not the (2-)class field tower of a given field \( k \) is infinite. However, it is known
by group theoretic results (see [2]) that if rank \( \text{Cl}_2(k^1) \leq 2 \), then the tower is
finite, in fact of length at most 3. (Here the rank means minimal number of
generators.) On the other hand, until now (see Table 1 and the penultimate
paragraph of this introduction) all examples in the mathematical literature
of imaginary quadratic fields with rank \( \text{Cl}_2(k^1) \geq 3 \) (let us mention in par-
ticular Schmithals [13]) have infinite 2-class field tower. Nevertheless, if
we are interested in developing a decision procedure for determining if the
2-class field tower of a field is infinite, then a good starting point would be
to find a procedure for sieving out those fields with rank \( \text{Cl}_2(k^1) \leq 2 \). We
have already started this program for imaginary quadratic number fields \( k \).
In [1] we classified all imaginary quadratic fields whose 2-class field \( k^1 \) has
cyclic 2-class group. In this paper we determine when \( \text{Cl}_2(k^1) \) has rank 2
for imaginary quadratic fields \( k \) with \( \text{Cl}_2(k) \) of type \( (2, 2^m) \). (The notation
\( (2, 2^m) \) means the direct sum of a group of order 2 and a cyclic group of or-
der 2^m.) The group theoretic results mentioned above also show that such
fields have 2-class field tower of length 2.
From a classification of imaginary quadratic number fields $k$ with $\text{Cl}_2(k) \simeq (2, 2^n)$ and our results from [1] we see that it suffices to consider discriminants $d = d_1 d_2 d_3$ with prime discriminants $d_1, d_2 > 0, d_3 < 0$ such that exactly one of the $(d_j/p_j)$ equals $-1$ (we let $p_j$ denote the prime dividing $d_j$); thus there are only two cases:

A) $(d_1/p_2) = (d_1/p_3) = +1, (d_2/p_3) = -1$;

B) $(d_1/p_3) = (d_2/p_3) = +1, (d_1/p_2) = -1$.

The $C_4$-factorization corresponding to the nontrivial 4-part of $\text{Cl}_2(k)$ is $d = d_1 \cdot d_2 d_3$ in case A) and $d = d_1 d_2 \cdot d_3$ in case B). Note that, by our results from [1], some of these fields have cyclic $\text{Cl}_2(k^1)$; however, we do not exclude them right from the start since there is no extra work involved and since it provides a welcome check on our earlier work.

The main result of the paper is that $\text{rank} \text{Cl}_2(k^1) = 2$ only occurs for fields of type B); more precisely, we prove the following:

**Theorem 1.** Let $k$ be a complex quadratic number field with $\text{Cl}_2(k) \simeq (2, 2^n)$, and let $k^1$ be its 2-class field. Then $\text{rank} \text{Cl}_2(k^1) = 2$ if and only if $\text{disc } k = d_1 d_2 d_3$ is the product of three prime discriminants $d_1, d_2 > 0$ and $-4 \neq d_3 < 0$ such that $(d_1/p_3) = (d_2/p_3) = +1, (d_1/p_2) = -1$, and $h_2(K) = 2$, where $K$ is a nonnormal quartic subfield of one of the two unramified cyclic quartic extensions of $k$ such that $\mathbb{Q}(\sqrt[4]{d_1 d_2}) \subset K$.

This result is the first step in the classification of imaginary quadratic number fields $k$ with rank $\text{Cl}_2(k^1) = 2$; it remains to solve these problems for fields with rank $\text{Cl}_2(k^1) = 3$ and those with $\text{Cl}_2(k) \supseteq (4, 4)$ since we know that $\text{rank} \text{Cl}_2(k^1) \geq 5$ whenever $\text{rank} \text{Cl}_2(k) \geq 4$ (using Schur multipliers as in [1]).

As a demonstration of the utility of our results, we give in Table 1 below a list of the first 12 imaginary quadratic fields $k$, arranged by decreasing value of their discriminants, with $\text{rank } \text{Cl}_2(k) = 2$ and noncyclic $\text{Cl}_2(k^1)$.

Here $f$ denotes a generating polynomial for a field $K$ as in Theorem 1, $r$ denotes the rank of $\text{Cl}_2(k^1)$. The cases where $r = 3$ follow from our theorem combined with Blackburn’s upper bound for the number of generators of derived groups (it implies that finite 2-groups $G$ with $G/G' \simeq (2, 4)$ satisfy $\text{rank } G' \leq 3$), see [3].

In order to verify that $\text{Cl}_2(k^1)$ has rank at least 3 for $k = \mathbb{Q}(\sqrt{-2379})$ it is sufficient to show that its genus class field $k_{\text{gen}}$ has class group $4, 4, 8$: In fact, $\text{Cl}_2(k^1)$ then contains a quotient of $4, 4, 8)$ by $(2, 2) \simeq \text{Gal}(k^1/k_{\text{gen}})$, and the claim follows.

We mention one last feature gleaned from the table. It follows from conditional Odlyzko bounds (assuming the Generalized Riemann Hypothesis) that those quadratic fields with $\text{rank } \text{Cl}_2(k^1) \geq 3$ and discriminant $0 > d > -2000$ have finite class field tower; unconditional proofs are not known. Hence, conditionally, we conclude that those $k$ with discriminants $-1015, -1595$ and
2. Group Theoretic Preliminaries.

Let $G$ be a group. If $x, y \in G$, then we let $[x, y] = x^{-1}y^{-1}xy$ denote the commutator of $x$ and $y$. If $A$ and $B$ are nonempty subsets of $G$, then $[A, B]$ denotes the subgroup of $G$ generated by the set $\{[a, b] : a \in A, b \in B\}$.

The lower central series $\{G_j\}$ of $G$ is defined inductively by: $G_1 = G$ and $G_{j+1} = [G, G_j]$ for $j \geq 1$. The derived series $\{G^{(n)}\}$ is defined inductively by: $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geq 0$. Notice that $G^{(1)} = G' = [G, G]$ the commutator subgroup, $G'$, of $G$.

Throughout this section, we assume that $G$ is a finite, nonmetacyclic, 2-group such that its abelianization $G^{ab} = G/G'$ is of type $(2, 2^m)$ for some positive integer $m$ (necessarily $\geq 2$). Let $G = \langle a, b \rangle$, where $a^2 = b^{2^m} = 1$ mod $G_2$ (actually mod $G_3$ since $G$ is nonmetacyclic, cf. [1]); $c_2 = [a, b]$ and $c_{j+1} = [b, c_j]$ for $j \geq 2$.

**Lemma 1.** Let $G$ be as above (but not necessarily metabelian). Suppose that $d(G') = n$ where $d(G')$ denotes the minimal number of generators of the derived group $G' = G_2$ of $G$. Then

$$G' = \langle c_2, c_3, \ldots, c_{n+1} \rangle;$$

### Table 1.

<table>
<thead>
<tr>
<th>disc $k$</th>
<th>factors</th>
<th>$\text{Cl}_2(k)$ type</th>
<th>$f$</th>
<th>$\text{Cl}_2(K)$</th>
<th>$r$</th>
<th>$\text{Cl}<em>2(k</em>{\text{gen}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1015$</td>
<td>$-7 \cdot 5 \cdot 29$</td>
<td>(2, 8) A $x^4 - 22x^2 + 261$</td>
<td>(4)</td>
<td>$\geq 3$</td>
<td>(2, 2, 8)</td>
<td></td>
</tr>
<tr>
<td>$-1240$</td>
<td>$-31 \cdot 8 \cdot 5$</td>
<td>(2, 4) B $x^4 - 6x^2 - 31$</td>
<td>(2)</td>
<td>$2$</td>
<td>(2, 2, 8)</td>
<td></td>
</tr>
<tr>
<td>$-1443$</td>
<td>$-3 \cdot 13 \cdot 37$</td>
<td>(2, 4) B $x^4 - 86x^2 - 75$</td>
<td>(2)</td>
<td>$2$</td>
<td>(2, 2, 8)</td>
<td></td>
</tr>
<tr>
<td>$-1595$</td>
<td>$-11 \cdot 5 \cdot 29$</td>
<td>(2, 4) A $x^4 + 26x^2 + 1445$</td>
<td>(4)</td>
<td>$\geq 3$</td>
<td>(2, 2, 8)</td>
<td></td>
</tr>
<tr>
<td>$-1615$</td>
<td>$-19 \cdot 5 \cdot 17$</td>
<td>(2, 4) B $x^4 + 26x^2 - 171$</td>
<td>(2)</td>
<td>$2$</td>
<td>(2, 2, 8)</td>
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</tr>
<tr>
<td>$-1624$</td>
<td>$-7 \cdot 8 \cdot 29$</td>
<td>(2, 4) B $x^4 - 30x^2 - 7$</td>
<td>(2)</td>
<td>$2$</td>
<td>(2, 2, 8)</td>
<td></td>
</tr>
<tr>
<td>$-1780$</td>
<td>$-4 \cdot 5 \cdot 89$</td>
<td>(2, 4) A $x^4 + 6x^2 + 89$</td>
<td>(4)</td>
<td>$3$</td>
<td>(2, 2, 4)</td>
<td></td>
</tr>
<tr>
<td>$-2035$</td>
<td>$-11 \cdot 5 \cdot 37$</td>
<td>(2, 4) B $x^4 - 54x^2 - 11$</td>
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<td>$3$</td>
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<tr>
<td>$-2067$</td>
<td>$-3 \cdot 13 \cdot 53$</td>
<td>(2, 4) A $x^4 + x^2 + 637$</td>
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<td>$3$</td>
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<tr>
<td>$-2072$</td>
<td>$-7 \cdot 8 \cdot 37$</td>
<td>(2, 4) B $x^4 + 34x^2 - 7$</td>
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<td>$\geq 3$</td>
<td>(2, 2, 8)</td>
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<tr>
<td>$-2379$</td>
<td>$-3 \cdot 13 \cdot 61$</td>
<td>(4, 4) A $x^4 + 18x^2 - 23$</td>
<td>(2)</td>
<td>$3$</td>
<td>(4, 4, 8)</td>
<td></td>
</tr>
<tr>
<td>$-2392$</td>
<td>$-23 \cdot 8 \cdot 13$</td>
<td>(2, 4) B $x^4 + 18x^2 - 23$</td>
<td>(2)</td>
<td>$2$</td>
<td>(2, 2, 8)</td>
<td></td>
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</tbody>
</table>

$-1780$ have finite $(2,2)$-class field tower even though rank $\text{Cl}_2(k^1) \geq 3$. Of course, it would be interesting to determine the length of their towers.

The structure of this paper is as follows: We use results from group theory developed in Section 2 to pull down the condition rank $\text{Cl}_2(k^1) = 2$ from the field $k^1$ with degree $2^{m+2}$ to a subfield $L$ of $k^1$ with degree 8. Using the arithmetic of dihedral fields from Section 4 we then go down to the field $K$ of degree 4 occurring in Theorem 1.
moreover,
\[ G_2/G_2^2 \simeq \langle c_2G_2^2 \rangle \oplus \cdots \oplus \langle c_{n+1}G_2^2 \rangle. \]

Proof. By the Burnside Basis Theorem, \( d(G_2) = d(G_2/\Phi(G)) \), where \( \Phi(G) \) is the Frattini subgroup of \( G \), i.e., the intersection of all maximal subgroups of \( G \), see [5]. But in the case of a 2-group, \( \Phi(G) = G^2 \), see [8]. By Blackburn, [3], since \( G/G^2 \) has elementary derived group, we know that \( G_2/G_2^2 \simeq \langle c_2G_2^2 \rangle \oplus \cdots \oplus \langle c_{n+1}G_2^2 \rangle \). Again, by the Burnside Basis Theorem, \( G_2 = \langle c_2, \ldots, c_{n+1} \rangle \). □

Lemma 2. Let \( G \) be as above. Moreover, assume \( G \) is metabelian. Let \( H \) be a maximal subgroup of \( G \) such that \( H/G' \) is cyclic, and denote the index \( (G':H') \) by \( 2^\kappa \). Then \( G' \) contains an element of order \( 2^\kappa \).

Proof. Without loss of generality, let \( H = \langle b, G' \rangle \). Notice that \( G' = \langle c_2, c_3, \ldots \rangle \) and by our presentation of \( H, H' = \langle c_3, c_4, \ldots \rangle \). Thus, \( G'/H' = \langle c_2H' \rangle \). But since \( (G':H') = 2^\kappa \), the order of \( c_2 \) is \( \geq 2^\kappa \). This establishes the lemma. □

Lemma 3. Let \( G \) be as above and again assume \( G \) is metabelian. Let \( H \) be a maximal subgroup of \( G \) such that \( H/G' \) is cyclic, and assume that \( (G':H') \equiv 0 \mod 4 \). If \( d(G') = 2 \), then \( G_2 = \langle c_2, c_3 \rangle \) and \( G_j = \langle c_2^{2^{j-2}}, c_3^{2^{j-3}} \rangle \) for \( j > 2 \).

Proof. Assume that \( d(G') = 2 \). By Lemma 1, \( G_2 = \langle c_2, c_3 \rangle \) and hence \( c_4 \in \langle c_2, c_3 \rangle \). Write \( c_4 = c_2^x c_3^y \) where \( x, y \) are positive integers. Without loss of generality, let \( H = \langle b, c_2, c_3 \rangle \) and write \( (G':H') = 2^\kappa \) for some \( \kappa \geq 2 \). Since \( c_3, c_4 \in H' \) we have, \( c_3^2 \equiv 1 \mod H' \). By the proof of Lemma 2, this implies that \( x \equiv 0 \mod 2^\kappa \). Write \( x = 2^\nu x_1 \) for some positive integer \( x_1 \). On the other hand, since \( c_4, c_2^{2x_1} \in G_4 \), we see that \( c_3^2 \equiv 1 \mod G_4 \). If \( y \) were odd, then \( c_3 \in G_4 \). This, however, implies that \( G_2 = \langle c_2 \rangle \), contrary to our assumptions. Thus \( y \) is even, say \( y = 2y_1 \). From all of this we see that \( c_4 = c_2^{2x_1} c_3^{2y_1} \). Consequently, by induction we have \( c_j \in \langle c_2^{2^{j-2}}, c_3^{2^{j-3}} \rangle \) for all \( j \geq 4 \). Since \( G_j = \langle c_2^{2^{j-2}}, c_3^{2^{j-3}}, \ldots, c_2^{j-1}, c_j, c_{j+1}, \ldots \rangle \), cf. [1], we obtain the lemma. □

Let us translate the above into the field-theoretic language. Let \( k \) be an imaginary quadratic number field of type A) or B) (see the Introduction), and let \( M/k \) be one of the two quadratic subextensions of \( k^1/k \) over which \( k^1 \) is cyclic. If \( h_2(M) = 2^{m+\kappa} \) and \( \text{Cl}_2(k) = (2, 2^m) \), then Lemma 2 implies that \( \text{Cl}_2(k^1) \) contains an element of order \( 2^\kappa \). Table 2 contains the relevant information for the fields occurring in Table 1. An application of the class number formula to \( M/\mathbb{Q} \) (see e.g., Proposition 3 below) shows immediately that \( h_2(M) = 2^{m+\kappa} \), where \( 2^\kappa \) is the class number of the quadratic subfield \( \mathbb{Q}(\sqrt{d_i d_j}) \) of \( M \), where \( (d_i/p_j) = +1 \); in particular, we always have \( \kappa \geq 2 \),
and the assumption \((G' : H') \geq 4\) is always satisfied for the fields that we consider.

**Table 2.**

<table>
<thead>
<tr>
<th>(M_1)</th>
<th>(\text{Cl}_2(M_1))</th>
<th>(M_2)</th>
<th>(\text{Cl}_2(M_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{Q}(\sqrt{5}, \sqrt{-7 \cdot 29}))</td>
<td>(2, 16)</td>
<td>(\mathbb{Q}(\sqrt{5 \cdot 29}, -7))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{2}, \sqrt{-5 \cdot 31}))</td>
<td>(4, 4)</td>
<td>(\mathbb{Q}(\sqrt{5}, \sqrt{-2 \cdot 31}))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{13}, \sqrt{-3 \cdot 37}))</td>
<td>(2, 16)</td>
<td>(\mathbb{Q}(\sqrt{37}, \sqrt{-3 \cdot 13}))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{-11}, \sqrt{5 \cdot 29}))</td>
<td>(2, 16)</td>
<td>(\mathbb{Q}(\sqrt{29}, \sqrt{-5 \cdot 11}))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{5}, \sqrt{-17 \cdot 19}))</td>
<td>(4, 4)</td>
<td>(\mathbb{Q}(\sqrt{17}, \sqrt{-5 \cdot 19}))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{29}, \sqrt{-2 \cdot 29}))</td>
<td>(2, 16)</td>
<td>(\mathbb{Q}(\sqrt{2}, \sqrt{-2 \cdot 29}))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{5 \cdot 89}, \sqrt{-1}))</td>
<td>(4, 4)</td>
<td>(\mathbb{Q}(\sqrt{5}, \sqrt{-89}))</td>
<td>(2, 8)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{37}, \sqrt{-5 \cdot 11}))</td>
<td>(4, 4)</td>
<td>(\mathbb{Q}(\sqrt{5}, \sqrt{-37 \cdot 11}))</td>
<td>(2, 32)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{53}, \sqrt{-3 \cdot 13}))</td>
<td>(4, 4)</td>
<td>(\mathbb{Q}(\sqrt{13 \cdot 53}, \sqrt{-3}))</td>
<td>(2, 2, 4)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{37}, \sqrt{-2 \cdot 29}))</td>
<td>(2, 16)</td>
<td>(\mathbb{Q}(\sqrt{2}, \sqrt{-2 \cdot 37}))</td>
<td>(2, 16)</td>
</tr>
<tr>
<td>(\mathbb{Q}(\sqrt{13}, \sqrt{-2 \cdot 23}))</td>
<td>(4, 4)</td>
<td>(\mathbb{Q}(\sqrt{2}, \sqrt{-13 \cdot 23}))</td>
<td>(2, 16)</td>
</tr>
</tbody>
</table>

We now use the above results to prove the following useful proposition.

**Proposition 1.** Let \(G\) be a nonmetacyclic 2-group such that \(G/G' \cong (2, 2^m)\); (hence \(m > 1\)). Let \(H\) and \(K\) be the two maximal subgroups of \(G\) such that \(H/G'\) and \(K/G'\) are cyclic. Moreover, assume that \((G' : H') \equiv 0 \mod 4\). Finally, assume that \(N\) is a subgroup of index 4 in \(G\) not contained in \(H\) or \(K\). Then

\[
(N : N') \begin{cases} 
= 2^m & \text{if } d(G') = 1 \\
= 2^{m+1} & \text{if } d(G') = 2 \\
\geq 2^{m+2} & \text{if } d(G') \geq 3
\end{cases}
\]

**Proof.** Without loss of generality we assume that \(G\) is metabelian. Let \(G = \langle a, b \rangle\), where \(a^2 \equiv b^{2^m} \equiv 1 \mod G_3\). Also let \(H = \langle b, G' \rangle\) and \(K = \langle ab, G' \rangle\) (without loss of generality). Then \(N = \langle ab^2, G' \rangle\) or \(N = \langle a, b^4, G' \rangle\).

Suppose that \(N = \langle ab^2, G' \rangle\).

First assume \(d(G') = 1\). Then \(G' = \langle c_2 \rangle\) and thus \(N' = \langle ab^2, c_2 \rangle\). But \([ab^2, c_2] = c_2^2 \eta_4\) for some \(\eta_4 \in G_4 = \langle c_4^3 \rangle\) (cf. Lemma 1 of [1]). Hence, \(N' = \langle c_2^0 \rangle\), and so \((G' : N') = 2\). Since \((N : G') = 2^{m-1}\), we get \((N : N') = 2^m\) as desired.
Next, assume that \( d(G') = 2 \). Then \( N = \langle ab^2, c_2, c_3 \rangle \) by Lemma 1. Notice that \( [ab^2, c_2] = c_2^5 \eta_4 \) and \( [ab^2, c_3] = c_2^5 \eta_5 \) where \( \eta_j \in G_j \) for \( j = 4, 5 \). Hence \( N' = \langle c_2^5 \eta_4, c_3^5 \eta_5, N_3 \rangle \) and so \( \langle c_2^5 \eta_4, c_3^5 \eta_5 \rangle \subseteq N' \). But then \( N'G_5 \supseteq \langle c_2^5, c_3^5 \rangle = G_4 \) by Lemma 3. Therefore, by [5], \( N' \supseteq G_4 \). But notice that \( N_3 \subseteq G_4 \). Thus \( N' = \langle c_2^5, c_3^5 \rangle \) and so \( (G' : N') = 4 \) which in turn implies that \( (N : N') = 2^{m+1} \), as desired.

Finally, assume \( d(G') \geq 3 \). Then \( d(G'/G_5) = 3 \). Moreover there exists an exact sequence

\[
N/N' \rightarrow (N/G_5)/(N/G_5)' \rightarrow 1,
\]
and thus \( \#N^{ab} \geq \#(N/G_5)^{ab} \). Hence it suffices to prove the result for \( G_5 = 1 \) which we now assume. \( N = \langle ab^2, c_2, c_3, c_4 \rangle \) and so, arguing as above, we have \( N' = \langle c_2^5 \eta_4, c_3^5 \eta_5, N_3 \rangle = \langle c_2^5 \eta_4, c_3^5 \eta_5, N_3 \rangle \), where \( \eta_j \in G_j \). But \( N_3 = \langle [ab^2, c_2^5 \eta_4] \rangle = \langle c_2^5 \rangle \). Therefore, \( N' = \langle c_2^5 \eta_4, c_3^5 \rangle \). From this we see that \( (G' : N') = 8 \) and thus \( (N : N') = 2^{m+2} \) as desired.

Now suppose that \( N = \langle a, b^4, G' \rangle \). Then the proof is essentially the same as above once we notice that \( [a, b^4] \equiv c_3^2 c_2^{-4} \mod G_5 \).

This establishes the proposition. \( \square \)

3. Number Theoretic Preliminaries.

**Proposition 2.** Let \( K/k \) be a quadratic extension, and assume that the class number of \( k, h(k) \), is odd. If \( K \) has an unramified cyclic extension \( M \) of order 4, then \( M/k \) is normal and \( \text{Gal}(M/k) \simeq D_4 \).

**Proof.** Rédei and Reichardt [12] proved this for \( k = \mathbb{Q} \); the general case is analogous. \( \square \)

We shall make extensive use of the class number formula for extensions of type \((2, 2)\):

**Proposition 3.** Let \( K/k \) be a normal quartic extension with Galois group of type \((2, 2)\), and let \( k_j \ (j = 1, 2, 3) \) denote the quadratic subextensions. Then

\[
(1) \quad h(K) = 2^{d-k-2-v} q(K) h(k_1) h(k_2) h(k_3)/h(k)^2,
\]

where \( q(K) = (E_K : E_1 E_2 E_3) \) denotes the unit index of \( K/k \) (\( E_j \) is the unit group of \( k_j \)), \( d \) is the number of infinite primes in \( k \) that ramify in \( K/k \), \( k \) is the \( \mathbb{Z} \)-rank of the unit group \( E_k \) of \( k \), and \( v = 0 \) except when \( K \subseteq k(\sqrt{E_k}) \), where \( v = 1 \).

**Proof.** See [10]. \( \square \)

Another important result is the ambiguous class number formula. For cyclic extensions \( K/k \), let \( \text{Am}(K/k) \) denote the group of ideal classes in \( K \) fixed by \( \text{Gal}(K/k) \), i.e., the ambiguous ideal class group of \( K \), and \( \text{Am}_2 \) its 2-Sylow subgroup.
Proposition 4. Let $K/k$ be a cyclic extension of prime degree $p$; then the number of ambiguous ideal classes is given by

$$\# \text{Am}(K/k) = h(k) \frac{p^{t-1}}{(E:H)},$$

where $t$ is the number of primes (including those at $\infty$) of $k$ that ramify in $K/k$, $E$ is the unit group of $k$, and $H$ is its subgroup consisting of norms of elements from $K^\times$. Moreover, $\text{Cl}_p(K)$ is trivial if and only if $p \nmid \# \text{Am}(K/k)$.

Proof. See Lang [9, part II] for the formula. For a proof of the second assertion (see e.g., Moriya [11]), note that $\text{Am}(K/k)$ is defined by the exact sequence

$$1 \longrightarrow \text{Am}(K/k) \longrightarrow \text{Cl}(K) \longrightarrow \text{Cl}(K)^{1-\sigma} \longrightarrow 1,$$

where $\sigma$ generates $\text{Gal}(K/k)$. Taking $p$-parts we see that $p \nmid \# \text{Am}(K/k)$ is equivalent to $\text{Cl}_p(K) = \text{Cl}_p(K)^{1-\sigma}$. By induction we get $\text{Cl}_p(K) \subseteq \text{Cl}_p(K)^p$. But then $\text{Cl}_p(K)$ must be trivial. \(\square\)

We make one further remark concerning the ambiguous class number formula that will be useful below. If the class number $h(k)$ is odd, then it is known that $\# \text{Am}_2(K/k) = 2^r$ where $r = \text{rank} \text{Cl}_2(K)$.

We also need a result essentially due to G. Gras [4]:

Proposition 5. Let $K/k$ be a quadratic extension of number fields and assume that $h_2(k) = \# \text{Am}_2(K/k) = 2$. Then $K/k$ is ramified and

$$\text{Cl}_2(K) \simeq \begin{cases} (2,2) \text{ or } \mathbb{Z}/2^n\mathbb{Z} \ (n \geq 3) & \text{if } \#\kappa_{K/k} = 1, \\ \mathbb{Z}/2^n\mathbb{Z} \ (n \geq 1) & \text{if } \#\kappa_{K/k} = 2, \end{cases}$$

where $\kappa_{K/k}$ denotes the set of ideal classes of $k$ that become principal (capitulate) in $K$.

Proof. We first notice that $K/k$ is ramified. If the extension were unramified, then $K$ would be the 2-class field of $k$, and since $\text{Cl}_2(k)$ is cyclic, it would follow that $\text{Cl}_2(K) = 1$, contrary to assumption.

Before we start with the rest of the proof, we cite the results of Gras that we need (we could also give a slightly longer direct proof without referring to his results). Let $K/k$ be a cyclic extension of prime power order $p^r$, and let $\sigma$ be a generator of $G = \text{Gal}(K/k)$. For any $p$-group $M$ on which $G$ acts we put $M_i = \{m \in M : m^{(1-\sigma)^i} = 1\}$. Moreover, let $\nu$ be the algebraic norm, that is, exponentiation by $1 + \sigma + \sigma^2 + \ldots + \sigma^{p^r-1}$. Then [4, Cor. 4.3] reads:
Lemma 4. Suppose that $M^\nu = 1$; let $n$ be the smallest positive integer such that $M_n = M$ and write $n = a(p - 1) + b$ with integers $a \geq 0$ and $0 \leq b \leq p - 2$. If $\# M_{i+1}/M_i = p$ for $i = 0, 1, \ldots, n - 1$, then $M \simeq (\mathbb{Z}/p^{n+1}\mathbb{Z})^b \times (\mathbb{Z}/p^b\mathbb{Z})^{p-1-b}$.

We claim that if $\kappa_{K/k} = 1$, then $M = \text{Cl}_2(K)$ satisfies the assumptions of Lemma 4: In fact, let $j = j_{K\rightarrow K}$ denote the transfer of ideal classes. Then $c^{1+\sigma} = j(N_{K/k}c)$ for any ideal class $c \in \text{Cl}_2(K)$, hence $M^\nu = j(\text{Cl}_2(k)) = 1$. Moreover, $M_1 = \text{Am}_2(K/k)$ in our case, hence $M_1/M_0$ has order 2. Since the orders of $M_{i+1}/M_i$ decrease towards 1 as $i$ grows (Gras [4, Prop. 4.1.ii]), we conclude that $\# M_{i+1}/M_i = 2$ for all $i < n$. Since $a = n$ and $b = 0$ when $p = 2$, Lemma 4 now implies that $\text{Cl}_2(K) \simeq \mathbb{Z}/2^n\mathbb{Z}$, that is, the 2-class group is cyclic.

The second result of Gras that we need is [4, Prop. 4.3]:

Lemma 5. Suppose that $M^\nu \neq 1$ but assume the other conditions in Lemma 4. Then $n \geq 2$ and

$$M \simeq \begin{cases} (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^{n-2} & \text{if } n < p; \\ (\mathbb{Z}/p\mathbb{Z})^p \text{ or } (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^{n-2} & \text{if } n = p; \\ (\mathbb{Z}/p^{n+1}\mathbb{Z})^b \times (\mathbb{Z}/p^b\mathbb{Z})^{p-1-b} & \text{if } n > p. \end{cases}$$

If $\kappa_{K/k} = 1$, then this lemma shows that $\text{Cl}_2(K)$ is either cyclic of order $\geq 4$ or of type $(2, 2)$. (Notice that the hypothesis of the lemma is satisfied since $K/k$ is ramified implying that the norm $N_{K/k} : \text{Cl}_2(K) \rightarrow \text{Cl}_2(k)$ is onto; and so the argument above this lemma applies.) It remains to show that the case $\text{Cl}_2(K) \simeq \mathbb{Z}/4\mathbb{Z}$ cannot occur here.

Now assume that $\text{Cl}_2(K) = \langle C \rangle \simeq \mathbb{Z}/4\mathbb{Z}$; since $K/k$ is ramified, the norm $N_{K/k} : \text{Cl}_2(K) \rightarrow \text{Cl}_2(k)$ is onto, and using $\kappa_{K/k} = 1$ once more we find $C^{1+\sigma} = c$, where $c$ is the nontrivial ideal class from $\text{Cl}_2(k)$. On the other hand, $c \in \text{Cl}_2(k)$ still has order 2 in $\text{Cl}_2(K)$, hence we must also have $C^2 = C^{1+\sigma}$. But this implies that $C^\sigma = C$, i.e., that each ideal class in $K$ is ambiguous, contradicting our assumption that $\# \text{Am}_2(K/k) = 2$. \hfill \Box

4. Arithmetic of some Dihedral Extensions.

In this section we study the arithmetic of some dihedral extensions $L/Q$, that is, normal extensions $L$ of $Q$ with Galois group $\text{Gal}(L/Q) \simeq D_4$, the dihedral group of order 8. Hence $D_4$ may be presented as $\langle \tau, \sigma \mid \tau^2 = \sigma^4 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$. Now consider the following diagrams (Galois correspondence):
In this situation, we let $q_1 = (E_L : E_1 E_2 E_K)$ and $q_2 = (E_L : E_2 E_2 E_K)$ denote the unit indices of the bicyclic extensions $L/k_1$ and $L/k_2$, where $E_i$ and $E_i'$ are the unit groups in $K_i$ and $K_i'$, respectively. Finally, let $\kappa_i$ denote the kernel of the transfer of ideal classes $j_{k_i - K_i} : \text{Cl}_2(k_i) \rightarrow \text{Cl}_2(K_i)$ for $i = 1, 2$.

The following remark will be used several times: If $K_1 = k_1(\sqrt{\alpha})$ for some $\alpha \in k_1$, then $k_1 = \mathbb{Q}(\sqrt{\alpha})$, where $a = \alpha \alpha'$ is the norm of $\alpha$. To see this, let $\gamma = \sqrt{\alpha}$; then $\gamma^\sigma = \gamma$, since $\gamma \in K_1$. Clearly $\gamma^{1+\sigma} = \sqrt{\alpha} \in K$ and hence fixed by $\sigma^2$. Furthermore,

$$\langle \gamma^{1+\sigma} \rangle = \gamma^{\sigma^2 + \sigma^2} = \gamma^{\sigma^3 + \sigma^2} = (\gamma^\sigma)^{\sigma^3 + \sigma^2} = \gamma^{1+\sigma},$$

implying that $\sqrt{\alpha} \in k_2$. Finally notice that $\sqrt{\alpha} \notin \mathbb{Q}$, since otherwise $\sqrt{\alpha} = \sqrt{\gamma} \in K_1$ implying that $K_1/\mathbb{Q}$ is normal, which is not the case.

Recall that a quadratic extension $K = k(\sqrt{\alpha})$ is called *essentially ramified* if $\alpha \mathcal{O}_K$ is not an ideal square. This definition is independent of the choice of $\alpha$.

**Proposition 6.** Let $L/\mathbb{Q}$ be a non-CM totally complex dihedral extension not containing $\sqrt{-1}$, and assume that $L/K_1$ and $L/K_2$ are essentially ramified. If the fundamental unit of the real quadratic subfield of $K$ has norm $-1$, then $q_1 q_2 = 2$.

**Proof.** Notice first that $k$ cannot be real (in fact, $K$ is not totally real by assumption, and since $L/k$ is a cyclic quartic extension, no infinite prime can ramify in $K/k$); thus exactly one of $k_1, k_2$ is real, and the other is complex. Multiplying the class number formulas, Proposition 3, for $L/k_1$ and $L/k_2$ (note that $\nu = 0$ since both $L/K_1$ and $L/K_2$ are essentially ramified) we find that $2q_1 q_2$ is a square. If we can prove that $q_1, q_2 \leq 2$, then $2q_1 q_2$ is a square between 2 and 8, which implies that we must have $2q_1 q_2 = 4$ and $q_1 q_2 = 2$ as claimed.

We start by remarking that if $\zeta \eta$ becomes a square in $L$, where $\zeta$ is a root of unity in $L$, then so does one of $\pm \eta$. This follows from the fact that the only nontrivial roots of unity that can be in $L$ are the sixth roots of unity $\langle \zeta_6 \rangle$, and here $\zeta_6 = -\zeta_3^2$.

Now we prove that $q_1 \leq 2$ under the assumptions we made; the claim $q_2 \leq 2$ will then follow by symmetry. Assume first that $k_1$ is real and let $\varepsilon$ be the fundamental unit of $k_1$. We claim that $\sqrt{\pm \varepsilon} \notin L$. Suppose
otherwise; then $k_1(\sqrt{\pm e})$ is one of $K_1$, $K'_1$ or $K$. If $k_1(\sqrt{\pm e}) = K_1$, then $K'_1 = k_1(\sqrt{\pm e'})$ and $K = k_1(\sqrt{ee'})$. (Here and below $x' = x^\sigma$.) This however cannot occur since by assumption $ee' = -1$ implying that $\sqrt{-1} \in L$, a contradiction. Similarly, if $k_1(\sqrt{\pm e}) = K$, then again $\sqrt{-1} \in L$.

Thus $\sqrt{\pm e} \notin L$, and $E_1 = \langle -1, e, \eta \rangle$ for some unit $\eta \in E_1$. Suppose that $\sqrt{u\eta} \in L$ for some unit $u \in k_1$. Then $L = K_1(\sqrt{u\eta})$, contradicting our assumption that $L/K_1$ is essentially ramified. The same argument shows that $\sqrt{u\eta'} \notin L$, hence either $E_L = \langle \zeta, e, \eta, \eta' \rangle$ and $q_1 = 1$ or $E_L = \langle \zeta, e, \eta, \sqrt{u\eta\eta'} \rangle$ for some unit $u \in k_1$ and $q_1 = 2$. Here $\zeta$ is a root of unity generating the torsion subgroup $W_L$ of $E_L$.

Next consider the case where $k_1$ is complex, and let $\varepsilon$ denote the fundamental unit of $k_2$. Then $\pm \varepsilon$ stays fundamental in $L$ by the argument above.

Let $\eta$ be a fundamental unit in $K_1$. If $\pm \eta$ became a square in $L$, then clearly $L/K_1$ could not be essentially ramified. Thus if we have $q_1 \geq 4$, then $\pm \varepsilon \eta = \alpha^2$ is a square in $L$. Applying $\tau$ to this relation we find that $-1 = \varepsilon \varepsilon'$ is a square in $L$, contradicting the assumption that $L$ does not contain $\sqrt{-1}$. \hfill \Box

**Proposition 7.** Suppose that $q_2 = 1$. Then $K_2/k_2$ is essentially ramified if and only if $\kappa_2 = 1$; if $K_2/k_2$ is not essentially ramified, then $\kappa_2 = \langle [b] \rangle$, where $K_2 = k_2(\sqrt{\beta})$ and $\langle \beta \rangle = b^2$.

*Proof.* First notice that if $K_2/k_2$ is not essentially ramified, then $\kappa_2 \neq 1$: In fact, in this case we have $\langle \beta \rangle = b^2$, and if we had $\kappa_2 = 1$, then $b$ would have to be principal, say $b = (\gamma)$. This implies that $\beta = \varepsilon \gamma^2$ for some unit $\varepsilon \in k_2$, which in view of $q_2 = 1$ implies that $\varepsilon$ must be a square. But then $b$ would be a square, and this is impossible.

Conversely, suppose $\kappa_2 \neq 1$. Let $a$ be a nonprincipal ideal in $k_2$ of absolute norm $a$, and assume that $a = (\alpha)$ in $K_2$. Then $\alpha^{1-\sigma^2} = \eta$ for some unit $\eta \in E_2$, and similarly $\alpha^{\sigma^2 - \sigma^3} = \eta'$, where $\eta'$ is a unit in $E'_2$. But then $\eta' = \alpha^{1+\sigma^2 - \sigma^3 - \sigma^2} = N_{L/k} \alpha = \pm N_{L/k} a = \pm \sigma^2 = \pm 1$ in $L^\times$, where $2^\pm$ means equal up to a square in $L^\times$. Thus $\pm \eta' \in L$ is a square in $L$, so our assumption that $q_2 = 1$ implies that $\pm \eta' \in k_2$. The same argument show that $\pm \eta'' = \pm \eta \eta' \in K_2$, hence we find $\eta 
\in k_2$. Thus $\alpha^{1-\sigma^2}$ is fixed by $\sigma^2$ and so $\beta := \alpha^2 \in k_2$. This gives $K_2 = k_2(\sqrt{\beta})$, hence $K_2/k_2$ is not essentially ramified, and moreover, $a \sim b$. \hfill \Box

From now on assume that $k$ is one of the imaginary quadratic fields of type A) or B) as explained in the Introduction. Let

- $k_1 = \mathbb{Q}(\sqrt{d_1})$ and $k_2 = \mathbb{Q}(\sqrt{d_2d_3})$ in case A), and
- $k_1 = \mathbb{Q}(\sqrt{d_3})$ and $k_2 = \mathbb{Q}(\sqrt{d_1d_2})$ in case B).
Then there exist two unramified cyclic quartic extensions of \( k \) which are \( D_4 \) over \( \mathbb{Q} \) (see Proposition 2). Let us say a few words about their construction. Consider e.g., case B); by Rédei’s theory (see [12]), the \( C_4 \)-factorization \( d = d_1 d_2 \cdot d_3 \) implies that unramified cyclic quartic extensions of \( k = \mathbb{Q}(\sqrt{d}) \) are constructed by choosing a “primitive” solution \((x, y, z)\) of \( d_1 d_2 x^2 + d_3 y^2 = z^2 \) and putting \( L = k(\sqrt{d_1 d_2}, \sqrt{\alpha}) \) with \( \alpha = z + x\sqrt{d_1 d_2} \) (primitive here means that \( \alpha \) should not be divisible by rational integers); the other unramified cyclic quartic extension is then \( L = k(\sqrt{d_1 d_2}, \sqrt{d_1 \alpha}) \). Since \( 4\alpha \beta = (x\sqrt{d_1 d_2} + y\sqrt{d_3} + z)^2 \) for \( \beta = \frac{1}{2}(z + y\sqrt{d_3}) \), we also have \( L = k(\sqrt{d_2}, \sqrt{\beta}) \) etc. If \( d_3 = -4 \), then it is easy to see that we may choose \( \beta \) as the fundamental unit of \( k_2 \); if \( d_3 \neq -4 \), then genus theory says that a) the class number \( h \) of \( k_2 \) is twice an odd number \( u \); and b) the prime ideal \( \mathfrak{p}_3 \) above \( d_3 \) in \( k_2 \) is in the principal genus, so \( \mathfrak{p}_3^2 = (\pi_3) \) is principal. Again it can be checked that \( \beta = \pm \pi_3 \) for a suitable choice of the sign.

**Example.** Consider the case \( d = -31 \cdot 5 \cdot 8 \); here \( \pi_3 = \pm (3 + 2\sqrt{10}) \), and the positive sign is correct since \( 3 + 2\sqrt{10} \equiv (1 + \sqrt{10})^2 \mod 4 \) is primary. The minimal polynomial of \( \sqrt{\pi_3} \) is \( f(x) = x^4 - 6x^2 - 31 \); Compare Table 1.

The fields \( K_2 = k_2(\sqrt{\alpha}) \) and \( \bar{K}_2 = k_2(\sqrt{d_2 \alpha}) \) will play a dominant role in the proof below; they are both contained in \( M = K(\sqrt{\alpha}) \) for \( F = k_2(\sqrt{d_2}) \), and it is the ambiguous class group \( \text{Am}(M/F) \) that contains the information we are interested in.

**Lemma 6.** The field \( F \) has odd class number (even in the strict sense), and we have \( \# \text{Am}(M/F) | 2 \). In particular, \( \text{Cl}_2(M) \) is cyclic (though possibly trivial).

**Proof.** The class group in the strict sense of \( k_2 \) is cyclic of order 2 by Rédei’s theory [12] (since \((d_2/p_3) = (d_3/p_2) = -1 \) in case A) and \((d_1/p_2) = (d_2/p_1) = -1 \) in case B). Since \( F \) is the Hilbert class field of \( k_2 \) in the strict sense, its class number in the strict sense is odd.

Next we apply the ambiguous class number formula. In case A), \( F \) is complex, and exactly the two primes above \( d_3 \) ramify in \( M/F \). Note that \( M = F(\sqrt{\alpha}) \) with \( \alpha \) primary of norm \( d_3 y^2 \); there are four primes above \( d_3 \) in \( F \), and exactly two of them divide \( \alpha \) to an odd power, so \( t = 2 \) by the decomposition law in quadratic Kummer extensions. By Proposition 4 and the remarks following it, \( \# \text{Am}_2(M/F) = 2/(E:H) \leq 2 \), and \( \text{Cl}_2(M) \) is cyclic.

In case B), however, \( F \) is real; since \( \alpha \in k_2 \) has norm \( d_3 y^2 < 0 \), it has mixed signature, hence there are exactly two infinite primes that ramify in \( M/F \). As in case A), there are two finite primes above \( d_3 \) that ramify in \( M/F \), so we get \( \# \text{Am}_2(M/F) = 8/(E:H) \). Since \( F \) has odd class number in the strict sense, \( F \) has units of independent signs. This implies that the
group of units that are positive at the two ramified infinite primes has $\mathbb{Z}$-rank 2, i.e., $(E : H) \geq 4$ by consideration of the infinite primes alone. In particular, $\#\text{Am}_2(M/F) \leq 2$ in case B. \hfill $\square$

Next we derive some relations between the class groups of $K_2$ and $\overline{K}_2$; these relations will allow us to use each of them as our field $K$ in Theorem 1.

**Proposition 8.** Let $L$ and $\overline{L}$ be the two unramified cyclic quartic extensions of $k$, and let $K_2$ and $\overline{K}_2$ be two quadratic extensions of $k_2$ in $L$ and $\overline{L}$, respectively, which are not normal over $\mathbb{Q}$.

a) We have $4 \mid h(K_2)$ if and only if $4 \mid h(\overline{K}_2)$;

b) If $4 \mid h(K_2)$, then one of $\text{Cl}_2(K_2)$ or $\text{Cl}_2(\overline{K}_2)$ has type $(2, 2)$, whereas the other is cyclic of order $\geq 4$.

**Proof.** Notice that the prime dividing $\text{disc}(k_1)$ splits in $k_2$. Throughout this proof, let $p$ be one of the primes of $k_2$ dividing $\text{disc}(k_1)$.

If we write $K_2 = k_2(\sqrt{\alpha})$ for some $\alpha \in k_2$, then $\overline{K}_2 = k_2(\sqrt{d_2\alpha})$. In fact, $K_2$ and $\overline{K}_2$ are the only extensions $F/k_2$ of $k_2$ with the properties

1. $F/k_2$ is a quadratic extension unramified outside $p$;
2. $kF/k$ is a cyclic extension.

Therefore it suffices to observe that if $k_2(\sqrt{\alpha})$ has these properties, then so does $k_2(\sqrt{d_2\alpha})$. But this is elementary.

In particular, the compositum $M = K_2\overline{K}_2 = k_2(\sqrt{d_2}, \sqrt{\alpha})$ is an extension of type $(2, 2)$ over $k_2$ with subextensions $K_2$, $\overline{K}_2$ and $F = k_2(\sqrt{d_2})$. Clearly $F$ is the unramified quadratic extension of $k_2$, so both $M/K_2$ and $M/\overline{K}_2$ are unramified. If $K_2$ had 2-class number 2, then $M$ would have odd class number, and $M$ would also be the 2-class field of $\overline{K}_2$. Thus $2 \parallel h(K_2)$ implies that $2 \parallel h(\overline{K}_2)$. This proves part a) of the proposition.

Before we go on, we give a Hasse diagram for the fields occurring in this proof:

```
  N
 / \ /
M---F1-----F2
 / \          /
K2   \         /
   \   \      K2
    \   /
 k2
```

Now assume that $4 \mid h(K_2)$. Since $\text{Cl}_2(M)$ is cyclic by Lemma 6, there is a unique quadratic unramified extension $N/M$, and the uniqueness implies at once that $N/k_2$ is normal. Hence $G = \text{Gal}(N/k_2)$ is a group of order 8 containing a subgroup of type $(2, 2) \simeq \text{Gal}(N/F)$: In fact, if $\text{Gal}(N/F)$
were cyclic, then the primes ramifying in \( M/F \) would also ramify in \( N/M \) contradicting the fact that \( N/M \) is unramified. There are three groups satisfying these conditions: \( G = (2, 4), G = (2, 2, 2) \) and \( G = D_4 \). We claim that \( G \) is nonabelian; once we have proved this, it follows that exactly one of the groups \( \text{Gal}(N/K_2) \) and \( \text{Gal}(N/\bar{K}_2) \) is cyclic, and that the other is not, which is what we want to prove.

So assume that \( G \) is abelian. Then \( M/F \) is ramified at two finite primes \( q \) and \( q' \) of \( F \) dividing \( d \) (in \( k_2 \)); if \( F_1 \) and \( F_2 \) denote the quadratic subextensions of \( N/F \) different from \( M \) then \( F_1/F \) and \( F_2/F \) must be ramified at a finite prime (since \( F \) has odd class number in the strict sense: See Lemma 6); since both \( F_1 \) and \( F_2 \) are normal (even abelian) over \( k_2 \), ramification at \( q \) implies ramification at the conjugated ideal \( q' \). Hence both \( q \) and \( q' \) ramify in \( F_1/F \) and \( F_2/F \), and since they also ramify in \( M/F \), they must ramify completely in \( N/F \), again contradicting the fact that \( N/M \) is unramified.

We have proved that \( \text{Cl}_2(K_2) \) and \( \text{Cl}_2(\bar{K}_2) \) contain subgroups of type \((4)\) and \((2, 2)\), respectively. Now we wish to apply Proposition 5. But we have to compute \# \( \text{Am}_2(\bar{K}_2/k_2) \). Since the class number of \( \bar{K}_2 \) is even, it is sufficient to show that \# \( \text{Am}_2(\bar{K}_2/k_2) \leq 2 \). In case \( \text{A} \), there is exactly one ramified prime (it divides \( d_1 \)), hence \# \( \text{Am}_2(\bar{K}_2/k_2) = 2/(E : H) \leq 2 \). In case \( \text{B} \), there are two ramified primes (one is infinite, the other divides \( d_3 \)), hence \# \( \text{Am}_2(\bar{K}_2/k_2) = 4/(E : H) \); but \(-1\) is not a norm residue at the ramified infinite prime, hence \((E : H) \geq 2\) and \# \( \text{Am}_2(\bar{K}_2/k_2) \leq 2 \) as claimed.

Now Proposition 5 implies that \( \text{Cl}_2(K_2) \) is cyclic of order \( \geq 4 \), and that \( \text{Cl}_2(\bar{K}_2) \simeq (2, 2) \). This concludes our proof.

\[ \]

**Proposition 9.** Assume that \( k \) is one of the imaginary quadratic fields of type \( \text{A} \) or \( \text{B} \) as explained in the Introduction. Then there exist two unramified cyclic quartic extensions of \( k \). Let \( L \) be one of them, and write

\[
\begin{align*}
k_1 &= \mathbb{Q}(\sqrt{d_1}) \text{ and } k_2 = \mathbb{Q}(\sqrt{d_2d_3}) \text{ in case } \text{A}, \\
k_1 &= \mathbb{Q}(\sqrt{d_3}) \text{ and } k_2 = \mathbb{Q}(\sqrt{d_1d_2}) \text{ in case } \text{B}.
\end{align*}
\]

Then \( h_2(L) = \frac{1}{4}h_2(k)h_2(K_1)h_2(K_2) \) unless possibly when \( d_3 = -4 \) in case \( \text{B} \).

*Proof.* Observe that \( v = 0 \) in case \( \text{A} \) and \( \text{B} \); Kuroda’s class number formulas for \( L/k_1 \) and \( L/k_2 \) gives

\[
h_2(L) = \frac{q_1h_2(K_1)^2h_2(K)}{2h_2(k_1)^2} = \frac{q_2h_2(K_2)^2h_2(K)}{4h_2(k_2)^2}
\]

in case \( \text{A} \) and

\[
h_2(L) = \frac{q_1h_2(K_1)^2h_2(K)}{4h_2(k_1)^2} = \frac{q_2h_2(K_2)^2h_2(K)}{2h_2(k_2)^2}
\]

in case \( \text{B} \).
in case B). Multiplying them together and plugging in the class number formula for $K/Q$ yields

$$h_2(L)^2 = \frac{q_1 q_2 h_2(K_1)^2 h_2(K_2)^2 h_2(k)^2}{h_2(k_1)^2 h_2(k_2)^2}.$$ 

Now $h_2(k_1) = 1$, $h_2(k_2) = 2$ and $q_1 q_2 = 2$ (by Proposition 6), and taking the square root we find $h_2(L) = \frac{1}{4} h_2(k) h_2(K_1) h_2(K_2)$ as claimed. □

5. Classification.

In this section we apply the results obtained in the last few sections to give a proof for Theorem 1.

Proof of Theorem 1. Let $L$ be one of the two cyclic quartic unramified extensions of $k$, and let $N$ be the subgroup of $\text{Gal}(k^2/k)$ fixing $L$. Then $N$ satisfies the assumptions of Proposition 1, thus there are only the following possibilities:

<table>
<thead>
<tr>
<th>$d(G')$</th>
<th>$h_2(L)$</th>
<th>$h_2(K_1) h_2(K_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2^m$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$2^{m+1}$</td>
<td>4</td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$\geq 2^{m+2}$</td>
<td>$\geq 8$</td>
</tr>
</tbody>
</table>

Here, the first two columns follow from Proposition 1, the last (which we do not claim to hold if $d_3 = -4$ in case B) is a consequence of the class number formula of Proposition 9. In particular, we have $d(G') \geq 3$ if one of the class numbers $h_2(K_1)$ or $h_2(K_2)$ is at least 8. Therefore it suffices to examine the cases $h_2(K_2) = 2$ and $h_2(K_2) = 4$ (recall from above that $h_2(K_2)$ is always even).

We start by considering case A); it is sufficient to show that $h_2(K_1) h_2(K_2) \neq 4$. We now apply Proposition 5; notice that we may do so by the proof of Proposition 8.

a) If $h_2(K_2) = 2$, then $\#\kappa_2 = 2$ by Proposition 5, hence $q_2 = 2$ by Proposition 7 and then $q_1 = 1$ by Proposition 6. The class number formulas in the proof of Proposition 9 now give $h_2(K_1) = 1$ and $h_2(L) = 2^m$.

It can be shown using the ambiguous class number formula that $\text{Cl}_2(K_1)$ is trivial if and only if $\varepsilon_1$ is a quadratic nonresidue modulo the prime ideal over $d_2$ in $k_1$; by Scholz’s reciprocity law, this is equivalent to $(d_1/d_2)_4 (d_2/d_1)_4 = 1$, and this agrees with the criterion given in [1].

b) If $h_2(K_2) = 4$, we may assume that $\text{Cl}_2(K_2) = (4)$ from Proposition 8.b). Then $\#\kappa_2 = 2$ by Proposition 5, $q_2 = 2$ by Proposition 7 and $q_1 = 1$ by Proposition 6. Using the class number formula we get $h_2(K_1) = 2$ and $h_2(L) = 2^{m+2}$. 


Thus in both cases we have $h_2(K_1)h_2(K_2) \neq 4$, and by the table at the beginning of this proof this implies that rank $\operatorname{Cl}(k^1) \neq 2$ in case A).

Next we consider case B; here we have to distinguish between $d_3 \neq -4$ (case $B_1$) and $d_3 = -4$ (case $B_2$).

Let us start with case $B_1$.

a) If $h_2(K_2) = 2$, then $\#\kappa_2 = 2$, $q_2 = 2$ and $q_1 = 1$ as above. The class number formula gives $h_2(K_1) = 2$ and $h_2(L) = 2^{m+1}$.

b) If $\operatorname{Cl}_2(K_2) = (4)$ (which we may assume without loss of generality by Proposition 8.b)) then $\#\kappa_2 = 2$, $q_2 = 2$ and $q_1 = 1$, again exactly as above. This implies $h_2(K_1) = 4$ and $h_2(L) = 2^{m+3}$.

Finally, consider case $B_2$.

Here we apply Kuroda’s class number formula (see [10]) to $L/k_1$, and since $h_2(k_1) = 1$ and $h_2(K_1) = h_2(K_1')$, we get $h_2(L) = \frac{1}{2}q_1h_2(K_1)^2h_2(k) = 2^mq_1h_2(K_1)^2$. From $K_2 = k_2(\sqrt{\varepsilon})$ (for a suitable choice of $L$; the other possibility is $K_2 = k_2(\sqrt{\bar{\varepsilon}})$), where $\varepsilon$ is the fundamental unit of $k_2$, we deduce that the unit $\varepsilon$, which still is fundamental in $k_1$, becomes a square in $L$, and this implies that $q_1 \geq 2$. Moreover, we have $K_1 = k_1(\sqrt{\pi})$, where $\pi, \lambda \equiv 1 \mod 4$ are prime factors of $d_1$ and $d_2$ in $k_1 = \mathbb{Q}(i)$, respectively. This shows that $K_1$ has even class number, because $K_1(\sqrt{\pi})/K_1$ is easily seen to be unramified.

Thus $2 \mid q_1, 2 \mid h_2(K_1)$, and so we find that $h_2(L)$ is divisible by $2^{m+3}$. In particular, we always have $d(G') \geq 3$ in this case.

This concludes the proof. 

\[\square\]

The referee (whom we’d like to thank for a couple of helpful remarks) asked whether $h_2(K) = 2$ and $h_2(K) > 2$ infinitely often. Let us show how to prove that both possibilities occur with equal density in case $B_1$.

Before we can do this, we have to study the quadratic extensions $K_1$ and $K_2$ of $k_1$ more closely. We assume that $d_2 = p$ and $d_3 = r$ are odd primes in the following, and then say how to modify the arguments in the case $d_2 = 8$ or $d_3 = -8$. The primes $p$ and $r$ split in $k_1$ as $pO_1 = pp'$ and $rO_1 = r\tau'$. Let $h$ denote the odd class number of $k_1$ and write $p^h = (\pi)$ and $r^h = (\rho)$ for primary elements $\pi$ and $\rho$ (this is can easily be proved directly, but it is also a very special case of Hilbert’s first supplementary law for quadratic reciprocity in fields $K$ with odd class number $h$ (see [7]): If $a^h = \alpha O_K$ for an ideal $a$ with odd norm, then $\alpha$ can be chosen primary (i.e., congruent to a square mod $4O_K$) if and only if $a$ is primary (i.e., $[\varepsilon/a] = +1$ for all units $\varepsilon \in O_K^*$, where $[\cdot/\cdot]$ denotes the quadratic residue symbol in $K$)). Let $[\cdot/\cdot]$ denote the quadratic residue symbol in $k_1$. Then $[\pi/\rho][\pi'/\rho] = [p/\rho] = [p/r] = -1$, so we may choose the conjugates in such a way that $[\pi/\rho] = +1$ and $[\pi'/\rho] = [\pi/r] = -1$. 

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Put $K_1 = k_1(\sqrt{\pi \rho})$ and $\tilde{K}_1 = k_1(\sqrt{\pi \rho'})$; we claim that $h_2(\tilde{K}_1) = 2$. This is equivalent to $h_2(\tilde{L}_1) = 1$, where $\tilde{L}_1 = k_1(\sqrt{\pi}, \sqrt{\rho'})$ is a quadratic unramified extension of $\tilde{K}_1$. Put $\tilde{F}_1 = k_1(\sqrt{\pi})$ and apply the ambiguous class number formula to $\tilde{F}_1/k_1$ and $\tilde{L}_1/\tilde{F}_1$: Since there is only one ramified prime in each of these two extensions, we find $\text{Am}(\tilde{F}_1/k_1) = \text{Am}(\tilde{L}_1/\tilde{F}_1) = 1$; note that we have used the assumption that $\left[\pi/\rho\right] = -1$ in deducing that $r'$ is inert in $\tilde{F}_1/k_1$.

In our proof of Theorem 1 we have seen that there are the following possibilities when $h_2(K_2) \mid 4$:

<table>
<thead>
<tr>
<th>$q_2$</th>
<th>$\text{Cl}_2(K_2)$</th>
<th>$q_1$</th>
<th>$h_2(K_1)$</th>
<th>$\tilde{q}_2$</th>
<th>$\text{Cl}_2(\tilde{K}_2)$</th>
<th>$h_2(L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(2)</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>(2)</td>
<td>$2^{m+1}$</td>
</tr>
<tr>
<td>2</td>
<td>(4)</td>
<td>1</td>
<td>4</td>
<td>?</td>
<td>(2, 2)</td>
<td>$2^{m+3}$</td>
</tr>
</tbody>
</table>

In order to decide whether $\tilde{q}_2 = 1$ or $\tilde{q}_2 = 2$, recall that we have $h_2(K_1) = 4$; thus $\tilde{K}_1$ must be the field with 2-class number 2, and this implies $h_2(\tilde{L}) = 2^{m+2}$ and $\tilde{q}_2 = 1$. In particular we see that $4 \mid h_2(K_2)$ if and only if $4 \mid h_2(K_1)$ as long as $K_1 = k_1(\sqrt{\pi \rho})$ with $[\pi/\rho] = +1$.

The ambiguous class number formula shows that $\text{Cl}_2(K_1)$ is cyclic, thus $4 \mid h_2(K_1)$ if and only if $2 \mid h_2(L_1)$, where $L_1 = k_1(\sqrt{\pi})$ is the quadratic unramified extension of $K_1$. Applying the ambiguous class number formula to $L_1/F_1$, where $F_1 = k_1(\sqrt{\pi})$, we see that $2 \mid h_2(L_1)$ if and only if $(E : H) = 1$. Now $E$ is generated by a root of unity (which always is a norm residue at primes dividing $r \equiv 1 \mod 4$) and a fundamental unit $\varepsilon$. Therefore $(E : H) = 1$ if and only if $\{\varepsilon/\mathfrak{R}_1\} = \{\varepsilon/\mathfrak{R}_2\} = +1$, where $r\mathfrak{O}_{F_1} = \mathfrak{R}_1 \mathfrak{R}_2$ and where $\{ \cdot/\cdot \}$ denotes the quadratic residue symbol in $F_1$. Since $\{\varepsilon/\mathfrak{R}_1\} \{\varepsilon/\mathfrak{R}_2\} = [\varepsilon/r] = +1$, we have proved that $4 \mid h_2(K_1)$ if and only if the prime ideal $\mathfrak{R}_1$ above $r$ splits in the quadratic extension $F_1(\sqrt{\varepsilon})$. But if we fix $p$ and $q$, this happens for exactly half of the values of $r$ satisfying $(p/r) = -1$, $(q/r) = +1$.

If $d_2 = 8$ and $p = 2$, then $2\mathfrak{O}_{k_1} = 2\mathfrak{R}'$, and we have to choose $\mathfrak{z}_h = (\pi)$ in such a way that $k_1(\sqrt{\pi})/k_1$ is unramified outside $\mathfrak{p}$. The residue symbols $[\alpha/2]$ are defined as Kronecker symbols via the splitting of $2$ in the quadratic extension $k_1(\sqrt{\alpha})/k_1$. With these modifications, the above arguments remain valid.

References


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ON THE SIGNATURE OF CERTAIN SPHERICAL REPRESENTATIONS

CARINA BOYALLIAN

In this work we prove for all simple real groups of classical type, except for $SO^*(n)$ and $Sp(p,q)$, that if $\nu \in \mathfrak{a}^*$ does not belong to $C_\rho$, the convex hull of the Weyl group orbit of $\rho$, then the signature of the Hermitian form attached to the irreducible subquotient of the principal spherical series corresponding to $\nu$, with integral infinitesimal character, is indefinite on $K$-types in $\mathfrak{p}$.

0. Introduction.

Let $G$ be a real semisimple Lie group. Let $\mathfrak{g}_0$ be the Lie algebra of $G$. We will denote the complexification of any vector space $V_0$ by $V$ and its dual by $V^*$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a fixed Cartan decomposition corresponding to the Cartan involution $\theta$ of $\mathfrak{g}_0$. Let $K$ be the corresponding maximal compact subgroup of $G$. Let us fix $T \subset K$ a maximal torus. Define $S \subset \mathfrak{t}^*$ as the set of weights of $T$ that are sums of distinct non-compact roots.

We say that $(\mu, V_\mu)$ an irreducible representation of $K$ is unitarily small if the weights of $\mu$ lie in the convex hull of $S$. Let us state the following:

Salamanca-Vogan Conjecture. Suppose $X$ is an irreducible Hermitian $(\mathfrak{g}, K)$-module containing a $K$ type in $S$. Then:

(1) If $X$ is unitary, then the real part of the infinitesimal character belongs to the convex hull of the Weyl group orbit $W \cdot \rho$, where $\rho$ is the semi-sum of positive roots.

(2) If $X$ is not unitarizable, then the invariant hermitian form must be indefinite on unitarily small $K$-types.

Using this conjecture the classification of unitary representation can be reduced to the unitarily small case.

If $X$ is a spherical representation, the statement (1) is true by [HJ]. In order to move towards (2) we may assume that real part of the infinitesimal character does not belongs to $W \cdot \rho$, and the hope is that if this holds, the invariant hermitian form is negative definite in the $K$-types in $\mathfrak{p}$. In this work we prove for all simple real groups of classical type, except for $SO^*(n)$ and $Sp(p,q)$, that if $\nu \in \mathfrak{a}^*$ does not belong to $C_\rho$, then the signature of the Hermitian form attached to the irreducible subquotient of the principal
spherical series corresponding to $\nu$, with integral infinitesimal character, is indefinite on $K$-types in $p$.

**Acknowledgment.** I am very grateful to David Vogan, for introducing me in this problem and for fruitful discussions. Special thanks to MIT for the hospitality during my stay there.

### 1. Notation and General Results.

Recall that the complexification of a Lie algebra $g_0$ will be denoted by $g$. Let $a_0 \subset p_0$ be a maximal Abelian subspace. Let $\Sigma(g_0, a_0) = \Sigma$ be the corresponding set of restricted roots and let $\Sigma\hat{}(g_0, a_0) = \Sigma$ be the reduced restricted roots. If $\alpha \in a^*$, then we will denote the corresponding weight space in $g_0$ by $g^\alpha_0$ and let $m_\alpha$ the dimension of this subspace. For a choice of positive roots $\Sigma^+ (g_0, a_0) = \Sigma^+$ we have $\Pi(g_0, a_0) = \Pi$ the set of simple roots.

Let $W \cong N_K(a_0) \backslash Z_K(a_0)$ be the corresponding Weyl group. If $w \in W$, then we define $\Sigma^+ (w) = \{ \alpha \in \Sigma^+: \alpha w \not\in \Sigma^+ \}$. We also denote the longest element in the Weyl group by $w_0$. If $\Omega \subseteq \Sigma^+$ then we will say that $\nu \in a^*$ is it positive (resp. negative) with respect to $\Omega$, if $\text{Re} \langle \alpha, \nu \rangle$ is positive (resp. negative) or zero for all $\alpha \in \Omega$.

If $X$ is an admissible representation of $G$, we will also denote the corresponding $(g, K)$-module by $X$.

Recall that a $(g, K)$-module $(\pi, H_\pi)$ is called spherical if the trivial $K$-type occurs in the restriction of $(\pi, H_\pi)$ to $K$, i.e., $H_\pi$ contains a non-trivial $K$-fixed vector. Then, we have the following:

**Theorem 1.1.** The irreducible spherical $(g, K)$-modules $(\pi, H_\pi)$ are in one-to-one correspondence with the $W$-orbits in $a^*$.

A proof of this Theorem appears in [K, BJ2].

This correspondence can be realized as follows: Set $n_0 = \sum_{\alpha \in \Sigma^+} g^\alpha_0$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$. Let $P=MAN$ the corresponding minimal parabolic subgroup of $G$. If $\nu \in a^*$, define

$$I^G_P(\nu) = \text{Ind}^G_{MAN}(1 \otimes \nu \otimes 1)$$

where the right-hand-side is the space

$$\{ f \in C^\infty(G, \mathbb{C}) : \forall \text{man} \in MAN \text{ s.t. } f(g\text{man}) = a^{-\nu-\rho}f(g), f \text{ is } K \text{- finite} \}$$

and the $g$- action on this induced module is the left regular action, i.e.,:

$$(X \cdot f)(g) = \frac{d}{dt}f(\exp(-t \cdot X)g)_{|t=0}, \quad g \in G, X \in g, f \in I^G_P(\nu)$$

and

$$(k \cdot f)(g) = f(k^{-1}g), \quad k \in K \text{ and } f \in I^G_P(\nu)$$
It is easy to see that $I^G_P(\nu)$ is a Harish-Chandra module of finite length. Observe that by the Iwasawa decomposition $G = KAN$ of $G$ we have

$$I^G_P(\nu)|_K = K - \text{finite part of } \text{Ind}^K_M(1)$$

and hence by Frobenius reciprocity

$$\dim I^G_P(\nu)^K = \dim \text{Ind}^K_M(1)^K = \dim \text{Hom}(\mathbb{C}, \mathbb{C}) = 1. \quad (1.2)$$

So $I^G_P(\nu)$ is a spherical $(g, K)$-module, and by (1.1) there exists a unique irreducible composition factor $J^G_P(\nu) = J^G_P(\nu')$ containing the trivial $K$-type in $I^G_P(\nu)$. It is well-known that $J^G_P(\nu) \simeq J^G_P(\nu')$ if and only if there exists $w \in W$ such that $\nu = w \cdot \nu'$, in particular $J^G_P(\nu)$ does not depend on the choice of the minimal parabolic subgroup containing $MA$. The $(g, K)$-modules $I^G_P(\nu)$ are called spherical principal series representations.

We denote the set of $\nu \in \mathfrak{a}^*$ such that $J^G_P(\nu)$ has integral infinitesimal character by $\mathfrak{a}^*_{\text{int}}$.

Let $X$ be a $(g, K)$-module. Then we say that $X$ admits an invariant Hermitian form if there exists a non-zero map $\omega = \omega^G : X \times X \rightarrow \mathbb{C}$ such that:

1. $\omega$ is linear in the first factor and conjugate linear in the second factor.
2. $\omega(x, y) = \overline{\omega(y, x)}$, \quad $x, y \in X$.
3. $\omega(k \cdot x, k \cdot y) = \omega(x, y)$, \quad $x, y \in X, k \in K$.
4. $\omega(H \cdot x, y) = \omega(x, -\overline{H} \cdot y)$, \quad $x, y \in X, H \in g$, where $\overline{H}$ stand for complex conjugation of $g$ respect to $g_0$.

If $\mu, \mu'$ are two different $K$-types, then, (3) implies that $\omega(X^\mu, X^{\mu'}) = 0$. Hence $\omega$ is completely described by its restriction to the $K$-isotopic spaces on $X$.

Let $\mu \in \hat{K}$. Then $X^\mu \simeq \text{Hom}_K(V_\mu, X) \otimes V_\mu$ and $\omega$ induces a Hermitian form, $\omega^\mu$, on the first factor.

**Definition 1.3.** Let $(p(\mu), q(\mu)) := (p_X(\mu), q_X(\mu)) := (p^G_X(\mu), q^G_X(\mu))$ be the signature of $\omega^\mu$, i.e., $p(\mu)$ (respectively $q(\mu)$) is the sum of the strictly positive (respectively negative) eigenspaces of $\omega^\mu$.

Let us see when there exists this Hermitian form in $J(\nu)$. This is the one being used in this article. Choose a minimal parabolic subgroup $P = MAN$ in $G$. Consider $w \in W$ and $\nu$ positive with respect to $\Sigma^+(w)$. Then there exist an intertwining operator $\Psi(w) : I^G_P(\nu) \rightarrow I^G_P(w\nu)$ so that $\Psi(w)$ is an isomorphism on the trivial $K$-type. If $w = w_0$ then the image is isomorphic to $J(\nu)$.

Recall that there is a natural non-zero Hermitian paring

$$I^G_P(\nu) \times I^G_P(-\nu) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}. $$
Here $I^G_{MAN}(\nu)$ is naturally identified with the $K$-finite part of $L^2(K/M)$ and with this identification, $\langle \, , \rangle$ is the inner product on $L^2(K/M)$. Now suppose there exists $w \in W$ such that $-\nu = w\nu$ and $\nu$ is positive with respect to $MAN$. Then we get the Hermitian pairing

$$I^G_P(\nu) \times I^G_P(\nu) \twoheadrightarrow \mathbb{C}$$

$$\langle v_1, v_2 \rangle \mapsto \langle v_1, \Psi(w)v_2 \rangle.$$ 

Since $\Psi(w\nu, w_kw^{-1}) : I^G_P(w\nu) \rightarrow I^G_P(w_0\nu)$ is an isomorphism, we get that $I^G_{MAN}(\nu)/\text{Rad}\omega \cong J^G(\nu)$. Hence $\omega$ induces a Hermitian form on $J(\nu)$. The same argument shows that $J(\nu)$ admits an invariant Hermitian form if and only if $-\nu$ is $W$-conjugate to $\nu$.

Now, we will define the set of $K$-types where we will work. So, let’s define as the $p_0$-representation of $K$ the homomorphism

$$K \rightarrow GL(p_0)$$

$$k \mapsto Ad(k)|_{p_0},$$

and the $p$-representation of $K$ the complexification of $p_0$-representation.

Recall that if $g_0$ is simple, then $p$ is either irreducible or it is a direct sum of two inequivalent irreducible representations. Consider the following set of $K$-types:

$$\mathfrak{p} = \{ \mu \in \hat{K} : \text{Hom}_K(\mu, p) \neq 0 \}. \tag{1.4}$$

Through this work, we will consider the signature over this set of $K$-types.

Finally we will define $C_\rho$ as the convex hull of points $w\rho$ with $w \in W$. This set is also characterized in terms of positive roots by the following:

**Proposition 1.5.** The set $C_\rho$ coincides with the set of all weights of the form

$$r = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha$$

where $-1/2 \leq c_\alpha \leq 1/2$, and each $\alpha$ is counted with multiplicity.

For a proof of this result we refer to [SV].

2. Problem.

Take $\nu \in a_{int}^*$ such that $J(\nu)$ has integral infinitesimal character and admits an invariant Hermitian form. Then, we will prove the following

**Theorem 2.1.** If $\nu \not\in C_\rho$ then $q_\mu(\nu) > 0$, where $\mu$ is a $K$-type in $p$ and $q$ as in Definition 1.3.
The main problem here is that this Hermitian form has no known general expression. However, Bang-Jensen [BJ], proves some useful Theorems, that give conditions for an invariant Hermitian form on an irreducible spherical representation, with integral infinitesimal character, to be positive definite on the $K$-types $\mu \in p$, in terms of Langlands data, for the simple groups of classical type except $SO^*(n)$ and $Sp(p,q)$.

These Theorems will be the main tool we will use in order to prove Theorem 2.1. This will be done case by case. Bang-Jensen’s results are used to get an explicit characterization of $\nu \in a^*$, that is crucial for the proof of Theorem 2.1.

3. Case $SL(n, F)$ with $F = \mathbb{R}$ or $\mathbb{C}$.

Let $\mu$ be a $K$-type in $p \cap [g, g]$. We identify $a^*$ with $\mathbb{C}^n$ such that $\Sigma(g_0, a_0) = \{ \pm (e_i - e_j) \mid 1 \leq i < j \leq n \}$. Then $\Sigma^+(g_0, a_0) = \{(e_i - e_j) \mid 1 \leq i < j \leq n \}$. Put $r = \dim_{g} F$. Then, $\dim g_0^\ast = r$. Then $a^*_{\text{int}} = \{ \nu \in \mathbb{C}^n \mid \nu_i \equiv 0 \mod r \mathbb{Z}, i = 1, \ldots, n \}$. For $\nu \in a^*_{\text{int}}$ and $i \in \mathbb{Z}$ we define

$$R(i) := R_\nu(i) := \# \{ \nu_j : \nu_j = r \cdot i \}.$$ 

Now if $\nu \in a^*_{\text{int}}$, then $J^G(\nu)$ admits an invariant Hermitian form if and only if $R_\nu(i) = R_\nu(-i)$ for all $i$.

Then we have the following:

**Theorem 3.1.** Assume $F = \mathbb{C}$ or $\mathbb{R}$. Suppose $\nu \in a^*_{\text{int}}$ and $J^G(\nu)$ admits an invariant hermitian form, $\omega$. Then $\omega$ is positive definite on $J^G(\nu)^\mu$ if and only if $R_\nu(i + 1) \leq R_\nu(i)$ for all $i \geq 0$.

**Proof.** See Theorem 5.2 in [BJ].

We will now prove the following proposition.

**Proposition 3.2.** Consider $\nu \in a^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If $R_\nu(i + 1) \leq R_\nu(i)$ for all $i \geq 0$ then $\nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.

**Proof.** We will give the proof only for $Sl(n, \mathbb{R})$. The case $Sl(n, \mathbb{C})$ follows immediately from the definition of $a^*_{\text{int}}$ and the fact that for this group, each root has multiplicity 2. Take $\nu \in a^*_{\text{int}}$ such that:

1. $R_\nu(i + 1) \leq R_\nu(i)$ for all $i \geq 0$,
2. $R_\nu(i) = R_\nu(-i)$ for all $i$;

thus, we are assuming that $\omega$ is positive definite on $J^G(\nu)^\mu$. By definition, $C_\rho$ is stable by Weyl group action, hence by (1) and (2) we can consider,

$$\nu = (k, \ldots, k, k - 1, \cdots, 1, \cdots, 1, 0, \cdots, 0, -1, \cdots, -1, \cdots, -(k - 1), -1, \cdots, -k)$$
where \( k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \).

In order to complete this proof, we will need the following:

**Lemma 3.4.** Take \( \nu \) as above and consider \( R(0) \geq 1 \) (otherwise, \( \nu \equiv 0 \) by (1)). Then

\[
2(k - i) - 2 < n - \left( 2 \sum_{j=0}^{i} R(k - j) \right) + 1 \quad \text{where } i = 0, \ldots, k - 1.
\]

**Proof.** Since \( n = 2 \sum_{j=0}^{k-1} R(k - j) + R(0) \) we have

\[
2(k - i) - 2 \leq 2 \sum_{j=i+1}^{k-1} R(k - j) + R(0) = n - 2 \sum_{j=0}^{i} R(k - j)
\]

\[
< n - 2 \sum_{j=0}^{i} R(k - j) + 1.
\]

Then, (3.5) has been proved. □

**Proof of Proposition 3.2 (Continuation).** Let define

\[
T(m) = \begin{cases} 
\sum_{t=0}^{m-1} R(k - t), & \text{if } m > 0 \\
0, & \text{if } m = 0
\end{cases}
\]

(3.6)

Now, it is easy to check, that Lemma 3.4 allows us to rewrite \( \nu \) as follows:

\[
\nu = \frac{1}{2} \sum_{m=0}^{k-1} \sum_{j=1}^{R(k-m)} \left\{ (e_{T(m)+j} - e_{n-T(m)-j+1}) \right. \\
+ \left. 2\sum_{s=T(m)+j}^{2k-m-2+j + T(m)+j} \left[ (e_{T(m)+j} - e_{s+1}) + (e_{s+1} - e_{n-T(m)-j+1}) \right] \right\}
\]

And so, Proposition 3.2 is proved. □

**Proof of Theorem 2.1** for \( Sl(n, \mathbb{F}) \) with \( \mathbb{F} = \mathbb{R}, \mathbb{C} \). This is immediate from Proposition 3.2, Theorem 3.1 and Proposition 1.5. □

4. **Case \( Sp(2n, \mathbb{F}) \), \( n > 2 \) with \( \mathbb{F} = \mathbb{R}, \mathbb{C} \).**

Now, assume that \( G = Sp(2n, \mathbb{R}) \) or \( G = Sp(2n, \mathbb{C}) \). In this case, \( \Sigma(\mathfrak{g}_0, \mathfrak{a}_0) \) is of type \( C_n \), and identifying \( \mathfrak{a} \cong \mathbb{C}^n \) we have that the restricted roots are \( \{ \pm e_i \pm e_j, \pm 2e_i \} \). Put \( r = \dim_{\mathbb{R}} \mathbb{F} \).

Here, \( J^G(\nu) \) has integral infinitesimal character if and only if \( \nu_i \in r\mathbb{Z} \). Take \( \nu \in \mathfrak{a}^* \), and replace it by a Weyl group conjugate, then we may assume
that
\begin{equation}
\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0
\end{equation}
and with this assumption define
\[ R(i) = R_\nu(i) = \# \{ j : \nu_j = ri \}, \quad i \geq 0. \]

Then we can state the following:

**Theorem 4.2.** If \( J(\nu) \) has integral infinitesimal character, then an invariant Hermitian form is positive definite on \( J(\nu)^\mu, \mu \in \mathfrak{p} \), if and only if the following conditions are satisfied:

1. \( R(i+1) \leq R(i) + 1 \), for \( i \geq 1 \);
2. If \( R(i+1) = R(i) + 1 \), for \( i \geq 1 \), then \( R(i) \) is odd;
3. \( R(1) \leq 2R(0) + 2 \).

**Proof.** See Theorem 8.4 in [BJ].

Now, we can prove the following:

**Proposition 4.3.** Consider \( \nu \in \mathfrak{a}_\text{int}^* \) such that \( J(\nu) \) has integral infinitesimal character. If conditions (1)-(3) above are satisfied then \( \nu = \sum_{\alpha \in \Sigma} c_\alpha \alpha \) with \( c_\alpha = 1/2 \) or 0.

**Proof.** Let us assume that \( G = Sp(2n, \mathbb{R}) \) and so \( r = 1 \). It follows by Condition (2) in Theorem 4.2, that if \( R(j) = 0 \) for any \( j \geq 1 \) then \( R(j+1) = 0 \). This implies that if \( R(1) \neq 0 \) then
\[
\nu = (k, \ldots, k, k-1, \ldots, 1,0,\ldots,0)
\]
or
\[
\nu = (k, \ldots, k, k-1, \ldots, 1, 1)
\]
where \( k \leq n \). Now, since \( n = \sum_{j=0}^k R(k-j) \), we have that
\begin{equation}
k - j \leq n - \sum_{i=0}^j R(k-i)
\end{equation}
with \( j = 0, \ldots, k-2 \). Let us define \( T(m) \) as in (3.6), and Inequality 4.4 allows us to write \( \nu \) as follows
\[
\nu = \frac{1}{2} \sum_{m=0}^{k-2} \sum_{j=1}^{R(k-m)} \left\{ 2e_{T(m)+j} \right. \\
\left. + \sum_{s=T(m)+j+1}^{(k-m)+T(m)+j} [(e_{T(m)+j} - e_s) + (e_{T(m)+j} + e_s)] \right\} \\
+ \frac{1}{2} \sum_{j=1}^{R(1)} 2e_{T(k-1)+j}.
\]
The case $G = \text{Sp}(2n, \mathbb{C})$ follows from this using that $r = 2$ and the multiplicities of the positive roots. Hence, in this way we have proved Proposition 4.3.

Now, we can give the:

Proof of Theorem 2.1 for $\text{Sp}(2n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$. It follows from Proposition 4.3, Theorem 4.2 and Proposition 1.5. □

5. Case $SO(p,q), p > q + 1$ and $SU(p,q), p > q$.

Let us assume that $G = SO(p,q), p > q + 1$ or $G = SU(p,q), p > q$; and as usual identify $a \simeq \mathbb{C}^q$. Then the restricted roots become $\{\pm e_i \pm e_j, \pm e_l\}$.

Define

$$r = \begin{cases} 1, & \text{if } G = SO(p,q) \\ 2, & \text{if } G = SU(p,q). \end{cases}$$

Define, $\epsilon = 0, 1$ by $\epsilon \equiv p - q - 1 + r \mod 2\mathbb{Z}$. In these cases, $J(\nu)$ has integral infinitesimal character if and only if

$$2\nu_i \equiv r\epsilon \mod 2r\mathbb{Z}, \quad i = 1, \ldots, q.$$

We can replace $\nu$ by a Weyl group conjugate, and assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0. \quad (5.1)$$

With this assumption we define

$$R(i) = R_\nu(i) = \# \left\{ j : \nu_j = r \left( i + \frac{\epsilon}{2} \right) \right\}, \quad i \geq 1 \quad (5.2)$$

and

$$R(0) = R_\nu(0) = (2 - \epsilon)\# \left\{ j : \nu_j = r \left( \frac{\epsilon}{2} \right) \right\}. \quad (5.3)$$

Take $s = \frac{p-q-1+r-\epsilon}{2}$. Now, we can state the following:

Theorem 5.4. If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu, \mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:

1. $R(i + 1) \leq R(i) + 1$, for $i \geq 0, i \neq s - 1$;
2. $R(i + 1) = R(i) + 1$, for $i \geq 0, i \neq s - 1$ then
   $$\begin{cases} R(i) \text{ is even}, & \text{if } i < s \\ R(i) \text{ is odd}, & \text{if } i > s; \end{cases}$$
3. $R(s) \leq R(s - 1) + 2$;
4. $R(s) = R(s - 1) + 2$, then $R(s - 1)$ is even.

Proof. Cf. Theorem 6.2 in [BJ].

Now, we can prove the following:
Proposition 5.5. Consider \( \nu \in a_{\text{int}}^* \) such that \( J(\nu) \) has integral infinitesimal character. If conditions (1)-(4) above are satisfied then \( \nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha \) with \( c_\alpha = 1/2 \) or 0.

Proof. Assume that \( r = 1 \) and \( \epsilon = 0 \), in other words that \( G = SO(p,q) \) with \( p-q \) even. So \( s = \frac{p-q}{2} \in \mathbb{Z} \). In this setting \( J(\nu) \) has integral infinitesimal character if and only if \( \nu_i \in \mathbb{Z} \) and

\[
R(i) = \# \{ n : \nu_n = i \} \quad i \geq 1
\]

and

\[
R(0) = 2\# \{ n : \nu_n = 0 \}.
\]

Then by conditions (1)-(4), Formula (5.1) and assuming \( R(s) \neq 0 \) and there exists \( j < s \) such that \( R(j) \neq 0 \) we can consider \( \nu \) as follows

\[
(5.6) \quad \nu = (s+k, \ldots, s+k, s+(k-1), \ldots, s+(k-1), \ldots, s, \ldots, s, \nu_j, \ldots, \nu_j, \ldots, \nu_1, \ldots, \nu_1)
\]

with \( 0 \leq \nu_i < s, i = 1, \ldots, j \).

In order to complete the proof of this proposition, we will need the following:

Lemma 5.7. Take \( \nu \) as above. Then

\[
k - i + 1 \leq q - \sum_{r=0}^i R(s + k - r).
\]

Proof. Since \( R(s) \neq 0 \) we have, by condition (2) in Theorem 5.4 \( R(s+m) \neq 0 \), for \( m = 1, \ldots, k \). Hence, using that \( q = \sum_{r=0}^k R(s+k-r) + \sum_{p=0}^{j-1} R(\nu_{j-p}) \) we have

\[
k - i + 1 \leq \sum_{r=i+1}^k R(s + k - r) + \sum_{p=0}^{j-1} R(\nu_{j-p})
\]

\[
= q - \sum_{r=0}^i R(s + k - r).
\]

So, the proof of the lemma is complete. \( \square \)

Proof Proposition 5.5. (Cont.) Recalling that each root \( e_i \pm e_j \) has multiplicity 1 and \( e_i \) has multiplicity \( p-q = 2s \), and defining

\[
T(m) = \begin{cases} \sum_{i=0}^{m-1} R(s + k - i), & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \end{cases}
\]

and

\[
S(n) = \begin{cases} \sum_{i=0}^{n-1} R(\nu_{j-i}), & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases}
\]
we can, since is easy to check that \( k - m \leq q - \sum_{j=0}^{m} R(s + k - j) \) for \( m = 0, \ldots, k - 1 \) and \( \nu_i < s \) for \( i = 1, \ldots, j \), rewrite \( \nu \) as follows

\[
\nu = \left( \sum_{m=0}^{k-1} \sum_{r=1}^{k-m} \frac{1}{2} \sum_{t=1}^{k-m} \left[ (e_{T(m)+r} - e_{T(m)+r+t}) + (e_{T(m)+r} + e_{T(m)+r+t}) \right] + \sum_{r=1}^{R(s)} s e_{T(s-1)+r} \right) + \sum_{n=0}^{j-1} \sum_{p=1}^{R(\nu_{j-n})} \nu_{j-n} e_{T(k+1)+S(n)+p}.
\]

Now let us keep \( r = 1 \) and consider \( \epsilon = 1 \), i.e., \( G = SO(p,q) \), but now, \( p - q \) is odd. Here \( s = \frac{p-q-1}{2} \) and \( J(\nu) \) has integral infinitesimal character if and only if \( \nu_n = \frac{2l_n + 1}{2} \) with \( l_n \in \mathbb{Z}_+ \) and \( n = 1, \ldots, q \). Then, we have

\[
R(i) = \# \left\{ n : \nu_n = \frac{2i + 1}{2} \right\} \quad i \geq 1
\]

and

\[
R(0) = \# \left\{ n : \nu_n = \frac{1}{2} \right\}.
\]

Again, as before, we can assume that

(5.8) \[
\nu = \left( \frac{2(s + k) + 1}{2}, \ldots, \frac{2(s + k) + 1}{2}, \frac{2(s + (k - 1)) + 1}{2}, \ldots, \frac{2(s + (k - 1)) + 1}{2}, \frac{2s + 1}{2}, \ldots, \frac{2s + 1}{2}, \frac{2l_j + 1}{2}, \ldots, \frac{2l_j + 1}{2}, \frac{2l_1 + 1}{2}, \ldots, \frac{2l_1 + 1}{2} \right)
\]

with \( 0 \leq l_i < s, i = 1, \ldots, j \). By Lemma 5.7 and since \( l_i + \frac{1}{2} < s \), we can rewrite \( \nu \) as follows

\[
\nu = \left( \sum_{m=0}^{k} \sum_{r=1}^{R(s+k-m)} \sum_{t=0}^{k-m} \frac{1}{2} \left[ (e_{T(m)+r} - e_{T(m)+r+t+1}) + (e_{T(m)+r} + e_{T(m)+r+t+1}) \right] + \sum_{r=1}^{R(s)} s e_{T(s-1)+r} \right) + \sum_{n=0}^{j} \sum_{u=1}^{R(\nu_{j-n})} \left( l_{j-n} + \frac{1}{2} \right) e_{T(k+1)+S(n)+u}.
\]

When \( R(s) = 0 \), condition (2) in Theorem 5.4 implies that \( R(s + j) = 0 \) for all \( j \). Then this case or when \( R(\nu_i) = 0 \) for all \( i \), can be easily deduced from the cases above, and hence, we have completed this proof for \( SO(p,q) \)
and \( r = 1 \) The cases corresponding to \( SU(p, q) \) follows almost immediately from this cases above using, as before, the multiplicities of the positive roots for this group. □

Then we can give the:

Proof of Theorem 2.1 for \( SO(p, q) \), \( p > q + 1 \) and \( SU(p, q) \), \( p > q \). Is immediate from Proposition 5.5, Theorem 5.4 and Proposition 1.5. □

6. Case \( SU(n, n) \), \( n \geq 2 \).

In this case, identifying \( a \simeq \mathbb{C}^n \) we have that the restricted roots are \( \{ \pm e_i \pm e_j, \pm 2e_i \} \). \( J^G(\nu) \) has integral infinitesimal character if and only if \( \nu_i \equiv \epsilon \mod 2\mathbb{Z} \), \( \epsilon = 0, 1 \). Take \( \nu \in a^* \). Again, we may assume that

\[
\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0.
\]

Hence we define

\[
R(i) = R(\nu)(i) = \# \{ j : \nu_j = 2i + \epsilon \}, \quad i \geq 1
\]

and

\[
R(0) = (2\epsilon)\# \{ j : \nu_j = \epsilon \}.
\]

Now, we can state the following:

Theorem 6.4. If \( J(\nu) \) has integral infinitesimal character, then an invariant Hermitian form is positive definite on \( J(\nu)^\mu, \mu \in p \), if and only if the following conditions are satisfied:

1. \( R(i + 1) \leq R(i) + 1 \), for \( i \geq 0 \);
2. \( R(i + 1) = R(i) + 1 \), then \( R(i) \) is odd.

Proof. See Theorem 7.1 in [BJ].

With this, we can prove the following:

Proposition 6.5. Consider \( \nu \in a^*_{\text{int}} \) such that \( J(\nu) \) has integral infinitesimal character. If conditions (1)-(2) above are satisfied then \( \nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha \) with \( c_\alpha = 1/2 \) or 0.

Proof. Let us first assume that \( \epsilon = 0 \). By (6.3) we have that \( R(0) = 0 \). Then it follows from condition (2) in Theorem 6.4 that \( \nu \equiv 0 \). So we can suppose that \( \epsilon = 1 \) and again, by condition (2), we can consider that

\[
\nu = (2k + 1, \ldots, 2k + 1, \ldots, 1, \ldots, 1)
\]

with \( R(0) \geq 2 \). Since \( n = \sum_{i=0}^{k-1} R(k - i) + \frac{1}{2} R(0) \), we have

\[
k - m < n - \sum_{i=0}^{m} R(k - i), \quad m = 0, \ldots, k - 1
\]
and this formula plus the fact that each root $e_i \pm e_j$ has multiplicity 2 and each $2e_l$ has multiplicity one, allows us to say that

$$\nu = \left( \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \left( \sum_{t=1}^{k-m} \left[ e_{T(m)+r} - e_{T(m)+r+1} \right] + \left[ e_{T(m)+r} + e_{T(m)+r+1} \right] \right) + e_{T(m)+r} \right) + \sum_{r=1}^{\frac{1}{2}R(0)} e_{T(k)+r}$$

and in this way, we have completed this proposition. \(\Box\)

Hence, we can give the:

**Proof of Theorem 2.1** for $SU(n,n)$, $n > 2$. It is immediate from Proposition 6.5, Theorem 6.4 and Proposition 1.5. \(\Box\)

7. **Case $SO(n+1,n)$ and $SO(2n+1,\mathbb{C})$, $n \geq 2$.**

Let us assume that $G = SO(n+1,n)$ or $G = SO(2n+1,\mathbb{C})$, and as usual identify $a \simeq \mathbb{C}^n$. Then the restricted roots become $\{\pm e_i \pm e_j, \pm e_l\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(n+1,n) \\ 2, & \text{if } G = SO(2n+1,\mathbb{C}) \end{cases}$$

In these cases, $J(\nu)$ integral infinitesimal character if and only if

$$\nu_i \in r\mathbb{Z} + r\frac{\epsilon}{2}, \text{ with } \epsilon = 0, 1.$$  

We can replace $\nu$ by a Weyl group conjugate, and assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0$$

and so, we define

$$R(i) = R_\nu(i) = \# \left\{ j : \nu_j = r \left( i + \frac{\epsilon}{2} \right) \right\}, \quad i \geq 1$$

and

$$R(0) = R_\nu(0) = (2 - \epsilon)\# \left\{ j : \nu_j = r\frac{\epsilon}{2} \right\}.$$

Let us see the following:

**Theorem 7.4.** If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:

1. $R(i + 1) \leq R(i) + 1$, for $i \geq 0$;
2. if $R(i + 1) = R(i) + 1$, for $i \geq 1$, then $R(i)$ is odd;
3. $R(0) > 2 - \epsilon$. 


Proof. Cf. Theorem 9.4 in [BJ]. The condition $R(0) > 2 - \epsilon$ does not appear in [BJ]. However, it is easy to see by inspection of the proof that this condition is needed.

Now, we can prove the following:

**Proposition 7.5.** Consider $\nu \in a^*_\text{int}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(3) above are satisfied then $\nu = \sum_{\alpha \in \Sigma} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.

**Proof.** Let us assume that $r=1$. The other case, $r=2$, follows from this one. And also consider $\epsilon = 0$, so here we have that $J(\nu)$ admits integral infinitesimal character if and only if $\nu_i \in \mathbb{Z}$. Hence by condition (2) in Theorem 7.4 we have that

\begin{equation}
(7.6) \quad \nu = (k, \ldots, k, k-1, \ldots, k-1, \ldots, 1, 0, \ldots, 0).
\end{equation}

As before, since $R(0) \geq 2$, we can prove that $k - i \leq n - \sum_{j=0}^i R(k - j)$, and this inequality allows us to write down

$$\nu = \frac{1}{2} \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \sum_{i=1}^{k-m} [(e_T(m)+r - e_T(m)+r+t) + (e_T(m)+r + e_T(m)+r+t)]$$

where $T(m)$ is defined in (3.6). The case $\epsilon = 1$ follows from this one, using that

$$\nu = \nu_1 + \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$$

with $\nu_1$ as in (7.6). Since we do not use $e_i$'s in the case above, we can put

$$\nu = \nu_1 + \frac{1}{2} \sum_{i=1}^n e_i$$

and we have completed the proof of this proposition. \hfill \Box

Then we have:

**Proof of Theorem 2.1** for $SO(n+1, n)$ and $SO(2n+1, \mathbb{C})$, $n \geq 2$. It follows from Proposition 7.5, Theorem 7.4 and Proposition 1.5. \hfill \Box

8. **Case $SO(n, n)$ and $SO(2n, \mathbb{C})$, $n \geq 4$.**

Let us assume that $G = SO(n, n)$ or $G = SO(2n, \mathbb{C})$. In this case $\Sigma(g_0, a_0)$ is of type $D_n$ and, as usual we identify $a \simeq \mathbb{C}^n$, then the restricted roots become $\{\pm e_i \pm e_j\}$. Define

$$r = \begin{cases} 
1, & \text{if } G = SO(n, n) \\
2, & \text{if } G = SO(2n, \mathbb{C}).
\end{cases}$$
Here, \( J(\nu) \) integral infinitesimal character if and only if
\[ \nu_i \in r\mathbb{Z} + r\frac{\epsilon}{2}, \quad \text{with } \epsilon = 0, 1. \]

Again, we can assume that
\[ \nu_1 \geq \nu_2 \geq \cdots \geq |\nu_n| \geq 0 \quad (8.1) \]
and define
\[ R(i) = R_\nu(i) = \# \{ j : \nu_j = r\left(i + \frac{\epsilon}{2}\right)\}, \quad i \geq 1 \quad (8.2) \]
and
\[ R(0) = R_\nu(0) = (2 - \epsilon)\# \{ j : \nu_j = r\frac{\epsilon}{2}\} + (1 - \epsilon). \quad (8.3) \]

Take \( \nu \in a^*_\text{int} \), then \( J(\nu) \) admits an invariant Hermitian form if and only if \( n \) is even, or \( n \) is odd, \( \epsilon = 0 \) and \( R(0) > 1 \). Let us see the following:

**Theorem 8.4.** If \( J(\nu) \) has integral infinitesimal character, then an invariant Hermitian form is positive definite on \( J(\nu)^\mu, \mu \in \mathfrak{p} \), if and only if the following conditions are satisfied:

1. \( R(i + 1) \leq R(i) + 1 \), for \( i \geq 0 \);
2. \( R(i + 1) = R(i) + 1 \), for \( i \geq 1 \), then \( R(i) \) is odd;
3. \( R(0) \) is odd;
4. \( R(0) > 1 \).

**Proof.** See Theorem 10.3 in [BJ]. The condition \( R(0) > 1 \) does not appear in the statement of this theorem in [BJ], but it is easy to see, checking the proof, that, otherwise, \( q_\mu(\nu) > 0 \).

Now, we can prove the following:

**Proposition 8.5.** Consider \( \nu \in a^*_\text{int} \) such that \( J(\nu) \) has integral infinitesimal character. If conditions (1)-(4) above are satisfied then \( \nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha \) with \( c_\alpha = 1/2 \) or 0.

**Proof.** Let us assume that \( r = 1, \epsilon = 0 \). Since the Weyl group is the group of permutation and sign changes involving an even number of signs of the set of \( n \) elements, by condition (2) and \( R(0) > 1 \), we can suppose that \( \nu_i \geq 0 \) and
\[ \nu = (k, \ldots, k, k - 1, \ldots, 1, \ldots, 1, 0, \ldots, 0). \]
Since \( R(0) > 1 \), we have \( k - i \leq n - \sum_{j=0}^i R(k - j) \), and thus
\[ \nu = \frac{1}{2} \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \sum_{t=1}^{k-m} \left[ (e_T(m)+r - e_T(m)+r+t) + (e_T(m)+r + e_T(m)+r+t) \right], \]
where $T(m)$ was defined in (3.6). Now, let us assume that $\epsilon = 1$ and $n$ even. So, by a conjugation by the Weyl group we can assume that $\nu_i \geq 0$ for $i = 0, \ldots, n - 1$ and moreover

$$\nu = \left( k + \frac{1}{2}, \ldots, k + \frac{1}{2}, k - 1 + \frac{1}{2}, \ldots, k - 1 + \frac{1}{2}, \ldots, 1 + \frac{1}{2}, \ldots, 1 + \frac{1}{2}, \ldots, 1 + \frac{1}{2}, \ldots, \pm \frac{1}{2} \right).$$

Since $R(0) > 1$ and by condition (3) in Theorem 8.4 we can prove that $k - i + 1 \leq n - \sum_{j=0}^{i} R(k - j)$, and defining $\delta = \begin{cases} 1, & \text{if } \nu_n = \frac{1}{2} \\ 0, & \text{if } \nu_n = -\frac{1}{2} \end{cases}$, we can rewrite $\nu$ as follows

$$\nu = \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)-m} \sum_{t=1}^{1} \left\{ \left( e_{T(m)+r} - e_{T(m)+r+t} \right) + \left( e_{T(m)+r} + e_{T(m)+r+t} \right) \right\} + \left( e_{T(m)+r} \right) + \left( -1 \right)^{T(m)+r+\delta} e_n + \sum_{j=1}^{R(0)-1} \frac{1}{2} (e_{T(j)+r} + (-1)^{j} e_n)$$

where $T(m)$ was defined in (3.6). And, since $n$ is even and $R(0)$ is odd, we have completed our proof. The case $SO(2n, \mathbb{C})$ follows as above using multiplicities of positive roots for this particular case.

Then we can give the:

**Proof of Theorem 2.1** for $SO(n, n)$ and $SO(2n, \mathbb{C})$, $n \geq 4$. It is immediate from Proposition 8.5, Theorem 8.4 and Proposition 1.5.

**References**


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NIELSEN ROOT THEORY AND HOPF DEGREE THEORY

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The Nielsen root number $N(f; c)$ of a map $f : M \to N$ at a point $c \in N$ is a homotopy invariant lower bound for the number of roots at $c$, that is, for the cardinality of $f^{-1}(c)$. There is a formula for calculating $N(f; c)$ if $M$ and $N$ are closed oriented manifolds of the same dimension. We extend the calculation of $N(f; c)$ to manifolds that are not orientable, and also to manifolds that have non-empty boundaries and are not compact, provided that the map $f$ is boundary-preserving and proper. Because of its connection with degree theory, we introduce the transverse Nielsen root number for maps transverse to $c$, obtain computational results for it in the same setting, and prove that the two Nielsen root numbers are sharp lower bounds in dimensions other than 2. We apply these extended root theory results to the degree theory for maps of not necessarily orientable manifolds introduced by Hopf in 1930. Thus we re-establish, in a new and modern treatment, the relationship of Hopf’s Absolutgrad and the geometric degree with homotopy invariants of Nielsen root theory, a relationship that is present in Hopf’s work but not in subsequent re-examinations of Hopf’s degree theory.

1. Introduction.

The goal of this paper is two-fold. We will extend results from Nielsen root theory for maps between orientable $n$-manifolds so as to remove the orientability hypothesis. Then we will use the extended theory to re-establish the connection between Nielsen root theory and two variants of the degree of a map, namely, Hopf’s Absolutgrad and the geometric degree. By using methods from present-day Nielsen theory, we will provide new ways of understanding some of the basic concepts of Hopf’s theory as well as more direct proofs for some of the results. We next describe these goals in more detail.

If $f : M \to N$ is a map between two manifolds and $c \in N$, then a root of $f$ at $c$ is a point in $f^{-1}(c)$. The Nielsen root number $N(f; c)$ is a lower bound for the cardinality of $f^{-1}(c)$, and it is homotopy invariant. While it is possible to define $N(f; c)$ even if $M$ and $N$ are not manifolds, it is usually not possible to compute it in such general settings. If, however, $M$ and $N$
are orientable \( n \)-manifolds, then a formula for computing \( N(f; c) \) is known. Following Hopf [H2], we write \( j \) to denote the cardinality of the coset space \( \pi_1(N)/f_\pi(\pi_1(M)) \) and state the following partial version of Theorem 3.13 below, which is due to Hopf [H2, Satz VIIa] and Lin [L, Proposition 5].

**Theorem 1.1.** If \( f: M \to N \) is a map of closed, connected oriented \( n \)-manifolds, then \( N(f; c) = 0 \) if the degree of \( f \) is zero and \( N(f; c) = j \) if the degree is not zero.

Our main extension of this theorem is to maps between closed \( n \)-manifolds that are not necessarily orientable, but we will also allow manifolds with boundary if \( f \) maps boundary to boundary, and non-compact manifolds if \( f \) is proper (see Theorem 3.11).

Nielsen root theory was used in the degree theory that Heinz Hopf initiated in 1930 [H2], and therefore his degree theory is quite different from others that existed at Hopf’s time. The degree in Theorem 1.1, the classical degree due to Brouwer [Bw, p. 105], is usually defined in terms of the homomorphism of integer homology induced by \( f \). The definition can be extended to proper boundary-preserving maps of orientable but not necessarily compact \( n \)-manifolds with boundary. But if at least one of \( M \) and \( N \) is non-orientable, then the homological degree can only be defined in terms of homology with coefficients in \( \mathbb{Z}/2 \) and the resulting mod 2 degree \( \deg(f; 2) \) tells little about the map \( f \), in particular about its geometric properties. To obtain such geometric information, and in particular to give an algebraic approach to the geometric degree which looks at counterimages of points, Hopf introduced a degree that he called the Absolutgrad (absolute degree). It does not require orientability and provides much better information about the map than does the mod 2 degree. Hopf’s Absolutgrad may be viewed as a variant of the Nielsen root number, in fact it is precisely the “transverse Nielsen root number” \( N_\cap(f; c) \) which is a lower bound for the cardinality of the set of roots of maps between \( n \)-manifolds that are transverse to \( c \) in a sense made precise in Definition 3.1 (see Theorem 5.3). In general, \( N_\cap(f; c) \geq N(f; c) \) and equality need not hold. In Theorems 3.12 and 3.13, we compute \( N_\cap(f; c) \) in the same setting in which we compute \( N(f; c) \) in Theorems 3.11 and 3.13.

An important reason for calculating the Nielsen root number \( N(f; c) \) of a map \( f \) of \( n \)-manifolds is that it contains geometric information: It is a sharp lower bound if \( n \neq 2 \), that is, there exists a map \( g \) homotopic to \( f \) such that \( g^{-1}(c) \) contains exactly \( N(f; c) \) points. The transverse Nielsen number is also sharp, even if \( n = 2 \), in the sense that there is a map \( g \) homotopic to \( f \) and transverse to \( c \) such that \( g^{-1}(c) \) contains exactly \( N_\cap(f; c) \) points. This is equivalent to saying that \( N_\cap(f; c) \) can be realized by a map \( g \) which has \( N_\cap(f; c) \) as its geometric degree, and so it follows from the fact that \( N_\cap(f; c) \)
is sharp that the Absolutgrad of a map between $n$-manifolds is equal to its geometric degree. This is the property of the Absolutgrad that motivated Hopf’s introduction of the concept in $\textbf{[H2]}$. Hopf was influenced in his 1930 study of the degree by work of Jakob Nielsen $\textbf{[N1, N2]}$ on the subject of fixed points that was published a few years earlier and, in particular, the Nielsen root number (called “wesentliche Schichtenzahl”) and the transverse root Nielsen number (called “Absolutgrad”) appear for the first time in Hopf’s paper.

Thus our paper may be viewed in part as a re-examination of Hopf’s degree theory from a present-day mathematical standpoint. Ours is by no means the first updating and extension of Hopf’s work, and in particular of Hopf’s very novel concept of the Absolutgrad. The first such studies are contained in two important papers, by Olum $\textbf{[O]}$ and Epstein $\textbf{[E]}$. In 1953, Olum $\textbf{[O]}$ considered maps between closed but not necessarily orientable manifolds and used cohomology with local coefficients to introduce an algebraically defined “group ring degree” in a way which is more closely related to the definition of the classical Brouwer degree (but not to that of the geometric degree) than Hopf’s definition of the Absolutgrad. Olum showed that it follows from his definition that Hopf’s Absolutgrad equals the absolute value of the group ring degree, and he calculated the group ring degree in terms of a “twisted” global degree, introduced earlier in his paper, and the mod 2 degree $\textbf{[O}, \text{p. 478]}$. In an influential paper $\textbf{[E]}$ that Epstein published in 1966, the calculations of Olum were interpreted in terms of cohomology degrees of lifts of the map $f$ and these degrees were used by Epstein, and subsequently by other authors, as the definition of the Absolutgrad for maps between not necessarily orientable manifolds (see $\textbf{[E}, \text{(1.8) p. 371]}$ and $\textbf{[Sk, Definition p. 416]}$). In the approach of Olum and Epstein, maps between $n$-manifolds are classified into three types and the Absolutgrad is defined separately for each type. This somewhat complicated definition makes the Absolutgrad more readily computable, but it obscures its meaning.

There have been several recent extensions of Hopf’s work. In 1986, Lin $\textbf{[L]}$ concentrated on the root theory component of Hopf’s work and provided a modern definition of the multiplicity of a root class and a modern proof, with techniques that we also use in this paper, of the sharpness of the root Nielsen number in the special case that $f$ is a map between closed orientable manifolds of dimension at least 3. But Lin did not consider non-orientable manifolds, nor did he re-establish the connection between Nielsen root theory and degree theory. In 1987, Skora $\textbf{[Sk]}$ provided a modern geometric treatment of the connection between the Absolutgrad and the geometric degree for boundary-preserving maps between surfaces, and thus re-proved and extended results from $\textbf{[H2]}$ which were proved even earlier by Kneser $\textbf{[Kn1, Kn2]}$, but Skora did not connect his results to Nielsen root theory.
An extension of Nielsen root theory to proper maps \( f : M \to N \) between \( n \)-manifolds, where the point \( c \in N \) is replaced by a connected \( k \)-manifold of dimension \( 0 \leq k \leq n \), was obtained in 1992 by Yongwu Rong and Shicheng Wang [RW], and in the case \( k = 0 \) the new and very geometric proof of their main result can be interpreted to show that the transverse root Nielsen number is sharp for proper maps, under the assumption that the manifolds \( M \) and \( N \) are closed and orientable and the homology degree of \( f \) (and hence \( N(f; c) \), see Theorem 3.13 below) is non-zero. Their paper makes reference to Hopf [H2], but not to Hopf’s Absolutgrad.

An important difference between the previous reinterpretations of Hopf’s theory and this paper is that we re-establish the connection between Hopf’s work and the ideas introduced by Nielsen. In particular, our approach is based on the concept of root class that Hopf used as the analogue, in the degree context, of Nielsen’s central notion of fixed point class, and our methods are influenced by techniques of modern Nielsen theory. Most of our results are not new. Although the formulae for the two Nielsen root numbers in Theorems 3.11 and 3.12 are not due to Hopf, they can be obtained, by a careful inspection, from Olum [O]. The sharpness of both Nielsen root numbers was first proved by Hopf [H2], and an updated proof of the fact that the Absolutgrad equals the geometric degree in dimension \( \geq 3 \), and hence of the sharpness of the transverse Nielsen root number, was the goal of Epstein’s paper [E]. On the other hand, some of our definitions and all our proofs are new and different from existing ones. We use local degree theory to define the integer-valued multiplicity of a root class and define the Nielsen root number to be the number of root classes with non-zero multiplicity. We introduce the transverse Nielsen root number, defined as the sum of the multiplicities of all the root classes. The calculation of \( N(f; c) \) and \( N(f; c) \) is obtained by combining results from local degree theory with the definition of root class in terms of a lift of \( f \), and it also uses the various lifts of \( f \) employed by Epstein [E]. A form of the Whitney Lemma due to Jezierski [Je] and the theory of microbundle transversality are used to establish the sharpness results. We have also included many examples. By these means we obtain not only an extension of Nielsen root theory for maps between \( n \)-manifolds which need no longer be orientable, but also a foundation for Hopf’s Absolutgrad and its connection to the geometric degree, which is in the spirit of Hopf but based in large part on developments in algebraic topology and manifold theory since the time of Hopf’s work.

Our paper is organized as follows. We define the multiplicity of a root class in Section 2, and in Section 3 we define and compute the two Nielsen root numbers for maps of \( n \)-manifolds. Sharpness of the two Nielsen root numbers for such maps, for \( n \neq 2 \), is established in Section 4. In Section 5, we relate the Nielsen root theory developed in the previous sections to
Hopf’s degree theory. Although the geometric results of Section 4 exclude maps of surfaces, much is known and we describe the Nielsen root theory and Hopf degree theory of maps of surfaces in Section 6.

A very readable introduction to Nielsen root theory can be found in Kiang’s book [Kg]. For the background from Nielsen fixed point theory, see in addition [Bu1] and [Jg2].

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2. The multiplicity of a root class.

Throughout this paper, $M$ and $N$ will be connected topological $n$-manifolds with (possibly empty) boundary. The manifolds are not necessarily compact and they can be orientable or not (a manifold with boundary is called orientable if its interior is an orientable manifold [Do, p. 257]). A map $f: M \to N$ is boundary-preserving if it is a map of pairs $f: (M, \partial M) \to (N, \partial N)$. The map $f$ is proper if $K \subset N$ is compact implies that $f^{-1}(K)$ is a compact subset of $M$. All homotopies in this paper are understood to be boundary-preserving, and so a proper homotopy is a proper map $f: (M, \partial M) \times I \to (N, \partial N)$.

We will be concerned with a proper map $f: (M, \partial M) \to (N, \partial N)$ of manifolds of the same dimension and with the set $f^{-1}(c)$, for a point $c \in \text{int } N$, when that set is non-empty. We choose $c$ as the basepoint for $N$ and some $x_0 \in f^{-1}(c)$ as the basepoint for $M$, so $f$ induces a homomorphism $f_\pi: \pi_1(M, x_0) \to \pi_1(N, c)$.

A proper map $f: (M, \partial M) \to (N, \partial N)$ induces a homomorphism $f^*: \check{H}^n(N, \partial N) \to \check{H}^n(M, \partial M)$ of Čech cohomology with compact supports and integer coefficients (see [Do, 6.26, p. 290]). Let $W = M - f^{-1}(\partial N)$ which is an open subset of int $M$. If $M$ and $N$ are oriented manifolds, then there is a class $[N] \in \check{H}^n(N, \partial N)$ corresponding to the fundamental class $[\text{int } N] \in \check{H}^n(\text{int } N)$ and a class $[M] \in \check{H}^n(M, \partial M)$ corresponding to $[W] \in \check{H}^n(W)$ obtained from a component of $W$ by restricting the orientation of int $M$. The cohomological degree of $f$, denoted $\deg(f)$, is defined by $f^*[N] = \deg(f)[M]$.

All manifolds are orientable with respect to $\mathbb{Z}/2$ coefficients so, just as in the previous paragraph, we may always obtain classes $[N] \in \check{H}^n(N, \partial N; \mathbb{Z}/2)$ and $[M] \in \check{H}^n(M, \partial M; \mathbb{Z}/2)$ and define the mod 2 cohomological degree, denoted $\deg(f, 2)$, by setting $f^*[N] = \deg(f, 2)[M]$.

Maps between not necessarily orientable manifolds are classified in the following manner. A map $f: M \to N$ is called orientation-true if it maps orientation-preserving loops in $M$ to orientation-preserving loops in $N$ and orientation-reversing loops in $M$ to orientation-reversing loops in $N$, otherwise it is called not orientation-true. The class of maps that are not orientation-true is subdivided to produce the following classification:
Definition 2.1. Let \( f : M \to N \) be a map of manifolds. Then three types of maps are defined as follows.

1. **Type I**: \( f \) is orientation-true.
2. **Type II**: \( f \) is not orientation-true but does not map an orientation-reversing loop in \( M \) to a contractible loop in \( N \).
3. **Type III**: \( f \) maps an orientation-reversing loop in \( M \) to a contractible loop in \( N \).

Further, a map \( f \) is defined to be **orientable** if it is of Type I or II, and **non-orientable** otherwise.

The term **orientable map** is sometimes used for Type I, that is orientation-true, maps; see [Do, Exercise 6, p. 271]. The characterisation of the three types is based on Olum [O, p. 475] (see also [Sk, p. 416]). An equivalent characterisation in terms of the orientability of covering spaces of \( M \) and \( N \), which we will use in §3, is given by Epstein [E, p. 371]. In essence, the characterisation of the three types of maps is already contained in Hopf’s paper [H2]. We shall see that maps of the first two types share many properties with regard to Nielsen root theory and, as Hopf was well aware of this, he considered maps of these first two types together and therefore introduced the concept of an orientable or non-orientable map [H2, Definition V, p. 579].

The following examples illustrate the three types of maps.

**Example 2.2** (Type I). (a) If \( M \) and \( N \) are orientable manifolds, then all maps \( f : M \to N \) are orientation-true. (b) For \( N \) a non-orientable manifold and \( M \) its orientable covering, the covering map \( p : M \to N \) is of Type I. (The case of \( N \) the projective plane is mentioned by Hopf [H2, p. 584].) (c) The identity map of a non-orientable manifold is an example of a Type I map between non-orientable manifolds.

**Example 2.3** (Type II). Let \( M_+ \) be the Möbius band and let \( p : M_+ \to S^1 \) be the fibration obtained by retracting \( M_+ \) to its central circle. Let \( i : S^1 \to S^1 \times I = N_+ \) be defined by setting \( i(x) = (x, 0) \), then \( f_+ = ip \) is a boundary-preserving map from the Möbius band to the annulus. Let \( f = 2f_+ : M = 2M_+ \to 2N_+ = N \) be the double of the map \( f_+ \), so \( M \) is the Klein bottle and \( N \) is the torus. The loops representing elements in the kernel of the induced homomorphism \( f_* : \pi_1(M, x_0) \to \pi_1(N, c) \) are orientation-preserving, so \( f \) is not Type III. Since a map from a non-orientable manifold to an orientable manifold cannot be orientation-true, we conclude that \( f \) is Type II.

**Example 2.4** (Type III). (a) For \( M \) a non-orientable manifold, a constant map \( f : M \to N \) is obviously of Type III. (b) For an example of a Type III map of \( M \) onto \( N \), let \( T^2 \) denote the torus and \( P^2 \) the projective plane. Let \( D \) be a disc in \( T^2 \) and let \( id : T^2 - \text{int} \, D \to T^2 - \text{int} \, D \) be the identity. Extend \( id \) to \( f : T^2 \# P^2 \to T^2 \) by extending the identity map on \( \partial D \) in \( P^2 - \text{int} \, D \).
as a map from $P^2 - \text{int } D$ to $D \subset T^2$. Since the generator of $\pi_1(T^2 \# P^2, x_0)$ represented by an orientation-reversing loop is mapped into the contractible set $D$, we see that $f$ is of Type III.

The remainder of this section will be devoted to defining, for $f: (M, \partial M) \to (N, \partial N)$ a proper map of any type, the multiplicity of a root class of $f$ at $c \in \text{int } N$.

Points $x_1, x_2 \in f^{-1}(c)$ are in the same root class of $f$ at $c$ if there is a path $w: I \to M$ from $x_1$ to $x_2$ such that $f \circ w$ is a contractible loop at $c$ (see [Kg, Chapter V.B]). (This definition goes back to Hopf [H2, Definition V, p. 575], where a root class is called a “Schicht”.) Since $f: (M, \partial M) \to (N, \partial N)$ is proper, the root classes are compact subsets of $M$ and there are only finitely many of them. Let $V \subset \text{int } N$ be a contractible neighborhood of $c$. Since $f$ is boundary-preserving, $f^{-1}(V)$ is contained in $\text{int } M$. Let $R$ be a root class of $f$ at $c$ and let $U$ be an open subset of $f^{-1}(V)$ such that $U \cap f^{-1}(c) = R$. Since $U$ is an open subset of $M$, it is a manifold, that is a space locally homeomorphic to $\mathbb{R}^n$, but it is not necessarily connected.

We shall first assume that $U$ is an oriented manifold. If $M$ is itself an oriented manifold, then the orientation of $U$ is the restriction of the orientation of $M$. The neighborhood $V$ is contractible, so it is an orientable manifold and we choose an orientation for it, selecting the restriction of the orientation of $N$ if that manifold is oriented. The integer-valued local degree of $f|U: U \to V$ over $c$ is defined; it is denoted by $\deg_c(f|U)$ [Do, Definition 4.2, p. 267]. If $U_0 \subset U$, an open subset containing $R$, is oriented by restricting the orientation of $U$, then $\deg_c(f|U_0) = \deg_c(f|U)$. Consequently, if $U_1$ and $U_2$ are open subsets of $f^{-1}(V)$ containing $R$ that are oriented so that their orientations agree on $U_1 \cap U_2$, then $\deg_c(f|U_1) = \deg_c(f|U_2)$. Moreover, if $V_0 \subset V$ is also a contractible neighborhood of $c$, and $U \subset f^{-1}(V_0)$ then, if $V_0$ is oriented by restricting the orientation of $V$, it follows that $\deg_c(f|U)$ has the same value if we view $f|U$ as a map into $V_0$ as it does if we view $f|U$ as a map into $V$.

The following remark describes the relationship between the cohomological degree and the local degree.

**Remark 2.5.** The definition above of the cohomological degree $\deg(f)$ of a proper map $f: (M, \partial M) \to (N, \partial N)$ of oriented manifolds made use of fundamental classes $[\text{int } N] \in H^n(\text{int } N)$ and $[W] \in \tilde{H}^n(W)$, where $W = M - f^{-1}(\partial N)$. Duality [Do, Prop. 7.14, p. 297] gives us corresponding elements of singular homology $\{ \text{int } N \} \in H_0(\text{int } N)$ and $\{ W \} \in H_0(W)$ so that, for the homology transfer homomorphism $f_!$ [Do, Equation 10.7, p. 310], we have $f_!(\text{int } N) = \deg(f)\{ W \}$. Consequently, for $f_*: H_0(W) \to H_0(\text{int } N)$, we see that $f_*f_!(\text{int } N) = \deg(f)\{ \text{int } N \}$. On the other hand, [Do, Prop. 10.10, p. 312] implies that $f_*f_!(\text{int } N) = \deg_c(f|W)\{ \text{int } N \}$. 

We conclude that if \( f: (M, \partial M) \to (N, \partial N) \) is a proper map of oriented manifolds, then \( \deg_c(f|W) = \deg(f) \).

If \( M \) and \( N \) are oriented manifolds, since the orientations of \( V \) and of \( U = U_R \) for each root class \( R \) were chosen to be restrictions of those orientations and \( f^{-1}(c) \) is the union of the root classes, the additivity property of the local degree [Do, Prop. 4.7, p. 269] implies that

\[
\sum (\deg_c(f|U_R): R \text{ is a root class of } f) = \deg_c(f|\bigcup U_R).
\]

By excision \( \deg_c(f|\bigcup U_R) = \deg_c(f|W) \) and therefore Remark 2.5 implies that

\[
\sum (\deg_c(f|U_R): R \text{ is a root class of } f) = \deg(f).
\]

We have assumed that \( U \) is orientable because the definition of \( \deg_c(f|U) \) required that the open subset \( U \) of \( f^{-1}(V) \) containing the root class \( R \) be an oriented manifold. Now we no longer assume that \( U \) is orientable. Then there may not be any open subset of \( f^{-1}(V) \) containing \( R \) that is orientable, for instance if \( M \) is a closed non-orientable manifold and \( f \) is the constant map to \( c \). However, we will now show in our Orientation Procedure (2.6) that if \( f \) is an orientable map, then \( U \) is also orientable, and we will describe an orientation procedure that we will always use to orient \( U \) when \( f \) is an orientable map.

**(2.6) Orientation Procedure.** We first note that if \( f: (M, \partial M) \to (N, \partial N) \) is a proper orientable map and \( U \subset f^{-1}(V) \) is an open set containing a root class \( R \), then \( U \) is an orientable manifold. For every loop in \( U \) maps to the contractible space \( V \), and hence every loop must be orientable since \( f \) is an orientable map. Thus the fundamental group of each component of \( U \) is generated by orientable loops, so each component of \( U \) is an orientable manifold and therefore \( U \) is orientable. We shall always use the following Orientation Procedure to orient \( U \): If \( M \) is an oriented manifold, we orient \( U \) by restricting the orientation of \( M \). Otherwise, we choose some \( x_R \in R \) and an orientation of \( U \) at \( x_R \). Let \( x \in R \) be any point, then there is a path \( w \) in \( M \) from \( x_R \) to \( x \) such that \( f \circ w \) is a contractible loop in \( N \) based at \( c \). Orient \( U \) at \( x \) by extending the chosen orientation of \( x_R \) along \( w \). The orientation is independent of the choice of the path \( w \) as follows. Let \( w' \) be another path in \( M \) from \( x_R \) to \( x \) such that \( f \circ w' \) is a contractible loop in \( N \) based at \( c \). The loop \( w^{-1} \cdot w' \) at \( x \) is orientable because \( f \circ (w^{-1} \cdot w') \) is contractible and the map \( f \) is orientable. Therefore, the orientation of \( U \) at \( x \) is independent of the choice of the path \( w \) as we claimed. Since each component of \( U \) is orientable, in this way the chosen orientation of \( x_R \) determines an orientation of each component of \( U \) that intersects \( R \). Choosing orientations of the remaining components arbitrarily, we make \( U \) an oriented manifold.
Unless $M$ and $N$ are oriented manifolds, there is no criterion for choosing orientations of $U$ and $V$, however, $\deg_c(f|U)$ is determined up to sign. This can be seen as follows: Suppose that $U$ is orientable and that we fix the orientation of $U$ for now. Having chosen an orientation for $V$, we have $\deg_c(f|U) = -\deg'_c(f|U)$. Now fix the orientation of $V$. If $f$ is an orientable map, choosing the opposite orientation of $U$ at $x_R$, the Orientation Procedure (2.6) changes the orientation of each component of $U$ that intersects $R$. The degree over $c$ of the restriction of $f$ to each of these components thus changes sign. Since $R$ is compact, the number of components of $U$ that intersect $R$ is finite. The degree over $c$ of the restriction of $f$ to each of the components that fails to intersect $R$ is zero, so applying the additivity property of the local degree to $f|U: U \to V$, which is a map of oriented manifolds, we see that changing the orientation of $U$ at $x_R$ reverses the sign of $\deg_c(f|U)$ in this case also. If the map $f$ is not orientable, $U$ may still be an orientable manifold, for instance if $R$ is finite and $U$ is taken to be a union of euclidean neighborhoods. However, if $f$ is not orientable, the orientation of $U$ obtained by the Orientation Procedure (2.6) would depend not only on the orientation of $U$ at $x_R$ but also on the choice of paths between $x_R$ and the other points of $R$. The facts that, (1) for orientable maps the integer-valued degree of $f|U$ over $c$ is defined only up to sign and (2) for maps that are not orientable the integer-valued degree may not be defined at all, motivate the following:

**Definition 2.7.** Let $f: (M, \partial M) \to (N, \partial N)$ be a proper map, let $c \in \text{int } N$ and let $R$ be a root class of $f$ at $c$. Let $U$ be any open subset of $f^{-1}(V)$ containing $R$ but no other roots of $f$ at $c$, where $V \subset \text{int } N$ is a contractible neighborhood of $c$. Then $|m(R)|$, the *multiplicity* of $R$, is defined by

$$|m(R)| = |\deg_c(f|U)|,$$

where, for orientable (that is, Type I and II) maps, $\deg_c(f|U)$ is the local degree with coefficients in $\mathbb{Z}$ and $U$ is oriented according to the Orientation Procedure (2.6) and, for non-orientable (that is, Type III) maps, $\deg_c(f|U)$ is the local degree with coefficients in $\mathbb{Z}/2$.

The definition is independent of choices because, up to sign, $\deg_c(f|U)$ is, as we noted in the paragraph preceding Remark 2.5.

If $M$ and $N$ are oriented manifolds without boundary, the multiplicity of Definition 2.7 agrees with the one used by Lin [L, p. 201] (see also [BS2, §3]) up to sign. Lin defined the “multiplicity” $m(R)$ of a root class $R$ as $m(R) = \deg_c(f|U)$, where $U$ is any open set of $M$ which contains $R$ but does not contain any roots of $f$ that do not lie in $R$. In the oriented case it is not necessary to use the absolute value sign in order to obtain a well-defined homotopy invariant. Hopf was aware of the fact that in general an absolute
value has to be used, for his definition of the multiplicity (called “Beitrag”) of a root is essentially the value of $|m(R)|$ in the following example. (See [H2, Definition VIIa, p. 581] for the definition of the “Beitrag”, and [H1, §5] for Hopf’s definition of the degree as the sum of the degree of $f$ on euclidean neighborhoods.)

**Example 2.8.** Suppose $f: (M, \partial M) \to (N, \partial N)$ is an orientable map such that $f^{-1}(c)$ is finite. For a root class $R = \{x_1, x_2, \ldots, x_k\}$, let $U = \bigcup_{\ell=1}^{k} U_\ell$, where the $U_\ell$ are disjoint euclidean neighborhoods of $x_\ell$ in $f^{-1}(V)$ that contain no other roots of $f$ at $c$, and orient $U$ as in the Orientation Procedure (2.6). The additivity property of the local degree, applied to the map of oriented manifolds $f|U: U \to V$, implies that

$$\deg_c(f|U) = \sum_{1 \leq \ell \leq k} (\deg_c(f|U_\ell)).$$

We conclude that, for an orientable map $f$ with $f^{-1}(c)$ finite, we have the following formula for the multiplicity of a root class

$$|m(R)| = \left| \sum_{1 \leq \ell \leq k} (\deg_c(f|U_\ell)) \right|.$$

If $f$ is non-orientable, then the same formula applies, but with coefficients in $\mathbb{Z}/2$.

If $f: M \to N$ is a Type I map of manifolds without boundary, then the multiplicity $|m(R)|$ of a root class at $c$ is the absolute value of the coincidence index $\text{ind}(c, f; R)$ of Gonçalves and Jeziorski [GJ, Definition 5.1, p. 19] for $c: M \to N$ the constant map. This can be proved by modifying the argument of Lemma 5.2 of [GJ]. Consequently, by Theorem 5.5 of [GJ], $|m(R)| = |\text{ind}(c, f; R)|$, the semi-index [GJ, p. 19] (see also [DJ] and [Je]).

3. The two Nielsen root numbers and their computation.

We now use the multiplicity of the previous section to define the Nielsen root number and the transverse Nielsen root number in our setting. The theory of these two numbers is closely linked, and the two numbers were already studied jointly by Hopf [H2]. The first, namely the Nielsen root number, is well-established [Kg, Definition 4.3, p. 129]. We shall see in §5 that the second, the transverse Nielsen root number, is closely related to degree theory, and it is for this reason that the number is introduced.

We define a root class $R$ of a proper map $f: (M, \partial M) \to (N, \partial N)$ at $c \in \text{int} N$ to be essential if its multiplicity $|m(R)| \neq 0$, and let the Nielsen root number $N(f; c)$ be the number of essential root classes of $f$. Hence $N(f; c)$ is a lower bound for the number of roots of $f$ at $c$. We write $f \sim g$ if $f$ is (properly) homotopic to $g$, and we will show in Corollary 3.14 that $N(f; c)$ is a proper homotopy invariant. Let $MR[f; c]$ denote the minimum
number of roots in the proper homotopy class of \( f \), that is,
\[
MR[f; c] = \min \{ \# \text{root} (g; c) : g \sim f \}.
\]
Corollary 3.14 will thus establish the inequality \( N(f; c) \leq MR[f; c] \).

We next introduce a minimum number for roots under the condition that the map \( f \) covers a neighborhood of \( c \) geometrically by local homeomorphisms, and thus has no multiple roots (that is, roots with a multiplicity \( \neq \pm 1 \)). More precisely, we will require that \( f \) is transverse to \( c \), for \( c \in \text{int} \, N \), according to the following definition which is used by Epstein [E, p. 375], and is equivalent to the concept of “glatt” used by Hopf [H1, p. 599].

**Definition 3.1.** A proper map \( f : (M, \partial M) \rightarrow (N, \partial N) \) is transverse to \( c \), where \( c \in \text{int} \, N \), if there exists a euclidean neighborhood \( V \) of \( c \) in \( N \) so that \( f^{-1}(V) \) consists of finitely many euclidean neighborhoods in \( \text{int} \, M \), and each of them is mapped by \( f \) homeomorphically onto \( V \).

For \( f : (M, \partial M) \rightarrow (N, \partial N) \) a proper map, we define the minimum number of transverse roots in the proper homotopy class of \( f \) by
\[
MR_{\text{tr}}[f; c] = \min \{ \# \text{root} (g; c) : g \sim f \text{ and } g \text{ is transverse to } c \}.
\]
We define \( N_{\text{tr}}[f; c] \), the transverse Nielsen root number by setting
\[
N_{\text{tr}}[f; c] = \sum (|m(R)| : R \text{ is a root class of } f).
\]
Definition (3.3) has the same structure as the transversal Nielsen number that was introduced for the fixed point setting in [Sm]. If \( f \) is transverse to \( c \in \text{int} \, N \), then clearly \( f^{-1}(c) \) is finite, and in Example 2.8 each summand \( \deg_{g_c}(f[U]) = \pm 1 \). So a root class \( R \) of such a map must contain at least \( |m(R)| \) roots. Thus \( N_{\text{tr}}[f; c] \) is a lower bound for the number of roots of \( f \) at \( c \in \text{int} \, N \) if \( f \) is transverse to \( c \), and we will show in Corollary 3.14 that it is also a proper homotopy invariant, so \( N_{\text{tr}}[f; c] \leq MR_{\text{tr}}[f; c] \). Further, we will see in Corollary 3.14 that both the Nielsen root number \( N(f; c) \) and the transverse Nielsen root number \( N_{\text{tr}}(f; c) \) are independent of the choice of \( c \in \text{int} \, N \).

We will prove in the next section that if the dimension of the manifolds is different from two, then these Nielsen numbers are sharp, which means that \( N(f; c) = MR[f; c] \) and \( N_{\text{tr}}(f; c) = MR_{\text{tr}}[f; c] \). Further, we shall see in Theorem 5.3 that the numbers defined in (3.2) and (3.3) can be interpreted as the geometric degree and the Absolutgrad, respectively, of maps between not necessarily orientable manifolds. Thus it is important to be able to calculate \( N(f; c) \) and \( N_{\text{tr}}(f; c) \) and, for this purpose, we next investigate the multiplicity \( |m(R)| \).
Let \( \hat{q}: \hat{N} \to N \) be the covering space corresponding to \( f_*(\pi_1(M, x_0)) \) in \( \pi_1(N, c) \). The space \( \hat{N} \) is a space of equivalence classes of paths in \( N \) based at \( c \). Let \( \hat{f}: (M, \partial M) \to (\hat{N}, \partial \hat{N}) \) be the lift of \( f \) that takes \( x_0 \) to the class of the constant path at \( c \). For each \( \hat{c} \in \hat{q}^{-1}(c) \), either \( \hat{f}^{-1}(\hat{c}) \) is empty or \( R = \hat{f}^{-1}(\hat{c}) \) is a root class of \( f \) at \( c \) ([Bk2, Lemma 2], see also [H2, Satz III, p. 576]). We will next describe \(|m(R)|\) in terms of the lift \( \hat{f} \).

Let \( V \subset \text{int} \ N \) be a contractible neighborhood of \( c \) that is an elementary neighborhood for the covering space \( \hat{q}: \hat{N} \to N \). Let \( R = \hat{f}^{-1}(\hat{c}) \) be a root class and let \( \hat{V} \) be the component of \( \hat{q}^{-1}(V) \) containing \( \hat{c} \), so \( \hat{q}|\hat{V} \) is a homeomorphism onto \( V \). Since \( f \) is boundary-preserving, \( \hat{f} \) has the same property and therefore \( \hat{f}^{-1}(\hat{V}) \) is an open subset of \( \text{int} \ M \), moreover, \( \hat{f}^{-1}(\hat{V}) \cap f^{-1}(c) = R \). Let \( U \) be an open subset of \( \hat{f}^{-1}(\hat{V}) \) containing \( R \). If \( f \) is an orientable map, and therefore \( U \) can be oriented by the Orientation Procedure 2.6, then \( \deg_{\mathbb{Z}}(\hat{f}|U) \), the integer-valued local degree of \( \hat{f}|U: U \to \hat{V} \) over \( \hat{c} \) is defined when an orientation is chosen for \( \hat{V} \). If the map \( f \) is not orientable, then we still have the local degree \( \deg_{\mathbb{Z}}(\hat{f}|U) \) defined if we use \( \mathbb{Z}/2 \) coefficients. Choose an orientation of \( V \). If the homeomorphism \( \hat{q}|\hat{V}: \hat{V} \to V \) is orientation-preserving, then \( \deg_{\mathbb{Z}}(\hat{f}|U) = \deg_{\mathbb{Z}}(f|U) \) whereas if \( \hat{q}|\hat{V} \) is an orientation-reversing homeomorphism, then \( \deg_{\mathbb{Z}}(\hat{f}|U) = -\deg_{\mathbb{Z}}(f|U) \). Thus we have the following alternate description of the multiplicity of a root class.

**Theorem 3.4.** Let \( f: (M, \partial M) \to (N, \partial N) \) be a proper map, let \( c \in \text{int} \ N \) and let \( \hat{c} \in \hat{q}^{-1}(c) \) such that \( R = \hat{f}^{-1}(\hat{c}) \) is non-empty and thus a root class of \( f \) at \( c \). Let \( U \) be any open subset of \( \hat{f}^{-1}(\hat{V}) \) containing \( R \), where \( \hat{V} \) is the component of \( \hat{q}^{-1}(V) \) containing \( \hat{c} \) for \( V \subset \text{int} \ N \) a contractible elementary neighborhood of \( c \). Then

\[
|m(R)| = |\deg_{\mathbb{Z}}(\hat{f}|U)|,
\]

where, for orientable maps, \( \deg_{\mathbb{Z}}(\hat{f}|U) \) is the local degree with coefficients in \( \mathbb{Z} \) and \( U \) is oriented according to the Orientation Procedure (2.6) and, for non-orientable maps, \( \deg_{\mathbb{Z}}(\hat{f}|U) \) is the local degree with coefficients in \( \mathbb{Z}/2 \).

We now have the tool we need to prove:

**Theorem 3.5.** Let \( R \) and \( R' \) be root classes of a proper map \( f: (M, \partial M) \to (N, \partial N) \) at \( c \in \text{int} \ N \), then \( |m(R)| = |m(R')| \).

**Proof.** Let \( \hat{c}, \hat{c}' \in \hat{N} \) so that \( R = \hat{f}^{-1}(\hat{c}) \) and \( R' = \hat{f}^{-1}(\hat{c}') \). If \( M \) and \( N \), and therefore \( \hat{N} \), are orientable manifolds, choose orientations for them so that \( \hat{q}: \hat{N} \to N \) is an orientation-preserving map. Since \( \hat{f}: M \to \hat{N} \) is a map of oriented manifolds and \( \hat{N} \) is connected, \( \deg_{\mathbb{Z}}(\hat{f}) = \deg_{\mathbb{Z}}(\hat{f}) \) by [Do, Proposition 4.5, p. 268]. Let \( V \subset \text{int} \ N \) be a contractible elementary neighborhood of \( c \). Since \( \hat{q}|\hat{V}: \hat{V} \to V \) is orientation-preserving, \( \deg_{\mathbb{Z}}(\hat{f}|U) = \deg_{\mathbb{Z}}(f|U) \) for any open subset \( U \) of \( \hat{V} \) containing \( R \). Therefore, \( \deg_{\mathbb{Z}}(\hat{f}|U) = \deg_{\mathbb{Z}}(f|U) \) and \( |m(R)| = |m(R')| \).
of $c$ and let $\hat{V}$ and $\hat{V}'$ be the components of $\hat{q}^{-1}(V)$ containing $\hat{c}$ and $\hat{c}'$, respectively. Let $U = \hat{f}^{-1}(\hat{V})$ and $U' = \hat{f}^{-1}(\hat{V}')$ and orient all of $U, U', \hat{V}, \hat{V}'$ and $V$ by restricting the orientations of the manifolds in which they lie, then \( \deg_c(f(U)) = \deg_c(f(U')) \) and therefore \( \deg_c(\hat{f}|U) = \deg_c(\hat{f}|U') \), so Definition 2.7 tells us that \( |m(R)| = |m(R')| \). In particular, since all manifolds are orientable with respect to $\mathbb{Z}/2$ coefficients, if \( f: (M, \partial M) \to (N, \partial N) \) is a non-orientable map, then \( |m(R)| = |m(R')| \).

Thus we now assume that at least one of the manifolds $M$ and $N$ is non-orientable and $f$ is an orientable map, and we define $\hat{c}, \hat{c}' \in \hat{N}$ as before. Let $E$ be a euclidean subset in int $\hat{N}$ and choose $\hat{b}, \hat{b}' \in E$. Using the construction of the homeomorphism in [V, p. 133-134], we obtain a homeomorphism \( h: \hat{N} \to \hat{N} \) such that \( h(\hat{b}) = \hat{c} \) and \( h(\hat{b}') = \hat{c}' \). Thus \( S = h(E) \) is an open subset of int $\hat{N}$ homeomorphic to euclidean space such that $S$ contains both $\hat{c}$ and $\hat{c}'$. Since $f$ is an orientable map, so also is $\hat{f}$ and therefore, since $S$ is simply-connected, $\hat{f}^{-1}(S)$ is an orientable submanifold of int $M$ (compare the proof in the Orientation Procedure 2.6). We extend the Orientation Procedure 2.6 to orient $\hat{f}^{-1}(S)$ in the following manner. If there is no component of $\hat{f}^{-1}(S)$ that intersects both $R$ and $R'$, choose any points $x_R \in R$ and $x_{R'} \in R'$, choose orientations at $x_R$ and $x_{R'}$, orient the components of $\hat{f}^{-1}(S)$ that intersect $R$ or that intersect $R'$ by means of 2.6, and orient the remaining components arbitrarily. Otherwise, let $C_0$ be a component of $\hat{f}^{-1}(S)$ that intersects both $R$ and $R'$, choose an orientation for $C_0$, and choose $x_R$ and $x_{R'}$ both in $C_0$. We orient the components of $\hat{f}^{-1}(S)$ that intersect at least one of $R$ and $R'$ by means of 2.6 using the orientations at $x_R$ and $x_{R'}$ obtained from the orientation of $C_0$, and orient the other components arbitrarily. We will show that this procedure is well-defined, that is, if $C$ is a component of $\hat{f}^{-1}(S)$ that intersects both $R$ and $R'$, then it has the same orientation from 2.6 whether we use $x_R$ or $x_{R'}$ to orient it. Let $x \in R \cap C$ and $x' \in R' \cap C$ and let $\zeta$ and $\zeta'$ be paths in int $M$ from $x_R$ to $x$ and from $x_{R'}$ to $x'$, respectively, such that $f \circ \zeta$ and $f \circ \zeta'$ are contractible loops in int $N$ at $c$. Let $\alpha$ be a path in $C_0$ from $x_R$ to $x_{R'}$ and let $\beta$ be a path in $C$ from $x$ to $x'$. Since the orientations at $x_R$ and $x_{R'}$ are determined by the orientation of $C_0$, extending the orientation at $x_R$ to $x_{R'}$ along $\alpha$ agrees with the chosen orientation at $x_{R'}$. According to 2.6, the orientations at $x$ and $x'$ are obtained by extending the orientations at $x_R$ and $x_{R'}$ along $\zeta$ and $\zeta'$, respectively. Now suppose that the orientation of $C$ determined according to 2.6 from the orientation at $x$ is not the same as the orientation of $C$ determined according to 2.6 from the orientation at $x'$. Then extending the orientation by means of 2.6 at $x$ along $\alpha$ would not agree with the orientation at $x'$ obtained by means of 2.6 and therefore the loop $\lambda$ at $x'$ defined by $\lambda = \zeta'^{-1} \cdot \alpha^{-1} \cdot \zeta \cdot \beta$ would be a non-orientable
loop in \( M \). Since \( f \circ \zeta \) and \( f \circ \zeta' \) are contractible loops at \( c \) in \( N \), then \( \hat{f} \circ \zeta \) and \( \hat{f} \circ \zeta' \) (and consequently \( \hat{f} \circ \zeta^{-1} \)) are contractible loops at \( \hat{c} \) and \( \hat{c}' \) respectively. Therefore, the loop \( \hat{f} \circ \lambda \) is homotopic to \( (\hat{f} \circ \alpha^{-1}) \cdot (\hat{f} \circ \beta) \), a loop in the contractible set \( S \), so \( \hat{f} \circ \lambda \) is a contractible loop in \( \hat{N} \). Since \( \hat{f} \) is an orientable map, the loop \( \lambda \) must therefore be orientable and we conclude that if the Orientation Procedure 2.6 is used, then the orientation of \( C \) obtained from the orientation at \( x \) is equal to the orientation of \( C \) obtained from the orientation at \( x' \), as we claimed.

Now choose an orientation for \( S \) then, for the map of oriented manifolds \( \hat{f}: \hat{f}^{-1}(S) \to S \), we again have \( \deg_{\hat{c}}(\hat{f}) = \deg_{\hat{c}'}(\hat{f}) \) by [Do, Proposition 4.5, p. 268]. Let \( S_0 \) and \( S_0' \) be neighborhoods in \( S \) of \( \hat{c} \) and \( \hat{c}' \) respectively such that \( \hat{q}S_0 \) and \( \hat{q}S_0' \) are homeomorphisms onto their images. Let \( V \) be a euclidean elementary neighborhood of \( c \) in \( \hat{q}(S_0) \cap \hat{q}(S_0') \), let \( \hat{V} \) and \( \hat{V}' \) be the components of \( \hat{q}^{-1}(V) \) containing \( \hat{c} \) and \( \hat{c}' \), respectively, and let \( U = \hat{f}^{-1}(\hat{V}) \) and \( U' = \hat{f}^{-1}(\hat{V}') \). Note that we have chosen \( V \) so that \( \hat{V} \cup \hat{V}' \subset S \) and therefore \( U \cup U' \subset \hat{f}^{-1}(S) \). Orienting \( U \) and \( U' \) by restricting the orientation of \( \hat{f}^{-1}(S) \) we obtained using 2.6 and orienting \( \hat{V} \) and \( \hat{V}' \) by restricting the orientation of \( S \), for the maps \( \hat{f}|U: U \to \hat{V} \) and \( \hat{f}|U': U' \to \hat{V}' \) we then have \( \deg_{\hat{c}}(\hat{f}|U) = \deg_{\hat{c}'}(\hat{f}|U') \). By 3.4, we conclude that \( |m(R)| = |m(R')| \). \( \square \)

In view of Theorem 3.5, when we calculate \( |m(R)| \) for a root class \( R \) of a map \( f \) we understand that the calculation is valid for all the root classes of the map. Next we will present, in Lemmas 3.6 and 3.7, two cases in which \( |m(R)| \) is easy to compute.

All manifolds, in particular \( M \) and \( \hat{N} \), are orientable with respect to \( \mathbb{Z}/2 \) coefficients, so the mod 2 cohomological degree \( \deg(\hat{f},2) \) of the map \( \hat{f}: (M, \partial M) \to (\hat{N}, \partial \hat{N}) \) is defined as in \( \S 2 \). If \( f \) is a proper Type III map, excision together with the argument of Remark 2.5 utilizing \( \mathbb{Z}/2 \) coefficients implies that \( \deg_{\hat{c}}(\hat{f}|U) = \deg(\hat{f},2) \), for \( U \subset W = M - f^{-1}(\partial N) \) as in Theorem 3.4. Consequently we have:

**Lemma 3.6.** If \( f: (M, \partial M) \to (N, \partial N) \) is a proper Type III map, then \( |m(R)| = \deg(\hat{f},2) \in \mathbb{Z}/2 \).

Following Hopf [H2, Definition 2, p. 573], we write \( j \) to denote the cardinality of \( \hat{q}^{-1}(c) \), that is, the cardinality of the set of cosets \( \pi_1(N,c)/f_\pi(\pi_1(M,x_0)) \).

**Lemma 3.7.** If \( f: (M, \partial M) \to (N, \partial N) \) is a proper map such that \( j \) is infinite then \( |m(R)| = 0 \).

**Proof.** If \( |m(R)| \neq 0 \) for some \( R = \hat{f}^{-1}(\hat{c}) \) then \( \deg_{\hat{c}}(\hat{f}|U) \neq 0 \) by Theorem 3.4 and thus, for any other \( \hat{c}' \in \hat{q}^{-1}(c) \), the proof of Theorem 3.5 shows that \( \deg_{\hat{c}'}(\hat{f}|U') \neq 0 \) so \( \hat{f}^{-1}(\hat{c}') \) is non-empty. Thus, for every \( \hat{c} \in \hat{q}^{-1}(c) \) there
is a root class \( R = \hat{f}^{-1}(\hat{c}) \), which is open in \( f^{-1}(c) \). Since the cardinality of \( \hat{q}^{-1}(c) \) is \( j \) and \( f \) proper implies that \( f^{-1}(c) \) is compact, there cannot be infinitely many root classes, so \( j \) is finite. \( \square \)

As a consequence of Lemmas 3.6 and 3.7 we can now restrict our attention to the remaining cases, in which \( f \) is an orientable map and \( j \) is finite.

For the proof of the next lemma, we will use the following set of covering spaces of \( M \) and \( N \), and of basepoint-preserving lifts of \( f \) to these covering spaces; compare [E, p. 370].

\[
\begin{array}{c}
M', x_0' \xrightarrow{f'} N', c' \\
\downarrow p' \quad \downarrow q' \\
\widetilde{M}, \tilde{x}_0 \xrightarrow{\tilde{f}} \tilde{N}, \tilde{c} \\
\downarrow \tilde{p} \quad \downarrow \tilde{q} \\
M, x_0 \xrightarrow{f} \hat{N}, \hat{c}
\end{array}
\]

The spaces and maps in this diagram are obtained in the following manner.

1. \( \tilde{p}: \widetilde{M} \to M \) is the orientable covering of \( M \). Hence \( \widetilde{M} = M \) if \( M \) is orientable, but \( \widetilde{M} \) is a 2-sheeted covering of \( M \) if \( M \) is not orientable. This covering space corresponds to the subgroup of \( \pi_1(M, x_0) \) which is generated by the orientation-preserving loop classes of \( M \).

2. \( \tilde{q}: \tilde{N} \to \tilde{N} \) is the minimal covering space of \( \tilde{N} \) with the property that \( \tilde{f} \circ \tilde{p}: \widetilde{M} \to \tilde{N} \) has a lift \( \tilde{f}: \widetilde{M} \to \tilde{N} \). Hence \( \tilde{N} \) corresponds to the subgroup of \( \pi_1(\tilde{N}, \tilde{c}) \) which is generated by the images under \( f \) of all orientation-preserving loop classes of \( M \), and the homomorphism \( \tilde{f}_\pi: \pi_1(\tilde{M}, \tilde{x}_0) \to \pi_1(\tilde{N}, \tilde{c}) \) is onto.

3. \( q': N' \to \tilde{N} \) is the orientable covering of \( \tilde{N} \). Hence \( N' \) corresponds to the subgroup of \( \pi_1(\tilde{N}, \tilde{c}) \) which is generated by the images under \( f \) of all orientation-preserving loop classes of \( M \) which have an orientation-preserving image in \( N \).

4. \( p': M' \to \tilde{M} \) is the minimal cover of \( \tilde{M} \) with the property that \( \tilde{f} \circ p': M' \to \tilde{N} \) has a lift to \( f': M' \to N' \). Hence \( M' \) corresponds to the subgroup of \( \pi_1(\tilde{M}, \tilde{x}_0) \) which is generated by all orientation-preserving loop classes of \( M \) which have an orientation-preserving image in \( N \) under \( f \), and the homomorphism \( f'_\pi: \pi_1(M', x_0') \to \pi_1(N', c') \) is onto. Equivalently, \( p': M' \to \tilde{M} \) is the pullback of \( q': N' \to \tilde{N} \) over \( \tilde{M} \) by means of \( f \).
Note that all the spaces in the diagram are $n$-manifolds with (possibly empty) boundary because $M$ and $N$ are, and all the maps are proper and boundary-preserving because $f$ is proper and boundary-preserving. It is easy to see from the fundamental groups to which the covering spaces correspond that each of these covering spaces is either 2-sheeted or the identity.

By construction, not only $\tilde{M}$, but also $M'$ and $N'$ are always orientable, and so the integer-valued cohomological degree $\deg(f')$ of the map $f' : (M', \partial M') \rightarrow (N', \partial N')$ is always defined.

**Lemma 3.8.** If $f : (M, \partial M) \rightarrow (N, \partial N)$ is a proper orientable map and $j$ is finite, then $|m(R)| = |\deg(f')|$.

**Proof.** The orientation of $U$ by the Orientation Procedure (2.6) corresponds to a cross-section $s : U \rightarrow \tilde{M}$ and we set $\tilde{U} = s(U)$. Define $\hat{R} = s(R)$ and let $\tilde{x}_R = s(x_R)$. Set $\tilde{f}(\tilde{x}_R) = c \in \tilde{q}^{-1}(c)$. We may assume that $\tilde{V}$ in Theorem 3.4 has been chosen so that $\tilde{V}$ is an elementary neighborhood of the covering space $\tilde{p} : \tilde{N} \rightarrow \tilde{N}$. Let $\hat{V}$ be the component of $\hat{V}$ that contains $\tilde{c}$, then the restriction of $\tilde{q}$ to $\hat{V}$ is a homeomorphism. We note that $\tilde{f}(U) \subset \hat{V}$ because $\tilde{f}(U) \subset \hat{V}$. Letting $\tilde{S} = \tilde{f}^{-1}(c)$, we claim that $\tilde{S} = \hat{R}$.

To verify the claim, we first show that $\hat{R} \subseteq \tilde{S}$. We have $\tilde{x}_R \in \tilde{S}$ by definition. Let $\tilde{x} \in \hat{R}$ be any other point. We obtained $\tilde{x}$ by taking a path $w$ in $\tilde{M}$ from $x_R$ to $x = \tilde{p}(\tilde{x})$ in $\tilde{M}$ such that $f \circ w$ is a contractible loop in $N$ and lifting $w$ to a path $\tilde{w}$ in $\tilde{M}$ starting at $\tilde{x}_R$. Since $\tilde{f} \circ \tilde{w}$ is a lifting of a contractible loop in $N$, it is a contractible loop and we conclude that $\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}_R) = \tilde{c}$ so $\tilde{x} \in \tilde{S}$.

In order to prove that $\tilde{S} \subseteq \hat{R}$ we will assume that there exists $\tilde{x} \in \tilde{S}$ that is not in $\hat{R}$ and arrive at a contradiction. As $\hat{f} \circ \hat{p}(\tilde{x}) = \tilde{q} \circ \tilde{f}(\tilde{x}) = \tilde{c}$, it follows that $x = \tilde{p}(\tilde{x}) \in R$. Thus, there is a path $w$ from $x_R$ to $x$ such that $f \circ w$ is contractible in $N$. Consider the path $w^{-1}$ defined by $w^{-1}(t) = w(1-t)$ and let $\tilde{w}^{-1}$ be the path in $\tilde{M}$ obtained by lifting $w^{-1}$ to a path that starts at $\tilde{x}$, then $\tilde{x}_R = \tilde{w}^{-1}(1)$ is in $\tilde{p}^{-1}(x_R)$ but $\tilde{x}_R \neq \tilde{x}_R$ because otherwise lifting $w$ to a path that starts at $\tilde{x}_R = \tilde{x}_R$ would show that $\tilde{x} \in \hat{R}$. However, $\tilde{x}_R \in \tilde{S}$ because the lifting the contractible loop $f \circ w$ we see that $\tilde{f} \circ \tilde{w}$ is a contractible loop and therefore, in particular, $\tilde{f}(\tilde{x}_R') = \tilde{c}$. As $\tilde{x}_R' \in \tilde{S}$, there exists a path $\tilde{v}$ in $\tilde{M}$ from $\tilde{x}_R$ to $\tilde{x}_R'$ so that $\tilde{f} \circ \tilde{p} \circ \tilde{v}$ is a contractible loop at $\tilde{c}$ in $\tilde{N}$. But the loop $v = \tilde{p} \circ \tilde{v}$ at $x_R$ in $M$ does not lift to a loop at $\tilde{x}_R$ in $\tilde{M}$, and so it is orientation-reversing. As $f = \tilde{q} \circ \tilde{f}$ is orientable, it cannot map the orientation-reversing loop $v$ to the contractible loop $\tilde{q} \circ \tilde{f} \circ v$, and so we have a contradiction. Hence $\hat{R} = \tilde{S}$.

Since we have homeomorphisms $\tilde{p}|\tilde{U} : (\tilde{U}, \partial \tilde{U} - R) \rightarrow (U, U - R)$ and $\tilde{q}|\hat{V} : (\hat{V}, \hat{V} - \hat{c}) \rightarrow (\hat{V}, \hat{V} - \hat{c})$ such that $(\hat{f}|\hat{U}) \circ (\tilde{p}|\tilde{U}) = (\tilde{q}|\hat{V}) \circ (\tilde{f}|\tilde{U})$, we conclude, using Theorem 3.4, that $|m(R)| = |\deg(\hat{f}|\hat{U})| = |\deg(\tilde{f}|\tilde{U})|$. 
We recall that $p': M' \to \tilde{M}$ is the pullback of the orientable cover $q': N' \to \tilde{N}$ over $\tilde{M}$ by $\tilde{f}$, so
\[
M' = \{ (\tilde{x}, y') \in \tilde{M} \times N' | \tilde{f}(\tilde{x}) = q'(y') \}
\]
and $p'$ is projection $p'(\tilde{x}, y') = \tilde{x}$. Furthermore, $\tilde{f}$ can be lifted to $f' : (M', \partial M') \to (N', \partial N')$ by setting $f'(\tilde{x}, y') = y'$. Choosing an orientation for the contractible set $\tilde{V}$ defines a cross-section on $\tilde{V}$ to $N'$ and we let $V'$ be the image of the cross-section. Let $c'_R = q'^{-1}(c) \cap V'$. We define an open subset $U'$ of $M'$ by letting
\[
U' = \{ (\tilde{x}, y') \in M' | \tilde{x} \in \tilde{U} \text{ and } y' \in V' \}.
\]
The restriction of $p$ to $U'$ is a homeomorphism $h$ of $U'$ onto $\tilde{U}$ with inverse $h^{-1} : \tilde{U} \to U'$ given by $h^{-1}(\tilde{x}) = (\tilde{x}, q^{-1}(f(\tilde{x})) \cap V')$. Let
\[
R' = \{ (\tilde{x}, y') \in M' | \tilde{x} \in \tilde{R} \text{ and } y' \in V' \}.
\]
Letting $S' = f'^{-1}(c')$, we claim that $R' = S'$. If $(\tilde{x}, y') \in R'$ then $\tilde{x} \in \tilde{R} = \tilde{S}$, that is, $\tilde{f}(\tilde{x}) = \tilde{c}$. Therefore $f'(\tilde{x}, y') = y' \in q'^{-1}(\tilde{c})$ but since $y' \in V'$ it must be that $f'(\tilde{x}, y') = c'$ and we see that $R' \subseteq S'$. On the other hand, $(\tilde{x}, y') \in S'$ means that $y' = c'$ and since $(\tilde{x}, y') \in M'$ we know that $f(\tilde{x}) = q(y') = \tilde{c}$, that is, $\tilde{x} \in \tilde{R}$, so $(\tilde{x}, y') \in R'$ and we have established the claim.

We have homeomorphisms $h = p|U' : (U', U' - R') \to (\tilde{U}, \tilde{U} - \tilde{R})$ and $q'|V' : (V', V' - c') \to (\tilde{V}, \tilde{V} - \tilde{c})$ such that $(\tilde{f} \tilde{U}) \circ (p|U') = (q|V') \circ (f'|U')$ so we conclude that $|\deg_\mathbb{C}(\tilde{f} \tilde{U})| = |\deg_\mathbb{C}(f'|U')|$. On the other hand, since $M'$ and $N'$ are orientable manifolds and $R' = f'^{-1}(c')$, then the fact that $\deg_\mathbb{C}(f'|U') = \deg(f')$ follows from Remark 2.5, so we have proved that $|m(R)| = |\deg(f')|$. \hfill \Box

If $f$ is a map of Type I, then $\tilde{M} = M'$ and $\tilde{N} = N'$, and hence $f' = \tilde{f}$. Thus Lemma 3.8 tells us that:

**Lemma 3.9.** Let $f : (M, \partial M) \to (N, \partial N)$ be a proper Type I map and let $j$ be finite, then $|m(R)| = |\deg(f)|$.

The following lemma is due to Epstein [E, Lemma 3.3]. We include the brief proof for the convenience of the reader.

**Lemma 3.10.** Let $f : (M, \partial M) \to (N, \partial N)$ be a proper Type II map, then $\deg(f') = 0$ and therefore $|m(R)| = 0$.

**Proof.** As $f$ is not orientation-true, there exists loops $v$ in $M$ and $w = f \circ v$ in $N$ such that one of $v$ and $w$ is an orientation-preserving loop and the other is an orientation-reversing loop. Let $T_v$ and $T_w$ be the covering transformations of $M'$ and $N'$ that are induced by $v$ and $w$ on these covering spaces of $M$ and $N$ respectively, then one of the these is an orientation-preserving homeomorphism and the other is orientation-reversing. Now $f' \circ
$T_v$ and $T_w \circ f'$ are lifts of $f$ that agree at one point, so they are equal and $\deg(f') \deg(T_v) = \deg(T_w) \deg(f')$. Since $\deg(T_v) \deg(T_w) = -1$, we have $\deg(f') = -\deg(f')$, and the lemma follows. \qed

We have now finished the calculation of the multiplicity $|m(R)|$ for all cases. To summarize the results of Lemmas 3.6, 3.7, 3.9 and 3.10: If $j$ is infinite or $f$ is of Type II then $|m(R)| = 0$. If $f$ is of Type I then $|m(R)| = |\deg(\tilde{f})|$ whereas if $f$ is of Type III then $|m(R)| = \deg(\hat{f}, 2)$. Since Theorem 3.5 tells us that all root classes of a map have the same multiplicity, the Nielsen root numbers can be calculated from their definitions as follows.

**Theorem 3.11.** Let $f: (M, \partial M) \to (N, \partial N)$ be a proper map between two $n$-dimensional manifolds. If $f$ is a Type I map, then $N(f; c) = j$ if $\deg(\tilde{f}) \neq 0$ and $j$ is finite. If $f$ is a Type III map, then $N(f; c) = j$ if $\deg(\hat{f}, 2) = 1$ and $j$ is finite. In all other cases, $N(f; c) = 0$.

**Theorem 3.12.** Let $f: (M, \partial M) \to (N, \partial N)$ be a proper map between two $n$-dimensional manifolds. If $f$ is a Type I map, then $N_\hat{f}(f; c) = j \cdot |\deg(\hat{f})|$ if $j$ is finite. If $f$ is a Type III map, then $N_\hat{f}(f; c) = j \cdot \deg(\hat{f}, 2)$ if $j$ is finite. In all other cases $N_\hat{f}(f; c) = 0$.

If both $M$ and $N$ are orientable, then all maps are of Type I and the results of Theorems 3.11 and 3.12 can be simplified by using $\deg(f)$ rather than $\deg(\tilde{f})$. For compact manifolds $M$ and $N$, the formula for $N(f; c)$ given in the next theorem can also be obtained from [BS2, Theorems 3.1, 3.4, 3.12 and 4.8]. For closed manifolds, this theorem can be found in [L, Proposition 5].

**Theorem 3.13.** Let $f: (M, \partial M) \to (N, \partial N)$ be a proper map between two $n$-dimensional orientable manifolds. Then

$$N(f; c) = \begin{cases} j, & \text{if } \deg(f) \neq 0 \\ 0, & \text{if } \deg(f) = 0, \end{cases}$$

and $N_\hat{f}(f; c) = |\deg(f)|$.

**Proof.** We first assume that $j$ is finite. It is easy to see that then $\hat{q}$ is proper, and so we can use [Do, Propositions 4.5, p. 268 and 4.7, p. 269] to show that $\deg(\hat{q}) = j > 0$. As $M$ and $N$ are orientable, we have $\tilde{f} = \hat{f}$, and therefore $|\deg(f)| = |\deg(\hat{q})| \cdot |\deg(\hat{f})| = j \cdot |\deg(\hat{f})|$, and thus Theorem 3.13 follows from Theorems 3.11 and 3.12.

Now we assume that $j$ is infinite. We claim that in this case $\deg(f) = 0$, and that therefore Theorem 3.13 follows again from Theorems 3.11 and 3.12. To verify our claim, we use Lemma 3.7 which states that for infinite $j$ all root
classes have multiplicity $|m(R)| = 0$, and so $\deg_c(f | U) = 0$ also. Therefore $\deg(f) = \sum (\deg_c(f | U_R) : R$ is a root class of $f) = 0$ as claimed. □

Maps that are homotopic by a proper homotopy induce the same homomorphism of Čech cohomology with compact supports [Do, p. 290], so the cohomological degree is a proper homotopy invariant. Therefore, Theorems 3.11 and 3.12 imply

**Corollary 3.14.** Let $f: (M, \partial M) \to (N, \partial N)$ be a proper map, then $N(f; c)$ and $N(f; c)$ are proper homotopy invariants. Moreover, the values of both $N(f; c)$ and $N(f; c)$ are independent of the choice of $c \in \text{int } N$.

We conclude this section with some examples of maps with non-zero Nielsen root numbers.

**Example 3.15.** Let $f: S^n \to S^n$ be a map of degree $d \neq 0$ between two $n$-spheres, where $n \geq 2$. As $j = 1$, Theorem 3.13 shows that $N(f; c) = 1$ and $N(f; c) = |d|$, and so $N(f; c) \geq N(f; c)$ if $|d| > 1$. More generally, it follows from Theorem 3.13 that $N(f; c) = |d|$ is strictly greater than $N(f; c) = 1$ for any map $f: (M, \partial M) \to (N, \partial N)$ between two orientable $n$-manifolds if $f_\pi$ is an epimorphism and $|\deg(f)| = |d| > 1$. Examples with $N(f; c) \neq N(f; c)$ for maps of non-orientable manifolds can readily be constructed by using cartesian products. To be specific, let $f: P^2 \times S^2 \to P^2 \times S^2$ be the product map $f = f_1 \times f_2$ with $f_1$ the identity and $f_2$ of degree $d$ with $|d| > 1$. Then $f$ is of Type 1, $j = 1$, $f = \hat{f}$, $M = N = S^2 \times S^2$ and $\tilde{f} = \tilde{f}_1 \times \tilde{f}_2$ where $\tilde{f}_1$ is the identity and $\tilde{f}_2 = f_2$. Now $|\deg(\tilde{f})| = |d| \neq 0$ so $N(f; c) = 1$ by Theorem 3.11 whereas Theorem 3.12 implies that $N(f; c) = |d| > 1$.

**Example 3.16.** A different type of example of a map $f$ of non-orientable manifolds for which $N(f; c) \neq N(f; c)$ is illustrated by the following. Represent the Klein bottle by $K = P^2 \# S^2 \# P^2$, then a rotation of $S^2$ interchanging the copies of $P^2$ defines an action of $\mathbb{Z}/2$ on $K$ with two fixed points. Therefore, the homomorphism of fundamental groups induced by the quotient map $f: K \to P^2$ is onto [Bd, Cor. 6.3, p. 91]. Thus $\tilde{P}^2 = P^2$ and $\tilde{f}$ is the lift $\tilde{f}: T^2 \to S^2$ of $f$ to the oriented covers. By inspection, $N(f; c) = 2$, and so we obtain from Theorem 3.13 that $\tilde{f}$ is a map of degree $\pm 2$. Since the map $f$ is of Type 1, Theorems 3.11 and 3.12 imply that $N(f; c) = 1$ and $N(f; c) = 2$.

**Example 3.17.** As in Example 2.2(b), let $f: M \to N$ be the covering map of the orientable cover of a non-orientable manifold $N$. Since $j = 2$ and $\deg(f) = 1$, Theorem 3.11 implies that $N(f; c) = 2$ and Theorem 3.12 implies that $N(f; c) = 2$ also.
Example 3.18. Let \( f: K \to K \) be a map of the Klein bottle, then, by [BO, Prop. 6.4], there are integers \( b, d \) and \( e \) such that \( f_\pi(\alpha) = \alpha^b \beta^d \) and \( f_\pi(\beta) = \beta^e \) for appropriately chosen generators \( \alpha \) and \( \beta \) of \( \pi_1(K, x_0) \). If \( e \neq 0 \) then \( b \) is odd and \( N(f; c) = |be| \neq 0 \). Thus Theorem 3.11 tells us that \( j = |be| \) and that \( f \) is not of Type II. From the proof of [BO, Prop. 6.4] we learn that \( f_\pi \) is a monomorphism in this case, so \( f \) cannot be of Type III and we conclude that \( f \) is of Type I. We claim that \( |\deg(f)| = 1 \) and therefore, by Theorem 3.12, that \( N_\parallel(f; c) = |be| \). The map \( f \) is Type I and \( M = K \) is non-orientable so \( \tilde{N} \) is non-orientable (see [E, page 371]) and it follows that the fiber of the covering space \( q = \tilde{q} \circ \tilde{q}: \tilde{N} \to N \) has cardinality \( 2j = 2|be| \). Let \( r: N^0 \to N = K \) be the oriented cover and let \( q^o: (\tilde{N}, \tilde{c}) \to (N^0, c^o) \) be the covering space such that \( r \circ q^o = q \). Since \( q^o \) is a covering map between closed oriented manifolds and the cardinality of the fiber is \( |be| \), we conclude that \( |\deg(q^o)| = |be| \). The Type I map \( f \) lifts to a map \( f^o: (\tilde{M}, \tilde{x}_0) \to (N^0, c^o) \) of the oriented covers and \( f^o = q^o \circ f \) by uniqueness of lifts. We see from the proof of Proposition 6.4 of [BO] that \( \deg(f^o) = be \), so \( |\deg(f)| = 1 \) as we claimed.

Example 3.19. Let \( A \) be a \( 2 \times 2 \) integer matrix with determinant \( d \) odd and let \( f': T^2 \to T^2 \) be the corresponding map of the torus. Let \( f'': T^2 \# P^2 \to T^2 \) be the map of Example 2.4(b) and define \( f = f' \circ f'': T^2 \# P^2 \to T^2 \). The map \( f \) is of Type III because \( f'' \) is. Since \( d \) is odd, \( \deg(f', 2) = 1 \). Now \( \deg(f'', 2) = 1 \) also, so \( \deg(f, 2) = 1 \) and therefore \( \deg(f, 2) = 1 \). Noting that \( f_\pi(\pi_1(T^2 \# P^2, x_0)) = f''_\pi(f_\pi(\pi_1(T^2 \# P^2, x_0))) = f'_\pi(f_\pi(\pi_1(T^2, f''(x_0)))) \), we have \( j = |d| \). Therefore, Theorems 3.11 and 3.12 tell us that \( N(f; c) = N_\parallel(f; c) = |d| \).

4. Sharpness.

In this section, we will prove the sharpness of the two Nielsen root numbers defined in §3, that is, that \( N(f; c) = MR[f; c] \) and \( N_\parallel(f; c) = MR_\parallel[f; c] \), using results from [Je]. As in [Je], we will work in the general setting of topological manifolds, and hence we use the microbundle transversality from [KS, Essay III, p. 84]. Recall that a microbundle \( \xi = \xi^n \) over a space \( X \) consists of a total space \( E(\xi) \supset X \) together with a retraction \( r: E(\xi) \to X \) that is a submersion near \( X \), and which has the property that, for all \( x \in X \), the fibres \( \xi_x = r^{-1}(x) \) are \( n \)-manifolds without boundary. Now consider a pair \((Y, P)\) of topological spaces, where \( P \) is closed in \( Y \) and is equipped with a normal microbundle \( \eta \), which means that the total space of \( \eta \) is an open subset of \( Y \) containing \( P \). If \( X \) is a topological manifold without boundary, then a map \( h: X \to Y \) is called topologically transverse to \( \eta \) if \( h^{-1}(P) \) is a topological submanifold of \( X \) admitting a normal microbundle \( \xi \) such that,
for every \( x \in h^{-1}(P) \), a neighborhood of \( x \) in the fibre \( \xi_x \) is mapped by \( h \) homeomorphically onto a neighborhood of \( h(x) \) in the fibre \( \eta_{h(x)} \). See [KS, p. 84] and [Je, p. 167].

In order to apply microbundle transversality to a proper map \( f: (M, \partial M) \to (N, \partial N) \) between two topological \( n \)-manifolds with (possibly empty) boundaries, we will have to assume that \( f \) has the additional property that \( f(\text{int } M) \subset \text{int } N \). We write \( f^o \) for the restriction of \( f \) to \( \text{int } M \), select \( c \in \text{int } N \), and we will in the remainder of this section choose \( X = \text{int } M \), \( Y = \text{int } M \times \text{int } N \), \( P = \text{int } M \times \{c\} \), and \( \eta \) as the normal microbundle of \( P \) in \( Y \) which has \( E(\eta) = Y \) as its total space and the projection \( r: Y \to P \) given by \( r(x,y) = (x,c) \) as its retraction. Let \( h: \text{int } M \to \text{int } M \times \text{int } N \) be defined by \( h = (e,f^o) \), where \( e: \text{int } M \to \text{int } M \) is any map, then clearly \( h^{-1}(P) = \text{root}(f;c) \). According to Definition 3.1, a proper map \( f: (M, \partial M) \to (N, \partial N) \) is transverse to \( c \), where \( c \in \text{int } N \), if there exists a euclidean neighborhood \( V \) of \( c \) in \( N \) so that \( f^{-1}(V) \) consists of finitely many euclidean neighborhoods in \( \text{int } M \), and each of them is mapped by \( f \) homeomorphically onto \( V \). Transversality of \( h \) to \( \eta \) and transversality of \( f \) to \( c \) are intimately related: A proper map of the form \( f: (M, \partial M, \text{int } M) \to (N, \partial N, \text{int } N) \) is transverse to \( c \in \text{int } N \) if and only if there exists a map \( e: \text{int } M \to \text{int } M \) so that the map \( h = (e,f^o): \text{int } M \to \text{int } M \times \text{int } N \) is topologically transverse to \( \eta \). To see that this is true, note that if \( h = (e,f^o) \) is topologically transverse to \( \eta \), then the map \( e \) must be a map that is constant on a neighborhood of each of the points in \( h^{-1}(P) \). Conversely, given a proper map \( f \) transverse to \( c \), such a map \( e \) can always be found in order to construct a map \( h \) which is topologically transverse to \( \eta \).

The most important step in the proof of the sharpness of Nielsen numbers consists in uniting two points in the same Nielsen class whenever possible, and for this step we want to apply results concerning Nielsen classes from [Je] to a map \( h = (e,f^o) \). This is possible as, according to the definition of the Nielsen relation in [Je, p. 168], two points of \( h^{-1}(P) \) are in such a relation if and only if they belong to the same root class of \( f \). We will use a Whitney type lemma to unite roots, and for this we need local orientations. By a local orientation \( O(x) \) of \( M \) at \( x \in \text{int } M \) we mean a generator \( O(x) \in H_n(M,M-x) \) and by a local orientation of \( N \) at \( y \in \text{int } N \) we mean a generator of \( O(y) \in H_n(N,N-y) \), where the homology groups have coefficients in \( \mathbb{Z} \) (see [Do, Definition 2.1, p. 252]).

If \( f: (M, \partial M) \to (N, \partial N) \) is a map which is transverse to \( c \in \text{int } N \) and if \( x \) is a root of \( f \), then \( f \) defines, by restriction, a homeomorphism of a euclidean neighborhood \( U_x \) of \( x \) in \( \text{int } M \) onto a euclidean neighborhood \( V \) of \( c \) in \( \text{int } N \). Using the excision isomorphisms \( H_n(M,M-x) \cong H_n(U_x,U_x-x) \) and \( H_n(N,N-c) \cong H_n(V,V-c) \) we see that, given a (local) orientation \( O(x) \) of \( M \) at \( x \), the restriction of \( f \) to \( U_x \) defines a corresponding orientation of \( N \) at \( c \) which, by abuse of notation, we denote by \( f_*(O(x)) \). As in [Je, p. 168]
we define a local orientation of the microbundle \( \eta \) at \((x, c) \in \text{int } M \times \text{int } N\) to be a generator of \( H_n(x \times N, x \times (N - c))\). Then it is straightforward to check that the \( \mathcal{R} \)-relation of [Je, Definition (1.2)], which characterises points of \( h^{-1}(P) \) that can be removed by a homotopy, takes in our case the following form: Let \( f: (M, \partial M, \text{int } M) \to (N, \partial N, \text{int } N)\) be a proper map that is transverse to \( c \), where \( c \in \text{int } N \), and let \( x_1, x_2 \in \text{root } (f; c)\). Then \( x_1, x_2 \) are \( \mathcal{R} \)-related with respect to a map \( h = (e, f^\circ)\): \( \text{int } M \to \text{int } M \times \text{int } N \) which is transverse to \( \eta \) if and only if there exists a path \( w: I \to \text{int } M \) from \( x_1 \) to \( x_2 \) such that \( f \circ w \) is a contractible loop at \( c \) and \( f_* (O(x_1)) = -f_* (O(x_2)) \), where \( O(x_1) \) is a local orientation of \( M \) at \( x_1 \) and \( O(x_2) \) is the local orientation of \( M \) at \( x_2 \) that is obtained from \( O(x_1) \) by continuation along \( w \). Note that this characterisation of the \( \mathcal{R} \)-relation is independent of the choice of the map \( e \) and the orientation \( O(x_1) \).

The following lemma gives information about the number of points in \( \mathcal{R} \)-relation to one another. Its proof helps to explain why the different behavior of orientable and non-orientable maps leads to different minimal root sets. The technique in the proof of “flipping” the local orientation when dealing with non-orientable maps was used by Hopf in the elimination of inessential root classes in [H2, p. 601].

**Lemma 4.1.** Let \( f: (M, \partial N, \text{int } M) \to (N, \partial N, \text{int } N)\) be a proper map between two \( n \)-dimensional manifolds and let \( c \in \text{int } N \). Let \( f \) be transverse to \( c \), and let \( R = \{x_1, \ldots, x_k\} \) be a root class of \( f \) at \( c \). If \( k > |m(R)| \), then there exist two points in \( R \) which are \( \mathcal{R} \)-related with respect to any map \( h: \text{int } M \to \text{int } M \times \text{int } N \) given by \( h = (e, f^\circ) \) that is transverse to \( \eta \).

**Proof.** We first assume that \( f \) is an orientable map. Since \( f^{-1}(c) \) is finite, as in Example 2.8 we may choose \( U \) to be the disjoint union of euclidean neighborhoods \( U_\ell \), each containing one point \( x_\ell \in R \). Furthermore, since \( f \) is transverse to \( c \), we may choose the \( U_\ell \) so that each restriction \( f|U_\ell \) is a homeomorphism onto the euclidean neighborhood \( V \) of \( c \). If we orient \( U \) as in the Orientation Procedure 2.6, then we see from Example 2.8 that

\[
|m(R)| = \left| \sum (\deg_c(f|U_\ell)): 1 \leq \ell \leq k \right|
\]

where the local degree is defined with integer coefficients. Since each \( f|U_\ell \) is a homeomorphism, the corresponding summand equals \( \pm 1 \) and therefore \( k > |m(R)| \) implies that not all local degrees can be equal. So, without loss of generality, we assume that \( \deg_c(f|U_1) = -\deg_c(f|U_2) \), where according to the Orientation Procedure 2.6 the orientation of \( U_2 \) is obtained from the orientation of \( U_1 \) by continuation along a path \( w \) from \( x_1 \) to \( x_2 \) in \( \text{int } M \) that is chosen so that \( f \circ w \) is a contractible loop in \( N \) at \( c \). The orientations of \( U_1 \) and \( U_2 \) define local orientations \( O(x_1) \) and \( O(x_2) \) of \( M \) at \( x_1 \) and \( x_2 \) and, according to our selection of orientations for \( U_1 \) and \( U_2 \), the local orientation \( O(x_2) \) is obtained from \( O(x_1) \) by continuation along \( w \). From
$\deg_c(f|U_1) = -\deg_c(f|U_2)$ it follows that $f_*(O(x_1)) = -f_*(O(x_2))$, and therefore $x_1$ and $x_2$ are $R$-related with respect to a map $h = (e, f^o)$ that is transverse at $\eta$.

Now let $f$ be non-orientable, that is, of Type III. We can choose the open set $U$ as before and obtain, by using Example 2.8 with coefficients in $\mathbb{Z}/2$,

$$|m(R)| = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

We first assume that $k > |m(R)| = 1$, and so there exist at least two roots in $R$, say $x_1$ and $x_2$, and a path $w$ in $\text{int } M$ from $x_1$ to $x_2$ so that $f \circ w$ is a contractible loop at $c$ in $N$. If the continuation of the local orientation $O(x_1)$ from $x_1$ to $x_2$ along $w$ leads to a local orientation $O(x_2)$ so that $f_*(O(x_1)) = -f_*(O(x_2))$, then it follows as before that $x_1$ and $x_2$ are $R$-related. Otherwise, we select an orientation reversing loop $\ell: I \to \text{int } M$ so that $f \circ \ell$ is a contractible loop in $N$, and a path $v: I \to \text{int } M$ from $x_1$ to $\ell(0)$. Then $w' = v \cdot \ell \cdot v^{-1} \cdot w$ is a path from $x_1$ to $x_2$ so that its image under $f$ in $N$ is a contractible loop at $c$. As the continuation of a local orientation along $w'$ gives the opposite value as the continuation of this local orientation along the path $w$, it follows once again that $x_1$ and $x_2$ are $R$-related with respect to $h$. This completes our proof if $k > |m(R)| = 1$. If $|m(R)| = 0$ then $k$ is even, and so $k > |m(R)|$ implies $k \geq 2$ and two $R$-related roots can be found in the same way. \qed

In the next theorem we prove the sharpness of the transverse Nielsen root number with respect to proper homotopies. This result is essentially due to Hopf (see Theorem 5.4).

**Theorem 4.2.** Let $f: (M, \partial M) \to (N, \partial N)$ be a proper map between two $n$-dimensional manifolds and let $c \in \text{int } N$. If $n \neq 2$, then there exists a map $g: (M, \partial M) \to (N, \partial N)$ which is homotopic to $f$ by a proper homotopy, is transverse to $c$, and has precisely $N_{|\eta|}(f; c)$ roots at $c$. Hence $N_{|\eta|}(f; c)$ is sharp, that is $N_{|\eta|}(f; c) = MR_{|\eta|}[f; c]$.

**Proof.** The theorem is obviously true if $n = 1$, and so we assume that $n \geq 3$. As $c \in \text{int } N$, all roots of $f$ lie in $\text{int } M$. Using the construction in [BS1], there is a map $f^+: (M, \partial M, \text{int } M) \to (N, \partial N, \text{int } N)$ that has the same root set as $f$, and it is easy to see from the construction that there exists a proper homotopy from $f$ to $f^+$ relative to the complement of the interior of a collar of $\partial M$.

As before, we choose $X = \text{int } M, Y = \text{int } M \times \text{int } N, P = \text{int } M \times \{c\}$, and let $\eta$ be the microbundle of $P$ in $W$ given by the projection $r(x, y) = (x, c)$. If we define the map $h: \text{int } M \to \text{int } M \times \text{int } N$ by $h(x) = (x, f^+(x))$, we can use the microbundle transversality of [KS] (see also [Je, Lemma 1.1]) to deform $h$ on a compact neighborhood of root $(f; c)$ in $\text{int } M$, and relative
to the boundary of this neighborhood, and then extend it over \( \text{int} \, M \) as the identity, to obtain a map \( h' : \text{int} \, M \to \text{int} \, M \times \text{int} \, N \) that is transverse to \( \eta \).

The map \( h' = (h'_1, h'_2) \) defines a map \( f' : (M, \partial M, \text{int} \, M) \to (N, \partial N, \text{int} \, N) \) by setting \( f'(c) = h'_2(x) \) for all \( x \in \text{int} \, M \) and \( f'(x) = f(x) \) for all \( x \in \partial M \), and we have \( h'^{-1}(P) = \text{root} \, (f'; c) \). As the map \( h' \) is transverse to \( \eta \), the map \( f' \) is transverse to \( c \).

Now let \( R \) be any root class of \( f' \) which consists of \( k > |m(R)| \) points. Then Lemma 4.1 implies that at least one pair of these roots is \( \mathcal{R} \)-related with respect to the map \( h' \). Let this pair be \( x_1, x_2 \). As we can choose the path \( w \) which establishes the \( \mathcal{R} \)-relation to be a flat arc in \( \text{int} \, M \), there exists an \( n \)-dimensional ball \( D \subset \text{int} \, M \) that contains \( w \) but no root of \( f' \) other than \( x_1, x_2 \). Therefore we can use the Whitney Type Lemma 3.1 from [Je] to obtain a homotopy of \( h''|D \), relative to the boundary of \( D \), to a map from \( D \) into \( (\text{int} \, M \times \text{int} \, N) - P \). We keep \( h' \) fixed on the complement of \( D \).

We repeat this process as many times as possible to remove pairs of points from \( h''^{-1}(P) \) that belong to the same root class of \( f' \), and so construct a map \( h'' = (h''_1, h''_2) \). The map \( h''_2 : \text{int} \, M \to \text{int} \, N \) extends to a map \( g : (M, \partial M) \to (N, \partial N) \) by setting \( g(x) = h''_2(x) \) for all \( x \in \text{int} \, M \), and \( g(x) = f(x) \) for all \( x \in \partial M \). It follows from Lemma 4.1 that each root class of \( g \) consists of exactly \( |m(R)| \) points. Hence \( g \) has exactly \( N_{|\eta|}(f; c) \) roots at \( c \) and, as \( h'' \) is transverse to \( \eta \), the map \( g \) is transverse to \( c \). Each modification of \( h \) has been made on a compact subset of \( \text{int} \, M \), so the sequence of the constructed homotopies defines a homotopy from \( f \) to \( g \) that is boundary-preserving and has a compact carrier, that is, it is constant outside a compact set. Therefore this homotopy from \( f \) to \( g \) is proper. \( \square \)

With regard to \( N(f; c) \), sharpness was first proved (under slightly different assumptions than those used here) in [H2, Satz XIIIb]. Techniques are now available that allow us to give a short proof of this fact, as follows.

**Theorem 4.3.** Let \( f : (M, \partial N) \to (N, \partial N) \) be a proper map between two \( n \)-dimensional manifolds and let \( c \in \text{int} \, N \). If \( n \neq 2 \), then there exists a map \( g : (M, \partial M) \to (N, \partial N) \) that is homotopic to \( f \) by a proper homotopy and has precisely \( N(f; c) \) roots at \( c \). Hence \( N(f; c) \) is sharp, that is \( N(f; c) = MR[f; c] \).

**Proof.** Again we can assume that \( n \geq 3 \). We proceed as in the proof of Theorem 4.2 to obtain a proper boundary-preserving map which has \( N_{|\eta|}(f; c) \) roots at \( c \) and is transverse to \( c \). The proof is completed by using the Creating and Cancelling Procedures which were developed to minimize the number of fixed points on differentiable manifolds by [Jg1] and used to minimize the number of roots on orientable PL manifolds by [L, §3, proof of Theorem B]. As these procedures consist of local changes that occur in \( \text{int} \, M \), and
use neither a global orientation nor a PL or differentiable structure on the
manifolds, they can be applied without change.

Here are some examples to illustrate Theorems 4.2 and 4.3.

**Example 4.4.** Let \( f : S^n \to S^n \) be a map between two \( n \)-spheres, let \( c \) be any point of \( S^n \), and let \(|\deg(f)| = |d| > 1\). If \( n \geq 3 \), then we see from Example 3.15 and Theorem 4.3 that \( f \) can be homotoped to a map \( g \) with only one root at \( c \), and it is easy to see directly that such a map \( g \) exists even if \( n = 2 \). But \( N\bigl( f; c \bigr) = |d| > 1 \), and so the map \( g \) cannot be transverse to \( c \) as any such map must have at least \(|d|\) roots at \( c \). Theorem 4.2 and Theorem 6.1 below show that there exists in fact a map that is homotopic to \( f \), is transverse to \( c \) and has \(|d|\) roots at \( c \). Similarly, \( f : P^2 \times S^2 \to P^2 \times S^2 \) from Example 3.15 is a map of 4-manifolds with \( N(f; c) = 1 \) and \( N\bigl( f; c \bigr) = |d| > 1 \) so it has the property, by Theorem 4.3, that there is a map \( g \) homotopic to \( f \) with just one root at \( c \) whereas any map homotopic to \( f \) and transverse to \( c \) must have at least \(|d|\) roots at \( c \). Theorem 4.2 tells us that there is a map \( g \) homotopic to \( f \) and transverse to \( c \) with exactly \(|d|\) roots at \( c \).

**Example 4.5.** Let \( f = f' \times f'' : M = K \times S^1 \to K \times S^1 = N \) where \( f' : K \to K \) is the map of Example 3.18 and \( f'' : S^1 \to S^1 \) is a map of degree \( r \neq 0 \). Since \( K \times S^1 \) is a closed aspherical manifold and \( f_\pi : \pi_1(M, x_0) \to \pi_1(N, c) \) is a monomorphism then, as in the proof of [BO, Prop. 6.4], the Nielsen number \( N(f; c) \) is the absolute value of the degree of a lift \( f_\circ \) of \( f \) to \( T \times S^1 \), the orientable covering of \( K \times S^1 \). In the notation of Example 3.18, that degree equals \( \text{ber} \) in this case, and so \( N(f; c) = |\text{ber}| \). Thus, Theorem 3.11 implies that \( f \) is of Type I and, as in Example 3.18, we can see that \(|\deg(f)| = 1 \). Therefore Theorem 3.12 shows that \( N\bigl( f; c \bigr) = |\text{ber}| \). Theorem 4.3 tells us that there is a map \( g \) homotopic to \( f \) with exactly \(|\text{ber}|\) roots at \( c \) and, by Theorem 4.2, the map \( g \) can be chosen so that it is transverse to \( c \).

**Example 4.6.** Let \( n \geq 4 \) be an even integer. Let \( A \) be an \( n \times n \) integer matrix with determinant \( d \) odd and let \( f' : T^n \to T^n \) be the corresponding map of the \( n \)-torus. We modify Example 2.4(b) to define a map \( f'' : T^n \# P^n \to T^n \), where \( P^n \) is \( n \)-dimensional real projective space. That is, \( f'' \) is the identity on \( T^n - B^n \), where \( B^n \) is an \( n \)-ball, and it maps \( P^n - B^n \) to \( B^n \). Since \( n \) is even, \( P^n \) is a non-orientable manifold and, as in Example 2.4(b), we see that \( f'' \) is a Type III map, so \( f = f' \circ f'' \) is also Type III. We have \( \deg(f, 2) = 1 \) (compare Example 3.19) and therefore \( N(f; c) = N_{\bigl( f; c \bigr)} = |d| \) by Theorems 3.11 and 3.12. If \( c \not\in B^n \), it is evident that \( f \) is transverse to \( c \) and has exactly \(|d|\) roots at \( c \). Theorem 4.3 tells us
that if $c \in B^n$, there is still a map $g$ homotopic to $f$ with exactly $|d|$ roots at $c$ and, by Theorem 4.2, we may require that $g$ be transverse to $c$ as well.

**Remark 4.7.** A root class $R$ of a map $f$ was defined in [Bk1, p. 24] to be *essential* if $R$ cannot be removed by a homotopy of $f$ (see [BB, p. 556] or [Bn]). It was shown in [BS2, Remark 3.2], that if $f: M \to N$ is a map of closed orientable $n$-manifolds, then a root class $R$ of $f$ is essential in this sense if and only if it has non-zero multiplicity. If follows from Theorem 4.3 that this is still true for boundary-preserving maps of not necessarily orientable manifold of dimension $\neq 2$, and from Theorem 6.1 below that is true for maps of closed surfaces. The reason is that Lemmas 3.6, 3.7, 3.9 and 3.10 establish the fact that the multiplicity $|m(R)|$ of a root class is a proper homotopy invariant and therefore a root class with a non-zero multiplicity must be preserved by proper homotopies. Now Theorems 4.3 and 6.1 imply that, if their assumptions are satisfied, then there exists a proper homotopy from $f$ to a map $g$ that has no root classes of multiplicity zero and hence a root class with zero multiplicity cannot be preserved by all proper homotopies.

5. Relations between Nielsen root numbers and the degree of a map.

The purpose of this section is to interpret some of the concepts and results of the previous sections in the language of Hopf’s degree theory. If $f: (M, \partial M) \to (N, \partial N)$ is a proper map between $n$-manifolds, then the classical integer-valued cohomological degree $\deg(f)$ only exists if both $M$ and $N$ are orientable (that is if their interiors are orientable). But if one or both of $M$ and $N$ are non-orientable, then one can only define the cohomological mod 2 degree $\deg(f; 2)$ by using coefficients in $\mathbb{Z}/2$, and this degree generally provides little information about the geometric properties of the map $f$. In particular, it does not relate to the intuitive geometric concept of the degree, namely, the number of times the image $f(M)$ of $f$ covers the range $N$. This lack of an algebraic invariant which can characterise the geometric concept of the degree in all cases was drawn to the attention of Hopf by P. Alexandroff. That is the reason Hopf developed Nielsen root theory; he wanted to obtain algebraic, homotopy invariant information about the least number of “nice” (i.e., transverse) counter-images of points, and thus about the numbers of times $f(M)$ covers $N$, and relate his algebraic invariant to the geometric degree (see Definition 5.2 below). Thus he proposed in [H2] a very different kind of degree that he called the “Absolutgrad”, which provides geometric information even if one or both of the manifolds $M$, $N$ are not orientable. If $M$ and $N$ are both orientable and $n \neq 2$, then the Absolutgrad agrees with $|\deg(f)|$, the absolute value of the cohomological
degree (see Theorems 5.4 and 5.5 below). Here is the original definition of Hopf’s degree.

**Definition 5.1 ([H2, Definition VIIc, p. 582]).** Let \( f : (M, \partial M) \to (N, \partial N) \) be a proper map between two \( n \)-manifolds and let \( c \in \text{int } N \). Then the **Absolutgrad or absolute degree** \( \text{A}(f) \) of \( f \) is the sum of the multiplicities, in the sense of Definition 2.7, of its root classes.

The reason for Hopf’s introduction of the absolute degree is that it provides an algebraic homotopy invariant that is closely linked to the very concrete concept of the geometric degree. This geometric interpretation of the absolute degree is based on the equality of the “algebraic” and “geometric” degrees which, as Hopf explained in the introduction to [H2] (see page 563), was the goal of that paper. (For maps of euclidean spaces, the equality had already been established in [H1, Satz IX, p. 590].) The following definition of the geometric degree is taken from [E, p. 372] and [Sk, p. 416]. It is a restatement of what Hopf understood by a geometric degree.

**Definition 5.2.** Let \( f : (M, \partial M) \to (N, \partial N) \) be a proper map between two \( n \)-manifolds. Then the **geometric degree** \( \mathcal{G}(f) \) of \( f \) is the least non-negative integer for which there exists a closed \( n \)-ball \( B^n \subset \text{int } N \) and a proper map \( g : (M, \partial M) \to (N, \partial N) \), which is homotopic to \( f \) under a proper homotopy, such that \( g^{-1}(B^n) \) has \( \mathcal{G}(f) \) components, and each component is mapped by \( g \) homeomorphically onto \( B^n \).

Note that for proper maps such an integer always exists, as \( \mathcal{G}(f) \) is bounded above by the number of roots of any transverse map homotopic to it.

The absolute and the geometric degree of a map are in essence concepts from Nielsen root theory. This fact follows immediately from the definition of these degrees in 5.1 and 5.2 and from the definition of the minimum number of transverse roots and the transverse Nielsen root number in (3.2) and (3.3), but we state it explicitly in the next theorem in order to emphasize and clarify the connection between Nielsen root theory and Hopf degree theory.

**Theorem 5.3.** Let \( f : (M, \partial M) \to (N, \partial N) \) be a proper map between two \( n \)-manifold and let \( c \) be any point in \( \text{int } N \). Then \( A(f) = N(f; c) \) and \( \mathcal{G}(f) = MR(f; c) \), that is, the absolute degree equals the transverse root Nielsen number and the geometric degree equals the least number of roots which are transverse at \( c \) for all maps in the proper homotopy class of \( f \).

One can see from the introduction of his paper [H2] that Hopf was very well aware of the fact brought out by Theorem 5.3, that the problem of finding the geometric degree is of a similar nature to the problem of finding
the least number of fixed points in the homotopy class of a map. Hopf clearly explains that his definition of the absolute degree uses an extension to root theory of concepts that had been used quite recently by J. Nielsen to study minimal sets of fixed points \([N1, N2]\). However, this motivation for Hopf’s introduction of the absolute degree \(A(f)\) is not mentioned in later studies and applications of \(A(f)\). As we mentioned in the introduction, Epstein in \([E]\) interpreted calculations of Olum \([O]\) in terms of degrees of maps of lifts of \(f\) to covering spaces, the covering spaces we described in Section 3. Olum obtained the values for \(N_\cap(f; c)\) computed here in Theorem 3.12, but in a very different way. Epstein made use of a classification of maps into types that is equivalent to Definition 2.1 and then defined the “absolute degree” \(A(f)\) separately for each type \([E, (1.8), p. 371]\). Epstein acknowledged that \(A(f)\) has a “complicated definition” \([E, p. 372]\) but pointed out that it is justified by its geometric significance from the equality between the geometric and absolute degrees proved by Hopf. Epstein’s paper is the one usually cited, so the connection with Nielsen root theory, that unified Hopf’s treatment of the absolutegrad, has not been preserved in Hopf degree theory.

The equality of the absolutegrad and the geometric degree for maps of \(n\)-manifolds, \(n \neq 2\), was first proved in \([H2, Satz IV, p. 607]\). A new proof of this equality was the aim of the paper of Epstein \([E, Theorem 4.1, p. 376]\). Because of the identifications described in Theorem 5.3, we see that Theorem 4.2 can be restated as the same result:

**Theorem 5.4.** Let \(f: (M, \partial M) \rightarrow (N, \partial N)\) be a proper map between two \(n\)-manifolds. If \(n \neq 2\), then the absolute degree \(A(f)\) equals the geometric degree \(G(f)\).

If \(M\) and \(N\) are both orientable, then Hopf degree theory can be related to the integer-valued cohomological degree that is defined in this case. From Theorems 3.13, 5.3 and 5.4 we have:

**Theorem 5.5.** Let \(f: (M, \partial M) \rightarrow (N, \partial N)\) be a proper map between two \(n\)-manifolds. If \(M\) and \(N\) are orientable, then \(A(f) = |\deg(f)|\). If, further, \(n \neq 2\), then \(G(f) = |\deg(f)|\) also.


Theorems 4.2, 4.3, 5.4 and 5.5, that are concerned with the sharpness of the two Nielsen root numbers for maps of \(n\)-manifolds, exclude the case \(n = 2\). We will now discuss sharpness results for maps of surfaces. It was not known at the time of Hopf’s work, but is now well known, that the Nielsen number for fixed points \(N(f)\) can be realized as the minimal set of fixed points for all maps in the homotopy class of \(f\) if \(f\) is a selfmap of a manifold of dimension \(\neq 2\), but that this is often not possible for selfmaps of surfaces. So it is not surprising that the two Nielsen root numbers \(N_\cap(f; c)\) and \(N(f; c)\) are
not sharp for surface maps in general. But it is quite surprising that, as we shall see, $N \cap |(f; c)|$ may well be sharp when $N(f; c)$ is not. In this section we describe what is known about sharpness for maps of surfaces, interpret it in the light of the results of this paper and add some consequences which follow from the results of prior sections.

Various authors have found conditions for the equality $A(f) = G(f)$ to hold, and Theorem 5.3 shows that this equality is equivalent to the sharpness of $N \cap |(f; c)|$. The basic result of this kind, which is stated as the next theorem, concerns closed surfaces.

**Theorem 6.1** (Kneser, Hopf, Skora). Let $f: M \to N$ be a map between two closed surfaces. Then $N \cap |(f; c)|$ is sharp and hence the geometric degree equals the absolute degree.

Theorem 6.1 tells us, for instance, that there is a selfmap of the Klein bottle homotopic to the map of Example 3.18 that is transverse to $c$ and has $|be|$ roots. In other words, both the absolute and the geometric degree of the map $f$ in Example 3.18 are equal to $|be|$. It further tells us that the map from the Klein bottle to the projective plane constructed in Example 3.16 has both absolute and geometric degree equal to 2.

Theorem 6.1 was first proved by H. Kneser [Kn1], [Kn2]. Hopf [H2, Satz XIVc, p. 605] showed that its assumptions can be weakened, as it is not necessary to assume that $N$ is closed. Both Kneser and Hopf proved more general results concerning the fact that the map $g$ which realizes $A(f) = N \cap |(f; c)|$ can be constructed in such a way that it realizes the absolute degree as the geometric degree not only at $c$, but at all points in an open subset of $N$ which is everywhere dense [H2, Satz XIVb, p. 605]. A modern proof of Theorem 6.1 was given by Skora in [Sk, Corollary 2.2].

Skora also found extensions of Theorem 6.1 to boundary-preserving maps. But additional assumptions are needed if the boundaries of the surfaces are not empty. One of these requires that $f|\partial M$ be allowable, which means that $f|\partial M$ is a $(G(f))$-fold covering. It is a somewhat awkward assumption as $G(f)$ may not be known. In [Sk, Theorem 2.1 and 2.5], Skora proved the following extension of Theorem 6.1.

**Theorem 6.2** (Skora). Let $f: M \to N$ be a map between two surfaces with $f^{-1}(\partial N) = \partial M$. Then $N \cap |(f; c)|$ is sharp, and hence $A(f) = G(f)$, given that either $f|\partial M$ is allowable and $M$ and $N$ are compact, or that $f$ is orientation-true and proper.

In addition, Skora showed that Theorem 6.2 is not true for $n = 2$ without additional assumptions. In [Sk, §3], he constructed, for every $d \geq 2$, a boundary-preserving and proper but neither orientation-true nor allowable map from the twice punctured projective plane to the annulus such that
\( G(f) = d \) but \( A(f) = 0 \) if \( d \) is even and \( A(f) = 1 \) if \( d \) is odd. The computation of the geometric degree for all surface maps is an open problem.

The sharpness of the Nielsen root number \( N(f; c) \) is a more delicate property as the conclusions of Theorems 6.1 and 6.2 do not hold with \( N(f; c) \) in place of \( N(f; c) \). This was already known to Hopf who constructed examples, for every \( j > 4 \) as well as for \( j = 1 \), of a map \( f \) from the double torus to the torus such that \( N(f; c) = j \) but \( f \) cannot be homotoped to a map with only \( N(f; c) \) roots (see [H2, Satz XVa, p. 610 and Satz XVb, p. 623]). Using results of Hopf, Lin [L, §4] constructed a map from the double torus to the torus with \( N(f; c) = 3 \) and \( MR[f; c] = 4 \). For maps between orientable closed surfaces, necessary and sufficient conditions for the sharpness of the root Nielsen number been recently been found by D.L. Gonçalves and H. Zieschang [GZ1] and [GZ2]. They proved:

**Theorem 6.3** (Gonçalves and Zieschang). Let \( M \) and \( N \) be closed orientable surfaces of genus \( h \) and \( k \) respectively. Let \( f: M \to N \) be a map such that \( j \) is finite, then \( N(f; c) \) is sharp if and only if

\[
|\deg(f)| \leq \frac{2h - 2 + j}{2k - 1}.
\]

Some new results concerning the sharpness of \( N(f; c) \) for maps of surfaces can easily be obtained from our calculations of the two Nielsen root numbers in §3. We can use the fact that if \( N(f; c) = N(f; c) \) and \( N(f; c) \) is sharp then \( N(f; c) \) must also be sharp. So an inspection of Theorems 3.11, 3.12 and 6.1 immediately yields:

**Theorem 6.4.** If \( f \) is a map between two closed surfaces which is not orientation-true, then \( N(f; c) \) is sharp.

The map \( f: T^2 \# P^2 \to T^2 \) of Example 3.19 has exactly \( |d| \) roots at \( c \) if \( c \notin D \). Theorem 6.4 implies that if \( c \in D \), then there is a map homotopic to \( f \) that has \( |d| \) roots at \( c \) (compare Example 4.6).

Theorems 6.3 and 6.4 give necessary and sufficient conditions for the sharpness of \( N(f; c) \) for all maps between closed surfaces, with the exception of the case of orientation-true maps between non-orientable surfaces. The Nielsen root number may be sharp in this case also. For instance, for any \( c \in P^2 \), the map \( f: K \to P^2 \) of Example 3.16 is homotopic to a map \( g \) with one root at \( c \) because \( f \) is one-to-one at two points and \( P^2 \) is homogeneous. However, it is likely that, for orientation-true maps between non-orientable manifolds, \( N(f; c) \) will only be sharp for some maps, as is true in the orientable case. We contribute a partial solution to this problem in the following theorem, which again follows immediately from Theorems 3.11, 3.12 and 6.1.
Theorem 6.5. Let \( f \) be an orientation-true map between two non-orientable closed surfaces. Then \( N(f; c) \) is sharp if either \( j \) is infinite or if \( |\deg(\tilde{f})| \leq 1 \). \( \square \)

Finally, Theorems 3.11, 3.12 and 6.2 yield some results concerning the sharpness of \( N(f; c) \) in the case where the boundaries of the surfaces \( M, N \) are non-empty. We omit the details.

References


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UN THÉORÈME DE L’INDICE RELATIF

Gilles Carron

We study Dirac type operator on non-compact Riemannian manifold. We give a general criterion which implies that the \( L^2 \)-kernel of such an operator has finite dimension. Moreover, we show that a relativ (extended) index theorem holds for such operators.

0. Introduction.

Soit \( D : C^\infty(E) \to C^\infty(E) \) un opérateur de type Dirac sur une variété riemannienne complète \( (M, g) \), on sait que lorsque la variété est compacte sans bord, l’espace des solutions \( L^2 \) à l’équation \( (D\sigma = 0) \) est de dimension finie. Si la variété n’est plus compacte, ceci n’est généralement plus le cas; cependant dans certains cadres géométriques, on peut montrer que cet espace est de dimension finie: Par exemple, c’est le cas lorsque les bouts de la variété sont cylindriques (\([A-P-S]\)), sont euclidiens (\([B-M-S]\)) ou sont certains produits tordus (\([B\], \[A2]\)). M. Gromov et H.B. Lawson ont étudié l’opérateur de Dirac d’une variété spin complète dont la courbure scalaire est uniformément strictement positive au voisinage de l’infini, ils montrent que cet opérateur est Fredholm sur son domaine, en particulier son noyau \( L^2 \) est de dimension finie (\([G-L]\)). Cette étude a été généralisée par N. Anghel (\([A1]\)), il montre notamment qu’un opérateur de type Dirac \( D : C^\infty(E) \to C^\infty(E) \) est Fredholm sur son domaine si et seulement si il est inversible à l’infini autrement dit:

**Théorème 0.1.** \( D : D(D) \to L^2(E) \) est Fredholm si et seulement si il existe un compact \( K \) de \( M \) et une constante strictement positive \( \Lambda \) tel que

\[
\Lambda \|\sigma\|_{L^2(M-K,E)} \leq \|D\sigma\|_{L^2(M-K,E)}, \quad \forall \sigma \in C^\infty_0(M-K,E).
\]

Dans \([C2]\), on montrait que l’opérateur de Gauss-Bonnet était Fredholm d’un espace de Sobolev dans \( L^2 \), ceci sous conditions qu’une inégalité de Sobolev soit vérifiée et qu’une intégrale de courbure soit finie. Motivé par ces travaux et par le résultat de N. Anghel, nous introduisons la notion suivante:

**Définition 0.2.** Soit \( D : C^\infty(E) \to C^\infty(E) \) un opérateur de type Dirac sur une variété riemannienne \((M^n, g)\), on dira que \( D \) est **non-parabolique**
à l’infini s’il existe un compact \( K \) de \( M \) tel que pour tout ouvert \( U \) relativement compact dans \( M – K \), il existe une constante strictement positive \( C(U) \) telle que
\[
C(U) \| \sigma \|_{L^2(U)} \leq \| D\sigma \|_{L^2(M - K)}, \quad \forall \sigma \in C_0^\infty(M - K, E).
\]
En quelque sorte, les opérateurs non-paraboliques à l’infini sont ceux qui sont faiblement inversibles à l’infini. Les opérateurs de type Dirac qui sont Fredholm sur leurs domaines sont évidemment non-paraboliques à l’infini. Cette définition est aussi inspirée du résultat de A. Ancona; dans \([An]\), il montre qu’une variété riemannienne complète connexe \((M^n, g)\) a des fonctions de Green positives si et seulement si pour tout (ou pour un) ouvert borné \( U \) de \( M \) il existe une constante strictement positive \( C(U) \) telle que
\[
C(U) \| f \|_{L^2(U)} \leq \| df \|_{L^2(M)}, \quad \forall f \in C_0^\infty(M).
\]
On dit alors que \((M, g)\) est non-parabolique. Autrement dit si \( H^1_0(M) \) est le complété de l’espace \( C_0^\infty(M) \) muni de la norme \( u \mapsto \| du \|_{L^2} \), alors la non-parabolicité de \((M, g)\) est équivalente au fait que l’inclusion de \( C_0^\infty(M) \) dans \( H^1_{loc}(M) \) se prolonge continûment à une injection \( H^1_0(M) \) dans \( H^1_{loc}(M) \).

Et de même, on a la définition équivalente suivante pour la non-parabolicité à l’infini.

**Définition 0.3.** Un opérateur de type Dirac \( D : C^\infty(E) \rightarrow C^\infty(E) \) sur une variété riemannienne complète \((M, g)\) est non-parabolique à l’infini si et seulement si il existe un compact \( K \) de \( M \) tel que si l’on complète \( C_0^\infty(E) \) avec la norme
\[
N^+_K(\sigma) = \sqrt{\| \sigma \|^2_{L^2(K)} + \| D\sigma \|^2_{L^2(M)}},
\]
afin d’obtenir \( W(E) \) alors l’injection \( C_0^\infty(E) \rightarrow H^1_{loc}(E) \) se prolonge par continuité en une injection \( W(E) \) dans \( H^1_{loc}(E) \).

Ceci montre que l’opérateur de type Dirac non-parabolique à l’infini et la métrique \( g \) détermine l’espace de Sobolev \( W \). Les opérateurs non-paraboliques à l’infini ont la propriété suivante:

**Théorème 0.4.** Si \( D : C^\infty(E) \rightarrow C^\infty(E) \) est un opérateur de type Dirac non-parabolique à l’infini alors
\[
D : W(E) \rightarrow L^2(E)
\]
est Fredholm. En particulier, la dimension du noyau \( L^2 \) d’un opérateur de type Dirac non-parabolique à l’infini est finie.

En fait, de façon analogue à la caractérisation (0.1) des opérateurs de type Dirac Fredholm sur leur domaine, nous verrons que cette propriété caractérise les opérateurs de type Dirac non-paraboliques à l’infini. Par définition, la non-parabolicité à l’infini d’un opérateur de type Dirac ne dépend que de la géométrie d’un voisinage de l’infini, ceci est satisfaisant.
au regard du résultat de J. Lott à propos de la $L^2$-cohomologie. Dans [L], l'auteur montre que si deux variétés riemanniennes sont isométriques sur un voisinage de l'infini, alors la dimension de leurs espaces des formes harmoniques $L^2$ est simultanément finie ou infinie.

La notion de non-parablicité à l'infini paraît assez maniable, nous donnerons dans la deuxième partie de nombreux exemples qui reposent sur la formule de Bochner-Weitzenböck-Lichnerowicz. Par exemple, nous généraliserons le résultat de M. Gromov et H.B. Lawson en montrant que si $(M,g)$ est une variété riemannienne spin complète dont la courbure scalaire est positive ou nulle au dehors d'un compact alors son opérateur de Dirac est non-parabolique à l'infini. On montrera aussi que les opérateurs de Gauss-Bonnet et de Signature sur une variété plate sur un voisinage de l'infini sont non-paraboliques à l'infini. Nous verrons ensuite d'autres exemples utilisant des inégalités de Sobolev. On retrouvera notamment nos précédents résultats sur la $L^2$-cohomologie réduite. Enfin, nous verrons qu'un opérateur de type Dirac sur une variété à bout cylindrique est non-parabolique à l'infini, en particulier son noyau $L^2$ est de dimension finie. Ces opérateurs avaient été étudiés par M. Atiyah, V.K. Patodi, I.M. Singer afin d'obtenir une formule pour la signature d'une variété compacte à bord ([A-P-S]). En fait, nous verrons que dans ce cas particulier, les solutions de l'équation $D\sigma = 0$ qui sont dans l'espace $W(E)$ sont exactement celle que M. Atiyah, V.K. Patodi, I.M. Singer appellent solutions étendues. C'est pourquoi lorsque $D : C^\infty(M,E^+ \oplus E^-) \to C^\infty(M,E^+ \oplus E^-)$ sera un opérateur de type Dirac non-parabolique à l'infini $\mathbb{Z}/2\mathbb{Z}$ gradué, on nommera indice étendu l'indice de $D^+ : W(E^+) \to L^2(E^-)$, i.e.,

$$\text{ind}_e D^+ = \dim \{ \sigma \in W(E^+), \ D^+ \sigma = 0 \} - \dim \{ \sigma \in L^2(E^-), \ D^- \sigma = 0 \}.$$ 

En fait, une solution $L^2$ à l'équation $(D\sigma = 0)$ est dans $W$, on note $h_\infty(D^+)$ la codimension de l'espace $\{ \sigma \in L^2(E^+), \ D^+ \sigma = 0 \}$ dans l'espace $\{ \sigma \in W(E^+), \ D^+ \sigma = 0 \}$. Ainsi on relie indice $L^2$ et indice étendu

$$\text{ind}_{L^2} D^+ = h_\infty(D^+) + \text{ind}_{L^2} D^+.$$ 

Un cas particulier d'opérateur non-parabolique à l'infini est celui où $0$ n'est pas dans le spectre essentiel de l'opérateur ou de façon équivalente celui où l'opérateur est Fredholm sur son domaine (cf. [G-L], [D], [A1], [A3], [B2]). Notamment, M. Gromov et H.B. Lawson puis H. Donnelly ont montré un théorème de l'indice relatif pour de tels opérateurs ([G-L], [D]): Si deux opérateurs de type Dirac $\mathbb{Z}/2\mathbb{Z}$ gradués sont Fredholm sur leurs domaines et s'ils sont isométriques sur un voisinage de l'infini, alors leurs indices $L^2$ diffèrent exactement de la différence de l'intégrale de leur forme caractéristique; c'est à dire que si $D_1$ et $D_2$ sont ces opérateurs, et s'ils sont
isométriques au delà de compacts $K_1$ et $K_2$, alors
\[ \text{ind}_{L^2} D_1^+ - \text{ind}_{L^2} D_2^+ = \int_{K_1} \alpha_{D_1^+} - \int_{K_2} \alpha_{D_2^+}. \]

En fait, le théorème de l’indice $L^2$ relatif n’est pas vrai pour les opérateurs non-paraboliques à l’infini, en effet on verra qu’il n’est pas vérifié pour l’opérateur de Gauss-Bonnet sur la surface obtenue en recollant deux plans euclidiens suivant des disques isométriques. En fait c’est le théorème de l’indice étendu relatif qui est vrai:

**Théorème 0.5.** Si $D_1, D_2$ sont deux opérateurs de type Dirac $\mathbb{Z}/2\mathbb{Z}$ gradués non-paraboliques à l’infini et isométriques à l’infini, alors on a
\[ \text{ind}_e D_1^+ - \text{ind}_e D_2^+ = \int_{K_1} \alpha_{D_1^+} - \int_{K_2} \alpha_{D_2^+}, \]
on où on a noté $\alpha_{D_i^+}$ est la forme caractéristique construite à l’aide du symbole principal de $D_i^+$. Nous pouvons alors relier les indices $L^2$ des deux opérateurs $D_1^+$ et $D_2^+$ :

**Corollaire 0.6.**
\[ \text{ind}_{L^2} D_1^+ - \text{ind}_{L^2} D_2^+ = \int_{K_1} \alpha_{D_1^+} - \int_{K_2} \alpha_{D_2^+} - (h_{\infty}(D_1^+) - h_{\infty}(D_2^+)), \]
où $h_{\infty}(D_i^+)$ est la dimension de l’espace $\ker W D_i^+ / \ker_{L^2} D_i^+$. 

Ainsi le théorème de l’indice $L^2$ relatif est vrai si et seulement si $h_{\infty}(D_1^+) = h_{\infty}(D_2^+)$. On a dit que ceci n’est pas toujours vrai; néanmoins la somme $h_{\infty}(D_1^+) + h_{\infty}(D_2^-)$ ne dépend que de la géométrie à l’infini, i.e.:

**Corollaire 0.7.** Dans le même contexte qu’au théorème 0.5, on a
\[ h_{\infty}(D_1^+) + h_{\infty}(D_2^-) = h_{\infty}(D_2^+) + h_{\infty}(D_2^-), \]
en particulier si $h_{\infty}(D_1^+) + h_{\infty}(D_2^-) = 0$, alors le théorème de l’indice $L^2$ relatif a lieu.

Le théorème de l’indice relatif permet, dans certains cas, de calculer certains indices; en effet de la valeur de l’indice d’un opérateur de type Dirac non-parabolique à l’infini, on en déduit l’indice de tout opérateur qui lui est isométrique à l’infini. C’est ainsi que N. Anghel et U. Bunke expriment l’indice de certains opérateurs Dirac plus potentiel sur les variétés de dimension impaires; ceci généralise un résultat de Callias (cf. [Ca], [A3] et [B2]). Ici nous obtiendrons le résultat suivant:

**Théorème 0.8.** Soit $(M^n, g)$ une variété riemannienne complète orientée telle qu’il existe un compact $K$ de $M$ avec
\[ M - K = \coprod E_i \]
où chaque bout \( E_i \) est isométrique au produit riemannien \( \Sigma_i \times (\mathbb{R}^{n_i} - B_{R_i}) \), \( \Sigma_i \) étant une variété riemannienne compacte et \( B_{R_i} \) la boule euclidienne de rayon \( R_i \). Notons \( D_{GB} \) l’opérateur de Gauss-Bonnet sur \( M \) et \( D_S \) l’opérateur de signature sur \( M \) (lorsque \( \dim M = 0 \mod 4 \)) alors ces opérateurs sont non-paraboliques à l’infini et on a

\[
\text{ind}_c D_{GB}^+ = \int_M \Omega + \sum_{n_i=1,2} n_i q(\Sigma_i)
\]

\[
\text{ind}_c D_S^+ = \int_M \xi + \sum_{n_i=1,2} n_i q(\Sigma_i)
\]

où on a noté \( q(\Sigma_i) \) la somme des nombres de Betti réels de \( \Sigma_i \):

\[
q(\Sigma_i) = \sum_{k=0}^{\dim \Sigma_i} b_k(\Sigma_i)
\]

et où \( \Omega \) est la \( n \)-forme d’Euler sur \( M \) et \( \xi \) la forme caractéristique de \( D_S^+ \), qui est la composante de plus haut degré du \( L \)-genre de \( M \). De plus, si pour chaque bout, on a \( n_i \geq 3 \), alors on a les égalités

\[
\text{ind}_{L^2} D_{GB}^+ = \int_M \Omega
\]

\[
\text{ind}_{L^2} D_S^+ = \int_M \xi.
\]

Dans le cas où les bouts de \((M^n, g)\) sont euclidiens, ce théorème est en partie dû à N. Borisov, W. Muller, R. Schrader ([B-M-S]), il est plus que possible que leur preuve, utilisant la théorie du scattering, s’adapte pour montrer ce résultat.

**Notation.** Soit \( E \to M^n \) est un fibré Hermitien sur une variété riemannienne complète. Un opérateur différentiel symétrique du premier ordre \( D : C^\infty(E) \to C^\infty(E) \) est dit de type Dirac lorsque le symbole principal de \( D^2 \) est la métrique:

\[
\sigma(D^2)(x,\xi) = g_x(\xi,\xi) \text{Id}_{E_x}, \ \forall (x,\xi) \in T^*M.
\]

En particulier, un tel opérateur est elliptique et l’algèbre de Clifford de \((T_x M, g_x), \text{Cl}_x(M)\) agit sur \( E_x \).

On notera \( H^1(E) \) le domaine de \( D \) lorsqu’il opère comme opérateur non-borné sur \( L^2(E) \), cet espace est aussi le complété de l’espace \( C_0^\infty(E) \) des sections lisses à support compact de \( E \) pour la norme

\[
\sigma \mapsto \sqrt{\|\sigma\|_{L^2}^2 + \|D\sigma\|_{L^2}^2}.
\]
Comme $D$ est elliptique l’espace des sections $H^1_{\text{loc}}$ de $E$ est aussi l’espace de sections de $E$ qui sont, elles et leur l’image par $D$, au sens des distributions, localement dans $L^2$. Lorsque $\nabla$ est une connexion orthogonale sur $E$, on notera $H^1_0(E)$ l’espace obtenu en complétant l’espace $C^\infty_0(E)$ par la norme $\sigma \mapsto \|\nabla \sigma\|_{L^2}$; remarquons que cet espace dépend de la connexion orthogonale mais nous omettons de signaler cette dépendance car il n’y aura pas d’ambiguïté dans cet article. Si $U$ est un ouvert borné de $M$ on notera $H^1(U,E)$ ou $H^1_0(U,E)$ l’espace obtenu en complétant l’espace $C^\infty_0(U,E)$ par la norme $\sigma \mapsto \|\nabla \sigma\|_{L^2}$.

Lorsque le fibré $E$ est muni d’une connexion orthogonale $\nabla$ telle que la multiplication par un vecteur unitaire de $T_xM$ soit une isométrie de $E_x$ et que la connexion se comporte comme une dérivation par rapport à l’opération du fibré de Clifford $Cl(M)$ sur $E$, i.e.,

$$\nabla(\sigma,\phi) = (\nabla \sigma)\phi + \sigma(\nabla \phi), \quad \forall \sigma \in C^\infty(Cl(M)), \phi \in C^\infty(E).$$

Alors l’opérateur différentiel défini par

$$D = \sum_{i=1}^n e_i \cdot \nabla e_i \quad (\text{où } \{e_i\} \text{ est un repère orthonormé local})$$

de ne dépend pas du choix de ce repère. Un tel opérateur est un opérateur de type Dirac, on parle alors d’opérateur de Dirac généralisé.

1. Non-parablicité à l’infini pour les opérateurs de type Dirac.

Le but de ce paragraphe est d’étudier une condition assez générale qui assure que le noyau $L^2$ d’un opérateur de type Dirac est de dimension finie.

1.a. Définition. Soit $D : C^\infty(E) \rightarrow C^\infty(E)$ un opérateur de type Dirac sur une variété riemannienne $(M^n,g)$, on dira que $D$ est non-parabolique à l’infini s’il existe un compact $K$ de $M$ tel que pour tout ouvert $U$ relativement compact dans $M - K$, il existe une constante strictement positive $C(U)$ telle que

$$(1.1) \quad C(U) \|\sigma\|_{L^2(U)} \leq \|D\sigma\|_{L^2(M - K)}, \quad \forall \sigma \in C^\infty(M - K, E).$$

1.b. Caractérisation d’un opérateur de type Dirac non-parabolique à l’infini. Une propriété importante d’un opérateur de type Dirac non-parabolique à l’infini est qu’il existe un opérateur de Green $G$ agissant continûment $G : L^2(E) \rightarrow H^1_{\text{loc}}(E)$. En fait, nous avons le résultat suivant qui caractérise ces opérateurs:

Théorème 1.2. Un opérateur de type Dirac $D : C^\infty(E) \rightarrow C^\infty(E)$ est non-parabolique à l’infini si et seulement si il existe un espace de Hilbert $W(E)$ telle que:
i) $\mathcal{C}_0^\infty(E)$ est dense dans $W(E)$.

ii) L’injection $\mathcal{C}_0^\infty(E) \rightarrow H^1_{\text{loc}}(E)$ se prolonge par continuité à $W(E)$.

iii) $D : W(E) \rightarrow L^2(E)$ est Fredholm (en particulier continu).

**Remarque.** Les opérateurs non-paraboliques à l’infini sont des opérateurs dont le noyau $L^2$ est de dimension finie: En effet, puisque $D$ est symétrique, on sait que le noyau $L^2$ de $D$ est l’orthogonal de l’image par $D$ de l’espace des sections lisses à support compact de $E$:

$$\{\sigma \in L^2, D\sigma = 0\} = (DC_0^\infty(M, E))^\perp;$$

or $C_0^\infty(M, E)$ est dense dans $W$, on a donc

$$\{\sigma \in L^2, D\sigma = 0\} = (DW)^\perp,$$

le noyau $L^2$ de $D$ est donc le conoyau de l’opérateur $D : W \rightarrow L^2$.

Il faut relier ce théorème avec le résultat de N. Anghel [A1] qui montrait qu’un opérateur de type Dirac $D : H^1(E) \rightarrow L^2(E)$ est Fredholm si et seulement si il existe un compact $K$ de $M$ et une constante strictement positive $\Lambda$ tel que

$$\Lambda \|\sigma\|_{L^2(M-K,E)} \leq \|D\sigma\|_{L^2(M-K,E)}, \quad \forall \sigma \in C_0^\infty(M-K, E).$$

En fait ceci équival aussi à ce que le spectre essentiel de $D^2$ ne contient pas 0, ces opérateurs sont évidemment non-paraboliques à l’infini mais notre condition est beaucoup moins restrictive: Nous verrons des exemples d’opérateur de type Dirac non-parabolique à l’infini où 0 est dans le spectre essentiel de $D^2$.

En fait, ce théorème repose sur la proposition suivante:

**Proposition 1.3.** Soit $W(E)$ un espace de Hilbert constitué de sections de $E \rightarrow M$ tel que:

i) $\mathcal{C}_0^\infty(E)$ est dense dans $W(E)$, et

ii) L’injection $\mathcal{C}_0^\infty(E) \rightarrow H^1_{\text{loc}}(E)$ se prolonge par continuité à $W(E)$,

iii) $D : W(E) \rightarrow L^2(E)$ est continu,

alors $D : W(E) \rightarrow L^2(E)$ est Fredholm si et seulement si il existe un compact $K$ de $M$ et une constante strictement positive $C(K)$ tel que

$$C(K) \|\sigma\|_{W} \leq \|D\sigma\|_{L^2(M-K)}, \quad \forall \sigma \in C_0^\infty(M-K, E).$$

Avant de prouver cette proposition, montrons qu’elle implique le théorème (1.2). Tout d’abord, grâce à cette proposition, il est clair que si $D : W(E) \rightarrow L^2(E)$ est Fredholm alors $D$ est non-parabolique à l’infini car par hypothèse $W$ s’injecte continûment dans $H^1_{\text{loc}}$ et donc, selon la proposition (1.3), pour tout ouvert borné $U$ de $M-K$, il existe une constante $C(U) > 0$ telle que

$$C(U) \|\sigma\|_{L^2(U)} \leq \|\sigma\|_{W} \leq C' \|D\sigma\|_{L^2}, \quad \forall \sigma \in C_0^\infty(M-K, E).$$
Réciproquement si \( D : C^\infty(E) \to C^\infty(E) \) est non-parabolique à l’infini, alors on construit \( W(E) \) qui est le complété de \( C_0^\infty(E) \) pour la norme

\[
N(\sigma) = \sqrt{\|\sigma\|_{H^1(\tilde{K})}^2 + \|D(\rho\sigma)\|_{L^2(M)}^2},
\]

où \( \tilde{K} \) est un voisinage borné du compact \( K \) donné dans la définition de non-parablicité à l’infini; et \( 1 - \rho \) est une fonction lisse à support dans l’intérieur de \( \tilde{K} \). Alors, par construction, \( C_0^\infty(E) \) est dense dans \( W \); puisque \( D \) est non-parabolique à l’infini, la construction de cet espace \( W \) ne dépend pas du compact \( \tilde{K} \) ni de la fonction \( \rho \) choisi. Ainsi \( W(E) \) s’injecte continûment dans \( H^1_{\text{loc}} \). Puis bien-sûr pour une section \( \sigma \) de \( E \) ayant son support hors de \( \tilde{K} \) on a \( \|\sigma\|_W = \|D\sigma\|_{L^2} \). Ce qui achève de montrer le Théorème 1.2 à l’aide de la Proposition 1.3.

Prouvons maintenant la proposition: En fait on reprend plus ou moins les arguments de [A1] qui sont eux-mêmes assez proche de ceux utilisées par Fischer-Colbrie ([F]) à propos de l’indice des sous-variétés minimales.

La partie facile de cette proposition est de montrer que si \( D : W(E) \to L^2(E) \) est Fredholm alors il existe un compact \( K \) de \( M \) et une constante strictement positive \( C(K) \) tel que

\[
C(K) \|\sigma\|_W \leq \|D\sigma\|_{L^2(M-K)}, \quad \forall \sigma \in C_0^\infty(M-K,E).
\]

En effet, supposons que la conclusion n’ait pas lieu alors on trouve une suite de compact \( \{K_l\}_l \) exhaustant \( M \) et une suite \( \{\sigma_l\}_l \) de sections de \( E \) lisses à support compact dans \( M - K_l \) telles que

\[
\|\sigma_l\|_W = 1 \quad \text{et} \quad \|D\sigma_l\|_{L^2} \leq 1/l.
\]

Par construction, cette suite tend vers 0 dans \( L^2_{\text{loc}} \) et aussi dans \( W \) faiblement; car dans \( W \) c’est une suite bornée donc relativement faiblement-compact et à cause de l’injection continue de \( W \) dans \( L^2_{\text{loc}} \) la seule valeur d’adhérence est 0. Puis \( D : W(E) \to L^2(E) \) est Fredholm donc il existe un opérateur de Green \( G : L^2 \to W \) continu tel que

\[
G \circ D = \text{Id}_W - H,
\]

où \( H \) est la projection orthogonale sur le noyau de \( D \), c’est un opérateur de rang fini ainsi la suite \( \{H\sigma_l\}_l \) tend vers 0 fortement dans \( W \), or on a

\[
\|\sigma_l\|_W = 1 = \|GD\sigma_l + H\sigma_l\|_W \leq \|G\|/l + \|H\sigma_l\|_W,
\]

ce qui est une contradiction lorsque \( l \) tend vers l’infini.

Montrons maintenant la réciproque: Un point important est le suivant:
**Lemme 1.5.** Si $W$ est un espace de Hilbert vérifiant les propriétés de la Proposition 1.3, alors pour $\alpha \in C^1_0(T^*M)$, l’application
$$\alpha. : W \longrightarrow L^2(M) \quad \sigma \mapsto \alpha.\sigma,$$
est compacte.

**Preuve.** En effet, si $K$ un compact à bord lisse contenant le support de $\alpha$, c’est la composée des 5 applications continues suivantes:
- l’injection de $W$ dans $H^1_{loc}$,
- la restriction de $H^1_{loc}$ à $H^1(K)$,
- la multiplication par $\alpha$ de $H^1(K)$ dans lui-même,
- l’injection compacte de $H^1(K)$ dans $L^2(K)$,
- l’extension par 0 de $L^2(K)$ dans $L^2(M)$.

Ainsi cette application est bien compacte. □

A partir de l’hypothèse, nous allons construire un opérateur continu $Q : L^2(E) \longrightarrow W$ tel que $Q \circ D - Id_W$ soit un opérateur compact, ceci montrera que $D : W \longrightarrow L^2$ a son noyau de dimension finie et que son image est fermée, ceci impliquera que son conoyau, que l’on identifie au noyau $L^2$, est de dimension finie; en effet une solution de carré intégrable à l’équation $(D\sigma = 0)$ est dans l’espace de Sobolev $W$: Ceci se montre de la façon suivante. Si $(\rho_l)$ est une suite de fonctions Lipschitz à support compact dans la boule géodésique de rayon $2l$ et valant 1 sur la boule géodésique de rayon $l$, on peut choisir cette suite afin que
$$|d\rho_l|(x) \leq \frac{1}{l}, \quad \forall x \in M, \quad l \in N.$$ 
Alors on a $D(\rho_l\sigma) = d\rho_l.\sigma$ et donc
$$\|D\rho_l\sigma\|_{L^2} \leq \|\sigma\|_{L^2}/l.$$ 
Cette inégalité et l’inégalité
$$C(K)\|\sigma\|_W \leq \|D\sigma\|_{L^2(M\setminus K)}, \quad \forall \sigma \in C^\infty_0(M - K, E)$$ 
permettent de démontrer que $\sigma$ est dans $W$, i.e., la suite $\rho_l\sigma$ est de Cauchy dans $W$, elle converge donc vers $\sigma$.

Soit $W(M - K)$ le complété de $C^\infty_0(M - K, E)\) pour la norme de $W$, l’hypothèse dit que cet espace est aussi le complété de $C^\infty_0(M - K, E)\) pour la norme $\sigma \mapsto \|D\sigma\|_{L^2}$. Autrement dit l’opérateur $D : W(M - K) \longrightarrow L^2(M - K)$ est injectif et d’image fermée, il y a donc un opérateur continu $P : L^2(M - K, E) \longrightarrow W(M - K)$ qui est un inverse à gauche de $D$, i.e.,
$$P \circ D\sigma = \sigma, \quad \forall \sigma \in W(M - K)$$
$$D \circ P\sigma = \sigma - H\sigma, \quad \forall \sigma \in L^2(M - K),$$
où \( H \) est la projection orthogonale de \( L^2(M - K) \) sur \( \{ \sigma \in L^2(M - K, E), \ D\sigma = 0 \} \); c'est aussi la projection sur l'orthogonal de \( DC_0^\infty(M - K, E) \). Puis soit \( \Gamma : L^2_\text{comp}(M, E) \longrightarrow H^1_\text{loc}(M, E) \) une paramétrice pour \( D \), i.e., on a
\[
\Gamma \circ D\sigma = \sigma - S_1\sigma, \\
D \circ \Gamma\sigma = \sigma - S_2\sigma, \ \forall \sigma \in C_0^\infty(M, E),
\]
où \( S_1, S_2 \) sont des opérateurs à noyau lisse. Soient:

i) \( \rho \) une fonction lisse à support dans \( M - K \) et valant 1 sur \( M - \bar{K} \) où \( \bar{K} \) est un voisinage compact de \( K \),

ii) \( \varphi \) une fonction lisse à support dans \( M - K \) valant 1 sur le support de \( \rho \),

iii) \( \phi \) une fonction lisse à support compact dans \( \bar{K} \) valant 1 sur le support de \( 1 - \rho \).

Formons alors
\[
G = \phi\Gamma(1 - \rho) + \varphi P\rho,
\]
et montrons que \( G \) convient. Par construction \( G \) opère continûment de \( L^2 \) dans \( W \), car \( \phi\Gamma(1 - \rho) \) opère continûment de \( L^2(M) \) dans \( H^1_\text{loc}(\bar{K}) \) et \( P\rho \) opère continûment de \( L^2 \) dans \( W(M - K) \) qui est sous-espace fermé de \( W(M, E) \) et enfin l'opérateur multiplication par \( 1 - \varphi \) est un opérateur continu de \( W \) sur lui-même et il en est donc de même de l'opérateur de multiplication par \( \varphi \).

On calcule alors:
\[
G \circ D = Id_W - \phi S_1(1 - \rho) + \phi\Gamma d\rho - \varphi Pd\rho.
\]
L'opérateur \( \phi S_1(1 - \rho) \) est un opérateur compact car son noyau est lisse à support compact. Puis les opérateurs \( \phi\Gamma d\rho, \varphi Pd\rho \) sont aussi compact car d'après le lemme \((1.5)\) les opérateurs \( d\rho, d\varphi : W \longrightarrow L^2 \) sont compacts.

Remarquons qu'une norme équivalente à celle définie par \((1.4)\) est
\[
N_K(\sigma) = \sqrt{\|\sigma\|^2_{L^2(\bar{K})} + \|D\sigma\|^2_{L^2(M)}},
\]
ainsi nous avons cette autre caractérisation d'un opérateur non-parabolique à l'infini:

**Corollaire 1.6.** Un opérateur de type Dirac \( D : C^\infty(E) \longrightarrow C^\infty(E) \) sur une variété riemannienne complète \( (M, g) \) est non-parabolique à l'infini si et seulement si il existe un compact \( K \) de \( M \) tel que si l'on complète \( C_0^\infty(E) \) avec la norme
\[
N_K(\sigma) = \sqrt{\|\sigma\|^2_{L^2(K)} + \|D\sigma\|^2_{L^2(M)}},
\]
afin d'obtenir \( W(E) \) alors l'injection \( C_0^\infty(E) \longrightarrow H^1_\text{loc}(E) \) se prolonge continûment en une injection de \( W(E) \) dans \( H^1_\text{loc} \).
Remarque 1.7. Ceci montre que l’espace de Sobolev $W(E)$ est uniquement déterminé par $D$ et les métriques de $E$ et de $M$. Et aussi ceci montre que l’espace $H^1(E)$ s’injecte continûment dans $W(E)$. Puisque ce corollaire montre que l’injection
\[
(C_0^\infty(E), H^1) \longrightarrow (C_0^\infty(E), W(E))
\]
est continue, elle se prolonge donc par densité à une injection continue de $H^1(E)$ dans $W(E)$.

1.c. Indices des opérateurs non-paraboliques à l’infini. Supposons que $D : C^\infty(M, E^+ \oplus E^-) \longrightarrow C^\infty(M, E^+ \oplus E^-)$ soit un opérateur de type Dirac $\mathbb{Z}/2\mathbb{Z}$-gradué, i.e., le fibré $E$ admet la décomposition $E = E^+ \oplus E^-$ et $D$ se décompose en
\[
\begin{pmatrix}
 0 & D^- \\
 D^+ & 0
\end{pmatrix} : C^\infty(M, E^+ \oplus E^-) \longrightarrow C^\infty(M, E^+ \oplus E^-).
\]
Alors si $D$ est non-parabolique à l’infini, on peut définir l’indice de $D^+ : W(E^+) \longrightarrow L^2(E^-)$, celui-ci est bien défini puisque cet opérateur est Fredholm.

Définition. On appellera indice étendu de $D^+$ cet indice et on notera
\[
\text{ind}_e D^+ = \dim\{\sigma \in W(E^+), \; D^+\sigma = 0\} - \dim\{\sigma \in L^2(E^-), \; D^-\sigma = 0\}.
\]
Nous adoptons ce terme d’indice étendu car on verra en 2.c, dans le cas des variétés à bouts cylindriques que les solutions de l’équation $\{D\sigma = 0, \; \sigma \in W\}$ sont exactement les solutions étendues au sens de [A-P-S].

Ensuite une solution $L^2$ à cette équation est aussi dans $W$, en effet une telle solution est dans $H^1(M, E)$ donc dans $W$ par (1.7). Notons $h_\infty(D^+)$ la codimension de l’espace $\{\sigma \in L^2(E^+), \; D^+\sigma = 0\}$ dans l’espace $\{\sigma \in W(E^+), \; D^+\sigma = 0\}$, en quelque sorte $h_\infty(D^+)$ est la dimension des valeurs à l’infini des solutions de cette équation puisque les solutions $L^2$ sont en un certain sens celle qui s’annulent à l’infini. Cette dernière remarque est encore justifiée par [A-P-S]. Ainsi on relie indice $L^2$ et indice étendu
\[
\text{ind}_e D^+ = h_\infty(D^+) + \text{ind}_{L^2} D^+.
\]

2. Exemples d’opérateurs non-paraboliques à l’infini.

Préambule. La condition de non-parabolicité à l’infini ne dépend que de la géométrie à l’infini de la variété et de l’opérateur de type Dirac, en conséquence on peut construire d’autres exemples en recollant par somme connexe les exemples que nous donnerons ici.
2.a. Exemples reposant sur la formule de Bochner-Weitzenbock.
Soit $D : C^\infty(E) \rightarrow C^\infty(E)$ un opérateur de Dirac généralisé sur une variété riemannienne complète $(M, g)$. Nous savons que l’opérateur $D^2$ admet la décomposition suivante

$$D^2 = \nabla^* \nabla + \mathcal{R},$$

où $\mathcal{R}$ est un champ d’endomorphisme symétrique de $E$ s’exprimant à l’aide de la courbure de $E$ et de $TM$, et où $\nabla$ est la connexion avec laquelle on a défini l’opérateur de Dirac généralisé $D$. On a la formule:

$$(\nabla^* \nabla \sigma)(x) = -\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \sigma(x),$$

où $\{e_i\}$ est un repère orthonormé local qui vérifie $\nabla_{e_i} e_i = 0$ en $x$. Dans [G-L], les auteurs montrent que s’il existe un compact $K$ de $M$ et une constante $c > 0$ tels que la plus petite valeur propre de $\mathcal{R}(x)$ est uniformément minorée par $c$ sur $M - K$ alors $D : H^1(E) \rightarrow L^2(E)$ est Fredholm. Une première généralisation de ce résultat est le suivant:

**Théorème 2.1.** Si $D : C^\infty(E) \rightarrow C^\infty(E)$ un opérateur de Dirac généralisé sur une variété riemannienne complète $(M, g)$ et s’il existe un compact $K$ de $M$, hors duquel la plus petite valeur propre du terme en potentiel-courbure est positive ou nulle, alors $D$ est non-parabolique à l’infini.

**Preuve.** Soit $K$ un compact hors duquel $\mathcal{R}$ est positive ou nulle alors on a

$$\|D\sigma\|^2_{L^2} \geq \|\nabla \sigma\|^2_{L^2}, \quad \forall \sigma \in C_0^\infty(M - K, E).$$

Or nous avons le lemme suivant:

**Lemme.** Soit $(M^n, g)$ une variété riemannienne complète connexe alors si $K$ est un compact d’intérieur non-vide, l’espace défini comme le complété de $C_0^\infty(M)$ muni de la norme $u \mapsto \sqrt{\|du\|^2_{L^2(M)} + \|u\|^2_{L^2(K)}}$ ne dépend pas du compact $K$, en particulier cet espace s’injecte naturellement dans $H^1_{\text{loc}}$.

La preuve de ce lemme est aisé, elle est laissée en exercice: Il suffit de comparer deux telles normes pour des compacts $K$ et $K'$ tel que $K \subset K'$ (en raisonnant par l’absurde). On conclut alors la preuve du théorème en utilisant le lemme de Kato qui dit que

$$|\nabla \sigma|(x) \geq |d|\sigma||(x).$$

Ainsi le Théorème (1.2) nous dit $D$ est non-parabolique à l’infini et $W(E)$ est le complété de $C_0^\infty(E)$ muni de la norme

$$\sigma \mapsto \sqrt{\|\sigma\|^2_{L^2(K)} + \|D\sigma\|^2_{L^2}}.$$
Dans ce cas, $W(E)$ est aussi le complété de $C_0^\infty(E)$ muni de la norme
$$
\sigma \mapsto \sqrt{\|\sigma\|^2_{L^2(K)} + \|\nabla\sigma\|^2_{L^2(M)}}.
$$
De plus, lorsque $(M, g)$ est non-parabolique, on a $W = H^1_0$ et donc $D : H^1_0(E) \to L^2(E)$ est Fredholm. On peut donner des exemples d’opérateurs qui satisfont à ces hypothèses, ce sont les opérateurs de signature et de Gauss-Bonnet sur des variétés plates au voisinage de l’infini ou encore l’opérateur de Dirac sur une variété spin à courbure scalaire positive ou nulle au voisinage de l’infini.

2.b. Avec une inégalité de Sobolev. Notation. Si $\lambda(x)$ est la plus petite valeur propre du terme en potentiel courbure apparaissant dans la formule de Bochner-Weitzenböck-Lichnerowicz, nous notons
$$
\mathcal{R}_-(x) = \max\{0, -\lambda(x)\}.
$$
Lorsque la variété vérifie une inégalité de Sobolev on peut raffiner le dernier théorème:

**Théorème 2.2.** Si pour un $p > 2$, $(M, g)$ est une variété riemannienne satisfaisant à l’inégalité de Sobolev
$$
\mu_p(M) \left(\int_M |u|^{2p-2}(x)dx\right)^{1-\frac{2}{p}} \leq \int_M |du|^2(x)dx, \forall u \in C_\infty(M),
$$
et si $D : C^\infty(E) \to C^\infty(E)$ est un opérateur de Dirac généralisé sur $M$ dont le terme $\mathcal{R}$ en potentiel courbure apparaissant dans la formule de Bochner-Weitzenbock vérifie
$$
\int_M |\mathcal{R}_-|^\frac{2}{p}(x)dx < \infty,
$$alors
$$
D : H^1_0(E) \to L^2(E)
$$est Fredholm.

Remarquons que suivant [C2], si $M^n \to \mathbb{R}^N$ est une sous-variété d’un espace euclidien dont le vecteur courbure moyenne est dans $L^n$ vérifie cette inégalité de Sobolev pour $p = n$, ceci améliorerait le résultat de [H-S]. Ce théorème redémontrerait en partie certains des résultats que nous avions montré dans [C1], [C2] à propos de la $L^2$-cohomologie. La preuve est ici plus simple, la difficulté réside bien sûr dans l’étude des opérateurs de type Dirac non-paraboliques à l’infini, étude faite en première partie.

**Preuve.** L’opérateur $D$ vérifie les hypothèses de la Proposition 1.3: En effet, l’inégalité de Sobolev et l’inégalité de Kato montre que $H^1_0(E)$ s’injecte dans $L^{\frac{2p}{p-2}}$ donc dans $L^2_{\text{loc}}$ et $H^1_{\text{loc}}$. Choisissons alors un compact $K$ de $M$ tel que
$$
\|\mathcal{R}_-\|_{L^\frac{2}{p}(M-K)} \leq \mu_p/2,
$$
alors si \( \sigma \in C_0^\infty(M - K, E) \), on a
\[
\int_M |D\sigma|^2 = \int_M |\nabla\sigma|^2 + \langle R\sigma, \sigma \rangle \\
\geq \int_{M-K} |\nabla\sigma|^2 - \|\sigma\|_{L^2(\mathbb{R}_+, E)}^2 \|R\|_{L^2(M-K)} \\
\geq \frac{1}{2} \int_{M-K} |\nabla\sigma|^2,
\]
ou on a utilisé dans la dernière minoration l'inégalité de Sobolev et l'hypothèse sur l'intégrale de \( R \).

\[ \square \]

Ceci peut aisément se généraliser aux inégalités de Sobolev-Orlicz (cf. [C3] pour une présentation de ces inégalités). Grâce à [C4], nous pouvons obtenir une version localisée de cette proposition ceci grâce au inégalités de Sobolev-Orlicz non uniforme.

**Théorème 2.3.** Il existe une constante universelle \( C \) telle que si \((M, g)\) est une variété riemannienne complète, dont \((P(t, x, y), t \in \mathbb{R}_+, x, y \in M)\) est le noyau de l'opérateur de la chaleur et si \( D : C^\infty(E) \longrightarrow C^\infty(E) \) est un opérateur de Dirac généralisé sur \( M \) dont le terme \( R \) en potentiel courbure apparaissant dans la formule de Bochner-Weitzenböck vérifie
\[
\int_M R_{-}(x) \left( \int_{\mathbb{R}_+} \frac{P(s, x, x)}{s} ds \right)^2 dx < \infty,
\]
alors
\[ D : H^1_0(E) \longrightarrow L^2(E) \]
est Fredholm.

**Preuve.** Soit \( \varphi : \mathbb{R}_+ \times M \longrightarrow \mathbb{R}_+ \) la fonction définie par
\[
\varphi(\lambda, x) = \lambda \left( \int_{\frac{1}{\lambda x}}^\infty \frac{P(s, x, x)}{s} ds \right)^2.
\]
A \( u \in C^\infty_0(M) \), on associe
\[
N(u) = \sup \left\{ \int_M u v, \ v \in C^\infty_0(M) \text{ avec } \int_M \varphi(|v|(x), x) dx \leq 1 \right\},
\]
alors \( N \) est une norme et le complété de \( C^\infty_0(M) \) muni de cette norme est un espace de Banach (de Orlicz) constitué de fonctions localement intégrables et pour une constante universelle \( C \), on a l’inégalité de Sobolev
\[
N(u^2) \leq C\|du\|_{L^2}, \ \forall u \in C^\infty_0(M).
\]
Ceci a été montré dans [C4]. Par définition, on a l’inégalité de Hölder
\[
\int_M u^2 v \leq N(u^2) \inf \left\{ \lambda > 0, \ \int_M \varphi \left( \frac{|v|(x)}{\lambda}, x \right) dx \leq 1 \right\}.
\]
Si $\int_M \varphi(2CR_-(x), x)dx < \infty$ alors il existe un compact $K$ de $M$ tel que $\int_{M-K} \varphi(2CR_-(x), x)dx \leq 1$. Ainsi si $\sigma \in C_0^\infty(M-K, E)$, on a

$$\int_M |D\sigma|^2 = \int_M |\nabla\sigma|^2 + \langle R\sigma, \sigma \rangle \geq \frac{1}{2} \int_{M-K} |\nabla\sigma|^2 + \frac{1}{2} \int_{M-K} |\nabla\sigma|^2 - \frac{1}{2C} N(|\sigma|^2)$$

$$\geq \frac{1}{2} \int_{M-K} |\nabla\sigma|^2.$$ 

Ce qui montre que $D : H^1_0(E) \rightarrow L^2(E)$ est Fredholm. □

Grâce aux inégalités de Hardy, obtenues dans [C5], nous pouvons trouver d’autres opérateurs non-paraboliques à l’infini, nous nous limitons ici au cas des sous-variétés d’un espace euclidien.

**Théorème 2.4.** Soit $M^{n>2} \rightarrow \mathbb{R}^N$ une immersion isométrique d’une variété riemannienne complète alors si $D : C^\infty(\Lambda^*T^*M) \rightarrow C^\infty(\Lambda^*T^*M)$ est l’opérateur de Gauss-Bonnet et si, pour un compact $K$ de $M$, la seconde forme fondamentale de l’immersion vérifie

$$\sup_{x \in M-K} \{||II||(x)||x||\} < c(n),$$

alors $D : H^1_0(\Lambda^*T^*M) \rightarrow L^2(\Lambda^*T^*M)$ est non-parabolique à l’infini.

Si $M$ est spin et si, pour un compact $K$ de $M$, la seconde forme fondamentale de l’immersion vérifie

$$\sup_{x \in M-K} \{||II||(x)||x||\} < (n-2)/(2n+1),$$

alors l’opérateur de Dirac de $(M, g)$ est non-parabolique à l’infini.

**Preuve.** Elle repose sur l’inégalité de Hardy que nous avions montré dans [C5]: Si $M^n \rightarrow \mathbb{R}^N$ est une immersion isométrique dont le vecteur courbure moyenne est $k$, alors $(M^n, g)$ vérifie l’inégalité de Hardy suivante

$$\left(\frac{n-2}{2}\right)^2 \int_M \left(\frac{u}{r}\right)^2 (x) dx \leq \int_M |du|^2 (x) + \frac{n-2}{2} \frac{|k|}{r} u^2 dx, \forall u \in C_0^\infty(M),$$

où on a noté $r(x) = ||x||$. On développe alors la même preuve que précédemment: Si $D : C^\infty(E) \rightarrow C^\infty(E)$ est un opérateur de Dirac généralisé sur une telle variété alors pour $\sigma \in C_0^\infty(M, E)$ on a

$$\int_M |D\sigma|^2 = \int_M |\nabla\sigma|^2 + \langle R\sigma, \sigma \rangle \geq \int_M \left[\left(\frac{n-2}{2}\right)^2 - |k| r \left(\frac{n-2}{2}\right) - R_- r^2\right] \frac{|\sigma|^2}{r^2} dx.$$
Ainsi s’il existe un compact \( K \) de \( M \) tel que
\[
\sup_{x \in M - K} \left\{ |k|r \left( \frac{n-2}{2} \right) + R_- r^2 \right\} < \frac{(n-2)^2}{4},
\]
alors \( D \) est non-parabolique à l’infini. On applique alors ce critère à l’opérateur de Gauss-Bonnet. On a d’abord \( |k| \leq n|I| \) puis pour l’opérateur de Gauss-Bonnet on a
\[
|R|(x) \leq \alpha(n)|I|^2(x)
\]
et donc que si \( |I|/\|x\| < \frac{n-2}{n+\sqrt{n^2+4\alpha(n)}} \) alors
\[
|k|r \left( \frac{n-2}{2} \right) + |R_-|r^2 < \frac{(n-2)^2}{4},
\]
ce qui assure que l’opérateur est non-parabolique à l’infini. Puis, on applique ce critère à l’opérateur de Dirac sur une variété spin, on utilise le fait que
\[
R = \frac{\text{scal}_g}{4} \text{Id}_S
\]
et que \( \text{scal}_g = |I|^2 - |k|^2 \) pour conclure de la même façon.

\[\square\]

2.c. Étude du cas d’une variété riemannienne à bout cylindrique.

Nous exposons dans ce paragraphe comment les résultats obtenus par Atiyah, Patodi et Singer ([A-P-S]) peuvent s’interpréter dans notre cadre.

Soit \((M, g)\) une variété riemannienne à bout cylindrique, c’est à dire qu’il existe un compact \( K \) de \( M \) tel que \((M - K, g)\) soit isométrique au produit riemannien \( \mathbb{R}_+ \times \partial K \). On considère un opérateur de type Dirac \( D : C^\infty(E) \rightarrow C^\infty(E) \) agissant sur les sections d’un fibré hermitien qui respecte cette géométrie. C’est à dire que la métrique de \( E|_{M-K} \) ne dépend pas de la distance à \( \partial K \) et on suppose que sur \( \mathbb{R}_+ \times \partial K \), \( D \) prend la forme suivante
\[
D = n. \left( \frac{\partial}{\partial r} + A \right),
\]
où \( n \) est la multiplication de Clifford par la normale extérieure à \( \{r\} \times \partial K \) et \( A \) est un opérateur elliptique auto-adjoint sur \( E|_{\partial K} \).

**Proposition 2.5.** Un tel opérateur est non-parabolique à l’infini.

**Preuve.** On peut montrer ce résultat de deux façons. D’abord à l’aide de la Proposition 2.5 de [A-P-S], où les auteurs construisent un opérateur de Green sur \( E \rightarrow \mathbb{R}_+ \times \partial K \)
\[
Q : L^2(\mathbb{R}_+ \times \partial K, E) \rightarrow H^1_{\text{loc}}(\mathbb{R}_+ \times \partial K, E),
\]
tel que
\[
Q \circ D\sigma = \sigma, \forall \sigma \in C^\infty_0(\mathbb{R}_+ \times \partial K, E),
\]
ainsi pour tout ouvert borné \( U \) de \( \mathbb{R}_+ \times \partial K \) on a
\[
\|\sigma\|_{L^2(U)} = \|QD\sigma\|_{L^2(U)} \leq \|Q\|_{L^2 \rightarrow H^1(U)} \|D\sigma\|_{L^2}, \forall \sigma \in C^\infty_0(\mathbb{R}_+ \times \partial K, E).
\]
L’autre méthode consiste simplement à écrire que pour \( \sigma \in C_0^\infty (\mathbb{R}_+ \times \partial K, E) \), on a

\[
\| D\sigma \|_{L^2}^2 = \left\| \frac{\partial \sigma}{\partial r} \right\|_{L^2}^2 + \| A\sigma \|_{L^2}^2,
\]
on conclut alors en minorant \( \left\| \frac{\partial \sigma}{\partial r} \right\|_{L^2}^2 \) à l’aide de l’inégalité de Cauchy-Schwarz:

\[
|\sigma(r, \theta)| = \left| \int_0^r \frac{\partial \sigma}{\partial r} \right| \leq \sqrt{r} \left\| \frac{\partial \sigma}{\partial r} \right\|_{L^2},
\]
d’où en intégrant

\[
\| \sigma \|_{L^2([0,T] \times \partial K)}^2 \leq \frac{T^2}{2} \left\| \frac{\partial \sigma}{\partial r} \right\|_{L^2}^2.
\]
Ce qui montre que \( D \) est non-parabolique à l’infini.

Les auteurs définissent alors la notion de solutions étendues à l’équation \( D\sigma = 0 \), c’est une solution \( \sigma \) qui est localement dans \( L^2 \) et qui vérifie sur \( \mathbb{R}_+ \times \partial K \)

\[
\sigma(y, \theta) = \sigma_0(y, \theta) + \sigma_{\infty}(\theta), \quad (y, \theta) \in \mathbb{R}_+ \times \partial K
\]
où \( \sigma_0 \) est dans \( L^2 \) et où \( \sigma_{\infty} \) est dans le noyau de \( A \). Ici \( \sigma_{\infty} \) est en quelque sorte la valeur à l’infini de \( \sigma \). En fait nous avons:

**Proposition 2.6.** Les solutions étendues de l’équation \( D\sigma = 0 \) sont exactement les solutions qui appartiennent à l’espace de Sobolev \( W \) défini par \( D \).

**Preuve.** On utilise la décomposition spectrale de l’opérateur \( A \), on écrit

\[
L^2(\partial K, E) = \bigoplus_{\lambda \in \text{Sp} A} C \varphi_\lambda,
\]
où

\[
D \varphi_\lambda = \lambda \varphi_\lambda,
\]

\[
\int_{\partial K} |\varphi_\lambda|^2 = 1.
\]
Une solution de l’équation \( D\sigma = 0 \) admet au-dessus de \( \mathbb{R}_+ \times \partial K \) la décomposition en séries de Fourier suivante:

\[
\sigma(y, \theta) = \sum_{\lambda \in \text{Sp} A} c_\lambda e^{-\lambda y} \varphi_\lambda(\theta);
\]
et un calcul élémentaire montre que \( \sigma \in W \) si et seulement si \( c_\lambda = 0, \forall \lambda < 0 \), ce qui caractérise aussi les solutions étendues.

Cette décomposition en série de Fourier a pour conséquence la proposition suivante (Prop. 3.11 de [A-P-S]):
Proposition 2.7. Notons $P_{\leq 0}$ (resp. $P_{< 0}$) le projecteur spectral de $A$ associé aux valeurs propres négatives (resp. strictement négative) de $A$ alors

i) $\sigma$ est une solution de l'équation $\{ D\sigma = 0, \sigma \in W \}$ si et seulement si
\[
\begin{cases}
D\sigma = 0 & \text{sur } K \\
P_{\leq 0}\sigma = 0 & \text{sur } \partial K.
\end{cases}
\]

ii) $\sigma$ est une solution de l'équation $\{ D\sigma = 0, \sigma \in L^2 \}$ si et seulement si
\[
\begin{cases}
D\sigma = 0 & \text{sur } K \\
P_{< 0}\sigma = 0 & \text{sur } \partial K.
\end{cases}
\]

Supposons maintenant que $D$ soit $\mathbb{Z}/2\mathbb{Z}$ gradué, et que $A = A^+ + A^-$. Notons $h(E^\pm)$ la dimension de l’espace des solutions $L^2$ de l’équation $D^\pm\sigma = 0$, et $h_\infty(E^\pm)$ est la codimension de l’espace des solutions $L^2$ dans l’espace des solutions étendues. On a donc
\[
\text{ind}_e D^+ = h_\infty(E^+) + h(E^+) - h(E^-).
\]

Une conséquence de la proposition précédente est que cette indice étendu est aussi l’indice de l’opérateur $D^+ : C^\infty(K, E^+ \oplus P_{< 0}) \to C^\infty(K, E^-)$, et le calcul de cet indice par $[A-P-S]$ nous permet d’énoncer:

Théorème 2.8. Si $(M, g)$ est une variété riemannienne à bout cylindrique et si $D : C^\infty(M, E^+ \oplus E^-) \to C^\infty(M, E^+ \oplus E^-)$ est un opérateur de type Dirac $\mathbb{Z}/2\mathbb{Z}$ gradué respectant cette géométrie alors
\[
\text{ind}_e D^+ = \int_K \alpha_D + \frac{\dim \ker A^+ - \eta_{A^+}(0)}{2},
\]

où:

a) $\alpha_D$ est la forme caractéristique définie par le symbole principal de $D^+$.

b) $A^+$ est l’opérateur elliptique du premier ordre agissant sur les sections de $E^+ \to \partial K$ tel que au dessus de $R_+ \times \partial K$, on ait $D^+ = n.(\frac{\partial}{\partial r} + A^+)$.

c) $\eta_{A^+}(0)$ est la valeur en $0$ de l’extension méromorphe de
\[
\eta_{A^+}(s) = \sum_{\lambda \in \text{Sp}A^+} \text{sign } \lambda |\lambda|^{-s},
\]
en fait, cette extension est holomorphe sur le demi-plan $\text{Res} > -1/2$.

3. Le théorème de l’indice relatif.

Le but de cette partie est de montrer que le théorème de l’indice relatif a lieu pour les opérateurs de type Dirac non-paraboliques à l’infini. L’approche utilisée est celle de M. Gromov et H.B. Lawson ([G-L]) qui ont montré ce théorème pour les opérateurs de type Dirac dont le potentiel courbure apparaissant dans la formule de Bochner-Weitzenböck-Lichnerowicz est uniformément strictement positif sur un voisinage de l’infini. Comme l’a montré
N. Anghel dans [A1], cette preuve peut s’étendre au cas où 0 n’est pas dans le spectre essentiel de $D^2$. La méthode consiste à employer soigneusement les opérateurs de Green plutôt que l’opérateur de la chaleur associé à $D^2$ (cf. [D], [B1], [B-M-S]).

Définition. Deux opérateurs de type Dirac

\[
D_1 : C^\infty(M_1, E_1) \to C^\infty(M_1, E_1), \quad D_2 : C^\infty(M_2, E_2) \to C^\infty(M_2, E_2)
\]

$\mathbb{Z}/2\mathbb{Z}$ gradués sont dit isométriques à l’infini, s’il existe deux compacts $K_1 \subset M_1$ et $K_2 \subset M_2$ et une isométrie $\iota : (M_1 - K_1, g_1) \to (M_2 - K_2, g_2)$ qui induit une isométrie graduée entre les fibrés et telle que

\[
D_1 \circ \iota^* = \iota^* \circ D_2,
\]

au dessus de $M_2 - K_2$.

Le théorème que nous montrerons est le suivant:

Théorème 3.1. Si $D_1$, $D_2$ sont deux opérateurs de type Dirac $\mathbb{Z}/2\mathbb{Z}$ gradués non-paraboliques à l’infini et isométrique à l’infini, alors on a

\[
\text{ind}_e D_1^+ - \text{ind}_e D_2^+ = \int_{K_1} \alpha_{D_1^+} - \int_{K_2} \alpha_{D_2^+},
\]

où on a noté $\alpha_{D_i^+}$ est la forme caractéristique construite à l’aide du symbole principal de $D_i^+$.

Nous pouvons maintenant grâce à la discussion de 1.c relier les indices $L^2$ des deux opérateurs $D_1^+$ et $D_2^+$:

Corollaire 3.2. 

\[
\text{ind}_{L^2} D_1^+ - \text{ind}_{L^2} D_2^+ = \int_{K_1} \alpha_{D_1^+} - \int_{K_2} \alpha_{D_2^+} - (h_\infty(D_1^+) - h_\infty(D_2^+)),
\]

où $h_\infty(D_i^+)$ est la dimension de l’espace $\ker W D_i^+ / \ker L^2 D_i^+$.

Selon U. Bunke lorsque les fibrés et les variétés riemanniennes sont à géométries bornées, alors cette différence entre les dimensions de l’espace des valeurs à l’infini pourrait s’interpréter comme un indice de scattering entre les opérateurs $D_1^- D_1^+$ et $D_2^- D_2^+$ ([B1]). Il serait intéressant de mieux comprendre ceci, afin de rendre calculable ces dimensions.

On peut se demander si le théorème de l’indice relatif $L^2$ a lieu pour les opérateurs non-paraboliques à l’infini. Ce qui revient à savoir si $h_\infty(D_1^+)$ ne dépend que de la géométrie à l’infini. En fait ceci n’est pas vrai, considérons la surface $C$ obtenue en recollant deux plans euclidien le long de deux disques isométriques: C’est donc la variété $S^1 \times \mathbb{R}$ muni de la métrique

\[
g_{r, \theta} = dr^2 + r^2 d\theta^2, \quad |r| > 1, \quad \theta \in S^1.
\]

Comme cette surface est orientée et de volume infini, il n’y a sur $C$ ni fonctions, ni 2–formes harmonique $L^2$ non-nulles. Puis cette surface est conformément équivalente à la sphère moins deux points; or, en dimension 2,
l'espace des 1−formes harmoniques $L^2$ est un invariant conforme et il n'y a pas de 1−formes harmoniques $L^2$ non-nulles sur la sphère privée de deux points, ainsi il n'y a pas de forme harmonique $L^2$ sur $\mathcal{C}$ et l'indice $L^2$ de l'opérateur de Gauss-Bonnet est nul. Si on pouvait appliquer le théorème de l'indice $L^2$ à l'opérateur de Gauss-Bonnet entre $\mathcal{C}$ et deux copies de $\mathbb{R}^2$, alors sur $\mathcal{C}$, on aurait égalité entre l'indice $L^2$ et l'intégrale de la forme $KdA/2\pi$ (puisque sur $\mathbb{R}^2$ on a égalité); or on a
\[
\int_{\mathcal{C}} \frac{KdA}{2\pi} = -2
\]
ce qui n'est pas égale à 0!
Cependant si $h_\infty(D^+)$ ne dépend pas que de la géométrie à l'infini, $h_\infty(D^+) + h_\infty(D^-)$ ne dépend que de la géométrie à l'infini, on obtient ce résultat en appliquant le théorème de l'indice relatif aux opérateurs $D_1 = D_1^+ + D_1^-$ et $D_2 = D_2^+ + D_2^-$. Puisque ces opérateurs sont autoadjoint, leurs indices $L^2$ sont nuls, mais leurs formes caractéristiques sont aussi nulles, on obtient:

**Corollaire 3.3.** Dans le même contexte qu'au Théorème 3.1, on a
\[
h_\infty(D_1^+) + h_\infty(D_2^+) = h_\infty(D_2^-) + h_\infty(D_2^-).
\]
En particulier, si $h_\infty(D_1^+) + h_\infty(D_2^-) = 0$, alors le théorème de l'indice $L^2$ relatif a lieu.

**3.b. Preuve du théorème.** Nous commençons par remarquer que les deux expressions dont nous voulons montrer l'égalité ne changent pas lorsqu'on déforme les métriques et les opérateurs de type Dirac sur une partie compacte; on suppose donc que les compacts hors desquels les opérateurs sont isométriques sont à bord lisse et qu'un voisinage de ce bord est isométrique au produit riemannien $[-\varepsilon, \varepsilon] \times \Sigma$ où $\Sigma = \partial K_1 = \partial K_2$; on suppose de plus que les opérateurs respectent cette géométrie. Pour chacun des opérateurs $D_1$ et $D_2$, on considère une paramétrice
\[
\Gamma_i : L^2_{\text{comp}}(E_i^+) \longrightarrow H^1_{\text{loc}}(E_i^-),
\]
ces paramétrices sont des opérateurs à noyau lisse au dehors de la diagonale de $M_i \times M_i$ et elles vérifient :
\[
D_i^+ \Gamma_i = Id - S_i^- \quad \Gamma_i D_i^+ = Id - S_i^+;
\]
où $S_i^+$ et $S_i^-$ sont des opérateurs dont le noyau de Schwarz est lisse.

Nous allons maintenant construire deux paramétrices $P_1$ et $P_2$ pour $D_1^+$ et $D_2^+$ qui agissent continûment de $L^2$ dans $W$ telle que
\[
D_i^+ P_i = Id_{L^2} - T_i \quad P_i D_i^+ = Id_W - S_i,
\]
où $T_i$, $S_i$ sont des opérateurs à trace. Pour conclure, on se servira alors du
lemme suivant du à Atiyah [At]:

**Lemme.** Soit $f : A \rightarrow B$ une application linéaire Fredholm entre deux
espaces de Hilbert alors si $g : B \rightarrow A$ est une application linéaire continue
telle que

$$g \circ f = \text{Id}_A - T_A$$
$$f \circ g = \text{Id}_B - T_B,$$

où $T_A$, $T_B$ sont des opérateurs à trace, alors

$$\text{ind} f = \text{Tr} T_A - \text{Tr} T_B.$$

Soit $\Omega$ l’ouvert $(M_1 - K_1, g_1)$ qui est isométrique à $(M_2 - K_2, g_2)$. On
fera souvent et abusivement l’identification entre les fibrés et les sections
fibrés et les opérateurs définis hors de $K_1$ et $K_2$, ceci afin d’éviter des
notations trop lourdes. Soit $H^+$ l’opérateur de projection orthogonale de
$W(M_2, E_2^+)$ sur le noyau de $D_2^+$ et $H^-$ le projecteur $L^2$-orthogonal sur le
noyau $L^2$ de $D_2^-$. Ce sont des opérateurs de rang fini donc évidemment à trace.
Puisque $D_2^+ : W(E_2^+) \rightarrow L^2(E_2^-)$ est Fredholm, il existe un opérateur de
Green continue $G_2 : L^2(E_2^-) \rightarrow W(E_2^+)$ tel que

$$D_2^+ G_2 = \text{Id}_{L^2} - H^-$$
$$G_2 D_2^+ = \text{Id}_{W} - H^+.$$

Soient $\rho$ une fonction lisse à support dans $\Omega$ qui vaut 1 hors de $]0, \varepsilon[ \times \Sigma$ et
$\varphi$ une fonction lisse à support dans $\Omega$ qui vaut 1 sur le support de $\rho$, et $\phi_i$
des fonctions lisses à support compacts dans $K_i \cup \{0, \varepsilon[ \times \partial K_i$ et qui valent 1
sur le support de $1 - \rho$.

On considère alors les opérateurs

$$P_i = \phi_i \Gamma_i (1 - \rho) + \varphi G_2 \rho.$$

Soit $W(\Omega)$ le complété de l’espace $C^\infty(\Omega, E_i^+)$ dans $W(\Omega, E_i^+)$, pour $i =
1, 2$, ces espaces sont les mêmes et l’opérateur $\varphi G_2 \rho$ agit continûment de
$L^2(M_i, E_i^-)$ dans $W(\Omega) \subset W(E_i^+)$. Puis comme $\Gamma_i$ agit continûment de
$L^2_{\text{loc}}$ dans $H^1_{\text{loc}}$, $\phi_i \Gamma_i (1 - \rho)$ agit continûment de $L^2(E_i^-)$ dans $H^1_0(K_i \cup \{0, \varepsilon[ \times \partial K_i, E_i^+)$ donc dans $W(E_i^+)$. $P_i$ est donc un opérateur borné de $L^2(E_i^-)$
dans $W(E_i^+)$. Puis on a les identités suivantes

$$D_i^+ P_i = \text{Id}_{L^2} - \phi_i S_i^- (1 - \rho) + d\phi_i \Gamma_i (1 - \rho) + d\varphi G_2 \rho - \varphi H^- \rho,$$
$$P_i D_i^+ = \text{Id}_{W} - \phi_i S_i^+ (1 - \rho) + \phi_i \Gamma_i d\rho - \varphi G_2 d\rho - \varphi H^+ \rho.$$

Montrons maintenant que les opérateurs apparaissant au second membre de
 cette expression sont à traces. Il est évident que les opérateurs $\phi_i S_i^\pm (1 - \rho)$
sont à trace puisque ce sont des opérateurs dont le noyau est lisse à support
compact. De même, les opérateurs $\varphi H^\pm \rho$ le sont puisque $H^\pm$ sont des opérateurs de rang fini.

Puis comme les opérateurs $\Gamma_i$ sont des opérateurs à noyau lisse au dehors de la diagonale et que $|d\phi_i|(1 - \rho) = 0$, les opérateurs $d\phi_i \Gamma_i(1 - \rho)$ sont à trace car leur noyau est lisse à support compact.

Prouvons maintenant que l’opérateur $d\varphi G_2\rho$ est à trace dans $L^2(\Omega, E_i^-)$.

On a

$$d\varphi G_2\rho \left(D_2^+ G_2\right) = d\varphi G_2\rho - d\varphi G_2\rho H^-$$

mais aussi

$$d\varphi G_2\rho \left(D_2^+ G_2\right) = d\varphi G_2\rho (\rho D_2^+) G_2$$

$$= d\varphi G_2 D_2^+ \rho G_2 - d\varphi G_2 d\rho G_2$$

$$= d\varphi (Id - H^+) \rho G_2 - d\varphi G_2 d\rho G_2,$$

or $|d\varphi| \rho = 0$ et donc on obtient finalement

$$d\varphi G_2\rho = d\varphi G_2\rho H^- - d\varphi H^+ \rho G_2 - d\varphi G_2 d\rho G_2.$$

Il suffit maintenant de montrer que $d\varphi G_2 d\rho G_2$ est à trace. Comme $|d\varphi| |d\rho| = 0$, l’opérateur $d\varphi G_2 d\rho$ est un opérateur dont le noyau est lisse à support compact, en effet $G_2$ est un opérateur à noyau lisse au dehors de la diagonale; puis si $\chi$ est une fonction lisse à support compact et valant 1 sur le support de $d\rho$ alors on a $d\varphi G_2 d\rho G_2 = d\varphi G_2 d\rho \chi G_2$; mais $\chi G_2$ agit continûment de $L^2(E_i^-)$ dans $L^2(E_i^+)$, et $d\varphi G_2 d\rho$ est de Hilbert-Schmidt $d\varphi G_2 d\rho G_2$ est bien un opérateur à trace.

Il reste à prouver que l’opérateur $T = \phi_i \Gamma_i d\rho - \varphi G_2 d\rho$ est à trace. On commence pour cela à écrire $\phi_i = \hat{\phi}_i + \check{\phi}_i$ où $\check{\phi}_i = 0$ sur le support de $d\rho$ et où le support de $\hat{\phi}_i$ est dans $]0, \varepsilon[ \times \partial K$, ainsi

$$T = \hat{\phi}_i \Gamma_i d\rho + \check{\phi}_i (\Gamma_i - G_2) d\rho + \left(\hat{\phi}_i - \varphi\right) G_2 d\rho.$$

Alors les opérateurs $\check{\phi}_i \Gamma_i d\rho$, $\check{\phi}_i (\Gamma_i - G_2) d\rho$ sont des opérateurs dont le noyau est lisse à support compact, donc ils sont à trace. Puis on montre que $\left(\hat{\phi}_i - \varphi\right) G_2 d\rho$ est à trace de la même façon que pour montrer que $d\varphi G_2\rho$ est à trace, ceci en considérant l’expression

$$G_2 D_2^+ \left(\hat{\phi}_i - \varphi\right) G_2 d\rho.
Nous pouvons donc appliquer le lemme et nous obtenons
\[
\text{ind}_e D_1^+ - \text{ind}_e D_2^+ = \text{Tr}_{W(E_i^+)} \phi_1 S_1^+(1 - \rho) - \phi_1 \Gamma_1 d\rho + \varphi G_2 d\rho + \varphi H^+ \rho \\
- \text{Tr}_{L^2(E_i^-)} \phi_1 S_1^-(1 - \rho) + d\phi_1 \Gamma_1 (1 - \rho) + d\varphi G_2 \rho - \varphi H^- \rho \\
- \text{Tr}_{W(E_i^+)} \phi_2 S_2^+(1 - \rho) - \phi_2 \Gamma_2 d\rho + \varphi G_2 d\rho + \varphi H^+ \rho \\
+ \text{Tr}_{L^2(E_i^-)} \phi_2 S_2^-(1 - \rho) + d\phi_2 \Gamma_2 (1 - \rho) + d\varphi G_2 \rho - \varphi H^- \rho.
\]

Puis en se servant du fait que \(|d\varphi| \rho = 0\) et \(|d\phi_1|(1 - \rho) = 0\) on a
\[
\text{ind}_e D_1^+ - \text{ind}_e D_2^+ = \text{Tr}_{W(E_i^+)} \phi_1 S_1^+(1 - \rho) - \text{Tr}_{L^2(E_i^-)} \phi_1 S_1^-(1 - \rho) \\
\text{Tr}_{W(E_i^+)} \phi_2 \Gamma_2 d\rho + \varphi G_2 d\rho - \text{Tr}_{W(E_i^+)} \phi_1 \Gamma_1 d\rho - \varphi G_2 d\rho \\
- \text{Tr}_{W(E_i^+)} \phi_2 S_2^+(1 - \rho) + \text{Tr}_{L^2(E_i^-)} \phi_2 S_2^-(1 - \rho).
\]

Puis on écrit comme précédemment \(\hat{\phi}_i = \phi_i + \phi_i \) de telle façon que \(\hat{\phi}_1 = \hat{\phi}_2\) sur \(|0, \varepsilon| \times \Sigma\). Alors on obtient
\[
\text{Tr}_{W(E_i^+)} [\phi_2 \Gamma_2 d\rho + \varphi G_2 d\rho] - \text{Tr}_{W(E_i^+)} [\phi_1 \Gamma_1 d\rho - \varphi G_2 d\rho + \varphi H^+ \rho] \\
= \text{Tr}_{W} \hat{\phi}_1 (\Gamma_2 - \Gamma_1) d\rho.
\]

Ainsi on obtient
\[
\text{ind}_e D_1^+ - \text{ind}_e D_2^+ = \text{Tr}_{W(E_i^+)} S_1^+(1 - \rho) - \text{Tr}_{L^2(E_i^-)} S_1^-(1 - \rho) \\
\text{Tr}_{W(E_i^+)} \hat{\phi}_1 (\Gamma_2 - \Gamma_1) d\rho \\
- \text{Tr}_{W(E_i^+)} S_2^+(1 - \rho) + \text{Tr}_{L^2(E_i^-)} S_2^-(1 - \rho).
\]

Maintenant, on remarque que cette expression ne dépend plus de la géométrie du voisinage de l’infini \(\Omega\). Si on considère les variétés riemanniennes compactes \(\tilde{M}_1, \tilde{M}_2\) où \(\tilde{M}_1\) est le double de \(K_1\), i.e., \(\tilde{M}_1 = K_1 \# (-K_1)\) et où \(\tilde{M}_2\) est la somme connexe de \(K_2\) et de \(K_1\) le long de leurs bords, i.e., \(\tilde{M}_2 = K_2 \# \Sigma (-K_1)\), alors nous avons naturellement des opérateurs de type Dirac sur \(\tilde{M}_1\) et \(\tilde{M}_2\)
\[
\hat{D}_i : C^\infty (\tilde{M}_i, \tilde{E}_i) \longrightarrow C^\infty (\tilde{M}_i, \tilde{E}_i)
\]
tel que sur \(K_i \subset \tilde{M}_i\), \(\hat{D}_i\) soit l’opérateur \(\hat{D}_i\) et qu’au dessus de \(\tilde{M}_1 - \tilde{K}_1\) et de \(\tilde{M}_2 - \tilde{K}_2\), les opérateurs \(\hat{D}_1\) et \(\hat{D}_2\) soient isométriques. Alors en refaisant la même construction que précédemment on obtient la même expression pour la différence entre les indices de \(\hat{D}_1^+\) et \(\hat{D}_2^+\), on a donc
\[
\text{ind}_e D_1^+ - \text{ind}_e D_2^+ = \text{ind} \hat{D}_1^+ - \text{ind} \hat{D}_2^+,
\]
le théorème de l’indice relatif est alors une conséquence du théorème de l’indice d’Atiyah-Singer. \(\square\)

Nous commençons par décrire un autre exemple d’opérateurs non-paraboliques à l’infini.

4.a. Produit riemannien et opérateurs non-paraboliques à l’infini. Soit
\[ D_1 : C^\infty(M_1, E_1) \to C^\infty(M_1, E_1) \] et
\[ D_2 : C^\infty(M_2, E_2) \to C^\infty(M_2, E_2) \] deux opérateurs de type Dirac, on peut construire, au dessus du
produit riemannien \( M_1 \times M_2 \), le fibré \( \pi_1^*(E_1) \otimes \pi_2^*(E_2) \), où \( \pi_i : M_1 \times M_2 \to M_i \) est la projection sur le
i-ième facteur. Sur ce fibré agit l’opérateur de
type Dirac \( D = D_1 + D_2 \) et on a la:

**Proposition 4.1.** Si la variété \( M_1 \) est compact et si \( D_2 \) est non-parabolique
à l’infini alors l’opérateur
\[ D = D_1 + D_2 : C^\infty(M_1 \times M_2, \pi_1^*(E_1) \otimes \pi_2^*(E_2)) \to C^\infty(M_1 \times M_2, \pi_1^*(E_1) \otimes \pi_2^*(E_2)) \]
est non-parabolique à l’infini et si \( D_1 \) et \( D_2 \) sont \( \mathbb{Z}/2\mathbb{Z} \) gradués alors en
notant \( h(D_1^+) = \dim \ker L^2_1 D_1^+ \), on a
\[
\text{ind}_e D = \text{ind}_L^2 D_1^+ \text{ind}_L^2 D_2^+ + h_\infty(D_2^+)h(D_1^+) + h_\infty(D_2^-)h(D_1^-).
\]

**Preuve.** En effet, soit \( K \) un compact de \( M_2 \) tel que pour tout ouvert
\( U \) relativement compact dans \( M_2 - K \), il existe une constante strictement
positive \( C(U) \) telle que
\[
C(U) \| \sigma \|_{L^2(U)} \leq \| D_2 \sigma \|_{L^2(M_2 - K)}, \quad \forall \sigma \in C^\infty_0(M_2 - K, E_2).
\]
Alors en utilisant le théorème de Fubini, on obtient pour
\[
(4.2) \quad \sigma \in C^\infty_0((M_1 \times M_2) - (M_1 \times K), \pi_1^*(E_1) \otimes \pi_2^*(E_2))
\]
\[
C(U) \| \sigma \|_{L^2(M_1 \times U)} \leq \| D_2 \sigma \|_{L^2(M_1 \times (M_2 - K))} \leq \| D \sigma \|_{L^2(M_1 \times (M_2 - K))},
\]
ce qui prouve que \( D \) est non-parabolique à l’infini.

Le calcul de l’indice étendu résulte alors du fait que
\[
\ker_{L^2} D \simeq \ker_{L^2} D_1 \otimes \ker_{L^2} D_2
\]
\[
\ker_{W} D \simeq \ker_{L^2} D_1 \otimes \ker_{W} D_2.
\]

Le premier résultat est classique, pour le second, on utilise la décomposition
spectrale de \( D_1 \):
\[
L^2(M_1, E_1) = \bigoplus_{\lambda \in \text{Sp} D_1} C_{\varphi_\lambda},
\]
où
\[
D_1 \varphi_\lambda = \lambda \varphi_\lambda,
\]
\[
\int_{M_1} |\varphi_\lambda|^2 = 1.
\]
Alors une solution de l’équation $D\sigma = 0$ sur $M_1 \times M_2$ admet la décomposition suivante

$$\sigma = \sum_{\lambda} \varphi_{\lambda} \otimes \sigma_{\lambda},$$

avec $\lambda \sigma_{\lambda} + D_2 \sigma_{\lambda} = 0$ car $D\sigma = 0$ et $\lambda^2 \sigma_{\lambda} + D_2^2 \sigma_{\lambda} = 0$ car $D^2 \sigma = (D_1^2 + D_2^2)\sigma = 0$. Si $\sigma$ est dans l’espace $W$ alors $D\sigma$ est dans $L^2$ donc $D_2\sigma$ aussi, ceci implique que pour $\lambda \neq 0$ alors $\sigma_{\lambda} = -D_2\sigma_{\lambda}/\lambda \in L^2$ mais alors la seconde égalité implique que $\sigma_{\lambda} = 0$. Ceci montre l’inclusion

$$\ker_W D \subset \ker_{L^2} D_1 \otimes \ker_W D_2;$$

l’autre inclusion s’obtient facilement grâce à l’inégalité 4.2.

\[\square\]

4.b. Un calcul d’indice.

**Théorème 4.3.** Soit $(M^n, g)$ une variété riemannienne complète orientée telle qu’il existe un compact $K$ de $M$ avec

$$M - K = \coprod E_i$$

où chaque bout $E_i$ est isométrique au produit riemannien $\Sigma_i \times (\mathbb{R}^{n_i} - B_{R_i})$, $\Sigma_i$ étant une variété riemannienne compacte et $B_R$ la boule euclidienne de rayon $R$. Notons $D_{GB}$ l’opérateur de Gauss-Bonnet sur $M$ et $D_S$ l’opérateur de signature sur $M$ (si $\dim M = 0$ mod 4) alors ces opérateurs sont non-paraboliques à l’infini et on a

$$\operatorname{ind} D_{GB}^+ = \int_M \Omega + \sum_{n_i=1,2} n_i q(\Sigma_i)$$

$$\operatorname{ind} D_S^+ = \int_M \xi + \sum_{n_i=1,2} n_i q(\Sigma_i)$$

où on a noté $q(\Sigma_i)$ la somme des nombres de Betti réels de $\Sigma_i$:

$$q(\Sigma_i) = \sum_{k=0}^{\dim \Sigma_i} b_k(\Sigma_i)$$

et où $\Omega$ est la $n$-forme d’Euler sur $M$ et $\xi$ la forme caractéristique de $D_S^+$, qui est la composante de plus haut degré du $L$-genre de $M$. De plus, si pour chaque bout on a $n_i \geq 3$, alors on a les égalités

$$\operatorname{ind}_{L^2} D_{GB}^+ = \int_M \Omega$$

$$\operatorname{ind}_{L^2} D_S^+ = \int_M \xi.$$
Dans le cas où les bouts de $M$ étaient euclidiens, N. Borisov, W. Muller, R. Schrader avaient déjà obtenu ce résultat. Signalons aussi que dans ce cas, on peut calculer la dimension de l’espace des $k$-formes harmoniques $L^2$ ([C1]).

**Preuve.** La proposition précédente et le Théorème 2.1 nous assurent que les opérateurs de Gauss-Bonnet et de Signature sont non-paraboliques à l’infini sur les variétés $\Sigma_i \times \mathbb{R}^{n_i}$, et qu’au voisinage de l’infini $M$ est isométrique à une union de telles variétés, les opérateurs considérés sont bien non-paraboliques à l’infini. On applique alors le théorème de l’indice relatif entre l’opérateur de Gauss-Bonnet sur $M$ et sur $\bigcup_i (\Sigma_i \times \mathbb{R}^{n_i})$, ceci donne

$$\text{ind}_e D_{\text{GB}}(M) - \sum_i \text{ind}_e D_{\text{GB}}(\Sigma_i \times \mathbb{R}^{n_i}) = \int_M \Omega.$$  

On utilise alors le fait que sur $\mathbb{R}^{n_i}$, il n’y a pas de formes harmoniques $L^2$ non-nilles, que si $n_i \geq 3$, il n’y en a pas dans $W = H^1$ et que pour $n_i = 1, 2$ les formes harmoniques qui sont dans $W$ sont les formes parallèles. L’analyse est la même pour l’opérateur de signature. □

**References**


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THE KERNEL OF FOCK REPRESENTATIONS OF WICK ALGEBRAS WITH BRAIDED OPERATOR OF COEFFICIENTS

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It is shown that the kernel of the Fock representation of a certain Wick algebra with braided operator of coefficients T, \( ||T|| \leq 1 \), coincides with the largest quadratic Wick ideal. Improved conditions on the operator T for the Fock inner product to be strictly positive are given.

1. Introduction.

The problem of positivity of the Fock space inner product is central in the study of the Fock representation of Wick algebras (see [2], [3], [5], [6]). The paper [6] presents several conditions on the coefficients of the Wick algebra for the Fock inner product to be positive. If the operator of coefficients of the Wick algebra T satisfies the braid condition and the norm restriction \( ||T|| \leq 1 \), then, as proved in [2], the Fock inner product is positive. Moreover if \(-1 < T < 1\), it was shown in [2] that the Fock inner product is strictly positive. In this article we prove that, for braided T with \( ||T|| \leq 1 \), the kernel of the Fock inner product coincides with the largest quadratic Wick ideal. In particular this implies that, for \(-1 < T \leq 1\), the Fock inner product is strictly positive definite, and the Fock representation is faithful.

This article is organized as follows. In Sec. 2 we present definitions of Wick algebras and the Fock representation and show that, in the braided case, the kernel of the Fock representation is generated by the kernel of the Fock inner product. In Sec. 3 we prove that if the operator T is braided and \( ||T|| \leq 1 \), then the kernel of the Fock inner product coincides with the two-sided ideal generated by \( \ker(1+T) \). In Sec. 4 we combine results obtained in Sec. 2 and Sec. 3 to examine the \( C^* \)-representability of certain Wick algebras or their quotients. All results are illustrated by examples of different kinds of \( q_{ij} \)-CCR.
2. Preliminaries.

For more detailed information about Wick algebras and the Fock representation we refer the reader to [6]. In this section we present only the basic definitions and properties.

1. The notion of a \(\ast\)-algebra allowing Wick ordering (Wick algebra) was presented in the paper [6] as a generalization of a wide class of \(\ast\)-algebras [7], including the twisted CCR and CAR algebras (see [10]), the \(q\)-CCR (see [4]) algebra, etc.

Definition 1. Let \(\mathbb{J} = \mathbb{J}_d = \{1, 2, \ldots, d\}, T_{ij}^{kl} \in C, i, j, k, l \in \mathbb{J}\), be such that \(T_{ij}^{kl} = T_{ji}^{lk}\). The Wick algebra with the set of coefficients \(\{T_{ij}^{kl}\}\) is denoted \(W(T)\), and is a \(\ast\)-algebra, defined by generators \(a_i, a_i^\ast, i \in \mathbb{J}\), which satisfy the basic relations:

\[
a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l a_k^*.
\]

Definition 2. Monomials of the form \(a_{i_1} a_{i_2} \cdots a_{i_m} a_{j_1}^* a_{j_2}^* \cdots a_{j_n}^*\) are called Wick ordered monomials.

It was proved in [6] that the Wick ordered monomials form a basis for \(W(T)\).

Let \(\mathcal{H} = \langle e_1, \ldots, e_d \rangle\). Consider the full tensor algebra over \(\mathcal{H}, \mathcal{H}^*\), denoted by \(T(\mathcal{H}, \mathcal{H}^*)\). Then

\[
W(T) \cong T(\mathcal{H}, \mathcal{H}^*) / \langle e_i^* \otimes e_j - \delta_{ij} 1 - \sum T_{ij}^{kl} e_l \otimes e_k^* \rangle.
\]

To study the structure of Wick algebras, and the structure of the Fock representation, it is useful to introduce the following operators on \(\mathcal{H} \otimes_n := \mathcal{H} \otimes \cdots \otimes \mathcal{H}\) (see [6]):

\[
\begin{align*}
T: \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \otimes \mathcal{H}, \quad Te_k \otimes e_l = \sum_{i,j} T_{ik}^{lj} e_i \otimes e_j, \quad T = T^*, \\
T_i: \mathcal{H}^{\otimes n} &\rightarrow \mathcal{H}^{\otimes n}, \quad T_i = 1 \otimes \cdots \otimes 1 \otimes T \otimes 1 \otimes \cdots \otimes 1, \\
R_n: \mathcal{H}^{\otimes n} &\rightarrow \mathcal{H}^{\otimes n}, \quad R_n = 1 + T_1 + T_1 T_2 + \cdots + T_1 T_2 \cdots T_{n-1}, \\
P_n: \mathcal{H}^{\otimes n} &\rightarrow \mathcal{H}^{\otimes n}, \quad P_2 = R_2, \quad P_{n+1} = (1 \otimes P_n) R_{n+1}.
\end{align*}
\]

In this article we suppose that the operator \(T\) is contractive, i.e., \(\|T\| \leq 1\), and satisfies the braid condition, i.e., on \(\mathcal{H}^{\otimes 3}\) the equality \(T_1 T_2 T_1 = T_2 T_1 T_2\) holds. It follows from the definition of \(T_i\) that then \(T_i T_j = T_j T_i\) if \(|i-j| \geq 2\), and for the braided \(T\) one has \(T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}\).
Remark 1. These conditions hold for such well-known algebras as $q_{ij}$-CCR, $\mu$-CCR, $\mu$-CAR (see [6]).

The Fock representation of a Wick $*$-algebra is determined by a vector $\Omega$ such that $a_i^*\Omega = 0$ for all $i = 1, \ldots, d$ (see [6]).

Definition 3 (The Fock representation). The representation $\lambda_0$, acting on the space $\mathcal{F}(\mathcal{H})$ by formulas

$$
\lambda_0(a_i)e_{i_1} \otimes \cdots \otimes e_{i_n} = e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad n \in \mathbb{N} \cup \{0\},
$$

$$
\lambda_0(a_i^*)1 = 0,
$$

where the action of $\lambda_0(a_i^*)$ on the monomials of degree $n \geq 1$ is determined inductively using the basic relations, is called the Fock representation.

Note that the Fock representation is not a $*$-representation with respect to the standard inner product on $\mathcal{F}(\mathcal{H})$. However, it was proved in [6] that there exists a unique Hermitian sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathcal{F}(\mathcal{H})$ such that $\lambda_0$ is a $*$-representation on $(\mathcal{F}(\mathcal{H}), \langle \cdot, \cdot \rangle_0)$. This form is called the Fock inner product on $\mathcal{F}(\mathcal{H})$.

The subspaces $\mathcal{H} \otimes \mathcal{H}^n, \mathcal{H} \otimes \mathcal{H}^m, n \neq m$, are orthogonal with respect to $\langle \cdot, \cdot \rangle_0$, and on $\mathcal{H} \otimes \mathcal{H}^n$ we have the following formula (see [6]):

$$
\langle X, Y \rangle_0 = \langle X, P_nY \rangle, \quad n \geq 2.
$$

So, the positivity of the Fock inner product is equivalent to the positivity of operators $P_n, n \geq 2$, and $J = \bigoplus_{n \geq 2} \ker P_n$ determines the kernel of the Fock inner product. It was noted in [6] that the Fock representation is the GNS representation associated with the linear functional $f$ on a Wick algebra such that $f(1) = 1$ and, for any Wick ordered monomial, $f(a_i \cdots a_i a_j^* \cdots a_j^*) = 0$. Then for any $X, Y \in \mathcal{F}(\mathcal{H})$ we have (see [6]):

$$
\langle X, Y \rangle_0 = f(X^*Y).
$$

2. In the following proposition we describe the kernel of the Fock representation of a Wick algebra with braided operator $T$ in terms of the Fock inner product.

Proposition 1. Let $W(T)$ be the Wick algebra with braided operator $T$, and let the Fock representation $\lambda_0$ be positive (i.e., the Fock inner product is positive definite). Then $\ker \lambda_0 = J \otimes \mathcal{F}(\mathcal{H}^*) + \mathcal{F}(\mathcal{H}) \otimes J^*$.

Proof. First, we show that $X \in \ker P_m$ implies $X \in \ker \lambda_0$. Indeed, let $Y \in \mathcal{F}(\mathcal{H}^n)$; then

$$
\lambda_0(X)Y = X \otimes Y.
$$

Note that for braided $T$ we have the following decomposition (see [2] and Sec. 3 for more details):

$$
P_{n+m} = P(D_m)(P_m \otimes 1_n),
$$

where
\[ P(D_m) = \tilde{R}_{n+m} \cdots \tilde{R}_{m+1}, \]
\[ \tilde{R}_k = 1 + T_{k-1} + T_{k-2}T_{k-1} + \cdots + T_1T_2 \cdots T_{k-1}, \quad k \geq 2. \]

Then
\[ P_{n+m}(\lambda_0(X)Y) = P_{n+m}(X \otimes Y) = P(D_m)(P_mX \otimes Y) = 0, \]
and \( \lambda_0(X) = 0 \) on \( \langle \mathcal{J}(\mathcal{H}), \langle \cdot, \cdot \rangle_0 \rangle \). Therefore \( \mathcal{J} \subset \ker \lambda_0 \), and since \( \ker \lambda_0 \) is a *-ideal,

(1) \[ \mathcal{J} \otimes \mathcal{J}(\mathcal{H}^*) + \mathcal{J}(\mathcal{H}) \otimes \mathcal{I}^* \subset \ker \lambda_0. \]

To prove the converse inclusion, we need a formula determining the action of \( \lambda_0(X^*) \) on \( \mathcal{T}(\mathcal{H}) \) for any \( X \in \mathcal{H}^\otimes k, k \in \mathbb{N} \). For \( k = 1 \), \( X = e_i, i = 1, \ldots, d \), it was proved in [6] that:

\[ \lambda_0(e_i^*)Y = \mu(e_i^*)R_nY, \quad \forall Y \in \mathcal{H}^\otimes n, \]
where \( \mu(e_i^*): \mathcal{T}(\mathcal{H}) \mapsto \mathcal{T}(\mathcal{H}) \) is the annihilation operator:

\[ \mu(e_i^*)e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} = \delta_{i_1i_2 \cdots i_n}. \]

Then, using the definition of \( P_n \), it is easy to see that, for \( X \in \mathcal{H}^\otimes n \) and \( Y \in \mathcal{H}^\otimes n \),

\[ \lambda_0(X^*)Y = \langle X, P_nY \rangle = \langle X, Y \rangle_0. \]

Let now
\[ Z = \sum_{i=1}^{n} Y_i X_i^* + \sum_{j=n+1}^{l} Y_j X_j^* \in \ker \lambda_0, \]
where \( Y_i \in \mathcal{T}(\mathcal{H}), i = 1, \ldots, l, \)
\[ X_i \in \mathcal{H}^\otimes m, i = 1, \ldots, n, \quad X_j \in \mathcal{H}^\otimes n_j, \quad n_j > m, \quad j = n+1, \ldots, l. \]

Now (1) implies that we can suppose that the elements \( X_i \) are linearly independent modulo \( \mathcal{J} \). Denote by \( \{ \hat{X}_i, \ i = 1, \ldots, n \} \subset \mathcal{H}^\otimes m \) a family dual to the \( \{ X_i, \ i = 1, \ldots, n \} \) with respect to \( \langle \cdot, \cdot \rangle_0 \), i.e., such that

\[ \langle X_i, P_m\hat{X}_j \rangle = \langle X_i, \hat{X}_j \rangle_0 = \delta_{ij}, \quad i, j = 1, \ldots, n. \]

Since, for any \( j = n+1, \ldots, l \) and \( i = 1, \ldots, n, \)
\[ \lambda_0(X_j^*)\hat{X}_i = 0, \]
we have, in \( \langle \mathcal{J}(\mathcal{H}), \langle \cdot, \cdot \rangle_0 \rangle \),
\[ 0 = \lambda_0(Z)\hat{X}_i = Y_i, \quad i = 1, \ldots, n, \]
which implies \( Y_i \in \mathcal{J}, i = 1, \ldots, n \). The proof can be completed by evident induction. \[ \square \]
Remark 2. In particular, we have shown, for braided $T$, and for any $X \in \mathcal{H} \otimes n$ and $Y \in \ker P_m$, that

$$X \otimes Y \in \ker P_{n+m}.$$ 

By similar arguments, $Y \otimes X \in \ker P_{n+m}$, i.e., $J = \left\langle \bigotimes_{n \geq 2} \ker P_n \right\rangle$ is a two-sided ideal in $\mathcal{T(H)}$.

The two-sided ideal $J \subset \mathcal{T(H)}$ is called a Wick ideal (see [6]) if it satisfies the following condition:

(2) $\mathcal{T}(\mathcal{H}^*) \otimes J \subset J \otimes \mathcal{T}(\mathcal{H}^*)$.

If $J$ is generated by some subspace of $\mathcal{H} \otimes n$, then $J$ is called a homogeneous Wick ideal of degree $n$.

We show that for Wick algebras with braided operator of coefficients, $J$ is a Wick ideal.

Proposition 2. Let $T$ satisfy the braid condition, and $J = \left\langle \bigoplus_{n \geq 2} \ker P_n \right\rangle$; then

(3) $\mathcal{H}^* \otimes J \subset J \otimes \mathcal{H}^*$.

Proof. Note that Conditions (2) and (3) are equivalent (see [6]). To prove the proposition, it is sufficient to show that, if $X \in \ker P_n$ for some $n \geq 2$, then for any $i = 1, \ldots, d$,

$$e_i^* \otimes X \in \ker P_{n-1} + \ker P_n \otimes \mathcal{H}^*.$$ 

Indeed, for any $X \in \mathcal{H} \otimes n$, we have the following formula (see [9]):

$$e_i^* \otimes X = \mu(e_i^*)R_nX + \mu(e_i^*) \sum_{k=1}^{d} T_1T_2 \cdots T_n(X \otimes e_k) \otimes e_k.$$ 

Then for $X \in \ker P_n$, we have

$$P_{n-1}\mu(e_i^*)R_nX = \mu(e_i^*)(1 \otimes P_{n-1})R_nX = \mu(e_i^*)P_nX = 0.$$ 

Note that, for braided $T$, for any $k = 2, \ldots, n$,

$$T_k(T_1T_2 \cdots T_n) = (T_1T_2 \cdots T_n)T_{k-1},$$

which implies that

$$(1 \otimes P_n)(T_1T_2 \cdots T_n) = (T_1T_2 \cdots T_n)(P_n \otimes 1).$$

Then for any $k = 1, \ldots, d$,

$$P_n\mu(e_i^*)T_1T_2 \cdots T_n(X \otimes e_k) = \mu(e_i^*)(1 \otimes P_n)T_1T_2 \cdots T_n(X \otimes e_k)$$

$$= \mu(e_i^*)T_1T_2 \cdots T_n(P_n \otimes 1)(X \otimes e_k) = 0.$$

$\square$
For Wick algebras with braided $T$, the largest homogeneous ideal of degree $n$ is generated by $\ker R_n$ (see [6] and [9]), i.e., the condition $\ker R_n \neq \{0\}$ is necessary and sufficient for the existence of homogeneous Wick ideals. In the following proposition we show that the same is true for arbitrary Wick ideals.

**Theorem 1.** If $\mathcal{J} \subset \mathcal{I}(\mathcal{H})$ is a non-trivial Wick ideal, then there exists $n \geq 2$ such that $\ker R_n \neq \{0\}$.

**Proof.** For any $X \in \mathcal{I}(\mathcal{H})$, by $\deg X$ we denote the highest degree of its homogeneous components. Let $Y \in \mathcal{J}$ be of minimal degree.

\[ Y = Y_1 + Y_2 + \cdots + Y_k, \quad Y_i \in \mathcal{H} \otimes \mathbb{N}^{n_i}, \quad i = 1, \ldots, k, \quad n_i \in \mathbb{N} \cup \{0\}. \]

Suppose that $\deg Y \geq 2$: Then for any $i = 1, \ldots, d$, we have

\[ e_i^* \otimes Y = \sum_{j=1}^{k} \mu(e_i^*) R_{n_j} Y_j + \mu(e_i^*) \sum_{j=1}^{k} \sum_{l=1}^{d} \tilde{T}_{n_j}(Y_j \otimes e_l) \otimes e_l^* , \]

where we put $R_0 = 1$, $R_1 = 1$, and

\[ \tilde{T}_k = \begin{cases} T_1 T_2 \cdots T_k, & k \geq 2, \\ T, & k = 1, \\ 1, & k = 0. \end{cases} \]

Then Condition (3) implies that for any $i = 1, \ldots, d$,

\[ \sum_{j=1}^{k} \mu(e_i^*) R_{n_j} Y_j \in \mathcal{J}. \]

Since the degrees of these elements are less than the degree of $Y$, we conclude that

\[ \sum_{j=1}^{k} \mu(e_i^*) R_{n_j} Y_j = 0, \quad i = 1, \ldots, d, \]

and the independence of the Wick ordered monomials then implies

\[ \mu(e_i^*) R_{n_j} Y_j = 0, \quad i = 1, \ldots, d. \]

Let $Y_k$ be the highest homogeneous component of $Y$; then, by our assumption, $\deg Y_k \geq 2$, and $\sum_{i=1}^{d} e_i \mu(e_i^*) R_{n_k} Y_k = R_{n_k} Y_k = 0$, i.e., $Y_k \in \ker R_{n_k}$.

To complete the proof, note that if $X = \beta + \sum_{i=1}^{d} \alpha_i e_i \in \mathcal{J}$, then for any $j$, we have

\[ e_j^* \otimes X = \alpha_j + \beta e_j^* + \sum_{i=1}^{d} \alpha_i \sum_{k,l=1}^{d} T_{j,k}^{l} e_l \otimes e_k^*, \]

and (3) implies $\alpha_j = 0$, $j = 1, \ldots, d$, $\beta = 0$. \qed
3. The structure of $\ker P_n$.

In this section, we show that for Wick algebras with braided $T$ satisfying the condition $-1 < T \leq 1$, the Fock representation is faithful, and for $-1 \leq T \leq 1$, the kernel of the Fock representation is generated by the largest quadratic Wick ideal (the largest quadratic Wick ideal is the largest homogeneous Wick ideal of degree 2).

To do this we need some properties of quasimultiplicative maps on the Coxeter group $S_n$ (for more detailed information we refer the reader to [2]).

1. Consider $S_{n+1}$ as a Coxeter group, i.e., a group defined as follows:

$$S_{n+1} = \langle \sigma_i : \sigma_i^2 = e, \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_i, i = 1, \ldots, n \rangle.$$

In order to study the invertibility of $P_n$ for any family of operators $\{T_i, i = 1, \ldots, n, T \in B(\mathcal{K})\}$, satisfying the conditions

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, T_i^* = T_i, -1 \leq T_i \leq 1,$$

where $\mathcal{K}$ is a separable Hilbert space, we may define (as in [2]) the function

$$\phi : S_{n+1} \mapsto B(\mathcal{K})$$

by the formulas

$$\phi(e) = 1, \phi(\sigma_i) = T_i, \phi(\pi) = T_{i_1} \cdots T_{i_k},$$

where $\pi = \sigma_{i_1} \cdots \sigma_{i_k}$ is a reduced decomposition. It was shown in [2] that

$$P_{n+1} = P(S_{n+1}) = \sum_{\sigma \in S_{n+1}} \phi(\sigma).$$

Denote by $S$ the set of generators of $S_{n+1}$ as a Coxeter group. Consider, for any $J \subset S$, the set

$$D_J = \{ \sigma \in S_{n+1} | |\sigma s| = |\sigma| + 1, \forall s \in J \}.$$

Let $W_J$ be a Coxeter group, generated by $J$. Then $S_{n+1} = D_JW_J$ (see [1]), and $P_{n+1} = P(D_J)P(W_J)$ (see [2]). Using the equalities $P_{n+1}^* = P_{n+1}$, $P(W_J)^* = P(W_J)$, we obtain $P_{n+1} = P(W_J)P(D_J)^*$, where for all $M \subset S_{n+1}$,

$$P(M) = \sum_{\sigma \in M} \phi(\sigma).$$

In what follows we use a quasimultiplicative analogue of the Euler-Solomon formula (see [2, Lemma 2.6]):

$$\sum_{\substack{J \subset S \\text{ s.t. } J \neq S, J \neq \emptyset}} (-1)^{|J|}P(D_J) = -(-1)^{|S|}1 + \phi(\sigma_0^{(n+1)}) - P(S_{n+1}),$$

(6)
where $\sigma_0^{(n+1)}$ is the unique element of $S_{n+1}$ with maximal possible length of the reduced decomposition.

**Remark 3.**
1. The element $\sigma_0^{(n+1)}$ of the group $S_{n+1}$ has the form

$$
\sigma_0^{(n+1)} = (\sigma_1 \cdots \sigma_n)(\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1 \sigma_2) \sigma_1.
$$

Set $U_n = \phi(\sigma_0^{(n+1)})$: Then

$$
U_n = (T_1 T_2 \cdots T_n)(T_1 T_2 \cdots T_{n-1}) \cdots (T_1 T_2) T_1.
$$

2. It is easy to see that the operator $U_n$ is selfadjoint, and, taking adjoints, we can rewrite (6) in the following form:

$$
\sum_{J \subset S, J \neq \emptyset} (-1)^{|J|} P(D_J)^* = (-1)^{n+1}1 + U_n - P_{n+1}.
\tag{7}
$$

3. Note also that, for all $J \subset S$, the group $W_J$ is isomorphic to $S_k$ for some $k < n$, or to the direct product of some such groups.

2. In what follows we shall use the following properties of the operator $U_n$.

**Proposition 3.** $\ker P_{n+1}$ is invariant with respect to the action of $U_n$.

**Proof.** First we show that for all $J \subset S$,

$$
P(D_J^*): \ker P_{n+1} \mapsto \ker P_{n+1}.
$$

It can be easily obtained from the equality

$$
P_{n+1} P(D_J)^* = P(D_J) P(W_J) P(D_J)^* = P(D_J) P_{n+1}.
$$

Then by (7), we have

$$
U_n - (-1)^n 1: \ker P_{n+1} \mapsto \ker P_{n+1}.
$$

\qed

**Proposition 4.** Let operators $\{T_i, i = 1, \ldots, n\}$ satisfy the braid condition $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $i = 1, \ldots, n-1$, and $T_i T_j = T_j T_i$, $|i - j| \geq 2$. Then

$$
T_k U_n = U_n T_{n+1-k}, \quad \forall k = 1, \ldots, n.
\tag{8}
$$

**Proof.**
1. For $n = 1$ the equality is evident.

2. Suppose that (8) holds for any $n \leq m$. Note that

$$
U_{m+1} = T_1 T_2 \cdots T_{m+1} U_m.
$$
Then, for $1 < k \leq m+1$, we have

$$T_k U_{m+1} = T_k (T_1 T_2 \cdots T_{m+1}) U_m$$

$$= T_1 T_2 \cdots T_{k-2} T_k T_{k-1} T_k T_{k+1} \cdots T_{m+1} U_m$$

$$= T_1 T_2 \cdots T_{k-2} T_{k-1} T_k T_{k+1} \cdots T_{m+1} U_m$$

$$= (T_1 T_2 \cdots T_{m+1}) T_{k-1} U_m$$

$$= T_1 T_2 \cdots T_{m+1} U_{m+1-(k-1)}$$

$$= U_{m+1} T_{m+2-k}.$$  

In particular, for $k = m+1$ we have $T_{m+1} U_{m+1} = U_{m+1} T_1$. Then taking adjoints, we obtain the required equality for $k = 1$. □

3. Now we can formulate the main result of this paper.

**Theorem 2.** Let $W(T)$ be a Wick algebra with braided operator $T$ satisfying the norm bound $\|T\| \leq 1$. Then for any $n \geq 2$, we have

$$\ker P_{n+1} = \sum_{k+l=n-1} 3^n \ker(1 + T) \otimes 3^n \ker(1 + T_k).$$

Proof. In fact, we shall prove the following: Let $T_1, T_2, \ldots, T_n \in B(K)$, where $K$ is a finite-dimensional Hilbert space, be selfadjoint contractions satisfying the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n-1, \quad T_i T_j = T_j T_i, \quad |i-j| \geq 2.$$  

Then

$$\ker P_{n+1} = \sum_{k=1}^n \ker(1 + T_k).$$

(9)

(It follows trivially from the decomposition $P_{n+1} = P(D_{(k)})(1 + T_k)$ that $\sum_{k=1}^n \ker(1 + T_k) \subset \ker P_{n+1}$.)

We proceed using induction.

**The case $n = 2$.**

In this case $P_2 = 1 + T$.

**The case $n \mapsto n + 1$.**

It follows from $P_{n+1} = P(W_J) P(D_J)^*$ that

$$P(D_J)^*: \ker P_{n+1} \mapsto \ker P(W_J),$$

i.e., $\text{ran}(P(D_J)^*|_{\ker P_{n+1}}) \subset \ker P(W_J)$. Moreover, it is obvious that for any $J \subset S$, $\ker P(W_J) \subset \ker P_{n+1}$. Therefore, by (7), we have the following inclusion:

$$\text{ran}(U_n - (-1)^n 1)|_{\ker P_{n+1}} \subset \sum_{J \subset S, J \neq \emptyset} \ker P(W_J).$$
Since, for \( J \subset S \), the group \( W_J = W_{j_1} \times \cdots \times W_{j_k} \), where \( W_{j_k} \simeq S_{n_l} \) with \( n_l < n + 1 \), we have a decomposition into the product of pairwise commuting selfadjoint operators
\[
P(W_J) = P(W_{j_1}) \cdots P(W_{j_k}).
\]
Therefore
\[
\text{ker} P(W_J) = \sum_{l=1}^k \text{ker} P(W_{j_l}) \subset \sum_{i=1}^n \text{ker}(1 + T_i),
\]
where the last inclusion is obtained from the assumption of induction. So,
\[
\text{ran}(U_n - (\mathbf{1})^n)_{|_{\text{ker} P_{n+1}}} \subset \sum_{i=1}^n \text{ker}(1 + T_i).
\]

Consider the operator \( 1 - U_n^2 \). Since \( U_n = U_n^* : \text{ker} P_{n+1} \mapsto \ker P_{n+1} \), then
\[
\text{ker} P_{n+1} = \text{ran}(1 - U_n^2)_{|_{\text{ker} P_{n+1}}} + \text{ker}(1 - U_n^2)_{|_{\text{ker} P_{n+1}}},
\]
Moreover, since \( \text{ran}(1 - U_n^2) \subset \text{ran}(U_n - (\mathbf{1})^n) \), using (10), we have the inclusion
\[
\text{ker} P_{n+1} \subset \sum_{k=1}^n \text{ker}(1 + T_i) + \text{ker}(1 - U_n^2)_{|_{\text{ker} P_{n+1}}}.
\]
To finish the proof it remains only to show that
\[
\text{ker}(1 - U_n^2) \cap \ker P_{n+1} \subset \sum_{i=1}^n \text{ker}(1 + T_i).
\]
To this end, we may present \( 1 - U_n^2 \) in the form
\[
1 - U_n^2 = 1 - T_1 T_2 \cdots T_n U_n^2 T_n \cdots T_2 T_1
= (1 - T_1^2) + T_1 (1 - T_2^2) T_1
+ \cdots
+ T_1 T_2 \cdots T_{n-1} (1 - T_n^2) T_{n-1} \cdots T_2 T_1
+ T_1 T_2 \cdots T_n (1 - U_{n-1}^2) T_n \cdots T_2 T_1.
\]
Since \( \|T\| \leq 1 \) implies that \( \|T_i\| \leq 1, i = 1, \ldots, n \), and \( \|U_k\| \leq 1, k \geq 2 \), then we have a sum of non-negative operators, and \( v \in \text{ker}(1 - U_n^2) \) implies that
\[
T_1^2 v = v,
T_2^2 T_1 v = T_1 v,
\]
\[
\vdots
T_n^2 T_{n-1} \cdots T_2 T_1 v = T_{n-1} \cdots T_2 T_1 v.
\]
However, \( T_k U_n = U_n T_{n+1-k} \) implies that \( T_k U_n^2 = U_n^2 T_k \), and, consequently,
\[
T_k : \text{ker}(1 - U_n^2) \mapsto \ker(1 - U_n^2), \quad k = 1, \ldots, n.
\]
Moreover, since the restriction of $T_1$ to $\ker(1 - U^2_n)$ is an involution,
\[ \text{ran}(T_1)|_{\ker(1 - U^2_n)} = \ker(1 - U^2_n), \]
and, for any $v \in \ker(1 - U^2_n)$, we have $T^2_2 v = v$. By the same arguments, we obtain that
\[ \forall \, v \in \ker(1 - U^2_n), \quad T^2_i v = v, \quad i = 1, \ldots, d. \]
Let now $v \in \ker(1 - U^2_n) \cap \ker P_{n+1}$; then, for any $k = 1, \ldots, n$,
\[
P_{n+1} T_k v = P(D_{(k)})(1 + T_k)T_k v
= P(D_{(k)})(T_k + T^2_k)v
= P(D_{(k)})(1 + T_k)v = P_{n+1} v = 0.
\]
Therefore $T_k$ maps $\ker(1 - U^2_n) \cap \ker P_{n+1}$ onto itself for any $k = 1, \ldots, n$. This fact implies that, for any $\sigma \in S_{n+1}$, we have $\phi(\sigma)v \in \ker(1 - U^2_n) \cap \ker P_{n+1}$, and
\[ \forall k = 1, \ldots, n, \forall \sigma \in S_{n+1}, \quad (1 - T^2_k)\phi(\sigma)v = 0. \]
For convenience, we fix the set $S_{n+1}$, and set $v_i := \phi(\pi_i)v$ for $\pi_i \in S_{n+1}$ ($\pi_1 := \text{id}$ and $v_1 = v$). Then the condition $P_{n+1} v = 0$ takes the form
\[
n! \sum_{k=1}^{n} v_k = 0.
\]
Finally, for any pair $i \neq j$ there exist generators $\sigma_{i_1}, \ldots, \sigma_{i_m} \in S$ such that
\[ \pi_j = \sigma_{i_1} \cdots \sigma_{i_m} \pi_i \]
and
\[ v_j = T_{i_1} \cdots T_{i_m} v_i. \]
Note that, if $v_k = T_r v_l$ for some $r = 1, \ldots, n$, then $T^2_r v_k = v_k$ implies that $v_k - v_l \in \ker(1 + T_r)$. Therefore, for any $i \neq j$,
\[ v_i - v_j \in \sum_{k=1}^{n} \ker(1 + T_k). \]
In particular, for any $j = 2, \ldots, n!$,
\[ v_1 - v_j \in \sum_{k=1}^{n} \ker(1 + T_k). \]
Then from (11), we have
\[ n! v = n! v_1 \in \sum_{k=1}^{n} \ker(1 + T_k), \]
and therefore
\[ v \in \sum_{k=1}^{n} \ker(1 + T_k). \]
Remark 4. Evidently the proof does not depend on the dimension of $\mathcal{K}$. Indeed, in the case when $\mathcal{K}$ is infinite-dimensional, the linear subspace in $(9)$ is replaced by its closure. I.e., if $\mathcal{K}$ is a separable Hilbert space and $\{T_i, i = 1, \ldots, n\}$ are selfadjoint contractions satisfying the braid conditions, then

$$\ker P_{n+1} = \sum_{k=1}^{n} \ker(1 + T_k).$$

As a corollary we have an improved version of the result of Bożejko and Speicher (see [2]).

Proposition 5. If the operator $T$ satisfies the braid condition, and $-1 < T \leq 1$, then $P_n > 0$, $n \geq 2$, i.e., the Fock inner product is strictly positive, and the Fock representation acts in the whole space $\mathcal{T}(\mathcal{K})$.

Proof. Recall that if $T$ is braided and $\|T\| \leq 1$ then $P_n \geq 0$ (see [2]). It remains only to show that $\ker P_n = \{0\}$ for $-1 < T \leq 1$. This fact trivially follows from our theorem since in this case $\ker(1 + T) = \{0\}$. \hfill $\square$

4. Corollaries and examples.

We summarize the results obtained above in the following proposition.

Proposition 6. If $W(T)$ is a Wick algebra with braided operator of coefficients $T$ satisfying the norm bound $\|T\| \leq 1$, then the following three statements hold.

1. The kernel of the Fock representation is generated by the largest quadratic Wick ideal. In particular, if $-1 < T \leq 1$, then the Fock representation is faithful.

2. For any $n \geq 2$ we have the inclusion $\mathcal{I}_n \subset \mathcal{I}_2$.

3. If $-1 < T \leq 1$, then $W(T)$ has no non-trivial Wick ideals.

Example 1. Consider the $q$-CCR algebra based on a Hilbert space $\mathcal{H}$ and the relations

$$a_i^* a_i = 1 + qa_i a_i^*, \quad i = 1, \ldots, d,$$

$$a_i^* a_j = qa_j a_i^*, \quad i \neq j, \quad 0 < q < 1.$$

We pick an orthogonal basis $(e_i)$ in $\mathcal{H}$, and then $T$ is determined on this basis by the formulas

$$Te_i \otimes e_j = qe_j \otimes e_i, \quad \|T\| < 1.$$

It is evident that $T$ is braided. Then by the proposition, we cannot have any Wick ideals in $W(T)$. 
It was proved in [2] that for braided $T$ satisfying the norm bound $\|T\| < 1$, the Fock representation is bounded. Therefore we may consider the $C^*$-algebra generated by operators of the Fock representation.

Recall that a $*$-algebra is called $C^*$-representable if it can be realized as a $*$-subalgebra of a certain $C^*$-algebra (see for example [7]). Combining the results of Theorem 2 and Proposition 5, we obtain the following statement.

**Proposition 7.** If $W(T)$ is a Wick algebra with braided operator of coefficients $T$ satisfying the norm bound $\|T\| < 1$, then $W(T)$ is $C^*$-representable.

Suppose that, in the case of braided $T$ with $\|T\| = 1$ and $\ker(1 + T) \neq \{0\}$, the Fock representation is bounded. Then Theorem 2 implies that the quotient $W(T)/I_2$ is $C^*$-representable.

**Example 2.** Consider the following type of $q_{ij}$-CCR (see [2]):

\[
\begin{align*}
a_i^*a_i &= 1 + q_i a_i a_i^*, \quad i = 1, \ldots, d, \quad 0 < q_i < 1, \\
a_i^*a_j &= \lambda_{ij} a_j a_i^*, \quad i \neq j, \quad |\lambda_{ij}| = 1, \quad \lambda_{ij} = \overline{\lambda_{ij}}.
\end{align*}
\]

The corresponding $T$ is braided, $\|T\| = 1$, and

\[
\ker(1 + T) = \langle a_j a_i - \lambda_{ij} a_i a_j, \ i < j \rangle.
\]

Moreover, the Fock representation of this algebra is bounded. Then, as noted above, the $*$-algebra generated by the relations

\[
\begin{align*}
a_i^*a_i &= 1 + q_i a_i a_i^*, \quad i = 1, \ldots, d, \quad 0 < q_i < 1, \\
a_i^*a_j &= \lambda_{ij} a_j a_i^*, \quad i \neq j, \quad |\lambda_{ij}| = 1, \quad \lambda_{ij} = \overline{\lambda_{ij}}, \\
a_j a_i &= \lambda_{ij} a_i a_j, \quad i < j
\end{align*}
\]

is $C^*$-representable.

A description of the irreducible representations of these relations can be found for example in [8, Sec. 2.4].

Note that, if $\|T\| = 1$, then the operators of the Fock representation can be unbounded.

**Example 3.** Consider the following Wick algebra:

\[
\begin{align*}
a_i^*a_i &= 1 + a_i a_i^*, \quad i = 1, \ldots, d, \\
a_i^*a_j &= qa_j a_i^*, \quad i \neq j, \quad -1 < q < 1.
\end{align*}
\]

The corresponding $T$ is determined by the formulas

\[
Te_i \otimes e_i = e_i \otimes e_i, \quad Te_j \otimes e_i = q e_i \otimes e_j, \quad i \neq j, \quad i = 1, \ldots, d.
\]

It is easy to see that $T$ is braided and $-1 < T \leq 1$. So, the Fock representation of this algebra is faithful. Note that, if we consider the complement of $T(3\mathfrak{f})$ with respect to the Fock inner product, then the operators of the Fock representation are unbounded.

For the definition and properties of representations of $*$-algebras by unbounded operators, see for example [11].
Unbounded representations of Wick algebras will be considered in more detail later.

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References


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CONSTRUCTING FAMILIES OF LONG CONTINUED FRACTIONS

Daniel J. Madden

This paper describes a method of constructing an unlimited number of infinite families of continued fraction expansions of the square root of $D$, an integer. The periods of these continued fractions all have identifiable sub patterns repeated a number of times according to certain parameters. For example, it is possible to construct an explicit family for the square root of $D(k, l)$ where the period of the continued fraction has length $2kl - 2$. The method is recursive and additional parameters controlling the length can be added.

Section 1.

In the last 20 years (starting with an example by D. Shanks), more and more examples of families of quadratic surds with unbounded continued fraction length have appeared. A long list of explicit continued fractions discovered by L. Bernstein, C. Levesque and G. Rind is given in [LR]. H.C. Williams [W], T. Azuhatu [A] and many others have added to this list over the years. In [vdP], A. van der Poorten gives several examples as he illustrates how matrices not only make such expansions more manageable, but also are an integral part of the theory of continued fractions. In this note, we describe a method of constructing an unlimited number of distinct families of expansions using two matrix identities.

Typically in the known examples, a family of surds is given in terms of several parameters of which one is explicitly connected with the length of the period in the partial quotients. For instance, an integer $d(k, n)$, $\sqrt{d(k, n)}$ might have the form

$$[n, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, a_3, b_2, c_2, d_2, \ldots, a_k, b_k, c_k, d_k, 2n]$$

where the partial quotients $a_i, b_i, c_i, d_i$ are given as formula in terms of $n$ and $i$. The internal structure of the repeating quotients contains a recurring pattern of length 4 repeated $k$ times.

In this paper we will describe methods for constructing families of integer square roots where two or more parameters control different aspects of the period length of the continued fraction expansion; i.e., $\sqrt{d(k_1, k_2, n)}$ produces a period containing a recurring pattern of length $k_2$ repeated $k_1$ times.
times. Further we will see how to produce surds with periods of ever increasing complexity. We will describe a method that takes the period of one example and inserts it inside a more complicated recurring pattern. For example, starting with a sequence $\vec{a}$ of length $k_2$ taken from one continued fraction, we can construct another continued fraction that involves a recurring pattern $b_i, \vec{a}, c_i$ that appears $k_1$ times in the repeating part. The process is recursive, and it can be used to produce ever more elaborate sequences of partial quotients. Finally, our method will allow us to choose the partial quotients to be of any size so our fractions can be chosen to limit the occurrence of small partial quotients. While there is no end to the number and complexity of families our methods can produce, they become increasingly difficult to write down explicitly in terms of the controlling parameters.

The notation that follows gets rather involved, and we will be forced to set some conventions just to make things readable. To start we will no longer overline the repeating quotients in the expansion a quadratic surd, and agree that

$$\sqrt{d} = [a_0; a_1, a_2, \ldots, a_n]$$

means that the quotients $a_1, a_2, \ldots, a_n$ repeat. We will still use $[a_0; a_1, a_2, \ldots, a_n]$ for the finite continued fraction in hopes that the context will allow the correct interpretation.

Suppose we have a matrix

$$N = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$ 

We will use the shorthand notation $N = \{a_1, a_2, \ldots, a_n\}$ to express such a product. We will write $\vec{N}$ to denote the sequence $a_1, a_2, \ldots, a_n$. We will denote the reverse sequence as $\vec{N}$ which could just as well be written as $\vec{N}^T$.

Any such matrix can be viewed as a fractional linear functions, and we will denote this action as

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} [x] = \frac{sx + t}{ux + v}.$$ 

Of course, the composition of functions corresponds to matrix multiplication.

To set out the method used in the proofs, we will begin with an example. This example will illustrate the steps used in this paper to evaluate a continued fraction, and allow readers to reconcile the layout of these steps with their own style. The example we will use is in one of the sequences we will construct later, $\sqrt{31}$. In this case, the expansion claimed is

$$\sqrt{31} = [5; 1, 1, 3, 5, 3, 1, 1, 10].$$
We can compute the convergents of the first cycle using the familiar PQ chart:

\[
\begin{array}{cccccccc}
Q & 1 & 0 & 1 & 1 & 2 & 7 & 37 & 118 \\
& 1 & 1 & 2 & 7 & 37 & 118 & 155 & 273 \\
& 1 & 1 & 3 & 5 & 3 & 1 & 1 & 10 \\
\end{array}
\]

For our purposes it is better to change the last quotient in this chart from 10 to 5, then the last two columns look like

\[
\begin{array}{cc}
1 & 5 \\
273 & 1520 \\
1520 & 8463 \\
\end{array}
\]

This calculation can be expressed in our matrix notation as

\[
M = \{5; 1, 1, 3, 5, 3, 1, 1, 5\} = \begin{pmatrix} 273 & 1520 \\ 1520 & 8463 \end{pmatrix}.
\]

Because the continued fraction repeats, the value of this fraction satisfies the quadratic equation

\[
M[x] = \frac{273x + 1520}{1520x + 8463} = \frac{1}{x}.
\]

This reduces to

\[
273x^2 = 8463
\]

or

\[
x^2 = 31.
\]

Further we can obtain the smallest solutions to Pell’s equation from these last two columns

\[
1520^2 - 273^2 \cdot 31 = 1.
\]

In general, if \( \sqrt{d} = [a_0; a_1, a_2, \ldots, 2a_0] \), then the corresponding matrix \( M = \{a_0, a_1, a_2, \ldots, a_0\} \) must have the form

\[
M = \begin{pmatrix} s & t \\ t & sd \end{pmatrix}
\]

where

\[
s^2d - t^2 = \pm 1.
\]

Then \( \sqrt{d} \) is the solution to the quadratic equation:

\[
M[x] = \frac{1}{x}.
\]
Further,
\[
\begin{pmatrix}
  s & t \\
  t & sd
\end{pmatrix}
\begin{pmatrix}
  \sqrt{d} & -\sqrt{d} \\
  1 & 1
\end{pmatrix}
= \begin{pmatrix}
  1 & 1 \\
  \sqrt{d} & -\sqrt{d}
\end{pmatrix}
\begin{pmatrix}
  t + s\sqrt{d} & 0 \\
  0 & t - s\sqrt{d}
\end{pmatrix}.
\]
If \(d \not\equiv 1 \pmod{4}\) and is square free, \(t + s\sqrt{d}\) is the fundamental unit of \(\mathbb{Q}(\sqrt{d})\).

**Section 2.**

All of our examples hinge on the reduction of a matrix product of the form
\[
M = A(N_{k-1}N_{k-2} \cdots N_2 N_1 N_0)A(N_{k-1}N_{k-2} \cdots N_2 N_1 N_0)^T A.
\]
Suppose we have a family of matrices of the form
\[
N_t = \begin{pmatrix}
  p & 2br^t \\
  br^{k-1-t} & q
\end{pmatrix}
\]
where
\[
pq - 2b^2 r^{k-1} = \epsilon = \pm 1
\]
and
\[
qr - p = 2bm \quad \text{for some } m.
\]
The product \(N_{k-1}N_{k-2} \cdots N_2 N_1 N_0\) can be reduced quickly. If
\[
\begin{pmatrix}
  1 & 0 \\
  0 & r
\end{pmatrix},
\]
it is easy to see that \(N_t = C^{-t}N_0C^t\). So our first reduction is
\[
N_{k-1} \cdots N_1 N_0 = C^{-k}(CN_0)^k.
\]
Now consider
\[
K = CN_0 = \begin{pmatrix}
  p & 2b \\
  br^k & qr
\end{pmatrix}.
\]
The fixed value of \(K^T[x]\), \(\alpha\), satisfies
\[
2b\alpha^2 + (qr - p)\alpha - br^k = 0.
\]
So if
\[
d = m^2 + 2r^k
\]
and
\[
\alpha = \frac{-m + \sqrt{d}}{2},
\]
then \(\alpha\) and its conjugate \(\bar{\alpha}\) are fixed values of \(K^T[x]\).
Next, let

\[ E = \begin{pmatrix} \sqrt{d} & -\sqrt{d} \\ 1 & 1 \end{pmatrix} \]

and

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}. \]

Now one simply verifies that

\[ C^{-k}AE = \begin{pmatrix} 1 & 1 \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\alpha} \\ \bar{\alpha}^{-1} & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \bar{\alpha}^{-1} \end{pmatrix}. \]

Next

\[ K^T \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2b\alpha^2 + qr\alpha & 2b\bar{\alpha}^2 + qr\bar{\alpha} \\ 2b\alpha q & 2b\bar{\alpha} + qr \end{pmatrix}. \]

So we have

\[ K^T \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix} \]

where \( \beta = 2b\alpha + qr = 2b\alpha + p = p + 2b(m + \alpha) = p - 2b\bar{\alpha}. \)

Now \( \beta \) and its conjugate \( \bar{\beta} \) are the eigenvalues of \( K^T \) (and consequently \( K \)). Immediately,

\[ (K^T)^k = \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \beta^k & 0 \\ 0 & \bar{\beta}^k \end{pmatrix}. \]

Next

\[ A \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ m + \alpha & m + \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\bar{\alpha} & -\alpha \end{pmatrix}. \]

This in turn diagonalizes \( K \):

\[ K \begin{pmatrix} 1 & 1 \\ -\bar{\alpha} & -\alpha \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix}. \]

Again

\[ K^k \begin{pmatrix} 1 & 1 \\ -\bar{\alpha} & -\alpha \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\bar{\alpha} & -\alpha \end{pmatrix} \begin{pmatrix} \beta^k & 0 \\ 0 & \bar{\beta}^k \end{pmatrix}. \]

Finally

\[ AC^{-k} \begin{pmatrix} 1 & 1 \\ -\bar{\alpha} & -\alpha \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \end{pmatrix} \begin{pmatrix} \frac{1}{2\alpha} & 0 \\ 0 & \frac{1}{2\alpha} \end{pmatrix}. \]
Now the whole point of this is

\[
N_{k-1}N_{k-2}N_{k-3}\cdots N_2N_1N_0 = C^{-k}(CN_0)^k = C^{-k}K^k.
\]

So if we define \( M \) as

\[
M = A(N_{k-1}N_{k-2}N_{k-3}\cdots N_2N_1N_0)A(N_{k-1}N_{k-2}N_{k-3}\cdots N_2N_1N_0)^T A
\]

then

\[
ME = AC^{-k}K^kA(C^{-k}K^k)^T AE = AC^{-k}K^k A(K^T)^k C^{-k} AE
\]

\[
= \begin{pmatrix}
1 & 1 \\
\sqrt{d} & -\sqrt{d}
\end{pmatrix}
\begin{pmatrix}
\beta^{2k} & 0 \\
0 & \beta^{2k}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2\alpha^2} & 0 \\
0 & \frac{1}{2\alpha^2}
\end{pmatrix}.
\]

This implies

\[
M \begin{pmatrix}
\sqrt{d} & -\sqrt{d} \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
\sqrt{d} & -\sqrt{d}
\end{pmatrix}
\begin{pmatrix}
\gamma & 0 \\
0 & \bar{\gamma}
\end{pmatrix},
\]

and so our identity is immediately useful in establishing the continued fraction expansion of \( \sqrt{d} \).

**Proposition 1.** Suppose we have a family of matrices of the form

\[
N_i(q,b,m,r) = \begin{pmatrix}
qr - 2bm & 2br^i \\
bm^{k-1-i} & q
\end{pmatrix} \quad i = 0, 1, 2, \ldots, k - 1.
\]

Further suppose that each \( N_i = \{a(i,1), a(i,2), \ldots, a(i,n)\} \) where \( a(i,j) \) are integers. If \( d(m,r) = m^2 + 2r^k \), then the following is a valid continued fraction expansion of \( \sqrt{d(m,r)} \):

\[
\sqrt{d(m,r)} = \left[ m; N_0, N_1, \ldots, \hat{N}_{k-1}, \ldots, N_1, N_0, 2m \right].
\]

Further if \( d(m,r) \) is square free and \( d \equiv 2 \) or \( 3 \pmod{4} \), then the fundamental unit of \( \mathbb{Q}(\sqrt{d(m,r)}) \) is

\[
\frac{\beta^{2k}}{2\alpha^2}
\]

where

\[
\alpha = \frac{-m + \sqrt{d(m,r)}}{2} \quad \text{and} \quad \beta = qr - 2bm - 2b\bar{\alpha}.
\]
Section 3.

The calculation above can be used directly to produce examples of continued fractions of quadratic surds with arbitrarily long repeating pattern. Let \( b, n \) and \( k \) be any natural numbers. We have

\[
\sqrt{(b(2bn + 1)^k + n)^2 + 2(2bn + 1)^k} = \left[ b(2bn + 1)^k + n; b, 2b(2bn + 1)^{k-1}, b(2bn + 1), 2b(2bn + 1)^{k-2}, b(2bn + 1)^2, \ldots, b(2bn + 1)^k - 1, b(2bn + 1)^1, b(2bn + 1)^{k-1}, 2b, b(2bn + 1)^{k-2}, 2b(2bn + 1)^1, b(2bn + 1)^k - 2, \ldots, b, 2b(2bn + 1)^{k-1}, b(2bn + 1), 2b(2bn + 1)^{k-2}, b(2bn + 1)^2, \ldots, b(2bn + 1)^k - 2, b(2bn + 1)^1, b(2bn + 1)^{k-1}, 2b, 2b(2bn + 1)^2, \ldots, \right].
\]

The length of the repeating pattern is \( 4k + 2 \). This expansion comes from the matrix product

\[
\begin{pmatrix} 0 & 1 \\ 1 & br^{k-1-t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2br^t \end{pmatrix} = \begin{pmatrix} 1 & 2br^t \\ br^{k-1-t} & 2br^{k-1-t} + 1 \end{pmatrix}
\]

which is in the form used in Section 2 provided \( qr - p = 2b^2r^k + r - 1 \) is divisible by \( 2b \). If we choose \( r = 2bn + 1 \), this condition will be met. In the proposition, we wrote \( qr - p = 2bn \). Keeping this notation

\[ m = br^k + n = b(2bn + 1)^k + n. \]

This leads us to the surd

\[ \sqrt{d} = \sqrt{m^2 + 2r^k} = \sqrt{(b(2bn + 1)^k + n)^2 + 2(2bn + 1)^k}. \]

Note that every partial quotient in this family is greater than or equal to \( b \), so this gives infinite families of quadratic surds with all partial quotients arbitrarily large.

In this example, \( r = 2bn + 1 \) is odd, so \( d = m^2 + 2r^k \equiv 2 \) or \( 1 \) (mod 4). If

\[ d = (b(2bn + 1)^k + n)^2 + 2(2bn + 1)^k \]

is square free, the fundamental unit of \( \mathbb{Q}(\sqrt{d}) \) comes directly from the continued fraction. This unit is

\[ \frac{\beta^{2k}}{2\alpha^2} \]

where

\[ \alpha = \frac{1}{2} (-b(2bn + 1)^k - n + \sqrt{d}), \]

and \( \beta \) is the solution to the equation

\[ \beta^2 + b\beta + 1 = 0. \]
and
\[ \beta = 1 - 2b\alpha = 1 + bn + b^2(2bn + 1)^k + b\sqrt{d}. \]

The next example is similar to examples found in [vdP]. Here \( \sqrt{d(k, b, n)} \) has length \( 6k + 2 \). Let \( b, n \) and \( k \) be any natural numbers,

\[
\sqrt{(b(2bn - 1)^k - n)^2 + 2(2bn - 1)^k} = \left[b(2bn - 1)^k - n; \ b - 1, \ 1, \ 2b(2bn - 1)^{k-1} - 1, \right.
\left. b(2bn - 1) - 1, \ 1, \ 2b(2bn - 1)^{k-2} - 1, \right.
\left. b(2bn - 1)^2 - 1, \ 1, \ 2b(2bn - 1)^{k-3} - 1, \right.
\left. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
Constructing families of long continued fractions

Next we provide a method for producing expansions where the repeating pattern contains recurring patterns of the form $A_i, a_1, a_2, \ldots, a_n, B_{k-i}$. Unfortunately, the $a_i$ cannot be chosen arbitrarily. We will see later exactly how to choose proper $a_i$.

Suppose you have an integral matrix product of the form

$$\{a_1, a_2, \ldots, a_n\} = \begin{pmatrix} u & v \\ 2v - \delta w & w \end{pmatrix},$$

where $\delta = 0$ or 1. The determinant of this matrix is

$$\epsilon = (-1)^n = uw - 2v^2 + \delta vw.$$

If we choose the following values for $b, r, m, A_i,$ and $B_i,$ then we can produce a family of matrices like those in Proposition 1 from

$$N_i = \{vA_{k-1-i}, a_1, a_2, \ldots, a_n, B_i\}.$$

For any value of $l \geq 0,$ let

$$b = b(u, v, w, \delta; l) = v$$

$$r = r(u, v, w, \delta; l) = \epsilon(w^2 - 2v^2) + 2lwv$$

$$A_i = A_i(u, v, w, \delta; l) = \frac{r^i - 1}{w}$$

$$B_i = B_i(u, v, w, \delta; l) = 2vA_i + \delta$$

and

$$m = m(u, v, w, \delta; l) = vA_k + \epsilon l.$$

Now it may not be immediately clear that the $A_i$ are all integers, but they are since

$$r - 1 = \epsilon(w^2 - 2v^2) + 2lwv - 1$$

$$= \epsilon(w^2 + \epsilon - uw - \delta vw) + 2lwv - 1$$

$$= \epsilon(w^2 - uw - \delta vw) + 2lwv.$$

Thus

$$\frac{r - 1}{w} = \epsilon(w - u - \delta v) + 2lv.$$

These values are set so that we have the required form:

$$\begin{pmatrix} 0 & 1 \\ 1 & vA_{k-1-i} \end{pmatrix} \begin{pmatrix} u & v \\ 2v - \delta w & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & B_i \end{pmatrix} = \begin{pmatrix} w & 2br^i \\ br^{k-1-i} & q \end{pmatrix}$$

where

$$q = u + (2v - \delta w)vA_{k-1-i} + B_ibr^{k-1-i}.$$
We can simplify $q$ using the fact that the determinant $q w - 2 b^2 r^{k-1} = \epsilon$. Then
\[
q = \frac{2 b^2 r^{k-1} + \epsilon}{w}.
\]
Thus it is independent of $i$. To use Proposition 1, we need to check that $q r - w = 2 b m$.

\[
q r - w = \frac{2 v^2 r^k + \epsilon r - w^2}{w} = \frac{2 v^2 r^k + (w^2 - 2 v^2) + 2 \epsilon lwv - w^2}{w} = 2 v \left( \frac{r^k - 1}{w} + \epsilon \right) = 2 v (v A_k + \epsilon) = 2 v m
\]
because $b = v$ and $m = v A_k + \epsilon$.

Once we have our matrix family, we have a continued fraction expansion for
\[
\sqrt{d} = \sqrt{m^2 + 2 r^k}
\]
where $m = v A_k + \epsilon$, and $A_k = \frac{r^k - 1}{w}$. We can write
\[
d = (v A_k + \epsilon)^2 + 2 w A_k + 2 = v^2 A_k^2 + 2(\epsilon lw + w) A_k + 2.
\]

Section 5.

Suppose we have a sequence of integers $a_1; a_2, \ldots, a_n$ for which
\[
\{a_1; a_2, \ldots, a_n\} = \left( \begin{array}{cc} u & v \\ 2 v - \delta w & w \end{array} \right)
\]
where $\delta = 0$ or 1. Let $\epsilon = uw - xv = (-1)^n$. In the last section we saw how to use this to construct a matrix of the form in Proposition 1, and so we are led to a formal continued fraction expansion of a quadratic surd. With the definitions,
\[
r = \epsilon w^2 - 2 \epsilon v^2 + 2 lwv.
\]
\[
A_t = \frac{r^t - 1}{w}
\]
\[
B_t = 2 v A_t + \delta
\]
\[
m = v A_k + \epsilon
\]
\[
d = m^2 + 2 r^k = v^2 A_k^2 + 2(\epsilon lw + w) A_k + 2
\]
\[
\alpha = \frac{-m + \sqrt{d}}{2}
\]
\[
\beta = w - 2 b \alpha,
\]
we saw that

\[ N_t = \{ vA_t, a_1, \ldots, a_n, B_{k-1-t} \} = \begin{pmatrix} w & 2b^r \\ br^{k-1-t} & q \end{pmatrix}. \]

This matrix is of the type in Proposition 1, so

\[ \sqrt{d} = \begin{pmatrix} \overrightarrow{N_0}, \overrightarrow{N_1}, \ldots, \overrightarrow{N_{k-1}}, m, \overleftarrow{N_{k-1}}, \ldots, \overleftarrow{N_1}, \overleftarrow{N_0}, 2m \end{pmatrix}. \]

This almost means that, as a continued fraction,

\[ \sqrt{d} = \left[ m; \ A_0, a_1, a_2, \ldots, a_n, B_{k-1}, \right. \]

\[ A_1, a_1, a_2, \ldots, a_n, B_{k-2}, \]

\[ A_2, a_1, a_2, \ldots, a_n, B_{k-3}, \]

\[ \ldots \]

\[ A_{k-2}, a_1, a_2, \ldots, a_n, B_1 \]

\[ A_{k-1}, a_1, a_2, \ldots, a_n, B_0 \]

\[ m, \]

\[ B_0, a_n, a_{n-1}, \ldots, a_1, A_{k-1}, \]

\[ B_1, a_n, a_{n-1}, \ldots, a_1, A_{k-2}, \]

\[ B_2, a_n, a_{n-1}, \ldots, a_1, A_{k-3}, \]

\[ \ldots \]

\[ B_{k-2}, a_n, a_{n-1}, \ldots, a_1, A_1, \]

\[ B_{k-1}, a_n, a_{n-1}, \ldots, a_1, A_0, \]

\[ 2m \].

If \( d \) is square-free and not equivalent to 1 mod 4, then the calculations also lead us directly to the fundamental unit of \( \mathbb{Q}[\sqrt{d}] \). It is

\[ \frac{\beta^{2k}}{2\alpha^2}. \]

There is a problem that may arise in the above; the partial quotients in the continued fraction may not all be positive. In fact, we always have \( A_0 = 0 \), and when \( \delta = 0 \), we also have \( B_0 = 0 \). Further, \( r \) could be negative. While the identity from Proposition 1 is valid, it may not represent the correct form of the continued fraction expansion.

The offending zeros are not a major problem, since they do not change the matrix identity. They can easily be dropped using the original meaning of a continued fraction. If the sequence \([ \ldots a, b, 0, d, e, \ldots ]\) occurs in a valid continued fraction identity, it can be replaced by \([ \ldots a, b + d, e, \ldots ]\). We can use this to change our identity into the standard continued fraction of the
surd, but we need borrow a bit from the next repeating cycle to get exactly what we want.

When \( \delta = 1 \), only \( A_0 = 0 \), and we end up with an expansion of length 2\((n + 2)k\).

(1) \[
\sqrt{d} = \left[ m + a_1; \quad a_2, \ldots, a_n, B_{k-1},
\begin{array}{c}
vA_1, \quad a_1, a_2, \ldots, a_n, B_{k-2}, \\
vA_2, \quad a_1, a_2, \ldots, a_n, B_{k-3}, \\
\ldots \\
vA_{k-2}, a_1, a_2, \ldots, a_n, B_1, \\
vA_{k-1}, a_1, a_2, \ldots, a_n, B_0,
\end{array}
\right]
\]

When \( \delta = 0 \), then \( A_0 = B_0 = 0 \). This adjust to an expansion with length 2\((n + 2)k - 2\).

(2) \[
\sqrt{d} = \left[ m + a_1; \quad a_2, \ldots, a_n, B_{k-1},
\begin{array}{c}
vA_1, \quad a_1, a_2, \ldots, a_n, B_{k-2}, \\
vA_2, \quad a_1, a_2, \ldots, a_n, B_{k-3}, \\
\ldots \\
vA_{k-2}, a_1, a_2, \ldots, a_n, B_1, \\
vA_{k-1}, a_1, a_2, \ldots, a_{n-1},
\end{array}
\right]
\]
The multiplier $r$ is a bit more trouble because it could be negative. This can be avoided with the right choice of $l$:

$$r = \epsilon w^2 - 2\epsilon v^2 + 2lwv.$$ 

Since $l$ can be any integer, $r$ can certainly be chosen positive. This quick fix will always work, but there is another method that will allow us to take $l = 0$. Recall that

$$\begin{pmatrix} u & v \\ x & w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

and that

$$\epsilon = (-1)^n.$$ 

If $r = \epsilon(w^2 - 2v^2)$ is negative, then we can change the parity of the length of the fraction by noting that

$$\begin{pmatrix} u & v \\ x & w \end{pmatrix} = \{a_1; a_2, \ldots, a_{n-1}, a_n\} = \{a_1; a_2, \ldots, a_{n-1}, a_n - 1, 1\}.$$ 

(We use this in whichever direction is necessary.) If we arrange the $a_1; a_2, \ldots, a_n$ so that

$$\begin{cases} 
    \text{if } w^2 > 2v^2, & \text{then } \epsilon = 1, \\
    \text{if } w^2 < 2v^2, & \text{then } \epsilon = -1.
\end{cases}$$

This will guarantee that $r > 0$.

All this has gotten a bit complicated; so we will summarize with our next proposition.

**Proposition 2.** Suppose we have a sequence of integers $a_1, a_2, \ldots, a_n$ for which

$$\{a_1, a_2, \ldots, a_n\} = \begin{pmatrix} u & v \\ 2v - \delta w & w \end{pmatrix} = U$$

and $\delta = 0$ or $1$. From the given values of $u, v, w$ and $\delta$ and for any values of $l \geq 0, k \geq 1$, let

$$\epsilon = \epsilon(u, v, w, \delta) = uw - (2v - \delta)v = (-1)^n$$

$$r = r(u, v, w, \delta; l) = \epsilon(w^2 - 2v^2) + 2lwv$$

$$A_i = A_i(u, v, w, \delta; l) = \frac{r^i - 1}{w}$$

$$B_i = B_i(u, v, w, \delta; l) = 2vA_i + \delta$$

$$m = m(u, v, w, \delta; l, k) = vA_k + \epsilon l$$
and
\[ d = d(u, v, w, \delta; l, k) = m^2 + 2r^k = v^2A_k^2 + 2(\epsilon v + w)A_k + 2. \]

We can always choose \( l \) large enough that \( r \) is positive. However, if we arrange the \( a_1, a_2, \ldots, a_n \) so that
\[
\begin{cases}
\epsilon = 1, & \text{if } w^2 > 2v^2, \\
\epsilon = -1, & \text{if } w^2 < 2v^2,
\end{cases}
\]
we can choose any \( l \geq 0 \).

From these choices \( \sqrt{d} \) has a continued fraction expansion
\[
\sqrt{d} = \left[ m, vA_0, \overline{U, B_{k-1}, vA_1, \overline{U, B_{k-2}, \ldots, vA_{k-1}, \overline{U, B_0, m, B_0, vA_{k-1}, \ldots, B_{k-1}, \overline{U, vA_0, 2m}}}} \right].
\]

After zeros are removed, this is of form (1) or (2) above. The length of the repeating pattern is \( 2k(n + 2) + 2 \) minus twice the number of zeros in the set \{\( A_0, B_0 \)\}.

If \( d \equiv 2 \) or \( 3 \) (mod 4) and is square free, then the fundamental unit of \( \mathbb{Q}(\sqrt{d}) \) is
\[
\frac{\beta^{2k}}{2\alpha^2}
\]
where
\[
\alpha = \frac{-m + \sqrt{d}}{2} \quad \text{and} \quad \beta = w - 2v\bar{\alpha}.
\]

Section 6.

Of course, there is now the problem of constructing a sequence of integers \( a_1, a_2, \ldots, a_n \) for which
\[
\begin{pmatrix}
0 & 1 \\
1 & a_1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & a_2
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
1 & a_n
\end{pmatrix} = 
\begin{pmatrix}
u & v \\
2v - \delta w & w
\end{pmatrix}
\]
with \( \delta = 0 \) or 1. (From this \( uw - xv = (-1)^n = \epsilon \).) There are two (roughly equivalent) approaches to this.

First we can choose any rational number which in reduced form has either an even numerator or denominator.
\[
\frac{w}{v} = [a_1; a_2, \ldots, a_n].
\]

First suppose \( v = 2v' \), and let
\[
[a_1; a_2, \ldots, a_n] = \begin{pmatrix} u & 2v' \\ x & w \end{pmatrix}.
\]
If we then expand the continued fraction $2\frac{w}{v}$

$$2\frac{w}{v} = [b_1; b_2, \ldots, b_n]$$

we can assume

$$\begin{pmatrix} u & v' \\ 2x & w \end{pmatrix} = [b_1; b_2, \ldots, b_m].$$

Then

$$\begin{align*}
\{b_1; b_2, \ldots, b_m, a_n; a_{n-1}, \ldots, a_1\} \\
= \begin{pmatrix} u & v' \\ 2x & w \end{pmatrix} \begin{pmatrix} u & 2v' \\ x & w \end{pmatrix} = \begin{pmatrix} u^2 + 2v'^2 & xu + wv' \\ 2(xu + wv') & 2x + w^2 \end{pmatrix}.
\end{align*}$$

If $v$ is odd, and $w = 2w'$, a similar trick works for $\frac{w}{v}$ and $\frac{w'}{2w'}$.

For an example, start with $\frac{1}{2}$. This leads to the matrix product

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix},$$

and in turn to values set in Proposition 2:

$$\begin{align*}
\epsilon &= 1 \\
r &= 7 + 6l \\
A_i &= \frac{r^i - 1}{3} \\
B_i &= 2A_i \\
m &= A_k + l
\end{align*}$$

and

$$d = m^2 + 2r^k = \left(\frac{(6l + 7)^k + 3l - 1}{3}\right)^2 + 2(6l + 7)^k.$$ 

Then

$$N_t = \left\{ \frac{(6l + 7)^t - 1}{3}, 2, 1, 2 \frac{(6l + 7)^{k-1-t} - 1}{3} \right\}$$

gives

$$\sqrt{\left(\frac{(6l + 7)^k + 3l - 1}{3}\right)^2 + 2(6l + 7)^k} = \left[m, \overline{N_0}, \overline{N_1}, \ldots, \overline{N}_{k-1}, m, \overline{N}_{k-1} \overline{N}_{k-2}, \ldots, \overline{N}_1, 2m\right].$$

After the zeros are removed, this is
\[
\begin{bmatrix}
\frac{(6l + 7)^k + 3l - 1}{3} + 2; 1, \frac{2(6l + 7)^{k-1} - 1}{3}, \overrightarrow{N_1}, \overrightarrow{N_2}, \ldots, \overrightarrow{N_{k-2}}, \\
\frac{(6l + 7)^{k-1} - 1}{3}, 2, \frac{(6l + 7)^{k-2} - 1}{3} + 2, \frac{(6l + 7)^{k-1} - 1}{3}, \\
\overrightarrow{N_{k-2}}, \overrightarrow{N_{k-3}}, \ldots, \overrightarrow{N_2}, \overrightarrow{N_1}, 2\frac{(6l + 7)^{k-2} - 1}{3} + 4
\end{bmatrix}.
\]

Since \( r = 6l + 7 \) is odd, \( d \) is not equivalent to 1 (mod 4); so when \( d \) is square free, we have the fundamental unit of \( \mathbb{Q}(\sqrt{d}) \):

\[
\beta^{2k}
\]

where \( \alpha = -\frac{(6l+7)^k+3l-1}{6} + \frac{1}{2}\sqrt{d} \) and \( \beta = 3 + \frac{(6l+7)^k+3l-1}{6} + \frac{1}{2}\sqrt{d} \).

If we begin with

\[
N = \{a_1; a_2, \ldots, a_n\} = \begin{pmatrix} u & v \\ 2v & w \end{pmatrix},
\]

then all the powers \( N^n \) will have the same matrix form. We can use the Chebychev polynomials to express these powers explicitly.

First we recall how Chebychev polynomials can be used to deal with quadratic units. (See page 355 in [LN].) We begin with the identity

\[
x^k + y^k = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{k-j} \binom{j}{k-j} (-xy)^j (x+y)^{k-2j}.
\]

If we have a quadratic unit \( \beta \) with norm \( \epsilon \),

\[
\beta^k + \overline{\beta}^k = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{k-j} \binom{j}{k-j} (-\epsilon)^j (\beta + \overline{\beta})^{k-2j}.
\]

Therefore if \( \beta = s + t\sqrt{d} \) has norm 1

\[
\beta^k + \overline{\beta}^k = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{k-j} \binom{j}{k-j} (-1)^j (2s)^{k-2j} = 2T_k(s)
\]

where \( T_k(s) \) is the Chebychev polynomial of the first kind of degree \( k \).

If

\[
N = \{a_1; a_2, \ldots, a_n\} = \begin{pmatrix} u & v \\ 2v & w \end{pmatrix}
\]

has determinant one, \( N^n \) can be explicitly given in terms of the rational part of the eigenvalues of \( N \). Let

\[
\beta = \frac{1}{2} \left( u + w + \sqrt{(u + w)^2 - 4} \right)
\]
and
\[
\beta = \frac{1}{2}\left(u + w - \sqrt{(u + w)^2 - 4}\right)
\]
be the eigenvalues, and note that $\beta \bar{\beta} = 1$. Then
\[
N^n = \begin{pmatrix} U & V \\ 2V & W \end{pmatrix}
\]
where
\[
U = \frac{(\beta^{n-2} + \bar{\beta}^{n-2}) - (\beta^n + \bar{\beta}^n) - u(\beta^{n-1} + \bar{\beta}^{n-1}) + u(\beta^{n+1} + \bar{\beta}^{n+1})}{(u + v)^2 - 4}
\]
\[
V = \frac{v(\beta^{n+1} + \bar{\beta}^{n+1}) - v(\beta^{n-1} + \bar{\beta}^{n-1})}{(u + w)^2 - 4}
\]
\[
W = \frac{(\beta^{n+2} + \bar{\beta}^{n+2}) - (\beta^n + \bar{\beta}^n) - u(\beta^{n+1} + \bar{\beta}^{n+1}) + u(\beta^{n-1} + \bar{\beta}^{n-1})}{(u + v)^2 - 4}
\]

So
\[
U = \frac{2T_{n-2}(\frac{u + w}{2}) - 2T_n(\frac{u + w}{2}) - 2uT_{n-1}(\frac{u + w}{2}) + 2uT_{n+1}(\frac{u + w}{2})}{(u + w)^2 - 4}
\]
\[
V = \frac{2v(T_{n+1}(\frac{u + w}{2}) - T_{n-1}(\frac{u + w}{2}))}{(u + w)^2 - 4}
\]
\[
W = \frac{2T_{n+2}(\frac{u + w}{2}) - 2T_n(\frac{u + w}{2}) - 2uT_{n+1}(\frac{u + w}{2}) + 2uT_{n-1}(\frac{u + w}{2})}{(u + w)^2 - 4}
\]

With these two expressions we can use Proposition 2. If we choose $l = 0$, $r$ will simplify just a bit:
\[
r = W^2 - 2V^2 = \frac{4v^2 + T_{2n+2}(\frac{u + w}{2}) - 2uT_{2n}(\frac{u + w}{2}) + (u^2 - 2v^2)T_{2n+1}(\frac{u + w}{2})}{(u + v)^2 - 4}
\]

After this we use the values set in Proposition 2:
\[
A_i = \frac{r^i - 1}{W}
\]
\[
B_i = 2VA_i
\]
\[
m = VA_k
\]
and
\[
d = d(u, v, w, \delta; l, k) = m^2 + 2r^k.
\]
Then the continued fraction expansion of $\sqrt{d}$ is

$$\left[ m, \ vA_0, \overrightarrow{N}, \overrightarrow{N}, \ldots, \overrightarrow{N}, B_{k-1},
\begin{array}{ccccccc}
vA_1, & \overrightarrow{N}, & \overrightarrow{N}, & \ldots, & \overrightarrow{N}, & B_{k-2} \\
\end{array}
\right]
$$

$$\begin{array}{ccccccc}
& \vdots \\
\vdots \\
& \vdots \\
& \vdots \\
\end{array}
$$

$$\begin{array}{ccccccc}
vA_{k-1}, & \overrightarrow{N}, & \overrightarrow{N}, & \ldots, & \overrightarrow{N}, & B_0, \\
\end{array}
$$

$$\left[ m, \ B_0, \overrightarrow{N}, \overrightarrow{N}, \ldots, \overrightarrow{N}, vA_{k-1},
\begin{array}{ccccccc}
& \vdots \\
\vdots \\
& \vdots \\
\vdots \\
\end{array}
\right]
$$

where each $N$ appears $n$ times.

In particular, if we begin with

$$N = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix},$$

then

$$U = \frac{T_{n-2}(2) - T_n(2) - T_{n-1}(2) + T_{n+1}(2)}{6},$$

$$V = \frac{T_{n+1}(2) - T_{n-1}(2)}{6},$$

$$W = \frac{T_{n+2}(2) - T_n(2) - T_{n+1}(2) + T_{n-1}(2)}{6},$$

$$r = \frac{4 + T_{n+2}(2) - 2T_{2n}(2) - 2T_{2n+2}(2)}{12}.$$

After this we use the values set in Proposition 2:

$$A_i = \frac{r^i - 1}{W},$$

$$B_i = 2VA_i,$$

$$m = VA_k$$

and

$$d = d(u, v, w, \delta; l, k) = m^2 + 2r^k.$$

This leads to

$$\sqrt{d} = \left[ A_k; \ A_0, \ 1, 2, 1, 2, \ldots, \ 1, 2, B_{k-1},
\begin{array}{ccccccc}
A_1, & 1, 2, 1, 2, & \ldots, & 1, 2, & B_{k-2}, \\
A_2, & 1, 2, 1, 2, & \ldots, & 1, 2, & B_{k-3},
\end{array}
\right]$$
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\[ A_{k-2}, 1, 2, 1, 2, \ldots, 1, 2, B_1, \]
\[ A_{k-1}, 1, 2, 1, 2, \ldots, 1, 2, B_1, \]
\[ A_k, \]
\[ B_0 \quad 2, 1, 2, \ldots, 2, 1, A_{k-1}, \]
\[ B_1 \quad 2, 1, 2, \ldots, 2, 1, A_{k-2}, \]
\[ B_2 \quad 2, 1, 2, \ldots, 2, 1, A_{k-3}, \]
\[ \ldots \]
\[ B_{k-2} \quad 2, 1, 2, \ldots, 2, 1, A_1, \]
\[ B_{k-1} \quad 2, 1, 2, \ldots, 2, 1, A_0, \]
\[ 2A_k \]

where in each line the (1,2) pair appears \( n \) times. After the zeros are removed the repeating part has length \( 2k(2n + 2) - 2 \).

Section 7.

The next technique for finding sequences to start the process involves factoring \( 2v^1 \pm 1 \). To begin, factor \( 2v^2 + \epsilon = ww_1 \). Write

\[ \frac{w}{v} = [a_1; a_2, \ldots, a_n]. \]

We can arrange it so that \((-1)^n = \epsilon\). Now

\[ \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} u & v \\ x & w \end{pmatrix}. \]

Since \( uw - xv = \epsilon \), and \( ww_1 - 2v^2 = \epsilon \). So \( x \equiv 2v \pmod{w} \). This leads us to Proposition 2.

In practice, this method of producing sequences \( a_1, a_2, \ldots, a_n \) can be used to produce very explicit, yet complicated, families. For example, let us start with the prime 3; if \( v \equiv 2 \pmod{3} \), then \( 2v^2 + 1 \equiv 0 \pmod{3} \). (\( \epsilon = 1 \).) Let \( v = 2 + 3n \). Choose

\[ w = \frac{2v^2 + 1}{3} = 6n^2 + 8n + 3. \]

Then

\[ \frac{w}{v} = \frac{6n^2 + 8n + 3}{3n + 2} = 2n + 1 + \frac{6n^2 + 8n + 3 - 6n^2 - 7n - 2}{3n + 2} \]
\[ = 2n + 1 + \frac{n + 1}{3n + 2} \]
\[ = 2 + \frac{n}{n + 1} \]
\[
\frac{n+1}{n} = 1 + \frac{1}{n}
\]
\[\frac{n}{1} = n.
\]
Thus
\[
\frac{6n^2 + 8n + 3}{3n + 2} = [2n + 1, 2, 1, n].
\]
The associated matrix is
\[
\begin{pmatrix}
u & v \\
2v & w
\end{pmatrix} = \begin{pmatrix}3n-1 & 3n+2 \\
6n+4 & 6n^2+8n+3\end{pmatrix}.
\]
Now we can set things up according to Proposition 2,
\[
r = w^2 - 2v^2 + 2lwv.
\]
Since
\[
\frac{w}{v} = \frac{6n^2 + 8n + 3}{3n + 2} = 2n + 1 + \frac{n+1}{3n+2} > 2,
\]
we can even take \(l = 0\). Returning to Proposition 2,
\[
r = 36n^4 + (96 + 36l)n^3 + (82 + 72l)n^2 + (24 + 50l)n + (1 + 12l)
\]
\[
A_i = \frac{v^i - 1}{w} \\
B_i = 2(3n+2)A_i
\]
\[
m = (3n+2)A_k + (3n+2)l
\]
and
\[
d = m^2 + 2r^k.
\]
Then
\[
\sqrt{d} = \left[ m+2n+1; \quad 2, 1, n, B_{k-1}, \right.
\]
\[
A_1, \quad (2n+1), 2, 1, n, B_{k-2}, \\
A_2, \quad (2n+1), 2, 1, n, B_{k-3}, \\
\ldots \\
A_{k-2}, \quad (2n+1), 2, 1, n, B_1, \\
A_{k-1}, \quad (2n+1), 2, 1, \\
2n+m+1, \\
2, (2n+1), A_{k-1}, \\
B_1, \quad n, 1, 2, (2n+1), A_{k-2}, \\
B_2, \quad n, 1, 2, (2n+1), A_{k-3}, \\
\ldots \\
B_{k-2}, \quad n, 1, 2, (2n+1), A_1,
\]
Section 8.

The final method of constructing continued fractions allows us to produce very intricate expansions by embedding the ones constructed above inside recurring patterns within the repeating quotients of larger fractions. As we have seen, the key to such embeddings is finding finite continued fractions \([a_1, a_2, \ldots, a_n]\) that lead to fractions of the form

\[
\begin{pmatrix}
    u & v \\
    2v - \delta w & w
\end{pmatrix}
\]

with \(\delta = 0\) or \(1\). As it turns out, the first half of all the repeating patterns above will do this.

Let us return to the calculation (and the notation) of Section 1. We started with a family of matrices

\[
N_i = \begin{pmatrix}
    p & 2br^i \\
    br^{k-1-i} & q
\end{pmatrix}.
\]

Without going into the details of the calculation, a return to original identity shows that

\[(N_{k-1} \ldots N_1 N_0)^T\]

has the form

\[
\begin{pmatrix}
    u & v \\
    2v & w
\end{pmatrix}.
\]

Since the \(N_i\) are products of matrices from continued fractions, we can embed the associated sequence of partial quotients inside a complementary pair \(A_i, B_i\) to recur any number of times within the repeating pattern.

Returning to our first example:

\[
\sqrt{(b(2bl + 1)^k + l)^2 + 2(2bl + 1)^k}
\]

\[
= \left[ b(2bl + 1)^k + l; \\
    b, 2b(2bl + 1)^k-1, b(2bl + 1)^k-2, b(2bl + 1)^2, \ldots, \\
    b(2bl + 1)^k-2, 2b(2bl + 1)^1, b(2bl + 1)^k-1, 2b, \\
    b(2bl + 1)^k + l, \\
    2b, b(2bl + 1)^k-1, 2b(2bl + 1), b(2bl + 1)^k-2, 2b(2bl + 1)^2, \ldots, \\
    2b(2bl + 1)^k-2, b(2bl + 1)^1, 2b(2bl + 1)^k-1, b, 2b(2bl + 1)^k + 2l \right].
\]
If we take \( b = 1 \) and \( l = 5 \), (and \( k = n \)), we get
\[
\sqrt{11^{2n} + 12 \cdot 11^n + 25}
= \left(11^n + 5; 1, 2 \cdot 11^n - 1, 11, 2 \cdot 11^n - 2, 11^2, 2 \cdot 11^n - 3, \ldots, 11^n - 2, 2 \cdot 11, 11^n - 1, 2, 11^n + 5, 2, 11^n - 1, 2 \cdot 11^n - 2, 2 \cdot 11^2, \ldots, 2 \cdot 11^n - 2, 11, 2 \cdot 11^n - 1, 1, 2 \cdot 11^n + 10\right).
\]
And in the notation above
\[
\beta = 6 + 11^n + \sqrt{11^{2n} + 12 \cdot 11^n + 25}.
\]
Then according to the calculation above
\[
\{1, 2 \cdot 11^n - 1, 11, 2 \cdot 11^n - 2, 11^2, 2 \cdot 11^n - 3, \ldots, 11^n - 2, 2 \cdot 11, 11^n - 1, 2\}
= \left(\frac{s - mt}{t} \frac{2t}{(11)^{-n}(s + mt)}\right) = U_n
\]
where \( \beta^n = s + t\sqrt{11^{2n} + 12 \cdot 11^n + 25} \). If we now reset our notation and use Proposition 2 and set \( l = 0 \), we define
\[
\epsilon = 1
\]
\[
r = w^2 - 2v^2
\]
\[
= \frac{(s^2 + 2(11^n + 5)st + (25 + 10 \cdot 11^n - 11^{2n})t^2)/11^{2n}}{11^n - 1}
\]
\[
A_i = \frac{r^i}{w}
\]
\[
B_i = 2vA_i + \delta
\]
\[
m_2 = vA_k + \epsilon l
\]
and
\[
d = m_2^2 + 2r^k = v^2A_k^2 + 2(\epsilon lv + w)A_k + 2.
\]
Then
\[
\sqrt{d} = \left[a_k; a_0, U_n, B_{k-1}, A_1, U_n, B_{k-2}, \ldots, A_{k-1}, U_n, B_0, A_k, B_0, U_n, A_{k-1}, B_0, U_n, A_{k-2}, \ldots, B_0, U_n, A_{k-1}, 2A_k\right].
\]
After removing the zeros, the repeating part of this expansion has length \( 2k(2n + 2) - 2 \).
We conclude with a few specific examples that can be constructed using our techniques. Let

\[ r = 2473892093033277097181734801 \]

\[ A'_t = 863109604528081677152276000 \sum_{i=0}^{t-1} r^i \]

\[ d = A'_k^2 + 2r^k. \]

Let \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \); \( V = \{5, 18, 4, 14, 3, 10, 2, 6\} \). Then

\[ \sqrt{d} = \left[ A'_k; A'_0, \overline{U}, \overline{V}, 2A'_{k-1}, A'_1, \overline{U}, \overline{V}, 2A'_{k-2}, \ldots, A'_k, \overline{U}, \overline{V}, 2A'_0, \right. \]

\[ A'_k, 2A'_0, \overline{U}, \overline{V}, A'_{k-1}, 2A'_1, \overline{U}, \overline{V}, A'_k, \overline{U}, A'_k, 2A'_k \].

Next consider

\[ U = \{1, 10, 1, 4, 1, \} = \begin{pmatrix} 54 & 65 \\ 2 \cdot 65 - 71 & 71 \end{pmatrix}. \]

This is an example of a matrix with, in the notation we have been using, \( \delta = 1 \). Using Proposition 2,

\[ \epsilon = -1 \]

\[ r = 9230l + 3409 \]

\[ A_i = (130l + 48) \sum_{j=0}^{i-1} (9230l + 3409)^j \]

\[ B_i = 130A_i + 1 \]

\[ m = 130A_k - l \]

and

\[ d = (71)^{-2}(16900r^{2k} - (9230l + 23718)r^k + (5041l^2 + 9230l + 16900)). \]

Then

\[ \sqrt{d} = \left[ m, vA_0, \overline{U}, B_{k-1}, vA_1, \overline{U}, B_{k-2}, \ldots, vA_{k-1}, \overline{U}, B_0, \right. \]

\[ m, B_0, \overline{U}, vA_{k-1}, \ldots, B_{k-1}, \overline{U}, vA_0, 2m \].

Next consider our very first family:

\[ \sqrt{(b(2bn + 1)^k + n)^2 + 2(2bn + 1)^k} \]

\[ = [b, (2bn + 1)^k + n; \]
If we take \( n = b \), we get

\[
\sqrt{b^2((2b^2 + 1)^k + 1) + 2(2b^2 + 1)^k}
= \sqrt{b^2(2b^2 + 1)^{2k} + 2(b^2 + 1)(2b^2 + 1)^k + 1}
= [b(2b^2 + 1)^k + b;
\]

\[
b, 2b(2b^2 + 1)^{k-1}, b(2b^2 + 1), 2b(2b^2 + 1)^{k-2}, b(2b^2 + 1)^2, \ldots,
b(2b^2 + 1)^{k-2}, 2b(2b^2 + 1)^{k-1}, 2b,
b(2b^2 + 1)^{k-1}, 2b,
b, 2b(2b^2 + 1)^{k-1}, 2b(2b^2 + 1), b(2b^2 + 1)^{k-2}, 2b(2b^2 + 1)^2, \ldots,
b(2b^2 + 1)^{k-2}, b(2b^2 + 1)^{k-1}, 1, 2b(2b^2 + 1)^{k-1}, 2b^2, 2b^2(2b^2 + 1)^{k-2}, (2b^2 + 1)^2, \ldots,
\]

We can multiply this by \( b \) to get a new expansion:

\[
\sqrt{b^4(2b^2 + 1)^{2k} + 2b^2(b^2 + 1)(2b^2 + 1)^k + b^2}
= [b^2(2b^2 + 1)^k + b^2;
\]

\[
1, 2b^2(2b^2 + 1)^{k-1}, (2b^2 + 1), 2b^2(2b^2 + 1)^{k-2}, (2b^2 + 1)^2, \ldots,
(2b^2 + 1)^{k-2}, 2b^2(2b^2 + 1)^{k-1}, 2b^2,
(2b^2 + 1)^{k-1}, 1, 2b^2, 2b^2(2b^2 + 1)^{k-2}, 2b^2(2b^2 + 1)^2, \ldots,
2b^2(2b^2 + 1)^{k-2}, (2b^2 + 1)^{k-1}, 2b^2(2b^2 + 1)^{k-1}, 1, 2b^2(2b^2 + 1)^k + b^2].
\]

References


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NOTES ON TANGLES, 2-HANDLE ADDITIONS AND
EXCEPTIONAL DEHN FILLINGS

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This paper is a set of notes concerning the following related
topics in 3-manifold topology: \( n \)-strand tangles, handle addi-
tions, the cabling conjecture, and exceptional Dehn fillings.

1. Introduction.

In this paper we consider several related problems in 3-manifold topology. Here
is a brief description of the content and organization of the paper. Definitions of terms which occur can be found in relevant sections.

Classical tangles, i.e., tangles in a 3-ball, have been heavily studied. Here
we consider tangles in a general compact 3-manifold with boundary a 2-
sphere. In Section 2 we give a complete classification of the set of irreducible,
non-split, completely tubing compressible tangles with two strands. To give
a complete description of such tangles with more than two strands in terms
of the mutual positions of the strands seems to be a difficult problem. A
conjecture is raised, which suggests possible pictures (classification) of such
tangles. Our study indicates that certain conditions posed on the comple-
ment of a tangle determines the tangle itself, up to topological equivalence.

The ambient space of a tangle can be considered as a union of the exterior
of the tangle and several 2-handles. Therefore, problems about tangles are
closely connected to problems about 2-handle additions, in particular to
problems about Dehn fillings. Through such a connection, we produce, in
Section 3, classes of hyperbolic knots in solid torus such that their exteriors
admit Dehn filling along their outside boundary torus producing manifolds
which are solid torus or Seifert fibered spaces. In the same section, we also
show that if a surgery on a hyperbolic knot in a solid torus produces a
Seifert fibered space, then the surgery slope, with respect to the standard
meridian-longitude coordinates of the knot, must be an integer slope.

One of our motivations of studying general tangles here is our concern
about the cabling conjecture. Recall that the cabling conjecture asserts that
if a knot in the 3-sphere is not a cabled knot (which includes the trivial knot),
then it does not admit any surgery resulting a reducible manifold \([GS]\). This
conjecture has been proven true for satellite knots \([Sch]\), strongly invertible
knots \([E]\), alternating knots \([MT]\), arborescent knots \([W]\), symmetric knots
\([HS]\), \([GL4]\) and genus one knots \([BZ]\). It is also known that if a surgery
on a non-trivial knot in $S^3$ produces a reducible manifold, then the surgery slope is not 0 [Ga], in fact it must be an integer [GL2] with absolute value larger than one [GL3], and the reducible manifold contains a lens space summand [GL3]. In Section 4, we look at the conjecture from the point of view of tangle sums. Namely, if some surgery on a knot in $S^3$ produces a reducible manifold, then a reducing 2-sphere will decompose the resulting manifold into a union of two tangles whose strands come from the core of the filling solid torus. We give some partial results along this line. Notably, we show that such tangle decomposition cannot give summands which are 2-strand tangles.

Our investigation of the cabling conjecture is extended to Section 5 but restricted to the class of knots in $S^3$ whose exteriors do not contain meridionally-incompressible closed essential surfaces. Note that this class of knots include all alternating knots [M], almost alternating knots, toroidally alternating knots and Montesinos knots [A]. We show that if such knot exterior admits a filling yielding a reducible manifold, then the reducible manifold must be a connected sum of two lens spaces. We also show that such knot exterior does not admit any Dehn filling producing a large Seifert fibered space. It is a conjecture that every knot in $S^3$ does not admit a surgery yielding a large Seifert fibered 3-manifold. This conjecture can be considered as an extension of the cabling conjecture, in some sense.

Studying exceptional Dehn surgery (filling) on hyperbolic knot (exterior), i.e., surgery (filling) which produces non-hyperbolic manifolds, is a basic subject in 3-manifold topology. Most of the problems and results described above belong to this subject. Given a knot $K$ in a compact orientable 3-manifolds $W$ with non-empty boundary such that the exterior of $K$ in $W$ is a simple manifold, an exceptional surgery on $W$ along $K$ will produce a manifold which contains either a reducing 2-sphere, or essential annulus, or $\partial$-reducing disk or essential torus. Sharp upper bounds on the distances (i.e., the geometric intersection numbers) between all these types of exceptional surgery slopes have been found. (See [GW, Introduction] for a table summary of these bounds.) But if one singles out Seifert Dehn surgery–surgery which yields a Seifert fibered space–as a special type of exceptional surgery, then it is still unknown what are the optimal upper bounds on the distances between a Seifert Dehn surgery slope and other types of exceptional surgery slopes. There are 5 bounds to be determined. In Section 6, the last section of the paper, we resolve this issue in four of the total five cases. We show that if one surgery on $W$ along $K$ produces a Seifert surgery and another surgery on $W$ along $K$ produces a reducible or $\partial$-reducible manifold, then the distance between the two surgery slopes is at most one. Previous known bounds in both cases were 2, obtained in [W2] and [GW] respectively. The new bound one is optimal as it can be realized by infinitely many examples found in [EW, Section 2]. We also show that if one surgery on $W$ along
Let $W$ be a connected compact orientable 3-manifold whose boundary is a 2-sphere. A tangle of $k$ strands in $W$ is a set of $k \geq 1$ mutually disjoint simple arcs properly embedded in $W$. We shall use $(W; \alpha_1, \alpha_2, \ldots, \alpha_k)$ to denote a tangle in $W$ with $k$ strands $\alpha_i, 1 \leq i \leq k$. Two tangles $(W; \alpha_1, \ldots, \alpha_k)$ and $(W'; \alpha'_1, \ldots, \alpha'_k)$ are considered the same (topologically equivalent) if there is a homeomorphism $h : W \to W'$ such that $h(\alpha_i) = \alpha'_i$ for each of $1 \leq i \leq k$. For a tangle $(W; \alpha_1, \ldots, \alpha_k)$, we shall always let $H_i$ denote a regular neighborhood of $\alpha_i$ in $W$ such that $H_1, \ldots, H_k$ are mutually disjoint, $A_i = \partial H_i - \partial W$, $c_i$ the center circle of $A_i$, $P = \partial W - (H_1 \cup \ldots \cup H_k)$ and $X = W - (H_1 \cup \ldots \cup H_k)$. Naturally $H_1, \ldots, H_k$ can be considered as 2-handles, which when attached to $X$ along $A_i$'s give the manifold $W$. A tangle $(W; \alpha_1, \alpha_2, \ldots, \alpha_k)$ is called reducible if and only if $X$ is a reducible 3-manifold, called split if and only if $P$ is compressible in $X$, called toroidal if and only if $X$ contains an incompressible torus and called annular if and only if $X$ contains a properly embedded annulus whose boundary lies in $P$ but is not isotopic in $(X, P)$ to any of $A_i$. Obviously the study of reducible or split tangles can be reduced to that of irreducible and non-split ones.

Let $(W; \alpha_1, \ldots, \alpha_k)$ be an irreducible non-split tangle. Then by definition, $P$ is an incompressible planar surface in $X$ which is irreducible. Let $F_i$ be the surface $P \cup A_i$, $i = 1, \ldots, k$. We call $F_i$ the surface obtained from $P$ by tubing along $A_i$. Then $F_i$ is a punctured torus (when $k > 1$) with two less boundary components than $P$ or a closed torus (when $k = 1$) embedded in $X$, but $F_i$ may or may not be incompressible in $X$. If $F_i$ is incompressible, we call the surface $P - A_i$-tubing incompressible. If $F_i$ is compressible in $X$, we perform compressing operations on $F_i$ as much as possible. Then eventually we either get an incompressible surface embedded in $X$ which has less boundary components than $P$ but is not isotopic to any of $A_j$ or every component of the surface resulting from compressing $F_i$ is isotopic to some of the annuli $A_j$. In the latter case, we call $P$ completely $A_i$-tubing compressible. If $P$ is completely $A_i$-tubing compressible for each of $i = 1, \ldots, k$, then we call such tangle completely tubing compressible. The
following lemma gives a characterization of the ambient spaces of irreducible non-split tangles which are completely tubing compressible.

**Lemma 1.** If \((W; \alpha_1, \ldots, \alpha_k)\) is an irreducible non-split completely tubing compressible tangle, then either \(W\) is a punctured lens space or \(W\) is a 3-ball, \(k = 1\) and \(\alpha_1\) is a trivial strand in \(W\).

A lens space in this paper is always assumed to be non-trivial, i.e., it is neither \(S^3\) nor \(S^2 \times S^1\). We use the standard notation \(L(p, q)\) to denote the lens space whose Heegaard diagram is a \((q, p)\) curve on the boundary of a solid torus \(V = D^2 \times S^1\) where \(q\) is the meridian coordinate of the curve and \(p\) is the longitude coordinate with respect to a fixed longitude of \(V\). So under this convention, we have \(\pi_1(L(p, q)) = \mathbb{Z}_{|p|}\) with \(1 < |p| < \infty\) and with \((p, q) = 1\). By a punctured lens space \(L^0(p, q)\), we mean the complement of an open 3-ball in \(L(p, q)\). It is well known that \(L^0(p, q)\) is well defined, i.e., is independent of the choice of the 3-ball.

The proof of Lemma 1 is essentially contained in [CGLS]. For the convenience of the reader, we give a sketch of proof in our current setting. We also need some more properties about such tangles (Corollary 2 below) obtained during the proof.

**Proof.** Note that \(P\) is incompressible, \(X\) is irreducible, \(\partial X\) has genus \(k\) and \(F_i\) is a punctured torus with \(2(k-1)\) boundary components. Consider the maximal compression body \(X_i\) in \(X\) based on the surface \(F_i\); i.e., \(X_i\) is a regular neighborhood in \(X\) of the union of \(F_i\) and all possible compressing disks for \(F_i\) in \(X\), with all possible 2-sphere boundary components capped off by 3-balls. The frontier of \(X_i\), i.e., \(\partial X_i - F_i\), is a set (possibly empty) of disjoint embedded surfaces each of which must be incompressible in \(X\). Since \(F_i\) is completely compressible, each component of \(\partial X_i - F_i\) is isotopic to one of the annuli \(A_j\). Therefore \(X_i\) is really the \(X\) minus some regular neighborhood of all \(A_j\), \(j \neq i\), in \(X\). Hence in particular \(X\) is a handlebody of genus \(k\).

If \(k = 1\), then \(X = X_1\) is a solid torus and \(W\) is the union of \(X\) and the 2-handle \(H_1\). Hence \(W\) is a lens space unless \(P\) is isotopic to \(A_1\) in which case \(W\) is a 3-ball and \(\alpha_1\) is a trivial (boundary parallel) arc in \(W\). Hence we may assume that \(k > 1\).

The same argument as that of [CGLS, Lemma 2.1.2] shows that there exist mutually disjoint (properly embedded) disks \(E^j_i\) in \(X\), \(j \neq i\), such that \(E^j_i\) meets \(c_j\), the center circle of \(A_j\), transversely in a single point and is disjoint from \(c_m\) if \(m \neq i\) or \(j\). We remark that \(E^j_i\) must intersect \(c_i\) as otherwise it would imply that \(P\) be compressible. As \(W\) is obtained from \(X\) by attaching 2-handles \(H_j\) along \(A_j\), the existence of the disks \(E^j_i\) implies that attaching any collection of 2-handles \(H_j\) to \(X\) along \(A_j\), \(j \neq i\), always produces a handlebody. In particular, we have:
Corollary 2. Let \((W; \alpha_1, \ldots, \alpha_k)\) be an irreducible non-split tangle which is completely tubing compressible. Then \(W - H_i\) is a solid torus for each of \(i\).

We continue the proof of Lemma 1. We know now that for any \(i\) and \(j\), \(W - (H_i \cup H_j)\) is a handlebody of genus two. Also as in \([CGLS, \text{Lemma } 2.1.4]\) one can show that the planar surface \(\partial W - (H_i \cup H_j)\) (with four boundary components) is incompressible in \(W - (H_i \cup H_j)\). Now one is ready to apply \([CGLS, \text{Lemma } 2.3.2]\) to see that \(W\) must be a punctured lens space.

From now on \((L^0(p, q'); \alpha_1, \ldots, \alpha_k)\) denotes an irreducible non-split completely tubing compressible tangle in a punctured lens space \(L^0(p, q')\). We may assume that \(p\) be positive. It is well known that two lens spaces \(L(p, q)\) and \(L^0(p, q')\) are homeomorphic if and only if either \(q \equiv \pm q' \mod (p)\) or \(qq' \equiv \pm 1 \mod (p)\). We may assume that \(k \geq 2\) (\(k = 1\) case is completely determined). By Corollary 2, we may assume, for any fixed \(i\) of \(1, \ldots, k\), that \(L^0(p, q') = V \cup a_i H_i\) whose attaching annulus \(A_i\) is glued with a regular neighborhood of a \((q, p)\) curve in \(\partial V\), with respect to a fixed meridian-longitude basis on \(\partial V\).

Also within the topological equivalence class of the tangle, we may actually assume that \(1 \leq q < p/2\). This is because that there is a homeomorphism of the solid torus \(V = D^2 \times S^1\) and the 2-handle \(H_i\) whose center circle \(c_i\) of \(A_i\) is a \((q, p)\) curve on \(\partial V\) with respect to some fixed meridian-longitude basis whose orientations are chosen in such way that they are consistent with the right-hand rule (i.e., if you curve your right hand along the orientation of meridian, then your thumb should point to the orientation of the longitude). Note that a \((q, p)\) curve can be placed on \(\partial V\) as indicated in Figure 1 (the curve \(c_i\) in the figure).

Note that \(X = V - \bigcup_{j \neq i} H_j\) and \(F_i = \partial V - \bigcup_{j \neq i} H_j\). As \(F_i\) is completely compressible in \(X\), each arc \(\alpha_j, j \neq i\), is boundary parallel in \(V\). For each fixed \(j \neq i\), we may arrange \(\alpha_j\), by isotopy, to be properly embedded in a disk fiber \(D\) of \(V\). The boundary of \(\alpha_j\) separates \(\partial D\) into two arcs \(\delta\) and \(\delta'\). Then the number of components of \(\delta \cap c_i\) and the number of components of \(\delta' \cap c_i\) add to \(p\). Let \(b_j\) be the minimal of the two numbers. Then \(0 \leq b_j < p\).

We call \(b_j\) the bridge width of the arc \(\alpha_j\) with respect to the arc \(\alpha_i\) (it is obviously well defined by its canonical constructional definition).

Proposition 3. Under the setting as established above, we have that for each fixed \(i\), the bridge width \(b_j = 1\) or \(q\) for each \(j \neq i\).
Figure 1. The disks \( \{D_x, D_y\} \) of the genus two handlebody dual to a free basis \( \{x, y\} \).

**Proof.** Consider the manifold \( Y = V - H_j \) and let \( A'_i = \partial V - A_i \) and \( Q = A'_i - H_j \). From the proof of Lemma 1, we knew that \( Q \) is incompressible in \( Y \). It follows that the bridge width \( b_{ij} \) must be positive. Note that \( Y \) is a handlebody of genus two as shown by Figure 1. Choose two disks \( D_x \) and \( D_y \) as in Figure 1 which cuts \( Y \) into a 3-ball. So there is a free basis \( \{x, y\} \) for the fundamental group of \( Y \) dual to the disk system \( \{D_x, D_y\} \), i.e., \( x \) has a representing loop intersecting \( D_x \) transversely exactly once and \( y \) has a representing loop intersecting \( D_y \) transversely exactly once. Now attaching the 2-handle \( H_i \) to \( Y \) along \( c_i \) is a solid torus by Corollary 2. This happens if and only if the word represented by the loop \( c_i \) in \( x \) and \( y \) is a free basis element of the free group. Now we are in a setting to apply [D, Theorem 3.2.2] to see that the latter happens if and only if \( b_{ij} = 1 \) or \( q \). The proposition is proved. \( \square \)

Proposition 3 gives a complete classification of all irreducible non-split completely tubing compressible tangles with two strands. We also have:

**Corollary 4.** If \( (L^0(p,1); \alpha_1, \ldots, \alpha_k) \) is an irreducible non-split completely tubing compressible tangle, then any two strands \( \alpha_i \) and \( \alpha_j \) from \( \{\alpha_1, \ldots, \alpha_k\} \) are parallel in \( L^0(p,1) \).

**Proof.** Since \( q' = 1 \), we see that for a fixed \( i \), the attaching curve of the 2-handle \( H_i \) to \( V \) can be assumed to be the \((1,p)\) curve by the classification of lens spaces (go back to the discussion after Corollary 2). Now according to Proposition 3, each other stand \( \alpha_j, j \neq i \), has bridge width (with respect
to \( \alpha_i \) \( b^i_j = 1 \). Also \( \alpha_j \) is boundary parallel in \( V \) for each \( j \neq i \). It follows obviously that \( \alpha_j \) is parallel to \( \alpha_i \) in their embient space \( L(p, 1) \).

When there are more than two strands in such tangle, their mutual positions are much more complicated to analyse. We do not even know if \( \alpha_j \) and \( \alpha_m \) are parallel in \( V - A_i \) when \( b^i_j = b^i_m \). The following conjecture suggests a possible picture of strands in such a tangle.

**Conjecture 5.** Let \((L^0(p, q'); \alpha_1, \ldots, \alpha_k)\) be an irreducible non-split completely tubing compressible tangle. Then for each fixed \( i \), all strands \( \alpha_j \), \( j \neq i \), can be arranged by isotopy in \( V - A_i \) to form at most two parallel families as shown in Figure 2, one family with bridge width 1 and the other family with bridge width \( q \).

\[ V = D^2 \times S^1 \]

\[ \text{parallel arcs of bridge width } q \]

\[ \text{parallel arcs of bridge width } 1 \]

**Figure 2.** A conjectural picture of an irreducible non-split completely tubing compressible tangle.

### 3. Knots in solid torus.

We now give another interesting consequence of Proposition 3, which concerns 1-bridge braid knots in a solid torus. Recall that a non-trivial knot \( K \) in a solid torus \( V = D^2 \times S^1 \) is called a 0-bridge braid knot (also called torus knot) if it can be isotoped into \( \partial V \), and is called a 1-bridge braid knot if it is not a 0-bridge braid knot but can be isotoped as a union of two arcs \( \alpha \) and \( \beta \) such that \( \alpha \) lies in \( \partial V \) intersecting each disk fiber of \( V = D^2 \times S^1 \) transversely and \( \beta \) is properly embedded in a disk fiber of \( V \). In [Ga2], the class of 1-bridge braid knots are classified (up to orientation preserving
homeomorphisms) by the triple \((\omega, b, t)\), where \(\omega\) is the winding number of \(K\), \(b\) is the bridge width of \(K\) \((1 \leq b \leq \omega - 2)\) and \(t\) is the twist number of \(K\) \((1 \leq t \leq \omega - 2)\). Figure 3 explains how a 1-bridge braid knot with winding number 9, bridge width 2 and twist number 5 is defined (constructed). This example should suffice for seeing how the general triple \((\omega, b, t)\) is defined. We note that the convention of bridge width used here is different from that used [Ga2] by taking the mirror image; that is, the bridge width \(b\) used here is the number \(\omega - b - 1\) in the convention of [Ga2]. Knots in a solid torus which admit a non-trivial surgery yielding a solid torus are classified in [B]. Such a knot must be either a 0-bridge braid knot or a 1-bridge braid knot. However it seems not so easy to see that whether the exterior of a hyperbolic 1-bridge braid knot \(K\) in \(V\) admits a Dehn filling along \(\partial V\) yielding a solid torus. We shall give a class of such knots.

![Diagram](image)

**Figure 3.** 1-bridge braid knot with \(\omega = 9\), \(b = 2\) and \(t = 5\).

Let \(V = D^2 \times S^1\), \(K \subset V\) a non-trivial knot (i.e., not contained in a 3-ball in \(V\)), \(M\) the exterior of \(K\) in \(V\), \(T_0 = \partial V\), \(T_1 = \partial M - T_0\) (we call \(T_0\) the outer boundary torus of \(M\)). Fix a basis \(\{\mu_0, \lambda_0\}\) for \(T_0\) where \(\mu_0\) is the meridian slope of \(V\). Embed \(V\) in \(S^3\) as a trivial solid torus so that \(\lambda_0\) bounds a disk in \(S^3\). Let \(\{\mu_1, \lambda_1\} \subset T_1\) be the standard meridian-longitude basis of \(K\) when considered as a knot in \(S^3\). Slopes on \(T_1\) and on \(T_0\) will be parameterized with respect to these bases respectively. For a loop \(\gamma\) in a manifold, we use \([\gamma]\) to denote its homology class in the first homology group of the manifold. Again the orientations of \([\mu_0]\) and \([\lambda_0]\) in the outer boundary torus \(T_0\) follow the right hand rule. We orient \([\mu_1]\) and \([\lambda_1]\) such
that \([\lambda_1] = \omega[\lambda_0]\) and \([\mu_0] = \omega[\mu_1]\) with \(\omega \geq 0\) being the winding number of \(K\) in \(V\).

**Proposition 6.** Let \(K\) be a 1-bridge braid knot in \(V\) defined by the triple \((\omega, b, t)\) and let \(M\) be its exterior. If \(b < \omega/2\), \(t = \omega-b-1\) and \((\omega+1, b+1) = 1\), then Dehn filling \(M\) along \(\partial V\) with the slope \(\pm((b+1)[\mu_0] + (\omega+1)[\lambda_0])\) yields a solid torus and the slope \(\pm(\omega^2(b+1)[\mu_1] + (\omega+1)[\lambda_1])\) on \(T_1\) becomes the meridian slope of the resulting solid torus.

**Proof.** A 1-bridge braid knot \(K\) satisfying the conditions of the proposition has a representative as shown in Figure 4 (a). Now consider the \((b+1, \omega+1)\) curve \(c\) on \(\partial V\) and let \(\alpha\) be a boundary parallel simple arc in \(V\) with bridge width \(b+1\) as shown in Figure 4 (b). Let \(Y = V - N(\alpha)\). By Proposition 3, attaching a 2-handle to \(Y\) along the curve \(c\) results a solid torus. On the other hand, one can slide the two end points of \(\alpha\) along \(\partial V - c\) in the way as indicated in Figure 4 (b). More precisely let \(\partial V - c\) have the induced foliation from the disk foliation of \(V\) (we may assume that \(c\) is transverse to the disk foliation of \(V\)), the sliding of \(\alpha\) along \(\partial V - c\) is always transverse to the interval leaves of the foliation. When the two end points eventually meet at the same interval leave for the first time, connect them along the interval. The resulting knot is the 1-bridge braid knot \(K\) in \(V\) (after pushed into the interior of \(V\)) with the triple \((\omega, b, \omega - b - 1)\). Now one only needs to observe that the manifold obtained by attaching a 2-handle to \(Y\) along \(c\) is the same as that obtained by Dehn filling \(M\) along \(T_0\) with the slope represented by \(c\). To see the meridian slope of the resulting solid torus, note that \(H_1(M; \mathbb{Z})\) is a free rank two abelian group generated by \([\mu_1]\) and \([\lambda_0]\), and \([\lambda_1] = \omega[\lambda_0]\), \([\mu_0] = \omega[\mu_1]\). The Dehn filling along \(T_0\) gives the homology relation \((b+1)[\mu_0] + (\omega+1)[\lambda_0] = (b+1)[\omega][\mu_1] + (\omega+1)[\lambda_0] = 0\). Hence a slope \(\pm(m[\mu_1] + n[\lambda_1]) = \pm(m[\mu_1] + n\omega[\lambda_0])\) on \(T_1\) can be the meridian slope of the resulting solid torus only if \(m = \omega^2(b+1)\) and \(n = \omega + 1\).

We remark that a 1-bridge braid knot \(K\) in a solid torus is not a 0-bridge braid knot and hence the exterior of \(K\) is not Seifert fibered. Also if the winding number \(\omega\) of a 1-bridge braid knot \(K\) is a prime number, then the exterior of \(K\) is hyperbolic [GW2, Corollary 7.4]. In fact applying results of [MS] or [E] one can show that a 1-bridge braid knot \(K\) in a solid torus \(V\) has its exterior containing essential torus if and only if \(K\) is an \((1, m)\) cable of a \((g, p)\) torus knot in \(V\) (with \(|m| > 1\) and \(|p| > 1\)).

**Problem 7.** Classify 1-bridge braid knots in a solid torus whose exterior admits Dehn filling along the outer boundary torus \(T_0\) yielding solid torus.

One may further consider which Dehn filling of \(M\) along the outer boundary torus \(T_0\) gives Seifert fibered space different from solid torus. Our next proposition provides a class of such knots which again include infinitely many hyperbolic knots.
Proposition 8. Let $K$ be a 1-bridge braid knot in $V$ defined by the triple $(\omega, b, t)$ and let $M$ be its exterior. If for some integer $m > 1$ and $n \geq 1$, $b = mn$, $t = \omega - 1 - n$ and $(\omega + m, n + 1) = 1$, then Dehn filling $M$ along $\partial V$ with the slope $\pm((n + 1)[\mu_0] + (\omega + m)[\lambda_0])$ yields a Seifert fibered space whose base orbifold is a disk with two singular points.

Proof. The argument is similar to that of Proposition 6 but applying another result of [D]. A 1-bridge braid knot $K$ satisfying the conditions of the proposition has a representative as shown in Figure 5 (a). Now consider the $(n + 1, \omega + m)$ curve $c$ on $\partial V$ and let $\alpha$ be a boundary parallel simple arc in $V$ with bridge width $n(m + 1)$ as shown in Figure 5 (b). Let $Y = V - N(\alpha)$. By [D, Proposition 3.3.1], attaching a 2-handle to $Y$ along the curve $c$ results a Seifert fibered space whose base orbifold is a disk with two singular points. On the other hand, one can slide the two end points of $\alpha$ along $\partial V - c$ (in directions as indicated in Figure 5 (b)) to get a 1-bridge braid knot $K$ in $V$ such that $K$ is of the given triple $(\omega, mn, \omega - n - 1)$. Again the manifold obtained by attaching a 2-handle to $Y$ along $c$ is the same as that obtained by Dehn filling $M$ along the outer boundary torus $T_0$ with the slope represented by $c$. \qed
Figure 5. Slide the arc $\alpha$ in part (b) to obtain the 1-bridge braid knot in part (a).

Here is perhaps a convenient place to make a note on Dehn surgery on a hyperbolic knot in a solid torus resulting a Seifert fibered space.

**Proposition 9.** Let $M$ be the exterior of a hyperbolic knot $K$ in a solid torus $V$. If some Dehn filling on $M$ along the inner boundary torus $T_1$ with slope $\beta$ produces a Seifert fibered space, then $\beta$ is an integer slope.

**Proof.** Let $U = M(T_1, \beta)$ denote the manifold obtained by Dehn filling $M$ along the inner boundary torus $T_1$ with the slope $\beta$. Let $F$ be the base orbifold of a Seifert fibration of $U$. Then $F$ has a single boundary component. We may assume that $F$ is not a disk with at most one cone point since otherwise $U$ is a solid torus in which case $\beta$ is known to be an integer slope.

Suppose that $F$ is a disk with exactly two cone points. Let $\sigma$ be the slope in $T_0 = \partial U$ represented by a fiber in $T_0$ of the Seifert fibration of $U$. One can now choose a slope $\delta$ in $T_0$ satisfying the conditions: (1) $\Delta(\delta, \sigma) = 1$ (here and later $\Delta$ is used to denote the distance, i.e., the geometric intersection number, of two slopes); (2) The manifold $N = M(T_0, \delta)$, obtained by Dehn filling $M$ along its outer boundary torus $T_0$ with the slope $\delta$, is a hyperbolic manifold. Such $\delta$ exists because of the well known hyperbolic surgery theorem of Thurston [T] which assures that except for finitely many slopes on $T_0$, all remaining slopes on $T_0$ will produce hyperbolic 3-manifolds, and because of the fact that there are infinitely many slopes on $T_0$ which
are distance one from a given slope. The manifold $N(\mu)$, obtained by Dehn filling $N$ along $\partial N = T_1$ with the meridian slope $\mu$, is the manifold $V(\delta)$, obtained by Dehn filling the solid torus $V$ along $\partial V = T_0$ with the slope $\delta$, and thus is a lens space or $S^3$ or $S^2 \times S^1$. And the manifold $N(\beta)$, obtained by Dehn filling $N$ along $\partial N = T_1$ with the slope $\beta$, is the manifold $U(\delta)$, obtained by Dehn filling $U$ along $\partial U = T_0$ with the slope $\delta$, is a Seifert fibered space with a Seifert fibration whose base orbifold is a 2-sphere with exactly two cone points since $\Delta(\sigma, \delta) = 1$, and thus is a lens space or $S^3$ or $S^2 \times S^1$. Hence applying the cyclic surgery theorem of [CGLS] to the hyperbolic manifold $N$, we get $\Delta(\beta, \mu) = 1$, i.e., $\beta$ is integer slope.

So we may assume that the base orbifold $F$ is not a disk with at most two cone points. We may also assume that $F$ is not a Mobius band since otherwise $U$ has another Seifert fibration whose base orbifold is a disk with two cone points (of indices 2 and 2) [J]. We still use $\sigma$ to denote the slope in $T_0 = \partial U$ represented by a fiber in $T_0$ of the Seifert fibration of $U$. One can now choose a slope $\delta$ in $T_0$ satisfying the conditions: (1) $\Delta(\sigma, \delta) > 1$; (2) $\Delta(\sigma, \delta) > 1$ implies that the Seifert fibration of $U(\delta)$ coming from an extension of $U$ will have an extra singular fiber which is the core of the filling solid torus. Hence we may apply [BZ2, Theorem 1.5] to the hyperbolic manifold $N$ to conclude that $\Delta(\beta, \mu) = 1$.

If a hyperbolic knot in a solid torus admits a non-trivial surgery yielding a solid torus, then the surgery slope is an integer, say $m$ [CGLS], [Ga2]. By Proposition 9, such knot admits at most two other non-trivial surgeries yielding Seifert fibered spaces and the surgery slope(s) must be $m-1$ or/and $m+1$. It is proved in [B] that there is a unique (up to topological equivalence) hyperbolic knot in a solid torus which admits two non-trivial surgeries yielding solid torus. By Proposition 9, this knot admits no other non-trivial surgery yielding Seifert fibered space.

4. Tangle sums and the cabling conjecture.

We now continue our study of tangles and retain notations established earlier. Given two tangles of the same number of strands, $(W; \alpha_1, \ldots, \alpha_k)$ and $(W'; \alpha'_1, \ldots, \alpha'_k)$, and any homeomorphism

$$h : (\partial W; \cup \partial \alpha_i) \rightarrow (\partial W'; \cup \partial \alpha'_i),$$

one can construct a manifold pair $(Q, L)$ such that $Q$ is the usual connected sum of $W$ and $W'$ and $L$ is a link in $Q$ obtained by identify end points of the strands $\alpha_1, \ldots, \alpha_k$ with the end points of the strands $\alpha'_1, \ldots, \alpha'_k$. 
using the map $h$. We call $(Q, L)$ the $h$-sum of the two tangles. Recall that $X$ (resp. $X'$) denotes the exterior of the strands $\alpha_i$’s (resp. $\alpha'_i$’s) in $W$ (resp. $W'$) and $P$ (resp. $P'$) is the planar surface $\partial W - (H_1 \cup \ldots \cup H_k)$ (resp. $\partial W' - (H'_1 \cup \ldots \cup H'_k)$). Now the map $h$ induces through restriction a homeomorphism map $h|_P : P \rightarrow P'$ such that the manifold $M = X \cup_h X'$ is the link exterior of $L$ in $Q$. If both of the tangles are irreducible and non-split, then $M$ is an irreducible manifold such that $P$ is an incompressible planar surface properly embedded in $M$, and moreover if in addition both of the tangles are atoroidal and annular, then $M$ is hyperbolic. The question of interest to us is when can $M$ be homeomorphic to a knot exterior in $S^3$ and if so must both of $W$ and $W'$ be the 3-ball? This is another formulation of the cabling conjecture. The (stronger) cabling conjecture asserts that if $M$ is the exterior of a knot in $S^3$, then it has no properly embedded incompressible planar surface with more than two boundary components and with its boundary slope different from the meridian slope of the knot.

We remark that if such planar surface, denoted $P$, exists in $M$, then $M(r_0)$ is either a reducible manifold or a lens space, where $r_0$ is the boundary slope of $P$ (for a slope $r$ on $\partial M$, we use $M(r)$ to denote the manifold obtained by Dehn filling $M$ along $\partial M$ with the slope $r$). This follows from [GL3]. We also note that if $M$ is a knot exterior in $S^3$ which is cabled, then $M$ does not contain incompressible properly embedded planar surface with more than two boundary components whose slope is different from the meridian slope.

**Proposition 10.** Suppose that $(W; \alpha_1, \ldots, \alpha_k)$ and $(W'; \alpha'_1, \ldots, \alpha'_k)$ are irreducible and non-split tangles with $k \geq 2$ strands such that $(W; \alpha_1, \ldots, \alpha_k)$ is $A_p$-tubing incompressible for some annulus $A_p$ and $(W'; \alpha'_1, \ldots, \alpha'_k)$ is $A'_q$-tubing incompressible for some annulus $A'_q$. Let $(Q, K)$ be any $h$-sum of the two tangles such that $K$ is a knot. Then the exterior $M$ of $K$ in $Q$ is not homeomorphic to a knot exterior in $S^3$ unless both $W$ and $W'$ are 3-balls, in which case the knot $K$ also satisfies the cabling conjecture.

**Proof.** Let $r_0$ be the boundary slope of $P$ in $\partial M$. The idea is to construct an essential branched surface in the interior of $M$ which remains essential in $M(r)$ for all filling slopes $r \neq r_0$ (see [GO] for the definition of an essential branched surface and for their topological implications to their ambient manifolds). Hence if one of $W$ and $W'$ is not a 3-ball, then the slope $r_0$ cannot be the canonical meridian slope of $\partial M$ if $M$ is a knot exterior in $S^3$. But then $M$ cannot admit a Dehn filling with a slope different from $r_0$ producing the 3-sphere as $S^3$ has no essential branched surface. If both $W$ and $W'$ are 3-balls, then $r_0$ is the meridian slope in $\partial M$. But then $M$ cannot admit a Dehn filling producing a reducible 3-manifold or a lens space as a reducible manifold or a lens space has no essential branched surface. Hence in such case, the knot $K$ satisfies the cabling conjecture.
The construction of the essential branched surface is a generalization of a method given in [W]. The planar surface \((P, \partial P) \subset (M, \partial M)\) has \(2k \geq 4\) boundary components. Up to re-ordering, we may assume that the strands \(\alpha_i\) and \(\alpha'_j\) occur in the knot \(K\) in the order \(\alpha_1, \alpha'_1, \alpha_2, \alpha'_2, \ldots, \alpha_k, \alpha'_k\) and we may also assume that \(p = 1\). Now construct a branched surface \(B\) in \(M\) as shown in Figure 6. Topologically \(B\) is the union of \(P\) and the annuli \(A'_1, A_2, A'_2, \ldots, A_k, A'_k\). The boundary circles of \(A_2, A_3, \ldots, A_k\) are the branch set of \(B\), which we denoted in Figure 6 by \(b_1, b_2, \ldots, b_{2k-2}\). Make cusps at \(b_1, \ldots, b_{2k-2}\) as indicated in Figure 6. Push \(B\) slightly into the interior of \(M\) and denote the resulting branched surface still by \(B\). Now following [W] with obvious modifications, one can show that \(B\) is essential in \(M\) and remains essential in \(M(r)\) for all slopes \(r \neq r_0\). □

**Figure 6.** The branched surface \(B\) in \(M\).

**Proposition 11.** Suppose that \((W; \alpha_1, \alpha_2)\) and \((W'; \alpha'_1, \alpha'_2)\) are irreducible and non-split tangles. Let \((Q, K)\) be any \(h\)-sum of the two tangles such that \(K\) is a knot. Then the exterior \(M\) of \(K\) in \(Q\) is not homeomorphic to a knot exterior in \(S^3\) unless both \(W\) and \(W'\) are 3-balls in which case the knot \(K\) also satisfies the cabling conjecture.

**Proof.** First we suppose that \(M\) is homeomorphic to a knot exterior in \(S^3\) and at least one of \(W\) and \(W'\) is not the 3-ball. We will get a contradiction. In this case, the boundary slope \(r_0\) of \(P\) on \(\partial M\) is not the canonical meridian slope. Let \(\mu\) be the meridian slope. Then \(\Delta(\mu, r_0)\), the geometric intersection number of \(\mu\) and \(r_0\) on \(\partial M\), is equal to one by [GL2]. By [GL3], there is a planar surface \((Q, \partial Q) \subset (M, \partial M)\) satisfying the following conditions: (1) Each component of \(\partial Q\) is an essential simple loop in \(\partial M\) with slope \(\mu\); (2)
Q intersects \( P \) transversely such that each component of \( \partial Q \) intersects each component of \( \partial P \) in exactly one point; (3) No arc component of \( Q \cap P \) is boundary parallel in either \( Q \) or \( P \). The arc components of \( Q \cap P \) give rise to two “dual” graphs \( \Gamma_Q \) and \( \Gamma_P \) in \( Q \) and \( P \) respectively. Namely one takes the boundary components of \( Q \) (resp. \( P \)) as vertices of \( \Gamma_Q \) (resp. \( \Gamma_P \)) and arc components of \( Q \cap P \) as edges of \( \Gamma_Q \) (resp. \( \Gamma_P \)). In our present case, \( \Gamma_P \) has four vertices coming from the components of \( \partial A_1 \) and \( \partial A_2 \). It follows from [GL3] that the graph \( \Gamma_Q \) must contain a Scharlemann cycle of order \( n > 1 \) (cf. [GL3] for the definition of Scharlemann cycle). Let \( e_1, \ldots, e_n \) be the edges of the Scharlemann cycle and let \( D \) be the disk bounded by the Scharlemann cycle. Then \( D \) is properly embedded in \( X \) or \( X' \), say \( X \). Further in the surface \( P \), the edges \( e_1, \ldots, e_n \) connect two boundary components of \( A_1 \) or \( A_2 \), say \( A_1 \), and divide \( P \) into \( n > 1 \) regions.

Now consider the two components of \( \partial A_2 \). If both of them are contained in the same region of \( P - (e_1 \cup \ldots \cup e_n) \), then the complement, denoted by \( P' \), of this region in \( P \) is a disk with two punctures \( a_1 \) and \( a_1' \), containing all the edges \( e_1, \ldots, e_n \). Let \( L^0 \) be a regular neighborhood of \( P' \cup D \cup H_1 \) in \( W \). Then \( L^0 \) is a punctured lens space of order \( n \) since \( D \) came from a Scharlemann cycle. Moreover the boundary 2-sphere of \( L^0 \) intersects \( \partial M \) exactly twice with the slope \( r_0 \), i.e., \( \partial L^0 \cap M \) is an annulus which we denote by \( A \). This annulus \( A \) must be essential in \( M \), i.e., it cannot be boundary parallel. This follows by considering two sides of \( A \) in \( M \); one side of \( A \) came from the punctured lens space \( L^0 \) and the other side of \( A \) contains the incompressible surface \( P \) with four boundary components. The existence of such annulus implies that \( M \) is cabled. But this gives a contradiction to the early remark immediately prior to Proposition 10 that a cabled knot exterior in \( S^3 \) cannot contain incompressible properly embedded planar surface with more than two boundary components whose slope is different from the meridian slope.

Hence the components of \( \partial A_2 \) are contained in different regions of \( P - (e_1 \cup \ldots \cup e_n) \). Let \( a_1, a_1' \) be the components of \( \partial A_1 \) and \( a_2, a_2' \) the components of \( \partial A_2 \). Note that there are exactly \( m \) end points of edges of \( \Gamma_P \) incident at each of vertices \( a_1, a_1', a_2 \) and \( a_2' \), where \( m \) is the number of boundary components of \( Q \). Since no edge in \( \Gamma_P \) is boundary parallel, edges of \( \Gamma_P \) which are incident at the vertex \( a_2 \) must connected \( a_2 \) to either \( a_1 \) or \( a_1' \) and thus there are \( m \) such edges. Suppose that there are \( p \) edges which connect \( a_2 \) to \( a_1 \) and \( m - p \) edges which connect \( a_2 \) to \( a_1' \). Similarly we may assume that there are \( q \) edges which connect \( a_2' \) to \( a_1 \) and \( m - q \) edges which connect \( a_2' \) to \( a_1' \). Also observe that there can be no loops at \( a_1 \) or \( a_1' \). The situation is depicted in Figure 7. Let \( n' \) be the number of edges of \( \Gamma_P \) connecting \( a_1 \) and \( a_1' \). Then \( n' \geq n \). Now counting the end points of edges around \( a_1 \) and \( a_1' \), we get \( p + q + n' = m \) and \( m - p + m - q + n' = m \). But these equations imply that \( n' = 0 \), a contradiction with \( n' \geq n > 1 \).
So we assume now that both of $W$ and $W'$ are 3-balls. Hence $Q = S^3$ and the boundary slope of $P$ is the meridian slope $\mu$ of the knot $K$. If the two strands $\alpha_1$ and $\alpha_2$ in $W$ are parallel, then it is not difficult to see that the tangle $(W; \alpha_1, \alpha_2)$ is toroidal since the tangle is non-split and its ambient space is a 3-ball. Hence by [W, Lemma 2.1], no Dehn filling on $M$ will produce a reducible manifold or a lens space unless $K$ is a cabled knot. So we may assume that the two stands of the tangle are not parallel. Now by [W, Theorem 2.3] no Dehn filling on $M$ will produce a reducible manifold or a lens space. Hence $K$ satisfies the cabling conjecture. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gamma_p.png}
\caption{The graph $\Gamma_P$.}
\end{figure}

We note that combining techniques from [Ma] and [Ho], one can show that any knot in $S^3$ which admits an irreducible non-split tangle decomposition of at most three strands satisfies the cabling conjecture.


Recall that a closed (embedded) essential (meaning orientable, incompressible and non-boundary parallel) surface $S$ in the exterior $M$ of a knot in $S^3$ is said to be meridionally-incompressible if there is no embedded annulus $A$ in $M$ such that $A \cap \partial M$ is one component of $\partial A$ with the meridian slope of the knot and $A \cap S$ is the other component of $\partial A$. Note that a knot in this class is either a composite knot, or a torus knot or a hyperbolic knot. In this section we make some notes concerning Dehn surgery on knots with no meridionally-incompressible closed essential surfaces. Our first observation concerns the cabling conjecture on this class of knots.
Proposition 12. Let \( M \) be the exterior of a non-trivial knot in \( S^3 \) without meridionally-incompressible closed essential surfaces. Suppose that \( M \) contains a properly embedded essential planar surface \( P \) whose boundary slope \( r_0 \) is not the meridian slope. Then \( M(r_0) \) is a connected sum of two lens spaces.

Proof. According to [GL3], \( M(r_0) \) contains a lens space as summand and \( r_0 \) is an integer slope but not the longitude slope. So \( M(r_0) \) is either a lens space or a reducible manifold. As \( r_0 \) is a boundary slope, [CGLS, Theorem 2.0.3] implies that either \( M(r_0) \) is a connected sum of two lens spaces, in which case we are done, or \( M \) contains a closed essential surface \( S \) such that \( S \) remains incompressible in \( M(r) \) for all slopes satisfying \( \Delta(r, r_0) > 1 \). We claim that \( S \) must be compressible in \( M(r_0) \). This is clearly true if \( M(r_0) \) is a lens space. In case that \( M(r_0) \) is reducible and \( S \) is incompressible in \( M(r_0) \), one can apply [Sch] and show that \( M \) is cabled. Since \( M \) contains essential closed surface, \( M \) cannot be a torus knot exterior. Therefore the cabled manifold \( M \) must contain an essential torus and thus must be a composite knot exterior. But it is known that a composite knot does not admit surgery producing a lens space or reducible manifold [G]. This contradiction completes the proof of the claim that \( S \) is compressible in \( M(r_0) \).

By our assumption, \( S \) is meridianly compressible. This, together with the condition that \( S \) is compressible in \( M(r_0) \), implies, by [CGLS, Theorem 2.4.3], that \( S \) is compressible in \( M(r) \) for all integer slopes \( r \). But most of these integer slopes \( r \) satisfy \( \Delta(r, r_0) > 1 \). This gives a contradiction to an early conclusion. \( \square \)

Hence, any reducible manifold resulting from Dehn filling a non-trivial knot exterior without meridionally-incompressible closed essential surfaces must be a connected sum of two lens spaces. We remark that by [GS], if a knot exterior in \( S^3 \) admits a filling producing a connected sum of two lens spaces, then the Alexander polynomial of the knot is divisible by the Alexander polynomial of a non-trivial torus knot.

Our next result concerns surgery producing Seifert fibered 3-manifolds. For convenience, in this paper we call a Seifert fibered 3-manifold large if its base orbifold is not a 2-sphere with at most 4 cone points or a projective plane with at most 2 cone points. It is conjectured that no knot in \( S^3 \) admits a surgery yielding a large Seifert fibered 3-manifold. We note, in contrast, that many of Seifert fibered spaces whose base orbifolds are 2-sphere with at most 4 cone points can be obtained by Dehn surgery on knots in \( S^3 \) [D], [KT], [MM].

Proposition 13. Let \( M \) be the exterior of a knot without meridionally-incompressible closed essential surfaces. Then no Dehn filling of \( M \) produces a large Seifert fibered space.
Proof. Suppose otherwise that for some slope \( r_0 \), \( M(r_0) \) is a large Seifert fibered space. Then the \( PSL(2, \mathbb{C}) \)-character variety \( \tilde{X}(M(r_0)) \) of \( M(r_0) \) (see [BZ2] for the definition of the \( PSL(2, \mathbb{C}) \)-character varieties of manifolds and for their basic properties) is at least two dimensional (such estimation of dimension can be found in [BZ2, Section 8]). Note that \( \tilde{X}(M(r_0)) \) is naturally contained in \( \tilde{X}(M) \), the \( PSL(2, \mathbb{C}) \)-character variety of \( M \). Recall that each element \( \chi_\bar{\rho} \) in \( \tilde{X}(M) \) is the character of a \( PSL(2, \mathbb{C}) \)-representation \( \bar{\rho} \) of \( \pi_1(M) \) and that each element \( \gamma \in \pi_1(M) \) defines a regular function \( f_\gamma \) on \( \tilde{X}(M) \), whose value at a character \( \chi_\bar{\rho} \) of \( X(M) \) is equal to \( \| \text{trace}(\Phi^{-1}(\bar{\rho}(\gamma))) \|^2 - 4 \) where \( \Phi \) is the canonical quotient map from \( SL(2, \mathbb{C}) \) to \( PSL(2, \mathbb{C}) \) (see [BZ2, p. 759] for details). The condition that \( \tilde{X}(M(r_0)) \subset \tilde{X}(M) \) is at least two dimensional implies that there is an algebraic curve \( X_0 \) in \( \tilde{X}(M(r_0)) \subset \tilde{X}(M) \) on which all the functions \( f_r \), \( r \in \pi_1(\partial M) \subset \pi_1(M) \), are constant functions. This implies that there is an essential closed surface \( S \) in \( M \) associated to an ideal point of \( \tilde{X}_0 \), where \( \tilde{X}_0 \) is the smooth projective completion of \( X_0 \) (see [BZ2, Proposition 4.7]).

We claim that the surface \( S \) must be compressible in the Seifert fibered space \( M(r_0) \). For otherwise \( S \) is either isotopic to a vertical torus (here vertical means consisting of fibers of the Seifert fibration of \( M(r_0) \)) or to a horizontal surface (meaning transverse to all fibers of the Seifert fibration of \( M(r_0) \)). Such isotopy of \( S \) can be arranged in \( M \subset M(r_0) \). If \( S \) is a horizontal surface, then it is either a non-separating surface or splits \( M(r_0) \) into two twisted \( I \)-bundles over a closed non-orientable surface. But no knot exterior in \( S^3 \) can contain either a non-separating closed orientable surface, or a closed non-orientable surface. Hence, \( S \) must be a vertical torus. It follows that \( M \) is the exterior of a satellite knot in \( S^3 \) and, thus, is the exterior of a composite knot in \( S^3 \). But Seifert surgery on composite knots have been classified in [KT]. Namely, if a surgery on a composite knot produces a Seifert fibered space, then the knot is a connected sum of two torus knots and the base orbifold of the Seifert fibered space is a 2-sphere with 4 cone points. But the existence of such a Seifert fibration contradicts the definition of large. This contradiction completes the proof of the claim that \( S \) is compressible in \( M(r_0) \).

Recall that \( S \) was associated to an ideal point of the curve \( \tilde{X}_0 \) and every function \( f_r \), \( r \in \pi_1(\partial M) \), was bounded (in fact constant) near the ideal point. Further, by construction, for each \( \chi_\bar{\rho} \in \tilde{X}_0 \), \( \bar{\rho}(r_0) = I \), since \( \tilde{X}_0 \subset \tilde{X}(M(r_0)) \subset \tilde{X}(M) \). Hence, we can apply [BZ2, Proposition 4.10] to conclude that the surface \( S \) remains incompressible in \( M(r) \), if \( \Delta(r, r_0) > 1 \). In particular, \( r_0 \) is an integer slope since \( S \) must be compressible in \( M(\mu) \), where \( \mu \) is the meridian slope in \( \partial M \).

We can now get a contradiction exactly as we did in the last paragraph of the proof of Proposition 12. \( \square \)
6. Seifert filling versus other types of exceptional fillings.

Let $M$ be a compact orientable simple 3-manifold with at least two boundary components one of which is a torus, denoted by $T$. Here the term simple means that $M$ contains no essential 2-sphere; essential annulus; essential torus; or $\partial$-reducing disk. It is shown in [W2] that if one Dehn filling on $M$ along $T$ with slope $\alpha$ produces a reducible manifold and another filling $M$ along $T$ with slope $\beta$ produces a manifold containing essential annulus, then $\Delta(\alpha, \beta) \leq 2$; it is shown in [GW] that if one Dehn filling on $M$ along $T$ with slope $\alpha$ produces a $\partial$-reducible manifold and another filling $M$ along $T$ with slope $\beta$ produces a manifold containing essential annulus, then $\Delta(\alpha, \beta) \leq 2$; it is shown in [W3] that two Dehn fillings on $M$ along $T$ with slopes $\alpha$ and $\beta$ produce $\partial$-reducible manifolds, then $\Delta(\alpha, \beta) \leq 1$; and it is shown in [Sch] that if one Dehn filling on $M$ along $T$ with slope $\alpha$ produces a reducible manifold, then there is no filling along $T$ that can produce a $\partial$-reducible manifold. We wish to consider Seifert Dehn filling on $M$ along $T$, i.e., fillings resulting in a Seifert fibered space. Note that if some filling on $M$ along $T$ produces a Seifert fibered space, then $\partial M$ must consist of tori and the resulting Seifert fibered space is irreducible (since the manifold has boundary) and contains an essential annulus unless it is a solid torus. Hence it follows from the results cited above that if one Dehn filling on $M$ along $T$ with slope $\alpha$ produces a reducible or $\partial$-reducible manifold, and another filling $M$ along $T$ with slope $\beta$ produces a Seifert fibered space, then $\Delta(\alpha, \beta) \leq 2$. The purpose of this section is to sharpen the bound from 2 to 1. As we mentioned in the introduction section, the bound 1 is best possible.

**Proposition 14.** Let $M$ be a compact orientable simple 3-manifold whose boundary has at least two components one of which is a torus, denoted by $T$. If one Dehn filling on $M$ along $T$ with slope $\alpha$ yields a reducible manifold and another Dehn filling on $M$ along $T$ with slope $\beta$ yields a Seifert fibered space, then $\Delta(\alpha, \beta) \leq 1$.

**Proof.** As we already noted, $\partial M$ must consist of tori. (Hence the condition that $M$ is simple is equivalent to the condition that $M$ admits a complete hyperbolic structure of finite volume, according to Thurston [T]). By [W2, Theorem 4.6] we may assume that $H_2(M, \partial M - T) = 0$. It follows that $\partial M$ consists of exactly two tori, one is $T$ and the other we denote by $T_0$. It also follows that $M$ does not contain any non-separating closed orientable surface. We denote by $M(T, \delta)$ the manifold obtained by Dehn filling $M$ along $T$ with slope $\delta$ in $T$; by $M(T_0, \delta)$ the manifold obtained by Dehn filling $M$ along $T_0$ with slope $\delta$ in $T_0$; and by $M((T, \delta), (T_0, \eta))$ the manifold obtained by Dehn filling $M$ along $T$ with slope $\delta$ in $T$ and along $T_0$ with slope $\eta$ in $T_0$. If $W$ is a 3-manifold whose boundary is a single torus, we always use $W(\delta)$ to denote the manifold obtained by Dehn filling $W$ along $\partial W$ with slope $\delta$ in $\partial W$. 
Let $P$ be a non-trivial connect summand of the reducible manifold $M(T, \alpha)$ such that $P$ does not contain $T_0$. We have $M(T, \alpha) = P \# Y$, where $Y$ is a $3$-manifold whose boundary is $T_0$. There are three cases to consider.

**Case 1.** $P$ is not a lens space.

Then we can choose a slope $\delta$ in $T_0$ such that it satisfies the following conditions: (1) $N = M(T_0, \delta)$ is a hyperbolic $3$-manifold; (2) The first Betti number of $N = M(T_0, \delta)$ is one; (3) $M((T_0, \delta), (T, \alpha))$ is a reducible manifold different from $S^2 \times S^1$; (4) There is no essential surface in $M$ disjoint from $T$ but intersecting $T_0$ with boundary slope $\delta$ in $T_0$. Condition (1) is guaranteed by Thurston’s hyperbolic surgery theorem. Condition (2) can be easily satisfied because the first Betti number of $M$ is two (since $H_2(M, \partial M - T) = 0$) and thus all slopes, except for one, in $T_0$ will produce manifolds with first Betti number equal to one. As $M((T_0, \delta), (T, \alpha)) = P \# Y(\delta)$ and $Y(\delta)$ is not a $3$-ball for infinitely many slopes $\delta$ in $T_0 = \partial Y$, condition (3) follows. Condition (4) is realized by applying the main result of \cite{Ha} which implies that for our manifold $M$, there are only finitely many slopes in $T_0$ which can be boundary slopes of essential surface in $M$ disjoint from $T$.

Since $N = M(T_0, \delta)$ is irreducible but $N(\alpha) = M((T_0, \delta), (T, \alpha))$ is reducible by condition (3), $\alpha$ is a boundary slope of $N$. Since the first Betti number of $N$ is one by condition (2) and $N(\alpha)$ is reducible but is not $S^2 \times S^1$ or a connected sum of two lens spaces (since $P$ is not a lens space by our assumption), we may apply \cite[Theorem 2.0.3]{CGLS} to see that $N$ must contain an essential closed surface $S$ which remains incompressible in $N(\eta) = M((T_0, \delta)(T, \eta))$ for all slopes $\eta$ in $T$ satisfying $\Delta(\alpha, \eta) > 1$. By condition (4), we may assume that the closed essential surface $S$ is disjoint from $T_0$, i.e., we have $S \subset M$. Since $M$ is hyperbolic $S$ is of genus at least two.

**Claim.** $S$ must be compressible in $M(T, \beta)$.

Suppose otherwise. Then $S$ is isotopic to either a horizontal or vertical surface in the Seifert fibered space $M(T, \beta)$. But $S$ cannot be isotopic to a horizontal surface since it is disjoint from $\partial M(T, \beta) = T_0$. Neither can $S$ be isotopic to a vertical surface since $S$ is closed and is not a torus. This proves the claim.

Since $S$ is compressible in $M(T, \beta) \subset N(\beta)$, we have $\Delta(\alpha, \beta) \leq 1$. Hence Proposition 14 holds in Case 1.

**Case 2.** $Y$ is not a solid torus.

Then we can choose a slope $\delta$ in $T_0$ such that it satisfies the following conditions: (1) $N = M(T_0, \delta)$ is a hyperbolic $3$-manifold; (2) The first Betti number of $N = M(T_0, \delta)$ is one; (3) The fundamental group of $Y(\delta)$ is not cyclic; (4) There is no essential surface in $M$ disjoint from $T$ but with boundary slope $\delta$ in $T_0$. Conditions (1) (2) (4) can be justified exactly as in
Case 1. Condition (3) can also be arranged to hold due to the fact that $Y$ is not a solid torus. So $N(\alpha) = P\#Y(\delta)$ is a reducible manifold but is not $S^2 \times S^1$ or a connected sum of two lens spaces. Now the rest of the proof in this case goes similarly as in Case 1.

Case 3. $P$ is a lens space and $Y$ is a solid torus.

The idea of proof is similar to that of Proposition 9, but more cases are involved. Let $U = M(T, \beta)$, which is Seifert fibered. Let $F$ be the base orbifold of a Seifert fibration of $U$. We may assume that $F$ is not a disk with at most one cone point since otherwise $U$ is a solid torus in which case the manifold $M$ must be cabled by [Sch], giving a contradiction with the assumption that $M$ is simple. Suppose that $F$ is a disk with exactly two cone points. Let $\sigma$ be the slope in $T_0 = \partial U$ represented by a fiber in $T_0$ of the Seifert fibration of $U$. One can now choose a slope $\delta$ in $T_0$ satisfying the conditions: (1) $\Delta(\sigma, \delta) = 1$; (2) The manifold $N = M(T_0, \delta)$ is a hyperbolic manifold. It follows from condition (1) that the manifold $N(\beta)$ has cyclic fundamental group since the manifold is a Seifert fibered space which has a Seifert fibration whose base orbifold is a 2-sphere with exactly two cone points. The manifold $N(\alpha) = P\#Y(\alpha)$ is either a lens space (when $Y(\alpha)$ is $S^3$) or a reducible manifold (when $Y(\alpha)$ is not $S^3$). If $N(\alpha)$ is a lens space, then we can apply the cyclic surgery theorem of [CGLS] to get $\Delta(\alpha, \beta) \leq 1$; and if $N(\alpha)$ is reducible, then we can apply [BZ2, Theorem 1.2] to get $\Delta(\alpha, \beta) \leq 1$. So we may assume that the base orbifold $F$ is not a disk with at most two cone points. We may also assume that $F$ is not a Mobius band since otherwise $U$ has another Seifert fibration whose base orbifold is a disk with two cone points.

If $F$ is not a 2-disk with three cone points, then we can choose a slope $\delta$ in $T_0$ satisfying the conditions: (1) $\Delta(\delta, \mu) = 1$ where $\mu \subset T_0 = \partial Y$ is the meridian slope of the solid torus $Y$; (2) The manifold $N = M(T_0, \delta)$ is a hyperbolic manifold, (3) $\delta$ is not the slope $\sigma$ (the fiber). Condition (1) implies that $N(\alpha) = P\#Y(\alpha) = P$ is a lens space. Also $N(\beta)$ is a Seifert fibered space which admits no Seifert fibration with base orbifold being a 2-sphere with at most three cone points. Hence we may apply [BZ2, Theorem 1.5 (1)] to the hyperbolic manifold $N$ to conclude that $\Delta(\alpha, \beta) \leq 1$.

So we may assume that $F$ is a 2-disk with exactly three cone points. If $\sigma$ is not the meridian slope $\mu$ of $Y$ in $\partial Y = T_0$, then we can choose a slope $\delta$ in $T_0$ satisfying the conditions: (1) $\Delta(\mu, \delta) = 1$; (2) The manifold $N = M(T_0, \delta)$ is a hyperbolic manifold; (3) $\Delta(\sigma, \delta) > 1$. Condition (1) implies that $N(\alpha) = P$ is a lens space. Condition (3) implies that $N(\beta)$ is a Seifert fibered space which admits no Seifert fibration with base orbifold being a 2-sphere with at most three cone points. Hence we may apply [BZ2, Theorem 1.5] to the hyperbolic manifold $N$ to conclude that $\Delta(\alpha, \beta) \leq 1$. 

Finally suppose that $\sigma$ is the meridian slope of $Y$. Then $M((T, \alpha), (T_0, \sigma)) = P\# Y(\sigma) = P\# S^2 \times S^1$ and $M((T, \beta), (T_0, \sigma)) = U(\sigma)$ is a connected sum of three lens spaces since the base orbifold $F$ of the Seifert fibered space $U$ is a 2-disk with exactly three cone points. Let $N = M(T_0, \sigma)$. If $N$ is irreducible, then we may apply the main result of [GL] to see that $\Delta(\alpha, \beta) \leq 1$ since $N$ admits two fillings each yielding a reducible manifold. So we assume that $N$ is reducible. We may write $N$ as $N = N_1 \# P_1$ with $N_1$ being irreducible and contains $T = \partial N = \partial N_1$. Since $N(\alpha)$ is a connected sum of a lens space and $S^2 \times S^1$, $P_1$ is either a lens space or $S^2 \times S^1$ or a connected sum of a lens space and $S^2 \times S^1$. But $P_1$ cannot contain $S^2 \times S^1$ since otherwise $N(\beta) = M((T, \beta), (T_0, \sigma))$ would also contain $S^2 \times S^1$, contradicting the fact that $N(\beta)$ was a connected sum of three lens spaces. So $P_1$ must be a lens space. Then $N_1(\alpha) = S^2 \times S^1$ and $N_1(\beta)$ is a connected sum of two lens spaces. So we can apply [GL] to $N_1$ to get $\Delta(\alpha, \beta) = 1$. This completes the proof of Case 3 and thus completes the proof of the proposition. \hfill $\square$

**Corollary 15.** Let $M$ be a compact orientable simple 3-manifold whose boundary has at least two components one of which is a torus, denoted by $T$. If one Dehn filling on $M$ along $T$ with slope $\alpha$ yields a $\partial$-reducible manifold and another Dehn filling on $M$ along $T$ with slope $\beta$ yields a Seifert fibered space, then $\Delta(\alpha, \beta) \leq 1$.

**Proof.** Again $\partial M$ must consist of tori. If $M(T, \alpha)$ is reducible, then we may apply Proposition 14 to get $\Delta(\alpha, \beta) \leq 1$. So we may assume that $M(T, \alpha)$ is irreducible. As $M(T, \alpha)$ is $\partial$-reducible and its boundary consists of tori, it must be a solid torus. Hence we may apply Proposition 9 to see that $\Delta(\alpha, \beta) \leq 1$. \hfill $\square$

**Proposition 16.** Let $M$ be a compact orientable simple 3-manifold whose boundary has at least two components one of which is a torus, denoted by $T$. If one Dehn filling on $M$ along $T$ with slope $\alpha$ yields a manifold which contains an essential torus or essential annulus, and another Dehn filling on $M$ along $T$ with slope $\beta$ yields a Seifert fibered space, then $\Delta(\alpha, \beta) \leq 3$.

**Proof.** Since $M(\beta)$ is Seifert fibered, it is either a solid torus or contains an essential annulus. If $M(\beta)$ is a solid torus, then $\Delta(\alpha, \beta) \leq 2$ by [GW] and [GL5]. So we may assume that $M(\beta)$ contains an essential annulus. By [GW2] and [GW3], $\Delta(\alpha, \beta) \leq 3$ unless $M$ is one of the three manifolds $M_1$, $M_2$ and $M_3$ given in [GW2, Section 7] and $\Delta(\alpha, \beta) = 4$ or 5. More precisely, if $M = M_1$ (which is the exterior of the Whitehead link in $S^3$) and $\Delta(\alpha, \beta) > 3$, then $\Delta(\alpha, \beta) = 4$ by [GW2, Theorem 1.1] and $M_1(\beta)$ is the double branched cover of one the two the tangles given in Figure 7.2 (d) and (e) of [GW2] (see [GW2, Lemma 7.1]). But one can easily check that the double branched cover of each of these two tangles is not Seifert fibered. So $M \neq M_1$. Similarly, if $M = M_2$ (which is the exterior in $S^3$ of
the link given in [GW2, Figure 7.1 (b)] and $\Delta(\alpha, \beta) > 3$, then $\Delta(\alpha, \beta) = 4$ by [GW2, Theorem 1.1] and $M_2(\beta)$ is the double branched cover of one of the two tangles given in Figure 7.4 (d) and (e) of [GW2] (see [GW2, Lemma 7.5]). But again one can verify that the double branched cover of each of these two tangles is not Seifert fibered. So $M \neq M_2$. Finally, if $M = M_3$ (which is the exterior in $S^3$ of the link given in [GW2, Figure 7.1 (c)]) and $\Delta(\alpha, \beta) > 3$, then $\Delta(\alpha, \beta) = 5$ by [GW2, Theorem 1.1] and $M_3(\beta)$ is the double branched cover of the tangle given in Figure 7.5 (d) or (e) of [GW2] (see [GW2, Lemma 7.5]). But again one can verify that the double branched cover of each of these two tangles is not Seifert fibered. □

![Figure 8](image-url)

**Figure 8.** Distance three between a Seifert slope and an annular and toroidal slope.

**Example 17.** We give here a family of infinitely many hyperbolic manifolds $N_n$ with $\partial N_n$ consists of two tori such that one of the tori contains two slopes, distance three apart, one producing a Seifert fibered manifold and the other producing a manifold containing an essential torus and an essential annulus. These examples are constructed based on [GW2, Lemma 7.2]. Here are the details. The manifold $M_3$ mentioned in the proof of Proposition 16 is the double branched cover of a twice punctured 3-ball $X$ whose branched set is a set of proper arcs shown in Figure 8 (a) (which is from [GW2, Figure 7.6 (c)]). $M_3$ is hyperbolic with $\partial M_3$ consists of three tori, which we denote...
by $T_1, T_2, T_3$. In Figure 8 (a), we have chosen framings for the two inside 2-spheres $S_1$ (the one on the left) and $S_2$ (the one on the right). We may assume that $T_1$ covers $S_1$ and $T_2$ covers $S_2$. Framings on $T_1$ and $T_2$ are lifts of those on $S_1$ and $S_2$ respectively. Now we let $N_n$ be the manifold obtained by Dehn filling $M_3$ along $T_2$ with slope $1/n$. Then it follows from Thurston's hyperbolic surgery theorem that the manifolds $N_n$'s are hyperbolic and are mutually different for infinitely many choices of $n$ (because that in the core of the filling solid torus is a geodesic whose length becomes arbitrarily small as $n$ becomes large). Each $N_n$ has its boundary consisting of two tori and is the double branched cover of a once-punctured 3-ball $Y_n$ obtained by filling $X$ along $S_2$ with a $1/n$-tangle, as shown in Figure 8 (b). Now Dehn filling $N_n$ along $T_2$ with the 0-slope will produces a Seifert fibered space (which is homeomorphic to the trefoil knot exterior in $S^3$) since the resulting manifold is the double branched cover of the tangle shown in Figure 8 (c), obtained by filling $S_1$ with a 0-tangle. On the other hand, Dehn filling $N_n$ along $T_2$ with the $-3/2$-slope will produces a manifold which contains an essential torus and an essential annulus since the resulting manifold is the double branched cover of the tangle shown in Figure 8 (d), obtained by filling $S_1$ with the $-3/2$-tangle.

There remains unsettled the case concerning the optimal bound on the distance between two Seifert Dehn filling slopes on a torus boundary component $T$ of a compact orientable simple manifold $M$ with at least two boundary components. Namely what is the minimal upper bound on $\Delta(\alpha, \beta)$ if both $M(T, \beta)$ and $M(T, \alpha)$ are Seifert fibered manifolds? The best known bound is 3 obtained recently in [GW3].

**Conjecture 18.** The optimal bound is 2.

The distance two can be realized on the Whitehead link exterior $M$; both the 1-slope and the 3-slope on any component of $\partial M$ (with respect to the standard framings) produce Seifert fibered spaces.

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**References**


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SOME CHARACTERIZATION, UNIQUENESS AND EXISTENCE RESULTS FOR EUCLIDEAN GRAPHS OF CONSTANT MEAN CURVATURE WITH PLANAR BOUNDARY

JAIME RIPOLL

We establish the existence and uniqueness of solutions to the Dirichlet problem for the cmc surface equation, including the minimal one, for zero boundary data, in certain domains of the plane. We obtain results that characterize the sphere and cmc graphs among compact embedded cmc surfaces with planar boundary satisfying certain geometric conditions. We also find conditions that imply that a compact embedded cmc surface which is a graph near the boundary is indeed a global graph.

0. Introduction.

In this paper we shall obtain some characterization, uniqueness and existence theorems to the Dirichlet’s problem for the constant mean curvature (cmc) equation with vanishing boundary data

\[ Q_H(u) := \text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} + 2H = 0, \quad u|_{\partial \Omega} = 0, \quad u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \] (1)

where \( \Omega \) is a domain in the plane and \( H \geq 0 \). Although being a very special case of boundary data, it is shown in [R] that the general case of arbitrary continuous boundary data can be reduced, in many situations, to the zero boundary data.

In the minimal case, it is well known that if \( \Omega \) is bounded and convex then there is a solution to \( Q_0 = 0 \) in \( \Omega \) which assumes any continuous prescribed data value at \( \partial \Omega \). For \( \Omega \) convex but not bounded, and for some special cases of non-convex unbounded domains, R. Earp and H. Rosenberg ([ER]) proved the existence of solutions which take on any continuous bounded boundary value if \( \Omega \) is not a half plane. This last case was treated by P. Collin and R. Krust ([CK]) who proved existence for a given continuous boundary data with growth at most linear. In [CK] a uniqueness theorem in arbitrary unbounded domains is also obtained.
Existence results in unbounded but not convex domains have been treated, starting with Nitsche ([N]) and more recently in [ET], [KT] and [RT], in the so-called (finite) exterior domains, that is, domains $\Omega$ such that $\mathbb{R}^2 \setminus \Omega$ is a (finite) union of pairwise disjoint bounded closed simply connected domains, and one can note a drastic difference between this case and the convex one. This can be seen in the non-existence theorem proved by N. Kutev and F. Tomi in [KT]: There are continuous necessarily non-zero boundary data $f$ on a given finite exterior domain, with arbitrarily small $C^0$ norm, for which no solution taking on the value $f$ on $\partial \Omega$ can exist. On the other hand, if one cuts a catenoid or, more generally, an embedded end of a minimal surface with finite total curvature by a plane, we get examples of minimal graphs on exterior of a closed curve, vanishing at the curve. This shows that one could expect the existence of solutions in finite exterior domains at least for special boundary data, for instance, zero boundary data. In fact, it is also proved in [KT] the existence of a solution in a finite exterior $C^{2,\alpha}$ domain for a generically small continuous boundary data $f$, that is, $f$ small in terms of bounds depending on the geometry of the boundary (see Theorem E in [KT]). Still, it subsisted the question of existence of minimal graphs on arbitrary (finite or not) exterior $C^0$ domains even for zero boundary data. We answer here this question positively requiring a “periodicity” of the domain when it is not finite. Precisely:

**Theorem 1.** Let $\Gamma$ be a subgroup of the isometry group of $\mathbb{R}^2$ acting properly discontinuously in $\mathbb{R}^2$, and let $D$ be a fundamental domain of $\Gamma$. Let $\gamma_1, \ldots, \gamma_m$ be Jordan curves bounding closed domains $G_i \subset D$, $i = 1, \ldots, m$ such that $G_i \cap G_j = \emptyset$ if $i \neq j$. Set

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{\phi \in \Gamma} \phi(G_1 \cup \ldots \cup G_m)$$

and let $s \geq 0$ be given. Then there is a non-negative function $u_s \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solving (1) with $H = 0$ in $\Omega$ such that

$$\sup_{\Omega} |\nabla u_s| = s.$$  

It follows that

$$\lim_{R \to \infty} \sup \frac{\max_{C_R} u_s}{R} < +\infty$$

where $C_R$ is the circle of radius $R$ centered at the origin.

Concerning the above theorem we note that the technique of previous works ([N], [ET], [KT], [RT]), of using catenoids as supersolutions, produces solutions having necessarily logarithm growth at infinity. Therefore, since in infinite exterior domains there are solutions with linear growth at infinity (Scherk’s second minimal surface for example), it seems that this
SOME CHARACTERIZATION, UNIQUENESS AND EXISTENCE RESULTS

Concerning the growth of a minimal graph at infinity, we recall that Collin and Krust ([CK]) proved that

\[
\lim_{R \to \infty} \inf_{C_R} \left( \max u / \ln R \right) > 0,
\]

(4)

where \( u \neq 0 \) is any non-negative solution of the minimal surface equation defined in a planar unbounded domain and vanishing at the boundary of the domain. The catenoid \( v = \cosh^{-1}(|x|) \) also shows that this estimate is optimal in the sense that

\[
\lim_{R \to \infty} \sup_{C_R} \frac{\max v}{\ln R} < +\infty.
\]

(5)

We remark that for the graphs constructed in Theorem 1 we can not, in general, improve (3) to (5), as one sees with the example of the second Sherck’s minimal surface.

Considering now the case \( H > 0 \), we recall the well-known result of J. Serrin stating that if \( \Omega \) is bounded and \( C^{2, \alpha}, \, 0 < \alpha < 1 \), then (1) has an unique solution provided that \( \partial \Omega \) has plane curvature bigger than or equal to \( 2H \) (Theorem 1 of [S]) (in fact, under this hypothesis, Serrin’s theorem asserts that the Dirichlet problem for the mean curvature equation is uniquely solvable for any continuous (not necessarily zero) boundary data). As far as we know, there are no existence results for domains which are not simply connected, or even simply connected but not convex. Examples given by pieces of Delaunay surfaces however (see Proposition 1), show that we could expect the existence of cmc graphs over domains which are not simply connected, at least for the especial case of zero boundary data. In fact, we were able to obtain here an existence theorem for arbitrary (bounded) domains, with zero boundary data, but assuming also some restrictions on the geometry of the domains (see Theorem 2 below for the general statement). As a corollary of this theorem, we obtain:

**Corollary 1.** Let \( \gamma, \gamma_1, \ldots, \gamma_k \) be \( C^{2, \alpha} \) convex curves bounding closed domains \( E, E_1, \ldots, E_k \) such that \( E_i \subset E \backslash \partial E \), \( E_i \cap E_j = \emptyset \) if \( i \neq j \), \( 0 < \alpha < 1 \). Given \( H > 0 \), we require that the curvatures \( \kappa \) and \( \kappa_i \) of \( \gamma \) and \( \gamma_i \) satisfy

\[
3H \leq a < \kappa < \kappa_i < a \left( \frac{a}{H} - 2 \right), \quad i = 1, \ldots, k
\]

for some \( a > 0 \). Then there exists a solution to (1) in \( \Omega := E \backslash \bigcup_{i=1}^k E_i \) belonging to \( C^{2, \alpha}(\Omega) \).

In the next results we study in more details the Dirichlet problem in convex domains of the plane. Recently, Rafael López and Sebastián Montiel proved the existence of a solution of (1) on a convex domain \( \Omega \) provided that the length \( L \) of \( \partial \Omega \) satisfies \( LH \leq \sqrt{3} \pi \) (Corollary 4 of [LM]). It follows from this result that if the curvature \( k \) of \( \partial \Omega \) satisfies \( k > (2/\sqrt{3})H \) then
(1) has a solution (see the remark after the proof of Corollary 4 of [LM]), improving Serrin’s theorem for zero boundary data. This same conclusion had already been obtained by L.E. Payne and G.A. Philippin in [PP] (see Theorem 5, Equation 3.11 of [PP]). As a corollary of our Theorem 2, we obtain the optimal estimate of the lower bound for the curvature, namely:

**Corollary 2.** If \( \Omega \) is convex, bounded and \( C^{2,\alpha} \), (1) is solvable provided that the curvature \( k \) of \( \partial \Omega \) satisfies \( k \geq H \). Furthermore, if \( k > H \) then the solution belongs to \( C^{2,\alpha}(\Omega) \).

The above improvement is optimal in the sense that given any \( 0 < \epsilon < 1 \), the curvature \( k \) of a disk of radius \( \epsilon H \) satisfies \( k \geq \epsilon H \) but there is no solution to \( \Delta_H u = 0 \) (with any boundary value) in this disk.

We obtained the following apparently technical result but which seems to be useful for applications:

**Theorem 3.** Let \( H > 0 \) be given and let \( \Omega \) be a \( C^{2,\alpha} \) bounded convex domain, \( 0 < \alpha < 1 \), satisfying the following condition: There exists \( a < 1/(2H) \) such that, given \( h \in [0,H] \), any solution \( u \in C^2(\Omega) \) to \( \Delta_h u = 0 \) in \( \Omega \) with \( u|_{\partial \Omega} = 0 \) satisfies the a priori height estimate \( |u| \leq a < 1/(2H) \). Then there is a solution \( u \in C^{2,\alpha}(\Omega) \) to \( \Delta_H u = 0 \) in \( \Omega \) with \( u|_{\partial \Omega} = 0 \).

As a corollary of Theorem 3, we obtain a result that extends Corollary 2 above and Corollary 4 of [LM] in other directions.

**Corollary 3.** Let \( H > 0 \) be given and let \( \Omega \) be a convex domain contained between two parallel lines \( 1/H \) far apart. Then (1) is solvable in \( \Omega \).

We observe that none two of these results, namely, Corollaries 2 and 3 above and Corollary 4 of [LM] are comparable.

It is proved in [EFR], Corollary 5, that any unbounded convex domain admitting a bounded solution to (1) is necessarily contained between two parallel lines \( 1/H \) far apart. Using this result and Corollary 3 above, we obtain:

**Corollary 4.** Let \( \Omega \) be a convex unbounded domain. Then (1) is solvable in \( \Omega \) if and only if \( \Omega \) is contained between two parallel lines \( 1/H \) far apart.

One can also use Theorem 3 to prove the existence of solutions to the Dirichlet problem with hypothesis on the area do the domain. For doing this, it is necessary to obtain an estimate of the area of the domain in terms of the height of the graph, what is done in the next result.

**Proposition 2.** Let \( \Omega \) be a bounded domain in the plane and let \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) be a non-negative solution to \( \Delta_H u = 0 \) in \( \Omega \) with \( u|_{\partial \Omega} = 0 \). Let \( G \) be the graph of \( u \) and set \( h = \max_{\overline{\Omega}} u \). Then

\[
\text{Area}(G) < (1 + 2hH) \text{Area}(\Omega)
\]
and
\[ \frac{2\pi h}{H(1+2hH)} < \text{Area}(\Omega). \]

**Corollary 5.** Let \( \Omega \) be a bounded convex domain in the plane such that 2\( \text{Area}(\Omega)H^2 < \pi \). Then there is a solution of (1) in \( \Omega \).

Corollary 3 implies that one has examples of bounded convex domains with arbitrarily large area (and therefore perimeter) where (1) has a solution. Therefore, there is no chance to get a converse on either Corollary 5 above or Corollary 4 of [LM]: The existence of a solution of (1) in a convex bounded domain \( \Omega \) does not imply the existence of an upper bound for \( \text{Area}(\Omega)H^2 \) (or \( L(\Omega)H \)). Nevertheless, we believe that Corollary 5 is not optimal. It implies the a priori height \( 1/(2H) \) for the cmc \( H \) graphs on \( \Omega \) which seems to be too “low”. We might expect that \( \text{Area}(\Omega)H^2 < \pi \) should enough to guarantee the solvability of (1) in \( \Omega \).

In the last years a number of papers have been written studying compact cmc surfaces whose boundary is a given Jordan curve in \( \mathbb{R}^3 \). Many related problems however remain still opened. For example, it is not known if an embedded cmc surface in \( \mathbb{R}^3_+ = \{ z \geq 0 \} \) whose boundary is a convex curve in the plane \( P = \{ z = 0 \} \) can have genus bigger than 0 (see [RR] and references therein). Using our previous results, we were able to prove that if the slope of the tangent planes of \( M \) at \( \partial M \) are not too small (in terms of the boundary data), then the surface is either a graph or part of a sphere. In particular, the surface is a topological disk. Precisely, we prove:

**Theorem 4.** Let \( M \) be an embedded connected cmc \( H > 0 \) surface with boundary \( \partial M \) in the plane \( P = \{ z = 0 \} \). We assume that \( M \) is contained in the half space \( z \geq 0 \) and that any connected component of \( \partial M \) is a convex curve. Assume furthermore that the closed interior of any two curves in \( \partial M \) have disjoint intersection. Set \( N = (1/H)\overline{H} \), where \( \overline{H} \) is the mean curvature vector of \( M \), and let \( n \) denote the unit vector along \( \partial M \) in the plane \( P \) normal to \( \partial M \) and pointing to the bounded connected components of \( P \setminus \partial M \).

Given any connected component \( C \) of \( \partial M \), if one of the alternatives below holds, then \( M \) is either a graph over \( P \) or a part of sphere of cmc \( H \). In particular, the genus of \( M \) is zero.

(a) Setting \( \kappa_0 = \min \kappa_C \), where \( \kappa_C \) is the curvature of \( C \) in the plane \( P \), we require that \( \kappa_0 \geq H \) and \( \langle N(p), n(p) \rangle \geq H/\kappa_0 \) if \( \langle N(p), e_3 \rangle \geq 0 \), \( p \in C \) (\( e_3 = (0,0,1) \)).

(b) Setting \( d = \inf d(l_1, l_2) \), where \( l_1, l_2 \) are any two parallel lines in \( P \) such that \( C \) is in between \( l_1 \) and \( l_2 \), and \( d(l_1, l_2) = \inf \{ ||p - q||, p \in l_1, q \in l_2 \} \),
$q \in \ell_2$, we require that $dH \leq 1$, and
\[
\langle N(p), n(p) \rangle \geq \frac{dH}{\sqrt{1 - d^2H^2}}
\]
if $\langle N(p), e_3 \rangle \geq 0, p \in C$.

(c) Denoting by $A$ the area of the region enclosed by $C$, we require that
\[
2AH^2 < \pi
\]
and
\[
\langle N(p), n(p) \rangle \geq \frac{H\sqrt{A(2\pi - 3H^2A)}}{\pi - 2H^2A}
\]
if $\langle N(p), e_3 \rangle \geq 0, p \in C$.

R. López and S. Montiel proved that an embedded compact cmc surface $M$ in $\mathbb{R}^3$ with area $A$ satisfying $AH^2 \leq \pi$ is necessarily a graph ([LM]).

The corollary of Theorem 4 that follows gives an additional characterization of a graph from a bound on the area of the surface, where the condition $AH^2 \leq \pi$ is not necessarily satisfied. Just as an example, one may deduce from Corollary 6 below that if $H^2A \leq (95/84)\pi$ and $\partial M$ is a convex planar curve bounding a domain with area $a$ satisfying $(5/12)\pi \leq aH^2 \leq \pi/2$, then $M$ is a graph.

**Corollary 6.** Let $M$ be a compact embedded surface of cmc $H$ whose boundary $\partial M$ is a convex curve in the plane $z = 0$, boundary of a planar domain $\Omega$. Denote by $A$ the area of $M$ and by $a$ the area of $\Omega$. Assume that $2aH^2 \leq \pi$ and that
\[
A \leq \frac{2\pi - aH^2}{\pi - aH^2}a.
\]
Then $M$ is a graph.

Next, we consider a more general situation: The boundary $\partial M$ of $M$ (compact embedded cmc $H$ surface) is not necessarily plane but has a $1-1$ projection over a convex curve $\gamma$ in the plane $z = 0$. By taking the right cylinder $C$ over $\gamma$, we assume that $M$ does not intersect the connected component of $C \setminus \partial M$ which is below $\partial M$. We prove that if there exists a neighborhood of $\partial M$ in $M$ which is a graph over a neighborhood of $\gamma$ in $\Omega$, where $\Omega$ in the interior of $\gamma$, then $M$ is a graph over $\Omega$ (Theorem 5). These hypothesis can be weakened when $\partial M$ is a plane curve: If there is a neighborhood of $\partial M$ in $M$ contained in the half space $z \geq 0$ which is a graph over a neighborhood of $\partial M$ in $\Omega$, then $M$ is a graph (Theorem 6).

1. **The minimal case.**

**Proof of Theorem 1.** If $s = 0$ then Theorem 1 has a trivial proof. Thus, let us assume that $s > 0$. We first consider the case that $\Omega$ is a $C^\infty$ domain. Given $n$ denote by $D^1_n$ the open disk centered at the origin with radius $n$ and
assume that \( n \) is such that \( D^1_n \) contains \( G_1 \cup \ldots \cup G_m \). Let \( E_1, \ldots, E_{k(n)} \) be the closed domains of \( \mathbb{R}^2 \setminus \Omega \) which are contained in \( D^1_n \), set \( a_i = \partial E_i \), \( \alpha_n = a_1 \cup \ldots \cup a_{k(n)} \).

Let \( L_n \) be the catenoid tangent to the cylinder \( H = C^1_n \times \mathbb{R} \), \( C^1_n = \partial D^1_n \), along the circle \( C^1_n \). Assume that \( L_n \cap \{ z \geq 0 \} \) is the graph of the function \( v_n \) in \( \mathbb{R}^2 \setminus D^1_n \). We choose \( R_n \geq n^2 \) such that \( |\nabla v_n| \leq s/2 \) at the circle \( C^2_n \) centered at the origin with radius \( R_n \). Let \( N \) be the unit normal vector to the catenoid that points to the rotational axis. Set \( I_n = H_n \cap L_n \cap \{ z \geq 0 \} \), where \( H_n := C^2_n \times \mathbb{R} \). Set \( \Omega_n = D^2_n \setminus (E_1 \cup \ldots \cup E_{k(n)}) \), where \( D^2_n \) is the disk bounded by \( C^2_n \).

We set

\[
T_n = \left\{ t \geq 0 \mid \exists u_t \in C^\infty(\Omega_n), \text{ such that } Q_0(u_t) = 0, \sup_{\Omega_n} |\nabla u_t| \leq s, \text{ and } u_t|_{\alpha_n} = 0, \ u_t|_{C^2_n} = t \right\).
\]

We have \( T_n \neq \emptyset \) since \( 0 \in T_n \) and obviously \( \sup T_n < +\infty \). Set \( t_n = \sup T_n \). We prove that \( t_n \in T_n \) and that \( \sup_{\alpha_n} |\nabla u_{t_n}| = s \). Given \( t \in T_n \), we first observe that \( \sup_{C^2_n} |\nabla u_t| \leq s/2 \). In fact: Let \( \eta \) denote the interior unit normal vector to \( C^2_n \), and denote also by \( \eta \) its extension to \( \mathbb{R}^3 \setminus \{ z - \text{axis} \} \) by radial translation in each plane \( z = c \), and let \( N_t \) be the unit normal vector to the graph \( G_t \) of \( u_t \) pointing upwards. Since \( G_t \) is contained in the convex hull of its boundary, we have \( \langle N_t, \eta \rangle \geq 0 \) at \( I_n \).

Moving \( L_n \) down if necessary, we have that \( L_n \cap G_t = \emptyset \). Going up with \( L_n \) until it touches the circle \( C^2_{n,t} \) centered at the \( z \)-axis, with radius \( R_n \), contained in the plane \( z = t \), we will obtain, from the maximum principle, that \( \langle N_t, \eta \rangle < \langle N, \eta \rangle < 1 \) at \( I_n \) (recall that any vertical translation of \( L_n \) does not intersect any curve \( \alpha_s \)). Since, by construction, \( L_n \) is given as a graph of a function \( v_n \) such that \( |\nabla v_n| \leq s/2 \) at \( C^2_n \) it follows that \( |\nabla u_t| \leq s/2 \) at \( C^2_n \).

Let \( \{ s_m \} \subset T_n \) be a sequence converging to \( t_n \) as \( m \to \infty \). Since the functions \( u_{s_m} \) are uniformly bounded having uniformly bounded gradient, standard \( C^k \) estimates guarantee us the existence of a subsequence of \( \{ u_{s_m} \} \) converging uniformly \( C^\infty \) on \( \Omega_n \) and to a solution \( w \in C^\infty(\Omega_n) \) of \( Q_0 = 0 \) in \( \Omega_n \). One of course has \( \sup_{\Omega_n} |\nabla w| \leq s, w|_{C^2_n} = t_n \) and \( w|_{\alpha_n} = 0 \), so that \( t_n \in T_n \) and \( w = u_{t_n} \).

Suppose that \( \sup_{\Omega_n} |\nabla u_{t_n}| < s \). Applying the implicit function theorem, one can guarantee the existence of a solution \( u' \in C^\infty(\Omega_n) \) to \( Q_0 = 0 \) vanishing at \( \alpha_n \) and taking on a value \( t_n + \epsilon \) on \( C^2_n \), \( \epsilon > 0 \). For \( \epsilon \) small enough, one still has \( \sup_{\Omega_n} |\nabla u'| < s \). It follows that \( t_n + \epsilon \in T \), a contradiction! Therefore, \( \sup_{\Omega_n} |\nabla u_{t_n}| = s \). Since \( \sup_{C^2_n} |\nabla u_{t_n}| \leq s/2 \), we obtain, from the gradient maximum principle, \( \sup_{\alpha_n} |\nabla u_{t_n}| = s \).
We define now a sequence \( \{u_n\} \) of non-negative solutions to \( Q_0 = 0 \) in the domain \( \Lambda_n := D_n^3 \cap \Omega \), where \( D_n^3 \) is an open disk centered at the origin with radius \( n^2 - n \), such that \( u_n|_{\partial \Lambda_n \setminus \partial D_n^3} = 0 \) and
\[
\sup_{\partial \Lambda_n \cap D} |\nabla u_n| = \sup_{D_n^3 \setminus \partial \Omega} |\nabla u_n| = s
\]
if \( D_n^3 \) contains \( D \), as follows. Given \( n \) such that the disk \( D_n^1 \) contains the fundamental domain \( D \), from what we have proved above there exists \( p_n \in \alpha_n \) such that \( |\nabla u_n(p_n)| = s \).

If \( p_n \in D \), then we set \( u_n = u_{1n}|_{\Lambda_n} \). If \( p_n \not\in D \), we take an isometry \( \phi \in \Gamma \) such that \( \phi(p_n) \in D \), and define \( u_n(p) = u_{1n}(\phi^{-1}(p)) \), for \( p \in \Lambda_n \). Since \( |p_n| < n \), it follows \( \phi^{-1}(p) \in D_n^2 \), for all \( p \in D_n^3 \), so that \( u_n \) is well defined and satisfies the stated conditions.

By standard \( C^k \) estimates, it follows that the sequence \( \{u_n\} \) contains a subsequence that converges uniformly on compacts of \( \overline{\Omega} \) to a solution \( u \in C^\infty(\overline{\Omega}) \) to \( Q_0 = 0 \) such that \( u|_{\partial\Omega} = 0 \) and (2) is obviously satisfied.

If \( \Omega \) is just \( C^0 \), we can take sequences \( \gamma_{i,n} \) of \( C^\infty \) curves contained in \( G_i \), for each \( 1 \leq i \leq m \) such that \( \gamma_{i,n} \) converges \( C^0 \) to \( \gamma_i \) as \( n \to \infty \). We can therefore apply the result obtained above and compactness results to guarantee the existence of a solution of (1) in \( \Omega \), concluding the proof of Theorem 1.

Remark. The proof of Theorem 1 is “experimental” in the sense that one can reproduce the arguments of the proof by using soap films. To obtain experimentally a solution \( u_{1n} \) of \( Q_0 = 0 \) in \( \Omega_n \), we represent the curves \( a_1, \ldots, a_n \) and the circle \( C_n^2 \) by wires and embed them in a soaped water, taking care that they are kept in the same plane. Then take them out from the water and drill the soap films that are enclosed by the wires \( a_1, \ldots, a_n \). One obtains the zero solution in \( \Omega \). Now, given \( s > 0 \), we lift the circle \( C_n^2 \) up until the soap film reaches the slope \( s \) at some curve \( a_i \). The resulting soap film is the graph of \( u_{1n} \).

2. The case \( H > 0 \).

In order to state and prove some of the next results, we need to introduce some notations and definitions.

Consider a bounded open domain \( \Omega \) in the plane whose boundary consists of a finite number of \( C^2 \) embedded Jordan curves. Set \( \Omega^c = \mathbb{R}^2 \setminus \Omega \) and let \( p \in \partial \Omega \) be given. If \( \Omega \) is globally convex at \( p \), let \( l_1(p, \Omega) \) be the tangent line to \( \partial \Omega \) at \( p \) and let \( l_2(p, \Omega) \) be the closest parallel line to \( l_1(p, \Omega) \) such that \( \Omega \) is between \( l_1(p, \Omega) \) and \( l_2(p, \Omega) \). We then set
\[
R_1(p, \Omega) = d(l_1(p, \Omega), l_2(p, \Omega)) = \inf\{|q_1 - q_2| \mid q_i \in l_i(p, \Omega), \ i = 1, 2\}.
\]

If \( \Omega \) is not globally convex at \( p \), denote by \( C_1(p, \Omega) \) the circle tangent to \( \partial \Omega \) at \( p \), contained in \( \Omega^c \), and whose radius is the biggest one among those
circles satisfying these properties. Denote by $C_2(p, \Omega)$ the circle with the same center as $C_1(p, \Omega)$ of smallest radius $R_2(p, \Omega)$ containing $\Omega$.

If $\partial \Omega$ admits a circle tangent to $\Omega$ at $p$ and containing $\Omega$ in its interior, let $R_3(p, \Omega)$ be the smallest radii of these circles. If $\Omega$ does not admit such a circle at $p$, we set $R_3(p, \Omega) = \infty$.

Finally, set
\[ W(p, \Omega) = \min\{R_1(p, \Omega), R_2(p, \Omega), R_3(p, \Omega)\}. \]

**Theorem 2.** Let $H_0 > 0$ be given and let $\Omega$ be a $C^{2,\alpha}$ bounded domain in the plane $z = 0$. Given a point $p$ of $\partial \Omega$, let $r(p, \Omega)$, $0 < r(p, \Omega) \leq \infty$, be the radius of the circle $C_1(p, \Omega)$ (the circle of biggest radius tangent to $\partial \Omega$ at $p$ and contained in $\mathbb{R}^2 \setminus \Omega$). We require that
\[ W(p, \Omega) \leq \frac{2}{H_0 \left(1 + \sqrt{1 + \frac{1}{r(p, \Omega)}}\right)} \]
for all $p \in \partial \Omega$. Then the Dirichlet problem (1) is solvable for any $0 \leq H \leq H_0$. Furthermore, if the inequality is strict in (7), for all $p \in \partial \Omega$, then the solution is in $C^{2,\alpha}(\Omega)$.

For proving Theorem 2 we will use, as barriers, rotational graphs with cmc described in the proposition below.

**Proposition 1.** Let $0 < r$ and $H > 0$ be given. Then there exists a rotational graph with cmc $H$ defined on an annulus in the plane whose boundary consists of two concentric circles of radii $r$ and $R$, where $R$ satisfies:
\[ R \geq \frac{2}{H \left(1 + \sqrt{1 + \frac{1}{rH}}\right)}. \]

**Proof.** It is well-known that there is a nodoid $N$ in the plane $x-z$ generating, by rotation around the $z$-axis, a cmc $H$ surface (with self intersections) whose distance to the rotational axis is $r$. We can assume that the point $A = (r, 0)$ belongs to $N$. We consider an embedded piece $N_r$ of $N$ from the point $A$ to the closest point $B$ of self intersection of $N$ and such that the $z$-coordinate of any point of $N_r$ is non-negative. The coordinates of $B$ are of the form $(0, R)$, $R > r$.

The rotation of $N_r$ around the $z$-axis is a graph over the plane $z = 0$ which vanishes along two circles centered at the origin with radius $r$ and $R$, being orthogonal to the circle of radius $r$. We prove that $R$ satisfies (8).

It is known that if $x = x(t)$, $z = z(t)$ represent a piece of a generating curve of a cmc $H$ rotational surface, then these functions satisfy the first order system of ordinary differential equations (see Lemma 3.15 of [doCD]):
\[
\begin{align*}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= 1 - \left(Hx - \frac{4}{H}\right)^2 \\
\left(\frac{dx}{dt}\right)^2 &= 1 - \left(Hx - \frac{4}{H}\right)^2
\end{align*}
\]
where $a$ is a constant. In the points where $x = x(z)$, we therefore get

$$x'^2 = \left(\frac{x}{a - Hx^2}\right)^2 - 1.$$  

We then have

$$z'(x) = \frac{1}{\sqrt{\left(\frac{x}{a - Hx^2}\right)^2 - 1}}$$  

for $r \leq x \leq x_1$, and

$$z'(x) = -\frac{1}{\sqrt{\left(\frac{x}{a - Hx^2}\right)^2 - 1}}$$  

for $x_1 \leq x \leq R$, where $x_1 \in (r, R)$ is such that $z'(x_1) = 0$. Since $z'(r) = \infty$, we have

$$\left(\frac{r}{a - Hr^2}\right)^2 - 1 = 0$$  

and $a = r(-1 + Hr)$ or $a = r(1 + Hr)$. In the case of the nodoids, we know moreover that $x'(z_1) = \infty$, where $z_1 = z(x_1)$. Therefore we have $a = Hx_1^2 \geq Hr^2$, and this implies that $a = r(1 + rH)$. It follows that

$$x_1 = \sqrt{\frac{r(1 + rH)}{H}}.$$  

Now, a computation shows that

$$z'(x_1 - x) \geq -z'(x_1 + x),$$  

for all $x \in [0, x_1 - r]$. It follows then that

$$R \geq x_1 + (x_1 - r) \geq \frac{2}{H \left(1 + \frac{1}{\sqrt{1 + rH}}\right)}$$  

proving the proposition.

Proof of Theorem 2. Considering the family of Dirichlet’s problems

$$\text{(9) } \text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = -2h, \quad u_{\partial\Omega} = 0, u \in C^2(\Omega), \; h \in [0, H_0)$$  

we first prove that

$$S := \{h \in [0, H_0) \mid \text{(9) has a solution}\}$$  

coincides with $[0, H_0)$. Since (9) has the trivial solution for $h = 0$, we have $S \neq \emptyset$. From the implicit function theorem, it follows that $S$ is open in $[0, H_0)$. For proving that $S$ is closed in $[0, H_0)$, let us consider a sequence $\{h_n\} \subset S$ converging to $H_1 \in [0, H_0)$. If $H_1 = 0$ then $H_1 \in S$ and we are done. Thus, let us assume that $H_1 > 0$. By contradiction, assume that
Given \( h \in S \). Let \( u_n \in C^2(\Omega) \) be a solution of (9) for \( h = h_n \). We claim that the gradient of the solutions \( u_n \) can not be uniformly bounded. In fact, otherwise, since by very known height estimates, \( u_n \) satisfies
\[
|u_n| \leq \frac{1}{H_1 - \epsilon}
\]
for all \( n \) bigger than a certain \( n_0 \), for some \( \epsilon > 0 \) smaller than \( H_1 \) and, by assumption, the gradient of the \( u_n \) are uniformly bounded, using standard \( C^k \) estimates, \( k \geq 2 \), one can extract a subsequence of \( u_n \) converging uniformly on compacts of \( \Omega \) to a solution \( u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) \) of \( Q_{H_1} = 0 \), such that \( u|_{\partial \Omega} = 0 \). Since \( \Omega \) is a \( C^2, \alpha \) domain, \( u \) is \( C^\infty \) in \( \Omega \) and has bounded gradient, \( u \) extends \( C^2, \alpha \) to \( \Omega \) (this is well known. It can be proved by reducing the problem to linear uniformly elliptic operators and applying standard arguments: See p. 249 paragraph 3 and Remark 1 p. 253 of [GT]. See also Lemma 1.1 of [EFR]). Therefore \( u \) solves (9) for \( h = H_1 \), implying that \( H_1 \in S \), a contradiction! This proves our claim.

Therefore, there is a sequence \( p_n \in \partial \Omega \) such that \( |\nabla u_n(p_n)| \to \infty \), as \( n \to \infty \). Without loss of generality, we may assume that \( p_n \) converges to \( p \in \partial \Omega \). Denote by \( G_n \) the graph of \( u_n \). Since \( H_1 < H_0 \), condition (7) allows us to take \( H' \in (H_1, H_0) \) and \( 0 < r' < r(p, \gamma) \) such that
\[
W(p, \Omega) < \frac{2}{H' \left(1 + \sqrt{1 + \frac{1}{r'H'}}\right)}.
\]
According to our previous notations, if \( W(p, \Omega) = R_3(p, \Omega) \), then \( \overline{\Omega} \) is contained in an open disk \( D \) of radius \( 1/H' \), since then
\[
R_3(p, \Omega) < \frac{2}{H' \left(1 + \sqrt{1 + \frac{1}{r'H'}}\right)} < \frac{1}{H'}.
\]
If \( W(p, \Omega) = R_1(p, \Omega) \), then \( \overline{\Omega} \) is contained in an open strip \( S \) having as boundary two parallel lines \( 1/H' \) far apart, since in this case there holds (10) with \( R_1(p, \Omega) \) in place of \( R_3(p, \Omega) \). Finally, in the case that \( W(p, \Omega) = R_2(p, \Omega) \) we conclude that \( \overline{\Omega} \) is contained in an annulus \( A \) of radius \( r' \) and \( R' := R_2(p, \Omega) \) with
\[
R' < \frac{2}{H' \left(1 + \sqrt{1 + \frac{1}{r'H'}}\right)}.
\]
In this last case, we can use Proposition 1 to guarantee the existence of a graph \( G \) of \( {\it {cmc}} \) \( H' \) (a piece of nodoid), defined in an annulus \( A' \) with \( A \subset A' \). Therefore, if \( U' \) denotes either \( D \), \( S \) or \( A' \), we may consider a \( {\it {cmc}} \) \( H' \) graph \( G' \) of a function \( u' \) defined in \( U' \) with \( \overline{\Omega} \subset U' \) such that \( G' \) is either a half sphere if \( U' = D \), a half cylinder if \( U' = S \) or \( G' = G \) if \( U' = A' \).
Moving \( G' \) slightly down in such a way that its domain \( U' \) still contains \( \Omega \), we may assume that

\[
|\nabla u'(p)| < C < \infty, \ p \in \partial U'.
\]

Now, we can move \( U' \) towards \( \Omega \) until the boundary of \( U' \) is tangent to \( \gamma \) at \( p \) and in such a way that \( \partial U' \) does not intersect \( \partial \Omega \) along this motion. By choosing \( n \) big enough, we have \( |\nabla u_n| > C \) so that the graph \( G_n \) of \( u_n \) and \( G' \) intersect themselves in interior points. But then, moving \( G_n \) vertically down until it reaches the last contact with \( G' \), we obtain a tangency between \( G_n \) and \( G' \) in an interior point, with \( G_n \) below \( G' \), a contradiction, since the mean curvature \( h_n \) of \( G_n \) satisfies \( h_n < H' \) for \( n \) big enough. Therefore, \( H_1 \in S \), and this proves the existence of a solution for (9) for \( H \in [0, H_0) \).

To prove the existence of a solution for \( H = H_0 \) of (1), let us consider a sequence \( H_n < H_0 \) with \( \lim_{n \to \infty} H_n = H_0 \). Given \( n \), let \( u_n \) be a solution of (1) in \( \Omega \) for \( H = H_n \). By the maximum principle, the sequence \( \{u_n\} \) is monotonically increasing, and standard compactness results guarantee us the \( \{u_n\} \) converges uniformly on compacts of \( \Omega \) to a solution \( u \in C^2(\Omega) \) of \( Q_H = 0 \) (given by \( u(p) = \lim_n u_n(p), \ p \in \Omega \)). To prove that \( u \in C^0(\Omega) \), let us consider a sequence \( p_n \in \Omega \) converging to \( p \in \partial \Omega \). As done above, we can consider a solution \( v \in C^2(\Lambda) \cap C^0(\Lambda) \) to \( Q_H = 0 \) where \( \Lambda \) is a domain containing \( \Omega \) with \( p \in \partial \Lambda \) and such that \( v|_{\partial \Lambda} = 0 \) (the graph of \( v \) is either part of sphere, a cylinder, or a Delaunay surface, and the gradient of \( v \) may have infinity norm at the boundary of the domain). By the maximum principle, \( u_m(p_n) \leq v(p_n) \), for all \( n \) and \( m \). It follows that \( 0 \leq \lim_n u(p_n) = \lim_n \lim_m u_m(p_n) \leq \lim_n v(p_n) = 0 \). This proves that \( u \in C^2(\Omega) \cap C^0(\Omega) \) and \( u|_{\partial \Omega} = 0 \), that is, \( u \) solves (1) for \( H = H_0 \) in \( \Omega \).

Now, if the inequality is strict in (7), we get a solution of (1) in \( \Omega \) whose gradient has bounded norm in \( \Omega \). Since \( \Omega \) is \( C^{2,\alpha} \), standard arguments already used above imply that the solution extends \( C^{2,\alpha} \) to \( \overline{\Omega} \), concluding the proof of Theorem 2.

**Proof of Corollary 1.** We have just to assure that \( \Omega \) satisfies the hypothesis of Theorem 2. Given \( p \in \partial \Omega \), if \( p \in \gamma \) then \( r(p, \Omega) = \infty \), so that we must have \( W(p, \Omega) < 1/H \), what is obviously true since the curvature of \( \gamma \) satisfies \( \kappa \geq 3H \). If \( p \in \gamma_i \) for some \( i \), then we can place a circle centered at \( z(p, \Omega) \) of radius \( r \) tangent \( \gamma_i \) and contained in \( E_i \), such that

\[
r > \frac{1}{a(\frac{a}{11} - 2)}.
\]

It follows from this that

\[
\frac{2}{a} < \frac{2}{H \left(1 + \sqrt{1 + \frac{1}{11}}\right)}.
\]
On the other hand, since the geodesic curvature $\kappa$ of $\gamma$ satisfies $\kappa > a$, $\Omega$ is contained in a circle of radius $2/a$ centered at $z(p, \Omega)$, so that $W(p, \Omega) < 2/a$, showing that (7) is satisfied and concluding the proof of the corollary.

**Proof of Corollary 2.** In the case that $\partial \Omega$ is a convex curve the second hand of (7) equals to $1/H$ for all $p \in \partial \Omega$, since $r(p, \partial \Omega) = \infty$, for all $p \in \partial \Omega$. Therefore, if one requires that the curvature of $\partial \Omega$ is bigger than or equal to $H$, (7) is everywhere satisfied, proving the corollary.

**Proof of Theorem 3.**

Set

$$T = \{ h \in [0, H] \mid \exists u \in C^2(\Omega) \text{ such that } Q_h(u) = 0 \text{ in } \Omega, \ u|_{\partial \Omega} = 0 \}.$$  

Then $0 \in T$ so that $T \neq \emptyset$. From the implicit function theorem, $T$ is open. Let $h_n \in T$ be a sequence converging to $h \in [0, H]$. Let $u_n$ be the solution to $Q_{h_n} = 0$ in $\Omega$ such that $u_n|_{\partial \Omega} = 0$, $u_n \in C^2(\Omega)$. We prove that the gradient of $\{u_n\}$ is uniformly bounded. From standard compactness results, it will follow that $\{u_n\}$ contains a subsequence converging uniformly on compacts of $\Omega$ to a solution $u \in C^{2,a} (\Omega)$ to $Q_h = 0$ in $\Omega$ with $u|_{\partial \Omega} = 0$. It follows that $h \in T$ and $T$ is closed. Thus, $T = [0, H]$ proving Theorem 3.

By contradiction, suppose the existence of $p_n \in \Omega$ such that $\lim_{n \to \infty} |\nabla u_n| = \infty$. Without loss of generality, we may assume that $p_n$ converges to $p \in \overline{\Omega}$. From interior gradient estimates, $\{u_n\}$ contains a subsequence, which we consider as being $\{u_n\}$ itself, converging uniformly on compacts of $\Omega$ to a solution $u \in C^2(\Omega)$ to $Q_h = 0$ in $\Omega$. This implies that $p \in \partial \Omega$. Let us consider the quarter of cylinder, say $C$, given by

$$z(x, y) = \sqrt{\frac{1}{4H^2} - x^2}, \quad -\frac{1}{2H} \leq x \leq 0$$

whose boundary consists of the straight lines $l_1 : \{ z = 0, x = -1/(2H) \}$ and $l_2 : \{ z = 1/(2H), x = 0 \}$. We apply a rigid motion on $C$ such that $l_1$ coincides with the tangent line to $\partial \Omega$ at $p$ and such that the projection of $l_2$ in the plane $z = 0$ contains points of $\Omega$. Call this cylinder $C$ again. Moving $C$ slightly down, we have that the height of the line $l_2$, after this motion, is still bigger than $a$, and the norm of the gradient of $z(x, y)$ at $l_1$ is finite, say $D$. By the assumption on the sequence $\{u_n\}$, we can get $n$ such that $|\nabla u_n(p)| > D$. It follows that the graph $G_n$ of $u_n$ is locally above $C$ in a neighborhood of $p$. By hypothesis, the height of $G_n$ for any $n$ is smaller than $a$ so that, if we move $G_n$ vertically down we will obtain a last interior contact point between $G_n$ and $C$, a contradiction with the maximum principle, proving Theorem 3.

**Proof of Corollary 3.** Assume that $\Omega$ is contained in the strip $\Lambda$ having as boundary two parallel lines $l_1$ and $l_2$ with

$$d(l_1, l_2) = \inf\{||x - y|| \mid x \in l_1, \ y \in l_2\} \leq \frac{1}{H}$$
and let us first consider the case that $\Omega$ is compact. Given $l > 0$, we consider the domain $\Lambda_l = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ given by

$$
\Lambda_1 = \left\{(x, y) \mid -l \leq x \leq l, \ |y| \leq \frac{1}{2H}\right\}
$$

$$
\Lambda_2 = \left\{(x, y) \mid (x+l)^2 + y^2 \leq \frac{1}{4H^2}, \ x \leq -l\right\}
$$

$$
\Lambda_3 = \left\{(x, y) \mid (x-l)^2 + y^2 \leq \frac{1}{4H^2}, \ x \geq l\right\}.
$$

By choosing $l$ big enough, we have $\Omega \subset \Lambda_l$. It follows from the proof of Theorem 3.2 of [EFR] the existence of a solution $v \in C^2(\Lambda_1) \cap C^0(\overline{\Lambda})$ to $Q_H = 0$ in $\Lambda_1$ such that $v|_{\partial \Lambda_1} = 0$ (obtained by using the method of Perron). Since on $\Lambda$ one can place a half cylinder of mean curvature $H$ whose boundary is the two lines $l_1$ and $l_2$, it follows immediately from the maximum principle that $a := \max_{\Lambda} u < 1/(2H)$. Using again the maximum principle, it follows that $\Omega$ satisfies the a priori height estimate $\max_{\Omega} u \leq a$ where $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is any solution to $Q_h = 0$ in $\Omega$ such that $u|_{\partial \Omega} = 0$, for any $h \in [0, H]$.

We consider now a sequence of convex bounded $C^{2,\alpha}$ domains $\Omega_n$ satisfying

$$
\Omega = \bigcup_{n=1}^{\infty} \Omega_n, \quad \overline{\Omega_n} \subset \Omega_{n+1}.
$$

Obviously the domains $\Omega_n$ satisfy the height estimate $\max_{\Omega_n} u \leq a$ so that we can get for any $n$, by Theorem 3, a solution $u_n \in C^{2,\alpha}(\overline{\Omega_n})$ to $Q_H = 0$ in $\Omega_n$ with $u_n|_{\partial \Omega_n} = 0$. Standard compactness results guarantee us the existence of a subsequence of $\{u_n\}$, which we assume to be $\{u_n\}$ again, converging uniformly on compacts of $\Omega$ to a solution $u \in C^2(\Omega)$ to $Q_H = 0$ in $\Omega$. By using quarter of cylinders as in the proof of Theorem 3 we have that the norm of the gradient of the family $\{u_n\}$ is uniformly bounded and therefore the gradient of $u$ is bounded on $\Omega$. It follows that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u|_{\partial \Omega} = 0$.

We suppose now that $\Omega$ is any convex domain in $\Lambda$ and set $\Omega_n = \Omega \cap \Lambda_n$. Choose $n_0$ such that $\Omega_n$ is non-empty for all $n \geq n_0$. Of course, $\{\Omega_n\}$ is a sequence of convex bounded domains contained in $\Omega$ satisfying the same conditions as above. According to what was proved before, there is, for each $n \geq n_0$ a solution $u_n \in C^2(\Omega_n) \cap C^0(\overline{\Omega_n})$ to $Q_H = 0$ in $\Omega_n$ such that $u_n|_{\partial \Omega_n} = 0$. Without loss of generality, we may assume that $\{u_n\}$ converges uniformly on compacts of $\Omega$ to a solution $u \in C^2(\Omega)$ to $Q_H = 0$ in $\Omega$. To prove that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u|_{\partial \Omega} = 0$, in view of $u_n|_{\partial \Omega_n} = 0$, it is enough to prove that the family $\{u_n\}$ has uniformly bounded gradient norm in any compact $K$ of $\overline{\Omega}$.
Let $K$ be a given compact of $\Omega$. Let $n_1$ be such that $K \subset \Omega_{n_1}$ for all $n \geq n_1$. Clearly, one has $a := \max_{\Omega_{n_1}} u < 1/(2H)$. Since, by the maximum principle, $u_n \leq u$ for all $n$, we have $\max_{\Omega_{n_1}} u_n \leq a$, for all $n \geq n_1$. Hence, one can apply again the argument using quarters of cylinders to conclude that the family $\{u_n\}$ has uniformly bounded norm of the gradient in $\Omega_{n_1}$, and therefore in $K$, finishing the proof of Corollary 3.

**Proof of Proposition 2.** We have, since $|\nabla u|$ is not identicaly zero,

\[
\text{Area}(G) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx < \int_{\Omega} \left(1 + \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}}\right) dx = \text{Area}(\Omega) + \int_{\Omega} \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} dx.
\]

Given $s \in [0, h)$ we set

\[\Omega_s = \{(x_1, x_2) \in \Omega \mid u(x_1, x_2) \geq s\}\]

and $\Gamma_s = \partial \Omega_s$. By the coarea formula ([F], 3.2.22), one has

\[
\int_{\Omega} \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} dx = \int_0^h \left[\int_{\Gamma_s} \frac{|\nabla u|}{\sqrt{1 + |\nabla u|^2}} dl_s\right] ds.
\]

Integrating Equation (1) (with $\Omega_s$ in place of $\Omega$) and using divergence's theorem we obtain

\[
\int_{\partial \Omega_s} \frac{|\nabla u|}{\sqrt{1 + |\nabla u|^2}} dl_s = 2H \text{Area}(\Omega_s),
\]

so that

\[
\text{Area}(G) < \text{Area}(\Omega) + 2H \int_0^h \text{Area}(\Omega_s) ds \leq (1 + 2hH)\text{Area}(\Omega),
\]

proving the first inequality of Proposition 2.

By Theorem 1 of [LM], we have

\[
\frac{2\pi h}{H} \leq \text{Area}(G)
\]

so that

\[
\frac{2\pi h}{H} < (1 + 2hH)\text{Area}(\Omega)
\]

what gives the second inequality of Proposition 2, concluding its proof.

**Proof of Corollary 5.** It follows from the second inequality of Proposition 2 that given any solution $u \in C^2(\Omega) \cap C^0(\Omega)$ to $Q_h = 0$ in $\Omega$ with $u|_{\partial \Omega} = 0$, $h \in [0, H]$ satisfies

\[
\max_{\Omega} u \leq a = \frac{\text{Area}(\Omega)H}{2(\pi - H^2\text{Area}(\Omega))}
\]
and, since $2\text{Area}(\Omega)H^2 < \pi$, it follows that $a < 1/(2H)$. Corollary 5 therefore follows from Theorem 3.

**Proof of Theorem 4.** Let $C_1, \ldots, C_n$ be the connected components of $\partial M$ and let $\Omega_i$ be the region enclosed by $C_i$. Anyone of the conditions (a), (b) or (c) guarantees that there is a graph $G_i$ over $\Omega_i$ with $\partial G_i = C_i$. We assume that $G_i \subset \{z \leq 0\}$, so that $N := M \cup G = G_1 \cup \ldots \cup G_n$ is a compact embedded topological surface without boundary which is $C^\infty$ differentiable with cmc $H$ in $N \setminus C$, $C = C_1 \cup \ldots \cup C_n$. Furthermore, anyone of the conditions (a), (b) or (c) also implies that the tangent planes of $M$ and $G$ at a common point $p \in C$ form a inner angle (that is, respect to the normal $N = (1/H)\vec{H}$) which is smaller than $\pi$ (see below).

We employ now the technique of Alexandrov to obtain a horizontal symmetry of $N$ (if $M$ is not a graph) considering a family of horizontal planes: Denote by $P_t$ the plane $\{z = t\}$. Set $U^+_t = \{z > t\}$ and $N^+_t = N \cap U^+_t$. Let $N^*_t$ be the reflection of $N^+_t$ in the plane $P_t$. Let $V$ be the bounded connected component of $\mathbb{R}^3 \setminus N$.

For $t$ big enough one has $U^+_t \cap N = \emptyset$. Set $t_0 = \inf\{t \mid U^+_t \cap N = \emptyset\}$. For $t$ slightly smaller than $t_0$, we have $N^*_t \subset V$. Therefore, the number

$$t_1 = \inf\{t \leq t_0 \mid N^*_t \subset V\}$$

is well defined and satisfies $-\infty < t_1 < t_0$. There are two possibilities: $t_1 \leq 0$ or $t_1 > 0$. In the first one, it follows that $M$ is a graph over $P$. In the second one, $N^-_t := N \setminus N^+_t$ and $N^*_t$ are either tangent at the boundary or they have a common point, say $p$, belonging neither to the boundary of $N^-_t$ nor to $N^*_t$. In the first case, it follows from the maximum principle at the boundary that $N^-_t \subset N^*_t$. It follows that $N^+_t \cup N^*_t$ is a compact embedded surface with cmc $H$ without boundary. By Alexandrov’s theorem, $N^+_t \cup N^*_t$ is a sphere, and this proves Theorem 4 in this case. In the second case, $p$ cannot belong to $C$ because the plane tangent of $M$ and $G$ at $p$ form an acute angle. Therefore, $p$ is an interior regular point in $N^*_t$ and $N^-_t$ so that $N^*_t$ and $N^-_t$ are tangent at $p$. The maximum again implies that $N^-_t \subset N^*_t$ and this implies, as before, that $M$ is part of a sphere.

We now remark why conditions (a), (b) or (c) imply that the tangent planes form an acute angle as proclaimed above. Setting, as before, $N = (1/H)\vec{H}|_M$, this is clearly the case if $\langle N, e_3 \rangle < 0$. Setting $N_G = (1/H)\vec{H}|_G$, one has therefore to prove that $\langle N(p), n(p) \rangle \geq \langle N_G(p), n(p) \rangle$ if $\langle N(p), e_3 \rangle \geq 0$, where $p \in \partial M$ and $n$ is the unit normal vector field along $\partial M$ in the plane $P$, orthogonal to $\partial M$.

From the hypothesis, in case (a), it follows that any point $p$ in a connected component $C$ of $\partial M$ belongs to a circle $L \subset P$ of radius $R_0 = 1/\kappa_0$ tangent to $C$ at $p$ and whose interior contains the interior of $C$. By the maximum principle, we have $\langle N_G, n \rangle \leq \langle N_S, n \rangle$, where $S$ is the graph a cup
sphere with cmc $H$ such that $\partial S = L$. Now, direct computations show that $\langle N_S, n \rangle = H/\kappa_0$, proving our claim in this case.

In case (b), since the convex domain $\Omega$ enclosed by $C$ is between two parallel lines $l_1, l_2$ with $d(l_1, l_2) = d$, considering the part of cylinder of cmc $H$ having as boundary $l_1$ and $l_2$, we conclude that the height of $G$ is at most

$$\frac{1 - \sqrt{1 - d^2H^2}}{2H}.$$ 

Therefore one can use, up to congruencies, the part of cylinder $v(x, y) = \sqrt{\frac{1}{4H^2} - x^2 - \frac{\sqrt{1 - d^2H^2}}{2H}}$, $-\frac{d}{2} \leq x \leq 0$ as a barrier at any point of $C$ so that we will have $\langle N_G, n \rangle \geq \langle N_S, n \rangle$, where $S$ now is the graph of $v$ in the given domain. A computation then shows that

$$\langle N_S, n \rangle = \frac{dH}{\sqrt{1 - d^2H^2}},$$

proving our claim in the case (b).

In case (c), we have that the height of $G$ (see the proof of Corollary 5) is at most

$$\frac{A H}{2(\pi - H^2 A)}$$

so that we can use the same reasoning of case (b) to conclude that

$$\langle N_S, n \rangle = \frac{\sqrt{H^2 A(2\pi - 3H^2 A)}}{\pi - 2H^2 A}$$

finishing the proof of Theorem 4.

**Proof of Corollary 6.** We first observe that the conditions $2aH^2 \leq \pi$ and $6$ imply that $AH^2 \leq 2\pi$ so that, by Corollary 3 of [LM], $M$ is contained in the half space $z \geq 0$. We prove that the condition

$$(11)\quad \langle N(p), n(p) \rangle \geq \frac{H \sqrt{a(2\pi - 3H^2 a)}}{\pi - 2H^2 a}$$

if $\langle N(p), e_3 \rangle \geq 0$, $p \in \partial M$, is satisfied (we are using the same notations of Theorem 4). For, choose a point $p \in \partial M$ where $\langle N(p), e_3 \rangle \geq 0$. Using reflections on vertical planes orthogonal to $n(p)$ and the maximum principle, we may conclude that $G_p$ is a graph over $P$, where $P$ is the plane orthogonal to $n(p)$ through $p$ and $G_p$ the closure of the connected component of $(\mathbb{R}^3 \setminus P) \cap M$ not containing $\partial M$. It is clear that the area $a_p$ of $G_p$ is smaller than $A - a$ and we may apply Theorem 1 of [LM] to conclude that

$$(12)\quad h_p \leq \frac{Ha_p}{2\pi} < \frac{H(A - a)}{2\pi},$$
where $h_p$ is the height of $G_p$. Let us assume that $G_p$ is given as the graph of a function $u_p \in C^2(\Omega_p)$ where $\Lambda_p$ is some domain in the plane $P$. Then (11) is implied by the inequality

\[(13) \quad |\nabla u_p(p)| \leq \frac{H \sqrt{a(2\pi - 3H^2 a)}}{\pi - 2H^2 a}.
\]

Observe that $\Lambda_p$ is globally convex at $p$ and it follows from (6), (12) and the condition $2aH^2 \leq \pi$ that

\[\frac{H(A - a)}{2\pi} < \frac{1}{2H}.
\]

By the maximum principle we may conclude that $G_p$ is below the piece $C_p$ of a cmc $H$ cylinder given as a graph over the plane $P$ having as boundary the straight tangent line $t$ to $\partial \Lambda_p$ at $p$ and a straight line $s$ above $P$ parallel to $P$ and $t$ at a height

\[\frac{H(A - a)}{2\pi}
\]

from $P$. Hence we can estimate the norm of the gradient of $u_p$ at $p$ by the gradient of $C_p$ at $t$ to obtain

\[|\nabla u_p(p)| \leq \frac{\sqrt{H(A - a)\left(\frac{1}{2\pi} - \frac{H(A - a)}{2\pi}\right)}}{\frac{1}{2\pi} - \frac{H(A - a)}{2\pi}}.
\]

It follows from (6) that

\[\frac{\sqrt{H(A - a)\left(\frac{1}{2\pi} - \frac{H(A - a)}{2\pi}\right)}}{\frac{1}{2\pi} - \frac{H(A - a)}{2\pi}} \leq \frac{H \sqrt{a(2\pi - 3H^2 a)}}{\pi - 2H^2 a}
\]

so that (13) and therefore (11) is satisfied. Corollary 6 then follows from Theorem 4, observing that condition (6) is never satisfied by a large cap sphere.

**Theorem 5.** Let $M$ be a compact embedded surface with cmc $H = 1$ in $\mathbb{R}^3$, with boundary $\partial M$ contained in $\mathbb{R}^3_+ = \{z \geq 0\}$ and having a $1 - 1$ projection over a closed $C^{2,\alpha}$ convex curve $\gamma$ on the plane $P = \{z = 0\}$. Set $C := \gamma \times \mathbb{R}$ (the right cylinder over $\gamma$), $\Omega := \text{int}(\gamma)$, and let $C_-$ be the connected component of $C \setminus \partial M$ which is below $\partial M$, that is, containing points with $z < 0$. We require that

(a) there exists a neighborhood $U$ of $\partial M$ in $M$ which is a graph over a neighborhood $\Lambda \subset \overline{\Omega}$ of $\partial \Omega$;

(b) $M \cap C_- = \emptyset$.

Then $M$ is a graph over $\overline{\Omega}$. 
Proof. We claim that, up to a reflection on the plane \( z = 0 \) followed by a translation along the \( z \)-axis, one can assume that the mean curvature vector of \( U \) points towards the plane \( z = 0 \). In fact, if the mean curvature vector points to the other direction, let us consider the topological compact (closed) surface

\[
K = M \cup T \cup \overline{\Omega},
\]

where \( T := \{ z \geq 0 \} \cap C_- \). Since \( M \cap C_- = \emptyset \), \( K \) is embedded. Therefore the mean curvature vector of \( M \) points to the unbounded component of \( \mathbb{R}^3 \setminus K \).

By the maximum principle, any plane coming from the infinity, approaching \( \partial M \), and not intersecting \( C_- \), cannot intersect \( M \setminus \partial M \). It follows that \( M \cap (C \setminus C_-) = \emptyset \). Therefore, reflecting \( M \) on the plane \( z = 0 \) and applying a translation along the \( z \)-axis, the new surface satisfies our claim.

Let us assume that \( U \) is the graph of a function \( v \in C^2(\Lambda \setminus \gamma) \cap C^0(\Lambda) \).

Let \( \Omega_0 \subset \Omega \) be a subdomain with smooth boundary close enough to \( \Omega \) so that \( \partial \Omega_0 \subset \Lambda \setminus \gamma \) and such that \( \partial \Omega_0 \) is \( C^{2,\alpha} \) and strictly convex.

We prove now the existence of a solution to Dirichlet’s problem for cmc 1 graphs:

\[
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = -2, \quad u \in C^2(\overline{\Omega_0}), \quad u|_{\partial \Omega_0} = \phi, \tag{14}
\]

where \( \phi := v|_{\partial \Omega_0} \). For, we consider a continuous homotopy between (14) and the Dirichlet problem for the minimal surface equation given by

\[
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = -2H, \quad u \in C^2(\overline{\Omega_0}), \quad u|_{\partial \Omega_0} = \phi, \tag{15}
\]

\( H \in [0, 1] \). The classical theorem of T. Radó about existence of a solution to the Dirichlet’s problem for the minimal surface equation in convex domains guarantees the existence of a solution of (15) for \( H = 0 \). It follows from the inverse function theorem, that (15) has a solution for \( 0 \leq H < H_1 \). To guarantee that (15) has a solution with \( H = 1 \), we prove that any solution of (15) for \( H_1 \leq H \leq 1 \) has a priori \( C^1 \) bound estimates in the whole domain \( \Omega_0 \). Thus, choose \( H \) with \( H_1 \leq H \leq 1 \) and let \( u \) be a solution of (15). As it is well known, one has

\[
|u| \leq \frac{1}{H_1} + \max_{\partial \Omega_0} \phi.
\]

Observe that orientation of the graph \( G \) of \( u \) is such that

\[
\vec{H} = \frac{H(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}},
\]

where \( \vec{H} \) is the mean curvature vector of \( G \). Therefore, if \( n_1 \) denotes the interior normal vector to \( \partial \Omega_0 \), then \( \langle \nabla u, n_1 \rangle = \sqrt{1 + |\nabla u|^2} \left\langle \vec{H}, n_2 \right\rangle \), where
n_2 \) is the interior unit normal vector to the cylinder \( D \) over \( \partial \Omega_0 \) at \( \partial G \).

Therefore, if there is \( p_0 \in \gamma \) such that

\[
\langle \nabla u(p), n_1(p) \rangle \to -\infty
\]
as \( p \to p_0, \ p \in \Omega_0 \), then

\[
\langle \tilde{H}(p), n_2(p) \rangle \to -1
\]
as \( p \to p_0 \). We observe that this possibility can not happen: Since \( \Omega_0 \) is
strictly convex one can place, at any point \( q \) in the boundary of the graph \( G \),
a circular cylinder \( E \) of constant mean curvature \( H \) tangent to the cylinder \( D \)
over \( \partial \Omega_0 \), so that \( C \cap D \) is a straight vertical line. Moreover, \( \Omega_0 \) and \( \phi \)
satisfy the bounded slope condition (see [GT], Remark 4, p. 255). Therefore, we
prove now that one can uniformly estimate this quantity from above.

Since \( \partial \Omega_0 \subset \Omega \) and \( d(\partial \Omega_0, \partial \Omega) > 0 \), it follows from gradient interior
bound estimates that \( |\nabla v| \) is uniformly bounded on \( \partial \Omega_0 \cap \Lambda \).
Therefore, since the maximum of \( |\nabla u| \) in \( \Omega_0 \) is assumed in \( \partial \Omega_0 \), it is enough to prove
that

\[
\langle \nabla u(p), n_1 \rangle \leq \langle \nabla v(p), n_1 \rangle
\]
for all \( p \in \partial \Omega_0 \). By contradiction, assume that

\[
\langle \nabla u(p), n_1 \rangle > \langle \nabla v(p), n_1 \rangle
\]
at some \( p \in \partial \Omega_0 \). From (17), it follows that \( M \) has points below \( G \) near
\( p \). Therefore, by moving \( G \) slightly down, we then have that \( \partial G \cap M = \emptyset \)
while \( G \cap M \neq \emptyset \). Therefore, by moving \( G \) down until it reaches the last point
of contact with \( M \), since \( M \cap C_0 = \emptyset \), we obtain a tangency between \( M \) and
\( G \) at an interior point of both \( G \) and \( M \), a contradiction, according to the
maximum principle. Hence, any solution \( u \) of (15) satisfies (16). Therefore,
using bound interior estimates for the gradient, it follows that any solution of
(15) is uniformly bounded in modulus by above and has uniformly bounded
gradient. Therefore, (10) admits a solution \( u \in C^2(\bar{\Omega}_0) \) for \( H = 1 \). Let \( G \) be
the graph of \( u \).

Denote by \( \lambda \) and by \( \eta \) the interior conormal to \( \partial G \) and \( \partial M' \) (\( \partial G = \partial M' \)),
respectively, where \( M' = M \setminus \text{graph}(v|_{\Omega \setminus \Omega_0}) \). We note that, as we have
explained in the first paragraph of this proof, \( M \) and \( G \) induce the same orientation
on their common boundary. By (16), we have

\[
\langle \eta(p), e_3 \rangle \geq \langle \lambda(p), e_3 \rangle
\]
for all \( p \in \partial M' \). On the other hand, if one considers a smooth surface \( T \) with
boundary \( \partial T = \partial M' \) such that \( M' \cup T \) and \( G \cup T \) are immersed two cycles,
and if $\rho$ denotes the unit normal to $T$ pointing to the bounded component of $\mathbb{R}^3 \setminus (G \cup T)$, then the balancing formula (Proposition I.1.8 of [K]) gives

$$\int_{\partial M'} \eta = -H \int_T \rho = \int_{\partial G} \lambda.$$ 

This implies that

$$\langle \eta(p), e_3 \rangle = \langle \lambda(p), e_3 \rangle$$

for $p \in \partial M'$ so that $G$ and $M'$ are tangent at the boundary. The maximum principle therefore implies that $M' = G$ and the theorem is proved.

Assuming that $\partial M$ is a plane curve, we can prove, using the same technique of Theorem 5, that $M$ is a graph, with weaker hypothesis than of Theorem 5, namely:

**Theorem 6.** Let $M$ be a compact embedded cmc $H \geq 0$ surface whose boundary is a planar curve in the plane $z = 0$, and assume the existence of a neighborhood $U$ of $\partial M$ in $M$ lying in $\{z \geq 0\}$ which is a graph over the plane $z = 0$. Then $M$ is a graph over $z = 0$.

**Proof.** Of course, if $H = 0$ then the result is trivial, so that let us assume $H > 0$. We first show that the mean curvature vector of $U$ points to the plane $z = 0$. Consider the immersed surface $K = M \cup \Omega$, where $\Omega$ is the planar domain bounded by $\partial M$. We choose a unit normal vector $N$ to $K$ in $K \setminus \partial M$, respecting the orientation of $K$, such that $N|_M = (1/H) \overline{H}$, where $\overline{H}$ is the mean curvature vector of $M$. At $\Omega$, one has $N = \pm (0,0,1)$. Therefore, our claim is proved if we show that $N|_{\Omega} = e_3 = (0,0,1)$.

It follows from Proposition I.1.8 of [K] that

$$\int_{\partial M} \langle v, e_3 \rangle \, ds = H \int_{\Omega} \langle N, e_3 \rangle \, dA$$

where $v$ is the unit interior conormal vector to $M$ at $\partial M$. Since $U$ lies in $\{z \geq 0\}$ we necessarily have $\langle v, e_3 \rangle \geq 0$. Since $H > 0$ we obtain $N|_{\Omega} = e_3$.

By the maximum principle, it follows that $U$ cannot be tangent to $z = 0$ along $\partial M$. It follows from Theorem 2 of [BEMR] that $M$ lies in $z \geq 0$. The proof now goes exactly as in Theorem 5: Using the continuity method we prove the existence of cmc $H$ graph $G$ defined in $\Omega$ and vanishing at $\partial \Omega$. Using again the balancing formula, we may therefore conclude that $M = G$, proving Theorem 6.

**References**


[K] R.B. Kusner, Global geometry of extremal surfaces in three space, Dissertation at University of California, Berkeley.


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SYMPLECTIC SUBMANIFOLDS FROM SURFACE FIBRATIONS

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We give a simple construction yielding homology classes in (non-simply-connected) symplectic four-manifolds which admit infinitely many pairwise non-isotopic symplectic representatives. Examples are constructed in which the symplectic curves can have arbitrarily large genus. The examples are built from surface bundles over surfaces and involve only elementary techniques. As a corollary we see that a blow-up of any simply-connected complex projective surface contains a connected symplectic surface not isotopic to any complex curve.

1. Symplectic tori.

The existence of symplectic submanifolds realising certain homology classes has had a significant impact on our understanding of symplectic topology, particularly in dimension four. This raises the natural question as to the uniqueness of symplectic representatives for homology classes in instances when they exist at all. In complex geometry the uniqueness (or finiteness) of complex representatives for (co)homology classes is well-known; for instance if $X$ is a projective surface then an element $\alpha$ of $H_2(X)$ is dual to a cohomology class $\alpha^*$, and complex representatives of $\alpha$ correspond to transverse holomorphic sections of one of finitely many line bundles with first Chern class $\alpha^*$. (If the surface is simply connected the line bundle is unique.) These sections are points of certain projective spaces $\mathbb{P}H^0(X, L_{\alpha^*})$; since the locus of non-transverse sections in each such space is a complex subvariety, its complement is connected, and there are at most finitely many isotopy classes of submanifold. Our main result shows that this cannot be true in the symplectic category:

Theorem 1.1.

• For every odd genus $g \neq 3$ there is a symplectic four-manifold $X$ and a homology class $C \in H_2(X, \mathbb{Z})$ such that $C$ may be represented by infinitely many pairwise non-isotopic connected symplectic surfaces $\{S_a\}_{a \in \mathbb{Z}}$ of genus $g$ inside $X$. 

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• For every odd $g \neq 3$ there is a simply-connected projective surface $Z_g$ which admits a connected symplectic surface of genus $g$ not isotopic to any complex curve. Indeed, some blow-up of any simply connected projective surface contains a symplectic non-complex curve. In fact it shall follow from the proof that there is no homeomorphism of pairs $(X, S_a) \equiv (X, S_b)$ where $S_a, S_b$ are a distinct pair of such symplectic surfaces. More precisely we shall show:

**Proposition 1.2.** The homology class $2m[\Sigma_g]$ in the four-manifold $\Sigma_g \times S^2$ (for $m \geq 2$ an integer and any $g \geq 1$) can be represented by infinitely many pairwise non-isotopic connected symplectic surfaces, each of which has genus $r$ where

$$2 - 2r = 2m(2 - 2g).$$

The restriction to curves of odd genus arises since our construction involves branched covers of even degree; it could be removed with the obvious adaptation. Similarly one can consider twisted sphere bundles. The first instance of the proposition contains all the essential ideas and is an immediate corollary of work of Geiges [5]. For any diffeomorphism $f$ of a surface $\Sigma$ we write $Y_f$ for the mapping torus of $f$; in particular for $A \in SL_2(\mathbb{Z})$ we have $Y_A = T^2 \times_A S^1$. Geiges notes that:

- If $A_1, A_2 \in SL_2(\mathbb{Z})$ satisfy $|\text{tr} (A_i)| > 3$ then the torus bundles $S^1 \times Y_{A_i}$ are diffeomorphic if and only if $A_1 = A_2$.
- For any four-manifold $Z = S^1 \times Y_A$ as above, and given any cohomology class $a \in H^2(Z)$ with $a^2 > 0$ we can find an integral symplectic form representing the class $a$. Moreover if $a(F) > 0$ where $F$ is a torus fibre, we can assume the symplectic structure is compatible with the fibration.
- None of the four-manifolds $Z_A$ above admit any complex structure.

Now the mapping class group $SL_2(\mathbb{Z})$ for the torus differs only by a central $\pm I$ from the braid group for four points in the sphere $S^2$. It follows that given any matrix $A \in SL_2$ we can define a braid on four strands; when the image of the braid is connected we may view this as a map $\beta_A : S^1 \rightarrow S^1 \times S^2$ with image $\Gamma_\beta$. (If the braid is not connected the domain of $\beta_A$ will be some disjoint union of circles, and we use the same notation for the image.)

**Lemma 1.3.** For any such braid, the product $S^1 \times \Gamma_\beta \subset S^1 \times (S^1 \times S^2)$ is a symplectic submanifold of $T^2 \times S^2$ with its standard product symplectic form.

**Proof.** This is a trivial computation. With co-ordinates $(\theta, \phi), (u, v)$ on $T^2 \times S^2$ the symplectic form locally has the shape $d\theta \wedge d\phi + du \wedge dv$ whilst the tangent space at any point to the submanifold is spanned (in a suitable normalisation) by $\partial/\partial \theta$ and $\partial/\partial \phi + \alpha \partial/\partial u + \gamma \partial/\partial v$ for some $\alpha, \gamma$ not both zero. The only non-zero term arises from the $\theta, \phi$ component of the symplectic form and this is always positive. □
For any braid $\beta_A$ we have built a symplectic submanifold in the homology class $4[Torus] \subset \mathbb{T}^2 \times S^2$. It is obviously possible to choose an infinite family of matrices $A_i$ for which the corresponding braids $\beta_{A_i}$ are connected and the matrices have trace $|\text{tr} A_i| > 3$. We claim that all of the associated symplectic submanifolds are non-isotopic (indeed not equivalent under any diffeomorphism of pairs). For given any such submanifold, we can form the double cover of $\mathbb{T}^2 \times S^2$ branched over this class, and the result is precisely the torus bundle $S^1 \times Y_A$. Since all of these manifolds are non-complex, but the branched cover of a complex manifold over a complex divisor does admit a complex structure, the submanifolds are not isotopic to complex curves. Again, since the $S^1 \times Y_A$ represent infinitely many diffeomorphism types, so we must have infinitely many isotopy classes of branch locus. The genus formula given in (1.2) holds since symplectic submanifolds satisfy the adjunction formula. This establishes the proposition in the case $m = 2, g = 1$. The second part of (1.1) for these special values is now immediate.

**Corollary 1.4.** Let $E_n$ denote the $n$-th fibre sum of the rational elliptic surface with itself. Then $E_n$ contains a connected symplectic surface not isotopic to any complex curve.

*Proof.* The $E_n$ are all elliptically fibred, and we may take a trivial fibre sum with $\mathbb{T}^2 \times S^2$ to exhibit a connected symplectic torus representing the homology class $4[Fibre]$. On the other hand, the (unique) complex representative for this homology class is a disjoint union of four parallel fibres, which is clearly not isotopic to any connected curve. \qed

For $n = 1, 2$ this gives connected symplectic non-complex submanifolds in $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ and $K3$. A conjecture due to Siebert and Tian asserts that there are no such symplectic surfaces in *minimal* rational ruled manifolds, and the example serves to demonstrate the necessity of the minimality assumption.

Via the same method of fibre summation, and taking higher $m$ in (1.2) one obtains a homology class on a simply connected complex surface with $N$ distinct symplectic representatives for any required $N$, distinguished by connectivity. However, Fintushel and Stern [4] have observed that for any braid $\beta_A$ if we include the associated symplectic submanifold $S^1 \times \Gamma_{\beta_A}$ into an elliptic surface $E$ for which $\pi_1(E\\setminus\{\text{Fibre}\}) = 0$ using this fibre sum trick, then the fundamental group of the complement is cyclic, independent of $A$. Hence we cannot use the same naive algebraic topological methods to give infinite families of representatives. With more sophistication, taking branched covers and using the Seiberg-Witten invariants, Fintushel and Stern have nonetheless shown that the above submanifolds do remain pairwise non-isotopic inside $E_i$. Their methods of computation do not apply, unfortunately, to distinguish curves of higher genus.
2. Generalisations.

In this section we will extend the previous discussion to cover the other cases of (1.2), in particular proving (1.1). The first step is to move from the homology class $4[Fibre]$ inside $T^2 \times S^2$ to $2m[Fibre]$. For this we just need an analogue of Geiges’ result: Then we can work with branched covers and the submanifolds $S^1 \times \Gamma_\beta$ for $\beta$ a braid corresponding to a hyperelliptic mapping class at some genus $h$, and the above proof will hold. (In particular precisely the same proof that the submanifolds are symplectic will apply.) It is straightforward to adapt Geiges’ proof. Since we have used the notation $\Gamma_\beta$ to denote a graph above, we will write $\text{Out}(\pi_1(\Sigma_g))$ to denote the mapping class group of a genus $g$ surface, typically denoted $\Gamma_g$.

**Proposition 2.1.**

Let $g \geq 1$ and $\gamma \in \text{Out}(\pi_1(\Sigma_g))$ be such that:
- $\gamma$ is not periodic,
- $1$ is not an eigenvalue of the action on cohomology $\gamma_* : H^1(\Sigma_g) \to H^1(\Sigma_g)$.

Then the manifold $X_\gamma = S^1 \times (S^1 \times_\gamma \Sigma_g)$ is a symplectic fibration with no compatible Kähler metric. Indeed the manifold has no complex structure at all.

**Proof.**

- The manifolds we are considering are smooth fibrations of surfaces over tori; since base and fibre are aspherical, it follows that so is the total space. Thus

$$H_*(\Sigma_g \times T^2; \mathbb{Z}) = H_*(\pi_1(\Sigma_g \times T^2); \mathbb{Z})$$

where we define the group homology in the usual way

1. There is an extension of groups

$$(2.2) \quad 0 \to \pi_1(\Sigma_g) \to \mathcal{G} \to \mathbb{Z}^2 \to 0$$

where $\mathbb{Z}^2$ acts on $\pi_1(\Sigma_g)$ via the monodromies $\gamma_i \in \text{Out}(\pi_1(\Sigma_g))$, and $\mathcal{G} = \pi_1(X)$. We will later assume one monodromy is trivial (so the four-manifold is $S^1$ times a mapping torus). In any case, there is a *Hochschild-Serre spectral sequence* - the group homology analogue of the Leray-Serre spectral sequence, valid because of the asphericity - for which

$$E^2_{p,q} = H_p(\mathbb{Z}^2; H_q(\pi_1 \Sigma_2)) \implies H_{p+q}(\mathcal{G}; \mathbb{Z}) = H_{p+q}(X_{\gamma_1,\gamma_2}; \mathbb{Z}).$$

The arguments of Thurston [8] show the total space of the fibration is symplectic iff $E^\infty_{0,2} = \mathbb{Z} = E^2_{0,2}$.

Riemann surfaces are Eilenberg-Maclane spaces so

$$H_*(\pi_1 \Sigma_g) = H_1(\Sigma_g) = \mathbb{Z}^{2g}.$$

$^1$That is, for a group $\Gamma$ take a projective resolution $\mathcal{R}_\Gamma$ of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ and define $H_*(\Gamma; M)$ to be the homology of the sequence $\mathcal{R} \otimes_\Gamma M$. 
We get the following free (projective!) resolution:

\[
0 \to \mathbb{Z}^{2g} \xrightarrow{\theta_*} \mathbb{Z}^4 \xrightarrow{\phi_*} \mathbb{Z}^{2g}
\]

(2.3)

where the maps are defined by

\[
\theta_* : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ([\gamma_1]_* - I)(x) \\ [I - (\gamma_2)_*]y \end{pmatrix}
\]

\[
\phi_* : \begin{pmatrix} (x, y) \\ (z, t) \end{pmatrix} \mapsto \begin{pmatrix} ([\gamma_1]_* - I)x \\ ([\gamma_2]_* - I)z \end{pmatrix}.
\]

From this we see that \(\ker \phi_* / \im \theta_* = E^{2,1}_{1,1}\) and since \(d_2 : E^{2,1}_{1,1} \to E^{2,2}_{1,2} = 0\) we have \(E^{2,2}_{1,1} = E^{\infty}_{1,1}\). Also \(E^{2,0}_{0,2} = \mathbb{Z}\) for \(H_2(\pi_1)\) is a trivial module. Since

\[
H_2 = E^{\infty}_{0,2} + E^{\infty}_{1,1} + E^{\infty}_{2,0}
\]

we see that if \(\rank(E_{1,1}^2) = b_2 - 2\) then there is a compatible symplectic structure. To find \(b_i(X)\) we make the assumption that the first monodromy \(\gamma_1\) is trivial; for then the \(E^2\)-page of the spectral sequence for the 3-manifold \(Y_\gamma = S^1 \times_\gamma \Sigma_g\) is trivial with all differentials zero. It follows from the analogous free resolution in the 3-dimensional case that

\[
H_1(Y_\gamma) = \mathbb{Z} \oplus \ker(\gamma_* - \text{id} : \mathbb{Z}^{2g} \to \mathbb{Z}^{2g})
\]

and in particular, if \(1 \notin \{\text{Eigenvalues} \gamma_*\}\) then \(b_1(X_\gamma) = b_1(Y_\gamma) + 1 = 2\).

But in this case, \(b_2(X) = 2\) and \(\rank[E^2_{1,1}] = 0 \Leftrightarrow \text{symplectic structures exist.} \) Equivalently, we require (2.3) to have rank zero when \((\gamma_2)_* = \text{id.}\) But the complex has \(H_1\) term with rank equal to \(2 \cdot \rank(\ker(\gamma_* - I))\), giving the result.

To see there are no possible complex structures, when \(g > 1\) one can most easily resort to the classification of surfaces, and check that these manifolds are not on the list; for they have \(c_2 = 0\) despite being minimal, and have \(b_1 = 2\) so cannot be Kodaira surfaces or minimal surfaces of class VII.

The \(g = 1\) case (already used in the first section) was covered by Geiges’ work. □

**Remark 2.4.** A few words on notation. By \(\Gamma_g^{\text{hyp}}\) we shall denote the hyperelliptic mapping class group; if we fix a hyperelliptic Riemann surface \(\Sigma_g\) then there is a distinguished hyperelliptic involution \(\iota \in \text{Out}(\pi_1(\Sigma_g))\) inside the mapping class group of \(\Sigma_g\), and \(\Gamma_g^{\text{hyp}}\) denotes the set of elements of the mapping class group commuting with \(\iota\). We also have the familiar groups \(\Gamma_0^{2g+2}\) and \(\text{Br}_{2g+2}(S^2)\) - the mapping class group of the marked sphere and the braid group on \((2g + 2)\)-strands in the sphere, respectively [2]. There are isomorphisms (for instance from explicit presentations)

\[
\Gamma_0^{2g+1} \sim \Gamma_g^{\text{hyp}} / \langle \iota \rangle \sim \text{Br}_{2g+2}(S^2) / \langle \Delta \rangle
\]
where $\Delta$ generates the centre $Z(\text{Br}_{2g+2}(\mathbb{S}^2)) \cong \mathbb{Z}_2$. In particular, given any braid in the sphere, we have a hyperelliptic mapping class defined uniquely up to composition with the involution $\iota$, and all mapping classes arise this way. (Indeed given a mapping class we can build a braid inside $\mathbb{S}^2 \times I$ and the central $\mathbb{Z}_2$-ambiguity corresponds to the different possible twistings of the sphere bundle over the circle $\mathbb{S}^1$ obtained by closing the ends of the interval $I$.) With these delicacies understood, we shall pass freely between braids and mapping classes henceforth; required choices can be made arbitrarily by the reader.

It is easy to obtain in this way infinitely many diffeomorphism types of symplectic non-complex surface bundle over a torus. Indeed, for different monodromies in general the homeomorphism types will differ, distinguished by the fundamental group. If we choose the non-trivial monodromy to be hyperelliptic then we can associate to the surface bundle a braid (on $2g + 2$ strands, where the fibre has genus $g$) such that the surface bundle is the double cover of $\mathbb{T}^2 \times \mathbb{S}^2$ over the surface $\mathbb{S}^1 \times \Gamma_g$ as before. For connected braids, this surface is a symplectic torus in the homology class $(2g+2)[\text{Fibre}]$. The genus one case of the Proposition (1.2) follows.

To obtain symplectic surfaces of higher genus we use the fibre sum operation. Recall that given two symplectic fibrations $Z_1 \to B_1$, $Z_2 \to B_2$ with diffeomorphic fibre, then after scaling the symplectic forms to give the fibres equal area we may glue tubular neighbourhoods of fixed fibres to obtain a new fibration $Z_1 \sharp F Z_2 \to B_1 \sharp B_2$ which covers the (usual) connected sum of the base surfaces. One of the choices in this construction is of a twisting diffeomorphism of the fibre with which we make the identification. In particular, given any two hyperelliptic genus $g$ surface bundles over a torus, we may take a fibre sum (twisted by an element of the hyperelliptic mapping class group) to produce a hyperelliptic surface fibration by genus $g$ surfaces over a genus two base; this new fibration still admits a symplectic structure compatible with the fibration.

Now think of our two surface bundles $Z_i$ as branched covers of $\mathbb{T}^2 \times \mathbb{S}^2 \to \mathbb{T}^2$ over symplectic torus multi-sections. We may fibre sum the two sphere bundles along a sphere fibre. Moreover, by the relative form of Gompf’s surgery, once we fix a fibre in each $Z_i$ we distinguish a set of $2g + 2$ points in the fibre which are the intersection points with the symplectic branch curve. View the hyperelliptic diffeomorphism as (induced by) an element of the braid group for $2g+2$ points on the sphere and choose a lift of this homotopy class of diffeomorphisms to a diffeomorphism fixing the marked points. We can glue the two sphere bundles with a twist by this diffeomorphism and Gompf’s theorems [6] provide a symplectic structure on the new sphere bundle over $\Sigma_2$ with respect to which the two tori glue to give a symplectic surface of genus $2g + 3$. This is an unramified cover of the base $\Sigma_2$ of degree
2g + 2. As we vary the original torus bundles and hyperelliptic gluing map, we obtain infinite families of symplectic fibrations over a genus two curve which we can view as branched covers of sphere bundles over connected symplectic surfaces.

**Lemma 2.5.** The above techniques yield infinite families of symplectic non-complex curves of any genus 2g + 3 with g ≥ 1 inside Σ₂ × S².

**Proof.** From the above, we need show first that the symplectic curves are not complex, and secondly that they are not pairwise isotopic; it will therefore be sufficient to show that the hyperelliptic surface bundles we construct represent infinitely many non-complex diffeomorphism types. Indeed once we know we have infinitely many diffeomorphism types the statement on complex structures will follow: Since a smooth complex fibre bundle will be determined by a representation π₁(Base) → Aut(Fibre) where Aut denotes the holomorphic automorphism group. For curves of genus g > 1 this is finite, and there can only be finitely many such automorphisms; for curves of genus one, the fibre bundles have c₂ = 0 and the classification of complex surfaces applies (over a genus two base we will have b₁ ≥ 4 and hence they cannot be minimal surfaces of class VII).

We are reduced to the homeomorphism classification of hyperelliptic surface bundles. Since base and fibre are again aspherical, the topology is completely encoded in the fundamental group; in the homotopy exact sequence

\[ 0 \rightarrow π_1(\text{Fibre}) \rightarrow π_1(Z) \rightarrow π_1(Σ₂) \rightarrow 0 \]

the extension is determined by the monodromies which can be viewed as relations amongst generators for π₁(Z). Indeed the bundle is given by a representation π₁(Σ₂) → Γₙhyp into the hyperelliptic genus g mapping class group. Given any two hyperelliptic mapping classes φ,ψ and a presentation

\[ π₁(Σ₂) = \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle \]

we may take a ↦→ φ, b ↦→ 1, c ↦→ q⁻¹ψq, d ↦→ 1 to obtain a representation as above. Here q is an arbitrary element of the hyperelliptic mapping class group, and as it varies we can obtain infinitely many different conjugacy classes of representation (since there are infinitely many conjugacy classes of element in Γₙhyp). It then follows from the homotopy exact sequence that we can obtain infinitely many distinct fundamental groups for the total space. The result follows. □

To complete the proof of (1.1) observe that there is a simply connected projective surface with a Lefschetz fibration of genus 2 curves - for instance blow up the K3 given by double covering P² over a sextic at the two preimages of the basepoint of a pencil of lines. We can fibre sum Σ₂ × S² into this genus two fibration, carrying with us a symplectic surface of arbitrarily large odd
genus from the above construction (choosing the braid to obtain the trivial sphere bundle $\Sigma_2 \times \mathbb{S}^2$ and not some twisted bundle). The symplectic surface represents a homology class $2m\langle \text{Fibre} \rangle$ for some $m$ and the unique complex representative of this fibre is again disconnected. The result of the fibre summation remains simply connected since in $K3\sharp 2\mathbb{CP}^2$ the fundamental group of the complement of a fibre is trivial. Now for any complex projective surface $Z$ we may find a Lefschetz pencil of curves of some large genus. Blowing up the base-points of this pencil we have a Lefschetz fibration, inside which we can identify a trivial fibration $\Sigma_r \times D$ for a complex disc $D \subset \mathbb{P}^1$. Iterating the fibre sum construction above, we can find a symplectic curve of any large odd genus inside $\Sigma_r \times \mathbb{S}^2$ which is disjoint from some fibre $\mathbb{S}^2$ and hence can be assumed to lie in the disc bundle $\Sigma_r \times D \subset Z$. The usual arguments show this is not the complex representative of the relevant homology class $2m\langle \text{Fibre} \rangle$ (which is unique when the surface is 1-connected). This completes the proof.

**Remark 2.6.** In complex geometry one is often as interested in nodal and cuspidal curves as smooth curves; for instance these appear as branch loci of generic projections of projective surfaces to $\mathbb{CP}^2$. Following Auroux’s work on symplectic four-manifolds [1] the symplectic geometry of these surfaces is now of interest. Moishezon [7] gave examples of nodal cuspidal curves in $\mathbb{CP}^2$ which were not isotopic to complex nodal cuspidal curves; all of his examples contained cusps. The surfaces we construct in $T^2 \times \mathbb{S}^2$ can be pushed down to $\mathbb{S}^2 \times \mathbb{S}^2$ where they acquire nodes (but not worse singularities). For careful braids (for instance of several components) the resulting surfaces can be made symplectic. It is not clear if this is of interest, however, since the symplectic surfaces built in this way appear always to have nodes of both positive and negative intersection.

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**References**


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PRIMITIVE COMPACT FLAT MANIFOLDS WITH
HOLONOMY GROUP $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Paulo Tirao

In this paper we determine and classify all compact Riemannian flat manifolds with holonomy group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and first Betti number zero. Also we give explicit realizations of all of them.

Introduction.

From an important construction of Calabi (see [Ca], [Wo]), it follows that the compact Riemannian flat manifolds with first Betti number zero are the building blocks for all compact Riemannian flat manifolds. It is, therefore, of interest to construct families of such objects. These are often called primitive manifolds.

Hantzsche and Wendt (1935) constructed the only existing 3-dimensional compact Riemannian flat manifold with first Betti number zero; this manifold has holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Cobb [Co] constructed a family of manifolds with these properties, for all dimensions $n \geq 3$. In [RT] a rather larger family of primitive ($\mathbb{Z}_2 \oplus \mathbb{Z}_2$)-manifolds was given.

The goal in this paper is the classification, up to affine equivalence, of all primitive manifolds with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We may notice that a similar project has been carried out in [RT2], where a full classification of 5-dimensional Bieberbach groups with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ was given.

The classification is achieved by following a classical result of Charlap ([Ch1]), which reduces the problem to:

(1) classification of faithful representations $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \text{Gl}(n, \mathbb{Z})$, without fixed points;

(2) computation of $H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \mathbb{Z})$ and enumeration of special classes modulo some equivalence relation.

All §2 is devoted to the solution of (1), which turns out to be in general a very difficult problem. The cohomological computations in §3 are standard. In §4 we prove the classification theorem stated below, which, together with the classification of all integral representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ having no fixed points (see Theorem 2.7), constitutes the main result.
Theorem. The affine equivalence classes of compact Riemannian flat manifolds with holonomy group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and first Betti number zero are in a bijective correspondence with the \( \mathbb{Z}[\mathbb{Z}_2 \oplus \mathbb{Z}_2] \)-modules \( \Lambda \), such that:

1. As abelian group \( \Lambda \) is free and of finite rank;
2. \( \Lambda^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} = 0 \);
3. \( \Lambda \) contains a submodule equivalent to the Hantzsche-Wendt module.

As it can be seen in the Theorem all primitive \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-manifolds are closely related to the Hantzsche-Wendt manifold. This relation will be more clear in section \( \S 5 \), where we construct all primitive \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-manifolds. We also identify those realizations which correspond to the classical examples of Cobb, those which correspond to the family of examples given in \( [RT] \) and which ones are newly found in the course of the classification. Finally, these explicit realizations allow us to compute their integral homology as well as their cohomology.

1. Preliminaries.

Let \( M \) be an \( n \)-dimensional compact Riemannian flat manifold with fundamental group \( \Gamma \). Then, \( M \cong \mathbb{R}^n/\Gamma \), \( \Gamma \) is torsion-free and, by Bieberbach’s first theorem, one has a short exact sequence

\[
0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1,
\]

where \( \Lambda \) is free abelian of rank \( n \) and \( \Phi \) is a finite group, the holonomy group of \( M \). This sequence induces an action of \( \Phi \) on \( \Lambda \) that determines a structure of \( \mathbb{Z}[\Phi] \)-module on \( \Lambda \) (or an integral representation of rank \( n \) of \( \Phi \)). Thus, \( \Lambda \) is a \( \mathbb{Z}[\Phi] \)-module which, moreover, is a free abelian group of finite rank. From now on we will refer to these \( \mathbb{Z}[\Phi] \)-modules as \( \Phi \)-modules.

As indicated by Charlap in \( [Ch2] \), the classification up to affine equivalence of all compact Riemannian flat manifolds with holonomy group \( \Phi \) can be carried out by the following steps:

1. Find all faithful \( \Phi \)-modules \( \Lambda \).
2. Find all extensions of \( \Phi \) by \( \Lambda \), i.e., compute \( H^2(\Phi; \Lambda) \).
3. Determine which of these extensions are torsion-free.
4. Determine which of these extensions are isomorphic to each other.

For each subgroup \( K \) of \( \Phi \), the inclusion \( i : K \longrightarrow \Phi \) induces a restriction homomorphism \( \text{res}_K : H^2(\Phi; \Lambda) \longrightarrow H^2(K; \Lambda) \).

Definition. A class \( \alpha \in H^2(\Phi; \Lambda) \) is special if for any cyclic subgroup \( K < \Phi \) of prime order, \( \text{res}_K(\alpha) \neq 0 \).

Step (3) reduces to the determination of the special classes by virtue of the following result.
Lemma 1.1 ([Ch1, p. 22]). Let $\Lambda$ be a $\Phi$-module. The extension of $\Phi$ by $\Lambda$ corresponding to $\alpha \in H^2(\Phi; \Lambda)$ is torsion-free if and only if $\alpha$ is special.

Definition. Let $\Lambda$ and $\Delta$ be $\Phi$-modules. A semi-linear map from $\Lambda$ to $\Delta$ is a pair $(f,A)$, where $f : \Lambda \rightarrow \Delta$ is a group homomorphism, $A \in \text{Aut}(\Phi)$ and

$$f(\sigma \cdot \lambda) = A(\sigma) \cdot f(\lambda), \quad \text{for } \sigma \in \Phi \text{ and } \lambda \in \Lambda.$$ 

The $\Phi$-modules $\Lambda$ and $\Delta$ are said to be semi-equivalent if $f$ is a group isomorphism. If $A = I$, then $\Lambda$ and $\Delta$ are equivalent via $f$.

Let $E(\Phi)$ be the category whose objects are the special pointed $\Phi$-modules, that is, pairs $(\Lambda,\alpha)$, where $\Lambda$ is a faithful $\Phi$-module and $\alpha$ is a special class in $H^2(\Phi; \Lambda)$. The morphisms of $E(\Phi)$ are the pointed semi-linear maps, that is, semi-linear maps $(f,A)$ from $(\Lambda,\alpha)$ to $(\Delta,\beta)$, such that $f_*(\alpha) = A^*(\beta)$, where $f_*$ is the morphism in cohomology induced by $f$ and $A^*$ is defined by $A^*(\beta)(\sigma,\tau) = \beta(A\sigma,A\tau)$ for any $(\sigma,\tau) \in \Phi \times \Phi$.

Theorem 1.2 ([Ch1, p. 20]). There is a bijection between the isomorphism classes of the category $E(\Phi)$ and connection preserving diffeomorphism classes of compact Riemannian flat manifolds with holonomy group $\Phi$.

It is well known that the first Betti number of $M$, where $M \simeq \mathbb{R}^n/\Gamma$ and $\Gamma$ is as in (1.1) can be computed by the formula

$$\beta_1(M) = \text{rk}(\Lambda^\Phi).$$

Thus, it is clear that the primitive $\Phi$-manifolds correspond to those objects $(\Lambda,\alpha)$ in $E(\Phi)$ satisfying $\Lambda^\Phi = 0$.

2. Integral Representations.

In this section we deal with the first of Charlap’s steps. That is, we determine (up to equivalence) all faithful $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-modules $\Lambda$, such that $\Lambda[\mathbb{Z}_2 \oplus \mathbb{Z}_2] = 0$.

Let $\Lambda$ be a $\Phi$-module. Since $\Lambda$ is free abelian of finite rank, say $n$, then $\Lambda \simeq \mathbb{Z}^n$ as abelian groups and the $\mathbb{Z}[\Phi]$-structure on $\Lambda$ induces an homomorphism $\rho : \Phi \rightarrow \text{Gl}(n,\mathbb{Z})$, i.e., an integral representation of $\Phi$ of rank $n$. Conversely, any integral representation of rank $n$ of $\Phi$ makes $\mathbb{Z}^n$ a $\Phi$-module. We will, then, identify $\Phi$-modules (of rank $n$) with integral representations of $\Phi$ (of rank $n$).

Under this identification, faithful modules correspond to faithful representations (monomorphisms); equivalence of modules corresponds to equivalence of representations, that is conjugation in $\text{Gl}(n,\mathbb{Z})$ by a fixed element $A \in \text{Gl}(n,\mathbb{Z})$ and invariants $(\lambda \in \Lambda[\mathbb{Z}_2 \oplus \mathbb{Z}_2])$, correspond to fixed points ($v \in \mathbb{Z}^n$, such that $\rho(\sigma)v = v$ for all $\sigma \in \Phi$).

Definition. An integral representation $\rho$ of a finite group $\Phi$ is decomposable if there are integral representations $\rho_1$ and $\rho_2$ of $\Phi$, such that $\rho$ is equivalent to $\rho_1 \oplus \rho_2$. The representation $\rho$ is indecomposable if it is not decomposable.
It follows from the previous definition that every integral representation \( \rho \) of a finite group \( \Phi \) is equivalent to a direct sum of indecomposable representations. However, the indecomposable summands are in general not uniquely determined (up to order and equivalence) by \( \rho \). That is, the Krull-Schmidt theorem does not hold for integral representations (see [Re]).

Since a fixed point for a representation \( \rho_1 \) is also a fixed point for the representation \( \rho_1 \oplus \rho_2 \) for any \( \rho_2 \), it follows that any integral representation without fixed points decomposes as a direct sum of indecomposable representations having no fixed points.

**\( \mathbb{Z}_2 \)-representations.** It is well known that there are three indecomposable representations of \( \mathbb{Z}_2 \), up to equivalence, which are given by:

\[
(1), \quad (-1), \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Moreover, Krull-Schmidt holds in this case.

**A useful invariant.** We introduce now an invariant for integral representations of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), which will allow us to prove the indecomposibility of some representations and also to prove that Krull-Schmidt holds for the sub-family of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-representations without fixed points.

**Proposition 2.1.** Let \( \rho \) and \( \rho' \) be two arbitrary integral representations of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and let \( S \) be any subset of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). If \( \rho \) and \( \rho' \) are equivalent via \( P \), then \( P \) induces an isomorphism of abelian groups

\[
\frac{\cap_{g \in S} \ker((\rho \oplus \rho')(g) \pm I)}{\cap_{g \in S} \im((\rho \oplus \rho')(g) \mp I)} \cong \frac{\cap_{g \in S} \ker(\rho(g) \pm I)}{\cap_{g \in S} \im(\rho(g) \mp I)} \cap \frac{\cap_{g \in S} \ker(\rho'(g) \pm I)}{\cap_{g \in S} \im(\rho'(g) \mp I)}.
\]

where the choice of signs is independent for each \( g \in S \).

**Proof.** We have \( \rho(g)^2 = I \) for all \( g \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), then \( \ker(\rho(g) \pm I) \supseteq \im(\rho(g) \mp I) \). Since \( P \in \text{Gl}(n, \mathbb{Z}) \), we obtain the following two equations

\[
P(\ker(\rho(g) \pm I))P^{-1} = \ker(P\rho(g)P^{-1} \pm I) = \ker(\rho'(g) \pm I)
\]

\[
P(\im(\rho(g) \mp I))P^{-1} = \im(P\rho(g)P^{-1} \mp I) = \im(\rho'(g) \mp I).
\]

Therefore, the restriction of \( P \) to \( \cap_{g \in S} \ker(\rho(g) \pm I) \) induces the claimed group isomorphism. \( \square \)

**Remark.** The proposition is still valid for representations of \( \mathbb{Z}_2^k \).

Notice that if \( \rho \) and \( \rho' \) are two integral representations of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), we have

\[
\frac{\cap_{g \in S} \ker((\rho \oplus \rho')(g) \pm I)}{\cap_{g \in S} \im((\rho \oplus \rho')(g) \mp I)} \cong \frac{\cap_{g \in S} \ker(\rho(g) \pm I)}{\cap_{g \in S} \im(\rho(g) \mp I)} \oplus \frac{\cap_{g \in S} \ker(\rho'(g) \pm I)}{\cap_{g \in S} \im(\rho'(g) \mp I)}.
\]
**Z₂ ⊕ Z₂-representations.** We adopt the following convention to describe a representation ρ of Z₂ ⊕ Z₂. We write B₁ = ρ(1,0), B₂ = ρ(0,1) and B₃ = B₁B₂ = ρ(1,1).

**Proposition 2.2.** There are three non-equivalent representations of Z₂ ⊕ Z₂, of rank 1 (characters), without fixed points, χᵢ for i = 1, 2, 3.

There are three equivalence classes of indecomposable representations of Z₂ ⊕ Z₂, of rank 2, without fixed points, ρⱼ for j = 1, 2, 3.

Representatives are given by,

\[
\begin{align*}
B₁ & = \rho(1,0), \\
B₂ & = \rho(0,1) \\
B₃ & = \rho(1,1).
\end{align*}
\]

\[
\begin{align*}
\chi₁ & : (1) (−1) (−1) \\
\chi₂ & : (−1) (1) (−1) \\
\chi₃ & : (−1) (−1) (1) \\
ρ₁ & : −I J −J \\
ρ₂ & : J −I −J \\
ρ₃ & : J −J −I
\end{align*}
\]

**Proof.** The determination of characters is straightforward.

Given any indecomposable representation of rank 2 we may assume that one of the matrices Bᵢ is J, otherwise the representation decomposes. From the identities BₖBₗ = BₗBₖ and B² = I it follows that Bᵢ = ±I or Bᵢ = ±J. On the other hand, a representation for which two matrices Bᵢ are equal to J (or −J) has a fixed point. Finally, by observing that J and −J are conjugate by \(( \begin{smallmatrix} 1 & 0 \\ 0 & −1 \end{smallmatrix} )\), the proof is complete.

Let us introduce two particular representations of rank 3, that will be referred to as μ and ν. They are defined by

\[
\begin{align*}
\mu & : B₁ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \\
\nu & : B₂ = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \\
\end{align*}
\]

(2.2)

It is easy to check that both representations have no fixed points.

For a fixed representation ρ and each triple of integers \(a₁, a₂, a₃\), with \(-1 \leq aᵢ \leq 1\), define the abelian group

\[
K^ρ_{(a₁,a₂,a₃)} = \frac{\bigcap_{aᵢ \neq 0} \text{Ker}(Bᵢ - aᵢI)}{\bigcap_{aᵢ \neq 0} \text{Im}(Bᵢ + aᵢI)}.
\]

**Example.** Let ρ₁ be the representation defined by B₁ = −I, B₂ = J and B₃ = −J as in Proposition 2.2. To determine the groups K^ρ₁(−1,0,1) and K^ρ₁(0,−1,1),

\[
K^ρ₁(−1,0,1) = \frac{\text{Ker}(B₁ + I) \cap \text{Ker}(B₃ - I)}{\text{Im}(B₁ - I) \cap \text{Im}(B₃ + I)},
\]

\[
K^ρ₁(0,−1,1) = \frac{\text{Ker}(B₁ - I) \cap \text{Ker}(B₂ - J)}{\text{Im}(B₁ + I) \cap \text{Im}(B₂ + J)}.
\]
we write down the kernels and the images in the canonical basis \{e_1, e_2\} of \(\mathbb{Z}^2\). We have

\[
\begin{align*}
\text{Ker}(B_1 + I) &= \langle e_1, e_2 \rangle & \text{Im}(B_1 - I) &= \langle 2e_1, 2e_2 \rangle \\
\text{Ker}(B_2 + I) &= \langle e_1 - e_2 \rangle & \text{Im}(B_2 - I) &= \langle e_1 - e_2 \rangle \\
\text{Ker}(B_3 - I) &= \langle e_1 - e_2 \rangle & \text{Im}(B_3 + I) &= \langle e_1 - e_2 \rangle,
\end{align*}
\]

from which it follows that

\[
K_{\rho}^{(0,-1,1)} \simeq \mathbb{Z}_2 \quad \text{and} \quad K_{\rho}^{(1,0,-1)} = 0.
\]

The computation of these invariants (Proposition 2.1) for all the representations in Proposition 2.2 and for the representations \(\mu\) and \(\nu\) in (2.2) are as simple as those performed above. Thus, we put together the results as a lemma, omitting the details.

**Lemma 2.3.** Let \(\rho\) be a representation equivalent to \(\chi_i\) or \(\rho_i\), \((1 \leq i \leq 3)\) as in Proposition 2.2 or equivalent to \(\mu\) or \(\nu\) as in (2.2). Then,

\[
\begin{align*}
\text{(a)} & \quad K_{\rho}^{(1,-1,-1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_1, \rho_2, \rho_3, \mu; \\
0, & \text{if } \rho \simeq \chi_2, \chi_3, \rho_1, \nu;
\end{cases} \\
\text{(b)} & \quad K_{\rho}^{(-1,1,-1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_2, \rho_1, \rho_3, \mu; \\
0, & \text{if } \rho \simeq \chi_1, \chi_3, \rho_2, \nu;
\end{cases} \\
\text{(c)} & \quad K_{\rho}^{(-1,-1,1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_3, \rho_1, \rho_2, \mu; \\
0, & \text{if } \rho \simeq \chi_1, \chi_2, \rho_3, \nu;
\end{cases} \\
\text{(d)} & \quad K_{\rho}^{(0,1,-1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_2, \rho_3, \mu; \\
0, & \text{if } \rho \simeq \chi_1, \chi_3, \rho_1, \rho_2, \nu;
\end{cases} \\
\text{(e)} & \quad K_{\rho}^{(0,-1,1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_3, \rho_2, \mu; \\
0, & \text{if } \rho \simeq \chi_1, \chi_2, \rho_1, \rho_3, \nu;
\end{cases} \\
\text{(f)} & \quad K_{\rho}^{(0,-1,-1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_1, \rho_3, \mu; \\
0, & \text{if } \rho \simeq \chi_2, \chi_3, \rho_1, \rho_2, \nu;
\end{cases} \\
\text{(g)} & \quad K_{\rho}^{(1,0,-1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_1, \rho_3, \mu; \\
0, & \text{if } \rho \simeq \chi_2, \chi_3, \rho_1, \rho_2, \nu;
\end{cases} \\
\text{(h)} & \quad K_{\rho}^{(-1,0,1)} = \begin{cases} 
\mathbb{Z}_2, & \text{if } \rho \simeq \chi_3, \rho_1; \\
0, & \text{if } \rho \simeq \chi_1, \chi_2, \rho_2, \rho_3, \mu, \nu;
\end{cases}
\end{align*}
\]
PRIMITIVE $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-MANIFOLDS

(i) $K^\rho_{(-1,0,-1)} = \begin{cases} \mathbb{Z}_2, & \text{if } \rho \simeq \chi_2, \rho_1, \rho_3, \mu; \\ 0, & \text{if } \rho \simeq \chi_1, \chi_3, \rho_2, \nu; \end{cases}$

(j) $K^\rho_{(-1,1,0)} = \begin{cases} \mathbb{Z}_2, & \text{if } \rho \simeq \chi_2, \rho_1; \\ 0, & \text{if } \rho \simeq \chi_1, \chi_3, \rho_2, \rho_3, \mu, \nu; \end{cases}$

(k) $K^\rho_{(1,-1,0)} = \begin{cases} \mathbb{Z}_2, & \text{if } \rho \simeq \chi_1, \rho_2, \mu; \\ 0, & \text{if } \rho \simeq \chi_2, \chi_3, \rho_1, \rho_3, \nu; \end{cases}$

(l) $K^\rho_{(-1,-1,0)} = \begin{cases} \mathbb{Z}_2, & \text{if } \rho \simeq \chi_3, \rho_1, \rho_2, \mu; \\ 0, & \text{if } \rho \simeq \chi_1, \chi_2, \rho_3, \nu. \end{cases}$

Corollary 2.4. The representations $\mu$ and $\nu$ are not equivalent.

Proof. The result follows from Proposition 2.1 and Equation (a) in the previous lemma. □

Corollary 2.5. The representations $\mu$ and $\nu$ are indecomposable.

Proof. Being $\nu$ of rank 3, if decomposable, it must be $\nu \simeq \chi_{j_1} \oplus \chi_{j_2} \oplus \chi_{j_3}$, for $1 \leq j_1, j_2, j_3 \leq 3$ or $\nu \simeq \chi_{j_1} \oplus \rho_{j_2}$, for $1 \leq j_1, j_2 \leq 3$. But Equations (a), (b) and (c) in Lemma 2.3 contradict both possibilities. The case of $\mu$ is similar. □

We had encountered 8 indecomposable representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ having no fixed points. The following theorem asserts that there are no more. Recall that they are given by

\[
\begin{align*}
B_1 & : \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\
B_2 & : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
B_3 & : \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}
\end{align*}
\]

(2.3)

\[
\begin{align*}
\chi_1 & : \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\
\chi_2 & : \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
\chi_3 & : \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
\rho_1 & : \begin{pmatrix} J \\ -I \end{pmatrix} \\
\rho_2 & : \begin{pmatrix} J \\ -J \end{pmatrix} \\
\rho_3 & : \begin{pmatrix} J \\ -I \end{pmatrix} \\
\mu & : \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
\nu & : \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
\end{align*}
\]

Theorem 2.6. Let $\rho$ be an indecomposable integral representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ having no fixed points. Then $\rho$ is equivalent to one and only one of the representations in (2.3).

The proof of Theorem 2.6 is elementary but not trivial. In order to make the paper more readable, we wrote the proof in the forthcoming subsection.

By assuming Theorem 2.6 one can skip the following subsection without losing the understanding of the whole paper.
Theorem 2.7. Let $\rho$ be an integral representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ having no fixed points. Let $\chi_i$ and $\rho_i$, for $1 \leq i \leq 3$, $\mu$ and $\nu$ be as in (2.3).

Then, there exist unique non-negative integers $m_i$ and $k_i$, for $1 \leq i \leq 3$, $s$ and $t$, such that

$$\rho \simeq m_1 \chi_1 \oplus m_2 \chi_2 \oplus m_3 \chi_3 \oplus k_1 \rho_1 \oplus k_2 \rho_2 \oplus k_3 \rho_3 \oplus s \mu \oplus t \nu.$$ 

Proof. From Theorem 2.6—and previous considerations—it follows that $\rho$ admits such a decomposition. So, it is clear that the groups $K_{(a_1,a_2,a_3)}^\rho$ will be all isomorphic to a power of $\mathbb{Z}_2$. Define $c(a_1,a_2,a_3) = \text{exponent of } K_{(a_1,a_2,a_3)}^\rho$.

One can compute, from Lemma 2.3, the 12 numbers $c(a_1,a_2,a_3)$, in particular

$$c(1,-1,-1) = m_1 + k_2 + k_3 + s,$$
$$c(-1,1,-1) = m_2 + k_1 + k_3 + s,$$
$$\vdots$$

All parameters but $t$ appear in these equations. It is not difficult to see that the linear system formed by those 12 equations and 7 unknowns has rank 7. Therefore, all but $t$ are uniquely determined by $\rho$. Finally, if $\rho$ has rank $n$ we have an extra equation,

$$n = m_1 + m_2 + m_3 + 2k_1 + 2k_2 + 2k_3 + 3s + 3t,$$

from which follows that also $t$ is uniquely determined by $\rho$. \qed

Remark. Theorem 2.7 says that the Krull-Schmidt theorem is valid for the sub-family of integral representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ without fixed points. Moreover, it follows from its proof that the multiplicities of the indecomposable summands can be effectively computed. This could be done by writing down the linear system in the proof and by computing the exponents $c(a_1,a_2,a_3)$ for the given $\rho$.

Indecomposable representations without fixed points of rank $n \geq 3$. As we said before, this subsection is devoted to the proof of Theorem 2.6.

While elementary, the proof is not trivial and since it is almost all technical, the reader can skip this part and continue with §3. In order to make it not too long not every single point will be explained. However, full details may be found in [T].

From now on, we follow some of the main ideas in [Na]. Unfortunately, as noted by Charlap ([Ch2, p. 135]), that paper “lacks complete proofs and is extremely laconic”. Moreover, low rank representations are not included.

We start with a pair of matrices $A, B \in \text{Gl}(n, \mathbb{Z})$ ($n \geq 3$), such that

$$A^2 = I = B^2 \quad \text{and} \quad AB = BA.$$ 

(2.4)

Notice that from such a pair one can define several integral representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (as many as $|\text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)| = 6$), possibly non-equivalent. It is
clear that all of them are decomposable or all of them are indecomposable simultaneously. By the representation defined by \( A \) and \( B \) we will refer to the representation \( \rho \) defined by

\[
\rho(1,0) = A \quad \text{and} \quad \rho(0,1) = B.
\]

In addition to (2.4) we assume that the representation defined by \( A \) and \( B \) is indecomposable and has no fixed points.

To achieve the result we will go through the following steps.

1. Show that \( A \) and \( B \) have some special canonical type (see Lemma 2.8).
2. Show that conjugating, in \( \text{Gl}(n, \mathbb{Z}) \), the special type of (1) is equivalent to performing some elementary operations on rows and columns (see Lemma 2.9).
3. Reduce matrices \( A \) and \( B \), according to (2), until it is clear whether the representation decomposes or not.

From (2.1) and the assumption \( n \geq 3 \) it follows that \( A \neq B \) and \( A \neq -B \). Otherwise, the representation defined by \( A \) and \( B \) decomposes. Observe that, in particular, the representation is faithful.

It is not difficult to show that for any integral matrix \( A \), of rank \( n \), the sub-lattice \( \Lambda = \text{Ker} A \subseteq \mathbb{R}^n \) admits a direct complement \( \Lambda^c \). That is, always there exists \( \Lambda^c \), such that \( \mathbb{Z}^n = \Lambda \oplus \Lambda^c \).

Let \( \Lambda = \text{Ker}(A - B) \). We have \( 0 \not\subset \Lambda \not\subset \mathbb{Z}^n \) and we can write

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.
\]

Since \( A - B|_{\Lambda} = 0 \), then \( A_{11} = B_{11} \) and \( A_{21} = B_{21} \). Moreover, from the equation \((A - B)(A + B) = A^2 - B^2 = 0\) we get \( 2A_{21} = 0 \) and \( A_{22} = -B_{22} \). Hence, we may assume that \( A \) and \( B \) are of the form

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} A_{11} & B_{12} \\ -A_{21} & -A_{22} \end{pmatrix}.
\]

The identity \( A^2 = I \) implies that \( A_{11}^2 = I = A_{22}^2 \). Therefore, both decompose as a direct sum of three blocks, an identity and a minus identity block and a matrix \( K \), being \( K \) a direct sum of matrices \( J \) (see (2.1)). Moreover, the condition of having no fixed points, for the representation defined by \( A \) and \( B \), forces \( A_{11} \) to be \(-I\). Finally, we get the following form for \( A \) and \( B \),

\[
(2.5) \quad A = \begin{pmatrix} -I & A_3 \\ I & -I \end{pmatrix}, \quad B = \begin{pmatrix} -I & 0 & B_2 & B_3 \\ -I & I & -K \end{pmatrix},
\]

where \( A_3 K = A_3 \) and \( B_3 K = -B_3 \).

**Remark.** Not all three blocks \( I, -I \) and \( K \) must be present in the decomposition of \( A_{22} \).
Notation: By $A = [\alpha_1 \ldots \alpha_n]$ we indicate the matrix with columns $\alpha_1, \ldots, \alpha_n$, being $\alpha_1$ the first one from the left.

It follows from the identities below (2.5) that $A_3 = [\alpha_1 \alpha_1 \alpha_2 \alpha_2 \ldots]$ and that $B_3 = [\alpha_1 (-\alpha_1) \alpha_2 (-\alpha_2) \ldots]$. We can then consider $B_3^\wedge = [\alpha_1 \alpha_2 \ldots]$, having half of the number of columns of $B_3$. Conversely, for any matrix $A = [\alpha_1 \alpha_2 \ldots]$ one can consider $A^\vee = [\alpha_1 (-\alpha_1) \alpha_2 (-\alpha_2) \ldots]$. Obviously, $B_3^\wedge \vee = B_3$.

**Lemma 2.8** (Canonical type). Let $A$ and $B$ be as in (2.4). Suppose the representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ defined by $A$ and $B$ is indecomposable and has no fixed points. Then, we may assume that $A$ and $B$ have the following type,

$$A = \begin{pmatrix} -I & A_1 & 0 & 0 \\ I & -I & K \end{pmatrix}, \quad B = \begin{pmatrix} -I & 0 & B_2 & B_3^\vee \\ -I & I & -K \end{pmatrix},$$

where $A_1$, $B_2$ and $B_3$ have entries in the ring $\mathbb{Z}_2 (= \{0, 1\})$.

**Proof.** We may first assume that $A$ and $B$ are as in (2.5). If $A_3 = [\alpha_1 \alpha_1 \alpha_2 \alpha_2 \ldots]$, take $C_3 = [(-\alpha_1)0(-\alpha_2)0\ldots]$. By conjugating $A$ and $B$ by

$$C = \begin{pmatrix} I & 0 & 0 \\ I & I \end{pmatrix},$$

one eliminates $A_3$.

Denote by $\tilde{P}$ the matrix obtained from $P$ by the canonical projection $\mathbb{Z} \longrightarrow \mathbb{Z}_2$. Let $\tilde{B}_3 = \tilde{B}_3^\wedge \vee$. Now the lemma follows by conjugating $A$ and $B$ by

$$C = \begin{pmatrix} I & \frac{\mathbf{A}_1 - A_1}{2} & \frac{\mathbf{A}_2 - B_2}{2} & \frac{\mathbf{B}_3 - B_3^\vee}{2} \\ I & I \end{pmatrix}.$$

□

**Lemma 2.9.** Let $A$ and $B$ be of the canonical type as in Lemma 2.8. If $A'$ and $B'$ are obtained from $A$ and $B$ by any of the following elementary row and column operations i-iv, then the representations given by $A, B$ and $A', B'$ are equivalent.

i. Column elementary operations on $A_1$.

ii. Column elementary operations on $B_2$.

iii. Column elementary operations on $B_3$.

iv. Simultaneous row elementary operations on $A_1$, $B_2$ and $B_3$.

**Proof.** Let $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ be unimodular. Recall that any elementary row (column) operation performed on an integral matrix $P$ can be realized
by multiplying $P$ on the left (right) by an adequate unimodular matrix. It is straightforward to see that if $C_4 K = K C_4$, then $A' = C A C^{-1}$ and $B' = C B C^{-1}$ are of the canonical type.

We notice that the matrix, which by right multiplication interchanges columns $2i+1$ and $2j+1$ and that simultaneously interchanges columns $2i+2$ and $2j+2$, commutes with $K$. Also the matrix, which replaces column $2i+1$ by the sum of columns $2i + 1$ and $2j + 1$ and that simultaneously replaces column $2i + 2$ by the sum of columns $2i + 2$ and $2j + 2$, commutes with $K$.

It is clear that performing on $B_3^\vee$ the latest operations is equivalent to performing on $B_3$ any elementary column operation. □

From now on we will concentrate on the sub-matrices $A_1$, $B_2$ and $B_3$. Recall that we can think of these as matrices with entries in the ring $\mathbb{Z}_2$.

**Lemma 2.10.** Let $A$ and $B$ be as in (2.5). If $A_{22} = \begin{pmatrix} I & -I \\ K & \end{pmatrix}$, then the representation given by $A$ and $B$ is decomposable.

**Proof.** Suppose the representation given by $A$ and $B$ (see (2.5)) is indecomposable. We may assume that $A$ and $B$ are of the canonical type. According to Lemma 2.9 we can reduce $A_1$ by row and column operations. Since a zero column in any of $A_1$, $B_2$ or $B_3$ would decompose the representation, we have

$$A_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

if $A_1$ is not square.

Consider the lower parts of $B_2$ and $B_3$ that correspond to the lower part of $A_1$ (the last zero rows). That one of $B_2$ can be reduced by row and column operations. It turns out that all these rows must be linearly independent. Otherwise one could obtain the following shapes for $A_1$, $B_2$ and $B_3$,

$$A_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} * \\ \vdots \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} * \\ \vdots \\ 0 \end{bmatrix},$$

from which it is clear that the representation decomposes.
Now is not difficult to see that one can obtain the following shape for $A_1$, $B_2$ and $B_3$,

$$
A_1 = \begin{bmatrix}
1 & & \\
& \ddots & \\
& & 1 \\
0 & & \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & * \\
1 & \ & 0 \\
& \ & 0 \\
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
* & * \\
1 & \ & 0 \\
& \ & 0 \\
\end{bmatrix}.
$$

To continue, we first reduce the $*$ block of $B_2$. If in this block the rows are linearly dependent, then we get a new partition of the upper part of $B_3$ and, after operating on $B_3$, we have

$$
A_1 = \begin{bmatrix}
1 & & \\
& \ddots & 1 \\
& & 0 \\
0 & & \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & I \\
0 & \ & 0 \\
I & 0 & \\
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
I & 0 \\
0 & I \\
0 & I \\
\end{bmatrix}.
$$

By writing down the corresponding matrices $A$ and $B$ it is clear that the representation decomposes.

Hence, it should be

$$
A_1 = \begin{bmatrix}
1 & & \\
& \ddots & \\
& & 1 \\
0 & & \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & 1 & & \\
& \ & \ddots & 1 \\
& & \ & 0 \\
I & 0 & & \\
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
0 & 1 & & \\
& \ & \ddots & 1 \\
& & \ & 0 \\
I & 0 & & \\
\end{bmatrix},
$$

which again gives a decomposable representation.

Therefore, $A_1$ must be square and in that case we obtain

$$
A_1 = \begin{bmatrix}
1 & & \\
& \ddots & \\
& & 1 \\
0 & & \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 & & \\
& \ddots & 1 \\
& & 1 \\
0 & & \\
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
1 & & \\
& \ddots & 1 \\
& & 1 \\
0 & & \\
\end{bmatrix}.
$$

The corresponding representation, given by $A$ and $B$, is clearly decomposable if any of $A_1$, $B_2$ or $B_3$ is of rank $m \geq 2$. So, it remains possibly indecomposable the representation defined by

$$
A = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
-1 & 0 & 1 & 1 & -1 \\
1 & -1 & 0 & -1 & 0 \\
-1 & 1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 \\
\end{pmatrix}.
$$

However, the unimodular matrix

$$
P = \begin{pmatrix}
1 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
$$
Lemma 2.11. Let $A$ and $B$ be as in (2.5). Suppose the lower block $A_{22}$ of $A$ decomposes as the sum of at most 2 blocks from among $I$, $-I$ and $K$. If the representation, defined by $A$ and $B$, is indecomposable, then we may assume that $A$ and $B$ have one of the following forms:

- $A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$;
- $A_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$.

Notice that the representations defined by the pairs $A_i$, $B_i$ ($i = 1, 2$) in Lemma 2.11 are exactly the representations $\mu$ and $\nu$ introduced in (2.2).

The proof of Lemma 2.11 is similar and easier than that of Lemma 2.10, even if there are several cases to be considered. For the details see [T].

The following proposition completes this sub-section.

Proposition 2.12. Let $\mu$ and $\nu$ be the representations defined respectively by the pairs of matrices $A_1$, $B_1$ and $A_2$, $B_2$ in Lemma 2.11. If $\sigma \in \text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, then $\mu \circ \sigma \sim \mu$ and $\nu \circ \sigma \sim \nu$.

Proof. Denote by $I_Q$ the conjugation by $Q$, i.e., $I_Q(A) = QAQ^{-1}$. Considering the first pair, $A_1$ and $B_1$, the matrix $P_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ satisfies

- $I_{P_1}(A_1) = B_1$, $I_{P_1}(B_1) = A_1$, $I_{P_1}(A_1B_1) = A_1B_1$.

On the other hand, the matrix $P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ satisfies

- $I_{P_2}(A_2) = A_2B_2$, $I_{P_2}(A_2B_2) = A_2$, $I_{P_2}(B_2) = B_2$.

Since $\text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \simeq S_3$, the result follows for the first pair.

The second case is analogous, we take $Q_1$ and $Q_2$ instead of $P_1$ and $P_2$, with

- $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

3. Cohomology Computations.

In this section we shall compute the second cohomology groups $H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda)$ for any $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-module $\Lambda$ satisfying $\Lambda^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} = 0$. In order to determine special classes in $H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda)$, we also investigate the restriction functions $\text{res}_K : H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \longrightarrow H^2(K; \Lambda)$ for each subgroup $K$ of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of order 2.
Regard $H^n(Z_2 \oplus Z_2; \Lambda)$ as the homology of the standard complex of functions $\{(F^n(Z_2 \oplus Z_2; \Lambda); \partial^n)\}_{n \geq 0}$ and recall that, in particular, we have

$$
\partial^2 f(x, y, z) = x \cdot f(y, z) - f(xy, z) + f(yz, x) - f(x, y);
\partial^1 g(x, y) = x \cdot g(y) - g(xy) + g(x).
$$

As is well known, we may assume that every (2-)cocycle $h$ is normalized, that is $h(x, I) = h(I, x) = 0$, for any $x \in Z_2 \oplus Z_2$.

If $\Lambda$ and $\Delta$ are two $Z_2 \oplus Z_2$-modules, then

$$
H^n(Z_2 \oplus Z_2; \Lambda \oplus \Delta) \simeq H^n(Z_2 \oplus Z_2; \Lambda) \oplus H^n(Z_2 \oplus Z_2; \Delta).
$$

If $\Lambda$ and $\Delta$ are semi-equivalent via the semi-linear map $(F, \sigma)$, then the map defined on the cocycles for $\Lambda$ by

$$
f(g_1, \ldots, g_n) \mapsto Ff(\sigma g_1, \ldots, \sigma g_n)
$$

induces an isomorphism

$$
H^n(Z_2 \oplus Z_2; \Lambda) \simeq H^n(Z_2 \oplus Z_2; \Delta).
$$

We will make use of the cohomology long exact sequence (3.3) induced by a short exact sequence of $\Phi$-modules as

$$
\cdots \rightarrow H^1(\Phi; \Lambda) \xrightarrow{\pi'} H^1(\Phi; \Lambda_2) \xrightarrow{\delta^1} H^2(\Phi; \Lambda_1) \xrightarrow{j'} H^2(\Phi; \Lambda_2) \xrightarrow{\delta^2} H^3(\Phi; \Lambda_1) \xrightarrow{j'} \cdots
$$

Recall that $j'[f] = [j \circ f]$ and $\pi'[g] = [\pi \circ g]$ for $f$ and $g$ cocycles. Also recall that $\delta^n : H^n(Z_2 \oplus Z_2; \Lambda) \rightarrow H^{n+1}(Z_2 \oplus Z_2; \Lambda)$ is defined by $\delta^n[f] = [h]$, for any cocycle $f$ if $h$ satisfies $j'h = \partial^n g$, where $g$ is any element in $F^n(Z_2 \oplus Z_2; \Lambda)$ for which $\pi'g = f$.

We come now to the computations.

Let $\Lambda$ be a $Z_2 \oplus Z_2$-module, such that $\Lambda^{Z_2 \oplus Z_2} = 0$. By (3.1) we can assume that $\Lambda$ is indecomposable. Therefore, by Theorem 2.6 we may restrict to the case $\Lambda$ is one of the modules in (2.3). Moreover, since clearly $\chi_1, \chi_2$ and $\chi_3$ are all semi-equivalent as $\rho_1, \rho_2$ and $\rho_3$ are all semi-equivalent, by (3.2) there remain 4 cases to be considered. Precisely those given by

$$
\begin{align*}
\text{I. } & \chi_1 : (1) & B_1 \\
\text{II. } & \rho_1 : -I & -J \\
\text{III. } & \mu : \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \\
\text{IV. } & \nu : \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}
\end{align*}
$$

Since the computations that follow are standard, we will only indicate how to get the results. However, full details may be found in [T].
Case i. In this case one can first determine all the (normalized) cocycles, that is, those functions \( f \in \mathcal{F}^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \) for which \( \partial^2 f = 0 \). Consider the linear system of 27 equations and 9 unknowns given by \( \partial^2 f = 0 \). It is not difficult to check that this system is equivalent to the following linearly independent set of equations:

\[
\begin{align*}
    h(B_2, B_2) &= 0, \\
    h(B_3, B_3) &= 0, \\
    h(B_2, B_3) &= h(B_2, B_1), \\
    h(B_3, B_2) &= h(B_3, B_1), \\
    h(B_2, B_3) &= -h(B_1, B_3), \\
    h(B_3, B_2) &= -h(B_1, B_2), \\
    h(B_1, B_1) &= h(B_1, B_2) + h(B_1, B_3).
\end{align*}
\]

Thus, it is clear that a general cocycle is of the form

\[
h = \begin{pmatrix}
    \alpha + \beta \\
    -\beta \\
    \alpha
\end{pmatrix}
\]

for some integers \( \alpha \) and \( \beta \). If we let \( h_\alpha \) (resp. \( h_\beta \)) be the cocycle obtained by setting \( \alpha = 1 \) and \( \beta = 0 \) (resp. \( \alpha = 0 \) and \( \beta = 1 \)), it is immediate that \( h_\alpha \) and \( h_\beta \) generates the group \( \mathbb{H}^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \) we are considering. Now it is not difficult to see that \( h_\alpha \sim h_\beta \), \( h_\alpha \not\sim 0 \) and \( 2h_\alpha \sim 0 \). Thus,

\[
\mathbb{H}^2(\langle B_1, B_2 \rangle; \Lambda) \cong \mathbb{Z}_2 \cong \langle [h_\alpha] \rangle \cong \langle [h_\beta] \rangle.
\]

Case ii. Consider the submodule \( \Lambda_1 = \langle e_1 + e_2 \rangle \). One can see that \( B_2 \) acts by \( 1 \) and \( B_1 \) by \( -1 \) on \( \Lambda_1 \). Let \( \Lambda_2 \) be the quotient \( \Lambda / \Lambda_1 \). On \( \Lambda_2 \), \( B_1 \) and \( B_2 \) both act as \( -1 \). Now we have the long exact sequence \((3.3)\). It follows from \((3.5)\) and \((3.2)\) that \( \mathbb{H}^2(\Phi; \Lambda_1) \sim \mathbb{Z}_2 \), as well as \( \mathbb{H}^2(\Phi; \Lambda_2) \sim \mathbb{Z}_2 \). Since we have explicit generators, one can show that the map \( j' \) is the zero map. On the other hand, one can also show that the map \( \delta^2 \) is injective. Hence, it follows that

\[
\mathbb{H}^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda_1) = 0.
\]

Case iii. Let \( \Lambda_1 \) be the submodule \( \langle e_1 \rangle \) and, as before, let \( \Lambda_2 \) be the quotient \( \Lambda / \Lambda_1 \). Then, \( \Lambda_1 \) is semi-equivalent to the module in Case i, while \( \Lambda_2 \) is semi-equivalent the module in Case ii. Since one can show that, in the corresponding long exact sequence \( j' \) is injective, then it follows from \((3.6)\) that

\[
j' : \mathbb{H}^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda_1) \longrightarrow \mathbb{H}^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda)
\]
is an isomorphism.

**Case IV.** If we let $\Lambda_1$ be the submodule $\langle e_1 + 2e_2 \rangle$ and $\Lambda_2$ as usual, it is not difficult to see that $\Lambda_1$ is given by the character $\chi_1$ and that $\Lambda_2$ decomposes as the direct sum of modules given by the characters $\chi_2$ and $\chi_3$.

In this case we need more information on the long exact sequence (3.3). We compute explicitly, as in Case 1, the groups $H^1(Z_2 \oplus Z_2; \Lambda)$ and $H^1(Z_2 \oplus Z_2; \Lambda_2)$. We find that $H^1(Z_2 \oplus Z_2; \Lambda) \simeq Z_1 \oplus Z_2$ and $H^1(Z_2 \oplus Z_2; \Lambda_2) \simeq Z_2 \oplus Z_2$, moreover we find that the map $\pi': H^1(Z_2 \oplus Z_2; \Lambda) \longrightarrow H^1(Z_2 \oplus Z_2; \Lambda_2)$ is defined by $\pi'(1, 0) = \pi'(0, 1) = (1, 1)$. Thus, we have the exact sequence

$$\cdots \longrightarrow Z_4 \oplus Z_2 \stackrel{\pi'}{\longrightarrow} Z_2 \oplus Z_2 \stackrel{\delta_1}{\longrightarrow} Z_2 \stackrel{j'}{\longrightarrow}$$

$$H^2(Z_2 \oplus Z_2; \Lambda) \stackrel{\pi'}{\longrightarrow} Z_2 \oplus Z_2 \stackrel{\delta_2}{\longrightarrow} H^3(Z_2 \oplus Z_2; \Lambda_1) \longrightarrow \cdots$$

It can be shown that $\delta_2$ is injective and therefore, $j'$ is onto. Since $\text{Im} \pi' = \langle (1, 1) \rangle = \ker \delta_1$, it follows that $\delta_1$ is also onto, from which it is immediate that $j'$ is the zero map. Hence,

$$H^2(Z_2 \oplus Z_2; \Lambda) = 0. \tag{3.8}$$

Putting together (3.5)-(3.8) we get the following.

**Proposition 3.1.** Let $\Lambda$ be a $Z_2 \oplus Z_2$-module, such that $\Lambda^{Z_2 \oplus Z_2} = 0$. If $\Lambda$ is equivalent to the $Z_2 \oplus Z_2$-module given by the representation $\rho = m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu$, then

$$H^2(Z_2 \oplus Z_2; \Lambda) \simeq Z_2^{m_1} \oplus Z_2^{m_2} \oplus Z_2^{m_3} \oplus Z_3^{3}.$$  

**Restriction functions.** We now investigate the restriction functions $\text{res}_K : H^2(Z_2 \oplus Z_2; \Lambda) \longrightarrow H^2(K; \Lambda)$, for any of the $Z_2 \oplus Z_2$-modules $\Lambda$ in which we are interested and where $K$ is any non-trivial subgroup of $Z_2 \oplus Z_2$. It is clear that it suffices to assume that $\Lambda$ is one of the modules in Cases i-iv in (3.4).

Set $K_i = \langle B_i \rangle$. Recall that there are three indecomposable $Z_2$-modules (see (2.1)). It is well known that $H^2(Z_2; \Lambda) \simeq Z_2 = \langle [f] \rangle$ if $\Lambda$ is the trivial module of rank 1, where $f : Z_2 \times Z_2 \longrightarrow \Lambda$ is defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x = y = 1; \\ 0, & \text{if } x = 0 \text{ or } y = 0; \end{cases}$$

and that $H^2(Z_2; \Lambda) = 0$ for the other two indecomposable modules.

We now come to a case by case analysis following the cases in (3.4).

**Case 1.** Since $H^2(K_i; \Lambda) = 0$ for $i = 2$ or $i = 3$, we only consider the case $i = 1$. We have (see (3.5))

$$h_\alpha |_{\langle B_1 \rangle \times \langle B_1 \rangle} (x, y) = \begin{cases} -1, & \text{if } (x, y) = (B_1, B_1); \\ 0, & \text{if } (x, y) \neq (B_1, B_1). \end{cases}$$
Notice that \( h_\alpha \mid_{\langle B_1 \rangle \times \langle B_1 \rangle} \sim f \), then
\[
\text{res}_{\langle B_i \rangle} = \begin{cases} 
\text{id}_{\mathbb{Z}_2}, & \text{if } B_i = B_1; \\
0, & \text{if } B_i = B_2, B_3.
\end{cases}
\]

**Case III.** We observe that each \( B_i \) \((1 \leq i \leq 3)\) decomposes as \((-1_j)\). In fact, there exist unimodular matrices \( P_i \) such that \( P_i B_i = (-1_j) P_i \). Precisely,
\[
P_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}; \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 \end{pmatrix}.
\]
Thus, we have \( H^2(K_i; \Lambda) = 0 \) and therefore, all the restriction functions are zero.

**Cases II and IV.** Since \( H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) = 0 \) (see (3.5) and (3.7)), there is nothing to be done.

4. **Classification.**

It is straightforward to deduce from the Preliminaries that the classification, up to affine equivalence, of all primitive \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-manifolds can be achieved by

(i) determining the semi-equivalence classes of faithful \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-modules \( \Lambda \) such that \( \Lambda \mathbb{Z}_2 \oplus \mathbb{Z}_2 = 0 \);

(ii) determining, for each \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-module \( \Lambda \) in (i), the equivalence classes of special cohomology classes in \( H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \).

Recall that \( \alpha \) and \( \beta \) in \( H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \) are equivalent \((\alpha \sim \beta)\) if and only if there exist a semi-linear map \((f, \phi) : \Lambda \longrightarrow \Lambda\), such that \( f_* \alpha = \phi^* \beta \).

We start dealing with (ii), postponing (i).

Let \( \Lambda \) be a fixed \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-module, such that \( \Lambda \mathbb{Z}_2 \oplus \mathbb{Z}_2 = 0 \). Since equivalence implies semi-equivalence, we may assume that \( \Lambda \) is given by
\[
\rho = m_1 \chi_1 \oplus m_2 \chi_2 \oplus m_3 \chi_3 \oplus k_1 \rho_1 \oplus k_2 \rho_2 \oplus k_3 \rho_3 \oplus s \mu \oplus t \nu,
\]
(see Theorem 2.7). Then, by Proposition 3.1, we have
\[
H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \cong \mathbb{Z}_{2}^{m_1} \oplus \mathbb{Z}_{2}^{m_2} \oplus \mathbb{Z}_{2}^{m_3} \oplus \mathbb{Z}_{2}^{s}.
\]
According to this decomposition, we may express a class \( \alpha \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \) as a 4-tuple, \( \alpha = (v_1, v_2, v_3, v_4) \), where \( v_i \in \mathbb{Z}_{2}^{m_i} \) for \( i = 1, 2, 3 \) and \( v_4 \in \mathbb{Z}_2 \).

**Proposition 4.1.** Let \( \alpha \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \), \( \alpha = (v_1, v_2, v_3, v_4) \) and let \( \delta_i = 1 \) if \( v_i \neq 0 \) or \( \delta_i = 0 \) if \( v_i = 0 \). Then
\[
\alpha \sim (\overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}),
\]
where \( \overline{\delta_i} = (\delta_i, 0, \ldots, 0) \in \mathbb{Z}_2^{m_i} \) \((i = 1, 2, 3)\) or \( \overline{\delta_4} \in \mathbb{Z}_2^{s} \).

Before proving this proposition, we state the following general lemma.
Lemma 4.2. Let $\Lambda_1$ be a $\Phi$-module and let $\alpha \in \text{H}^n(\Phi; \Lambda_1)$. If $\Lambda = \bigoplus_{i=1}^r \Lambda_1$, then for each $\beta_{(i_2, \ldots, i_r)} = (\alpha, i_2\alpha, \ldots, i_r\alpha) \in \text{H}^n(\Phi; \Lambda)$, with $i_j = 0, 1$, there exists a $\Phi$-morphism $f : \Lambda \to \Lambda$, such that 

$$f_*(\alpha, 0, \ldots, 0) = \beta.$$ 

Furthermore, for each $1 \leq j \leq r$, there exists a $\Phi$-morphism $g_j : \Lambda \to \Lambda$, such that 

$$g_j(\alpha, 0, \ldots, 0) = (0, \ldots, 0, \alpha, 0, \ldots, 0).$$ 

Proof. Given $\beta_{(i_2, \ldots, i_r)}$, we define $f : \Lambda \to \Lambda$ by the block matrix 

$$A = \begin{pmatrix} I \\ i_2I & I \\ \vdots \\ i_rI & I \end{pmatrix},$$ 

where each block corresponds to a $\Lambda_1$ summand.

If $\phi \in \Phi$, $\phi$ acts on $\Lambda$ by 

$$B = \begin{pmatrix} B_1 \\ B_1 \\ \vdots \\ B_1 \end{pmatrix}.$$ 

By checking that $AB = BA$, one shows that $f$ is a $\Phi$-morphism.

Suppose $\alpha_1 : \Phi \times \cdots \times \Phi \to \Lambda_1$ is a cocycle representing $\alpha$. Thus, the cocycle $\tilde{\alpha} : \Phi \times \cdots \times \Phi \to \Lambda$ defined by $p_1(\tilde{\alpha}) = \alpha_1$ and $p_i(\tilde{\alpha}) = 0$, for $2 \leq i \leq r$, where $p_i : \Lambda \to (\Lambda_1)_i$ is the $i$-th projection, is a representative of $(\alpha, 0, \ldots, 0)$.

Therefore, $f_*(\alpha, 0, \ldots, 0) = [f \circ \tilde{\alpha}]$. It is not difficult to see that the composition $f \circ \tilde{\alpha} : \Phi \times \cdots \times \Phi \to \Lambda_1 \oplus \cdots \oplus \Lambda_r$ satisfies $p_1(f \circ \tilde{\alpha}) = \alpha_1$ and $p_i(f \circ \tilde{\alpha}) = i_j\alpha_1$, therefore $f_*(\alpha, 0, \ldots, 0) = \beta_{(i_2, \ldots, i_r)}$.

Finally, we define $g$ by the matrix 

$$i \to \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix},$$

$$j \to \begin{pmatrix} 0 & \cdots & I \\ \vdots \\ I & \cdots & 0 \end{pmatrix},$$

and by arguing in a similar way as above, the lemma is proved. $\square$

Proof of Proposition 4.1. We shall define an adequate semi-linear map $(f, I)$. We define $f$ on each of its indecomposable submodules. Since $H^2(Z_2 \oplus Z_2; \chi_i) \simeq Z_2$ and $H^2(Z_2 \oplus Z_2; \mu) \simeq Z_2$, the result follows by applying Lemma 4.2 to the submodules corresponding to representations $m_i\chi_i$ and $s\mu$. In the other submodules define $f$ as the identity. $\square$
In light of the characterization in Proposition 4.1, it is not difficult to decide when a class \( \alpha \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \) is special.

**Proposition 4.3.** The class \( \alpha = (\delta_1, \delta_2, \delta_3, \delta_4) \), where \( \delta_i = 0 \) or \( \delta_i = 1 \), is special if and only if \( \delta_i = 1 \) for \( 1 \leq i \leq 3 \).

**Proof.** It suffices to consider the classes \( (\delta_1 \alpha_1, \delta_2 \alpha_2, \delta_3 \alpha_3, \delta_4 \alpha_4) \) with \( \alpha_i \) the chosen generators of \( H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \chi_i) \) (see (3.5) and (3.2)), for \( 1 \leq i \leq 3 \), and \( \alpha_4 \) the generator of \( H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \mu) \) given by (3.7).

From the computations of the restriction functions it follows directly that

\[
\text{res}(B_i) = \begin{cases} 
\delta_1, & \text{if } i = 1; \\
\delta_2, & \text{if } i = 2; \\
\delta_3, & \text{if } i = 3;
\end{cases}
\]

proving the proposition. \( \square \)

Thus, it is clear that in \( H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Lambda) \), there are at most two equivalence classes of special classes. Representatives for each of them are \( \alpha_1 = (1, 1, 1, 0) \) and \( \alpha_2 = (1, 1, 1, 1) \).

**Lemma 4.4.** The classes \( \alpha_1 \) and \( \alpha_2 \) are equivalent.

**Proof.** Let \( \Lambda = \langle e_1, \ldots, e_n \rangle \). The action of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is given by \( \chi_1 \) in \( \langle e_i \rangle \) if \( 1 \leq i \leq m_1 \), by \( \chi_2 \) in \( \langle e_j \rangle \) if \( m_1 + 1 \leq j \leq m_1 + m_2 \), etc. In particular \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) acts by \( \mu \) in \( \Delta_s = \langle e_{m_1+2k+1}, e_{m_2+2k+2}, e_{m_2+2k+3} \rangle \).

We shall define an additive \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-morphism \( f : \Lambda \to \Lambda \) (i.e., a \( \mathbb{Z} \)-isomorphism), such that \( f_\ast \alpha_1 = \alpha_2 \). Let \( \Delta \) be the submodule \( \Delta = \langle e_{m_1+m_2+1}, \Delta_s \rangle \). Define \( f : \Delta \to \Delta \) by \( f = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \) and notice that the matrix of \( f \) is unimodular. The action of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) on \( \Delta \) is given by

\[
B_1 = \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right), \quad B_2 = \left( \begin{array}{cc} -1 & 0 \\ -1 & 0 \end{array} \right).
\]

We verify that \( f : \Delta \to \Delta \) is in fact a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-morphism computing,

\[
\left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right); \\
\left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} -1 & -1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right).
\]

For \( \alpha' \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Delta) \), \( \alpha' = (1, 0) \) we compute \( f_\ast \alpha' \). If \( c : (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \to \Delta \) is a cocycle representing \( \alpha' \), then \( f_\ast \alpha' = f_\ast [c] = [f \circ c] \). It is clear from the previous section that we can choose the last three coordinates in \( c \) to be zero and in the first one...
The last two coordinates of the cocycle $f \circ c$ are zero, while the first two have values

\[
\begin{array}{c|ccc}
(c)_1 & B_1 & B_2 & B_3 \\
\hline
B_1 & 0 & 0 & 0 \\
B_2 & 1 & 0 & 1 \\
B_3 & -1 & 0 & -1 \\
\end{array}
\]

Now, it is clear that $[(f \circ c)_1] \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \langle e_{m_1+m_2+1} \rangle)$ does not vanish and that $[(f \circ c)|_{\Delta_s}] \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Delta_s)$ do not vanish either (see Case III in §3), therefore $[f \circ c] = (1, 1) \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \Delta)$. \(\square\)

(4.1) The Hantzsche-Wendt module. Consider the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-module $\Lambda$, of rank 3, given by the representation $\chi_1 \oplus \chi_2 \oplus \chi_3$. Notice that it is a faithful module and clearly $\Lambda_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} = 0$.

The Hantzsche-Wendt manifold (see Introduction) is built on this module (see §5). Thus we will call it the Hantzsche-Wendt module.

By Proposition 4.3 a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-module $\Lambda$ admits a special cohomology class if and only if $\Lambda$ contains a submodule equivalent to the Hantzsche-Wendt module.

We can state the main theorem which is now an immediate consequence of Proposition 4.3, Lemma 4.4 and (4.1).

**Theorem 4.5.** The affine equivalence classes of compact Riemannian flat manifolds with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and first Betti number zero are in a bijective correspondence with the $\mathbb{Z}[\mathbb{Z}_2 \oplus \mathbb{Z}_2]$-modules $\Lambda$, such that:

1. As abelian group $\Lambda$ is free and of finite rank;
2. $\Lambda_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} = 0$;
3. $\Lambda$ contains a submodule equivalent to the Hantzsche-Wendt module.

For completeness we should treat step (i) at the beginning of §4. Actually, after Theorem 4.5, it would suffice to determine the semi-equivalence classes of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-modules given by

\[
(4.2) \quad \rho = m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu
\]

\[
= (m_1, m_2, m_3, k_1, k_2, k_3, s, t)
\]

with $m_1, m_2, m_3 \geq 1$, which are already faithful.
For each \( \sigma \in S_3 = \text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \) denote by \( \sigma \rho \) the representation \( \rho \circ \sigma \). If \( \rho \) is given by matrices \( B_1, B_2 \) and \( B_3 \), then \( \sigma \rho \) is given by \( B_1' = B_{\sigma(1)}, B_2' = B_{\sigma(2)} \) and \( B_3' = B_{\sigma(3)} \).

It is clear that \( \rho \) and \( \rho' \) are semi-equivalent if and only if there exist \( \sigma \in S_3 \), such that \( \rho' \sim \sigma \rho \).

Notice that if \( \rho \) is as in (4.2), then

\[
\sigma \rho = m_1 \sigma \chi_1 \oplus m_2 \sigma \chi_2 \oplus m_3 \sigma \chi_3 \oplus k_1 \sigma \rho_1 \oplus k_2 \sigma \rho_2 \oplus k_3 \sigma \rho_3 \oplus s \sigma \mu \oplus t \sigma \nu.
\]

**Proposition 4.6.** Let \( \rho \) and \( \rho' \) be the representations given, respectively, by the 8-tuples \((m_1, m_2, m_3, k_1, k_2, k_3, s, t)\) and \((m'_1, m'_2, m'_3, k'_1, k'_2, k'_3, s', t')\). Then \( \rho \) and \( \rho' \) are semi-equivalent if and only if \( s = s' \), \( t = t' \) and there exist \( \sigma \in S_3 \), such that \( m_i = m_{\sigma(i)} \) and \( k_i = k_{\sigma(i)} \) for \( 1 \leq i \leq 3 \).

**Proof.** Observe that \( \sigma \chi_1 = \chi_{\sigma(i)} \) and, since \( J \sim -J \), then also \( \sigma \rho_i \sim \rho_{\sigma(i)} \).

Finally, from Proposition 2.12 it follows that for any \( \sigma \) it verifies \( \sigma \mu \sim \mu \) and \( \sigma \nu \sim \nu \). Thus, the proposition is a consequence of Theorem 2.6. \( \square \)

**Remark.** The total number of primitive \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-manifolds of dimension \( n \) \((e_n)\) can be estimated by estimating the number of 8-tuples \((m_1, m_2, m_3, k_1, k_2, k_3, s, t)\) with \( m_1, m_2, m_3 \geq 1 \) modulo the equivalence imposed by Proposition 4.6.

It can be shown that \( e_n \sim E n^7 \), as \( n \to \infty \), with \( E = \frac{1}{2^83^55^7} \) (c.f. Theorem 3.5 in [RT]).

5. Realizations.

In this section we construct discrete subgroups \( \Gamma \) of isometries of \( \mathbb{R}^n \) to exhibit each classified manifold as a quotient \( \mathbb{R}^n/\Gamma \).

Consider the group \( \Gamma = \langle \mathbb{Z}^n, B_1L_{b_1}, B_2L_{b_2} \rangle \), with \( B_1 \) and \( B_2 \) in \( O(n) \), \( b_1 \) and \( b_2 \) in \( \mathbb{R}^n \) and \( \langle B_1, B_2 \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), such that \( \Gamma \) satisfies the exact sequence

\[
0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow \langle B_1, B_2 \rangle \longrightarrow 1,
\]

where \( \langle B_1, B_2 \rangle \) acts on \( \mathbb{Z}^n \) by evaluation, \( B_1 \cdot \lambda = B_1(\lambda) \).

Let us construct \( \Gamma(m_1, m_2, m_3, k_1, k_2, k_3, s, t) \). Consider the following matrices, that we may assume orthogonal since the group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is finite. Set

\[
B_1 = \begin{pmatrix}
I_{m_1} & -I_{m_2} & -I_{m_3} & I_{2k_1} & K_{k_2} & K_{k_3} & M_s & N_t^1 \\
-1_{m_2} & I_{m_1} & -I_{m_3} & K_{k_2} & K_{k_3} & M_s & N_t^1 & \end{pmatrix},
\]

(5.1)
4.1 \text{it follows that it suffices to consider the submatrices}

\[
B_2 = \begin{pmatrix}
-I_{m_1} & I_{m_2} & -I_{m_3} \\
& K_1 & -I_{2k_2} \\
& & -K_{k_3} & M_2^2 \\
& & & N_2^2
\end{pmatrix},
\]

where \(I_m\) is the \(m \times m\) identity matrix, \(K_m\) is the direct sum of \(m\) matrices \(J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(M_s^1\) (resp. \(M_s^2\)) is the direct sum of \(s\) matrices \(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) (resp. \(\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}\)) and \(N_t^1\) (resp. \(N_t^2\)) is the direct sum of \(t\) matrices \(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\) (resp. \(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\)) (see 3.4).

We shall find translations \(L_{b_1}\) and \(L_{b_2}\) in order to have \(\Gamma\) torsionfree. We already know that every pair of translations producing a torsionfree \(\Gamma\), will give isomorphic groups (see Theorem 4.5).

From Proposition 4.3 it follows that it suffices to consider the submatrices (of \(B_1\) and \(B_2\))

\[
C_1 = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.
\]

Notice that \(\mathbb{Z}^3\) with the action of \(\langle C_1, C_2 \rangle\) is the Hantzsche-Wendt module (see (4.1)). Let \(\Gamma_C = \langle \mathbb{Z}^3, C_1 L_{b_1}, C_2 L_{b_2} \rangle\) with \(b_1 = (b_1^1, b_1^2, b_1^3)\) and \(b_2 = (b_2^1, b_2^2, b_2^3)\).

The extension class \(\alpha \in H^2((C_1, C_2); \mathbb{Z}^3)\) of \(\Gamma_C\) as an extension of \(\langle C_1, C_2 \rangle\) by \(\mathbb{Z}^3\) is \(\alpha = [f]\); where \(f : \langle C_1, C_2 \rangle \times \langle C_1, C_2 \rangle \to \mathbb{Z}^3\) is defined by \(f(x, y) = s(x) s(y) s(xy)^{-1}\), for \(x, y \in \langle C_1, C_2 \rangle\) and \(s\) is any section \(\langle C_1, C_2 \rangle \to \Gamma_C\).

Take the section \(s\) defined by \(s(I) = I, s(C_i) = C_i L_{b_i} (i = 1, 2)\) and \(s(C_1 C_2) = s(C_1) s(C_2)\). We have explicitly,

\[
\begin{array}{c|c|c|c}
\hline
f & C_1 & C_2 & C_3 \\
\hline
C_1 & \begin{pmatrix} 2b_1^1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 2b_1^1 \\ 0 \\ 0 \end{pmatrix} \\
C_2 & \begin{pmatrix} -2b_1^1 \\ 2b_2^1 \\ 2(b_1^1 - b_2^1) \end{pmatrix} & \begin{pmatrix} 0 \\ 2b_2^1 \\ 0 \end{pmatrix} & \begin{pmatrix} -2b_1^1 \\ 0 \\ 2(b_1^1 - b_2^1) \end{pmatrix} \\
C_3 & \begin{pmatrix} -2b_2^2 \\ 2(-b_2^1 + b_2^3) \\ 2(b_1^1 - b_2^3) \end{pmatrix} & \begin{pmatrix} 0 \\ -2b_2^1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2(-b_2^1 + b_2^3) \end{pmatrix} \\
\hline
\end{array}
\]

Since \([f] = [f]_1 + [f]_2 + [f]_3\) lies in

\[
H^3((C_1, C_2); \mathbb{Z}) \oplus H^2((C_1, C_2); \mathbb{Z}) \oplus H^2((C_1, C_2); \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,
\]
we should choose \( b_1 \) and \( b_2 \) in such a way that \([f]_1 = 1\), \([f]_2 = 1\) and \([f]_3 = 1\). Recalling the computations in §3, it turns out that 
\[
 b_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}
\]
fit our needs.

**Definition.** We will denote by 
\[
M(m_1, m_2, m_3, k_1, k_2, k_3, s, t),
\]
with \( m_i \geq 1 \) \((i = 1, 2, 3)\), the manifold \( \mathbb{R}^n/\Gamma \), where \( \Gamma = \langle \mathbb{Z}^n, B_1b_1, B_2b_2 \rangle \) being \( B_1 \) and \( B_2 \) as in (5.1) and
\[
(5.2) \quad b_1 = \left( \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \right) \quad \text{and} \quad b_2 = \left( 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \right).
\]

It follows from Theorem 4.5 that any primitive \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-manifold is affinely equivalent to one and only one of these manifolds.

**Cobb’s manifolds and other known families.** We first observe that \( M(1,1,1) \) is (must be) the Hantzsche-Wendt manifold.

To identify Cobb’s manifolds let \( X, Y, r_i, s_i, t_i, u_i \) be as in [Co], for \( 0 \leq i \leq m - 1 \). We have that \( V = \{u_i\} \) is a lattice \( \Lambda \) of rank \( m \). Notice that the vectors \( u_i \) are orthogonal and are of the same length. One can check that, in the basis \( V, X \) and \( Y \) act by
\[
X(r_i) = r_i, \quad X(s_i) = -s_i, \quad X(t_i) = -t_i; \\
Y(r_i) = -r_i, \quad Y(s_i) = s_i, \quad Y(t_i) = -t_i.
\]

After reordering, if necessary, we may assume
\[
V = \{r_0, \ldots, r_{m_2-1}, s_0, \ldots, s_{m-m_2-m_1-1}, t_0, \ldots, t_{m_1-1}\}
\]
hence,
\[
X = \begin{pmatrix} I_{m_1} & -I_{m_2} & -I_{m_3} \\ -I_{m_2} & I_{m_1} & -I_{m_3} \\ -I_{m_3} & -I_{m_3} & I_{m_1} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -I_{m_1} & I_{m_2} & -I_{m_3} \\ I_{m_2} & -I_{m_1} & -I_{m_3} \\ -I_{m_3} & -I_{m_3} & -I_{m_1} \end{pmatrix}.
\]

Cobb considered the elements \( x = B_1L_{r_0+s_0} \) and \( y = B_2L_{r_0+s_0} \) and proved that the group \( \Gamma = \langle x, y, \Lambda \rangle \) is a Bieberbach group.

Hence, it follows from Theorem 4.5 and Definition 5.1 that Cobb’s family is exactly \( \mathcal{C} = \{M(m_1, m_2, m_3)\} \).

It is even more clear, from Definition 5.1 and §3 of [RT], that the family considered in [RT] is exactly \( \mathcal{D} = \{M(m_1, m_2, m_3, k_1, k_2, k_3)\} \).

Then, it turns out that Cobb’s manifolds can be characterized as all the primitive manifolds, with holonomy group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), where the holonomy representation (§2) decomposes as a sum of 1-dimensional representations and that the family considered in [RT] consist of all primitive manifolds, with holonomy group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), such that the holonomy representation decomposes as a sum of 1-dimensional and 2-dimensional indecomposable representations,
being those in which the holonomy representation has indecomposable sum-mands of dimension 3 all new.

**Integral Homology.** Since we have explicit realizations for all the man-ifolds classified it is not difficult to compute their first integral homology group by the formula $H_1(M; \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$.

In all cases $\Gamma = \langle \omega_1, \omega_2, \Lambda \rangle$, where $\Lambda = \langle L_{e_1}, \ldots, L_{e_\nu} \rangle$, $\omega_1 = B_1 L_{b_1}$ and $\omega_2 = B_2 L_{b_2}$ (see (5.1) and (5.2)). Hence, $[\Gamma, \Gamma] = \langle [\omega_1, L_{e_1}]; [\omega_2, L_{e_1}]; [\omega_1, \omega_2] \rangle$.

We have,

$$[\omega_1, L_{e_1}] = B_1 e_1 - e_1 = (B_1 - I) e_1,$$

$$[\omega_2, L_{e_1}] = B_2 e_1 - e_1 = (B_2 - I) e_1.$$

Since $B_1$ and $B_2$ are block diagonal, with blocks of rank 1, 2 and 3, we proceed block by block.

**Rank 1.** Let $\Lambda = \langle e \rangle$ and suppose $B_1 = (1)$ and $B_2 = (-1)$. We then have

$$(B_1 - I) e = 0; \quad (B_2 - I) e = -2e.$$

**Rank 2.** Let $\Lambda = \langle e, f \rangle$ and suppose $B_1 = J$ and $B_2 = -J$. We get immediately

$$(B_1 - I) e = -e + f; \quad (B_1 - I) f = e - f;$$

$$(B_2 - I) e = -e - f; \quad (B_2 - I) f = -e - f.$$

**Rank 3.** Let $\Lambda = \langle e, f, g \rangle$.

Let $B_1$ and $B_2$ be the first two matrices describing $\mu$ as in (3.4). Then,

$$(B_1 - I) e = -2e,$$

$$(B_2 - I) e = -2e,$$

$$(B_1 - I) f = -f + g,$$

$$(B_2 - I) f = e - f - g,$$

$$(B_1 - I) g = f - g,$$

$$(B_2 - I) g = -e - f - g.$$

In this case writing

$$\Lambda \mathcal{N} = \frac{\langle e, f, g \rangle}{\langle \text{Im} (B_1 - I), \text{Im} (B_2 - I) \rangle} = \langle \overline{e}, \overline{f}, \overline{g} \rangle,$$

it turns out that $|\overline{e}| = 2$, $4\overline{f} = 0$, $\overline{f} = \overline{g}$ and $\overline{e} \neq \overline{f}$, from which we conclude that

$$\Lambda \mathcal{N} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

Now, let $B_1$ and $B_2$ be the first two matrices describing $\nu$ as in (3.4). Then,

$$(B_1 - I) e = -2e,$$

$$(B_2 - I) e = -2e,$$

$$(B_1 - I) f = e,$$

$$(B_2 - I) f = -2f,$$

$$(B_1 - I) g = -2g,$$

$$(B_2 - I) g = -e.
Writing
\[
\frac{\Lambda}{N'} = \frac{\langle e, f, g \rangle}{\langle \text{Im} (B_1 - I), \text{Im} (B_2 - I) \rangle} = \langle e, f, g \rangle,
\]
it turns out that \( \overline{e} = 0, 2\overline{f} = 2\overline{g} = 0 \) and \( \overline{f} \neq \overline{g} \), from which we conclude that
\[
\frac{\Lambda}{N'} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]

It only remains to compute
\[
[w_1, w_2] = B_1 B_2 L_{b_2} L_{b_1} B_1 L_{b_2} B_2 = B_1 B_2 L_{b_1 + b_2 - b_1} B_1 B_2 L_{b_2 - b_2} B_2
\]

Since \( b_1 \) and \( b_2 \) have only three non-zero coordinates and \( B_1 \) and \( B_2 \) preserve this subspace, we restrict our attention to this subspace. Suppose \( \Lambda = \langle e_1, e_2, e_3 \rangle \) and let
\[
B_1 = \begin{pmatrix} 1 & -1 & -1 \\ \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 1 & -1 \\ \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ \end{pmatrix}, \quad \text{and} \quad b_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \end{pmatrix}.
\]

We have that
\[
B_1 b_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{5}{2} \\ \end{pmatrix}, \quad B_1 B_2 b_2 = \begin{pmatrix} 0 \\ 0 \\ -\frac{5}{2} \\ \end{pmatrix}, \quad B_1 B_2 b_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{5}{2} \\ \end{pmatrix}, \quad \text{and} \quad B_2 b_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \end{pmatrix}.
\]

Hence, \( [w_1, w_2] = \begin{pmatrix} 1 \\ -1 \\ -1 \\ \end{pmatrix} \). By observing that \( \omega_1^2 = e_1 \) and \( \omega_2^2 = e_2 \) and by the previous computations we can state the final result.

**Proposition 5.1.** If \( M = M(m_1, m_2, m_3, k_1, k_2, k_3, s, t) \) then, its first homology group is
\[
H_1(M; \mathbb{Z}) = \mathbb{Z}_{m_3 - 3 + k_1 + s + 2t} \oplus \mathbb{Z}_2^{3 + s}.
\]

**Cohomology.** The cohomology of a compact Riemannian flat manifold coincides with the cohomology of its fundamental group (see [Ch2, p. 98]).

**Lemma 5.2 ([Hi]).** If \( \Gamma \) is a Bieberbach group and \( \Phi \rightarrow \text{Aut} (\Lambda) \) is its holonomy representation then,
\[
H^q(\Gamma; \mathbb{Q}) \simeq (\Lambda^q \Lambda^*_Q)^{\Phi},
\]
where \( \Lambda^q \) denotes the \( q \)-th exterior power and \( \Lambda^*_Q = \mathbb{Q} \otimes \Lambda \).

It is straightforward to prove the following lemma.

**Lemma 5.3.** The following \( \mathbb{Q} \)-equivalences hold,
\[
\begin{align*}
\rho_1 & \sim \chi_2 \oplus \chi_3, \\
\rho_2 & \sim \chi_1 \oplus \chi_3, \\
\rho_3 & \sim \chi_1 \oplus \chi_2,
\end{align*}
\[
\begin{align*}
\mu & \sim \chi_1 \oplus \chi_2 \oplus \chi_3, \\
\nu & \sim \chi_1 \oplus \chi_2 \oplus \chi_3.
\end{align*}
\]
Hence, if \( \rho = m_1 \chi_1 \oplus m_2 \chi_2 \oplus m_3 \chi_3 \oplus k_1 \rho_1 \oplus k_2 \rho_2 \oplus k_3 \rho_3 \oplus s \mu \oplus t \nu \) then,
\[
\rho \sim q \, n_1 \chi_1 \oplus n_2 \chi_2 \oplus n_3 \chi_3,
\]
where
\[
\begin{align*}
n_1 &= m_1 + k_2 + k_3 + s + t, \\
n_2 &= m_2 + k_1 + k_3 + s + t, \\
n_3 &= m_3 + k_1 + k_2 + s + t.
\end{align*}
\]

Let \( M = \Lambda Q \). Thus, \( M = M_1 \oplus M_2 \oplus M_3 \), where \( M_i \) is of rank \( n_i \) and \( Z_2 \oplus Z_2 \) acts by the character \( \chi_i \). Let \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \) be a basis for \( M \), so that \( \mathcal{B}_i \) is a basis for \( M_i \); denote by \( \mathcal{B}_1 = \{ f_i \}_{i=1}^{n_1}, \mathcal{B}_2 = \{ g_i \}_{i=1}^{n_2} \) and \( \mathcal{B}_3 = \{ h_i \}_{i=1}^{n_3} \). A generic element of the canonical basis of \( A^q M \) is
\[
e = f_{i_1} \wedge \cdots \wedge f_{i_{a_1}} \wedge g_{j_1} \wedge \cdots \wedge g_{j_{a_2}} \wedge \cdots \wedge h_{t_{a_3}} \cdot
\]
with \( a_1 + a_2 + a_3 = q \).

Since \( Z_2 \oplus Z_2 \) acts by characters, then
\[
\dim H^q(\Gamma; Q) = \left| \{ e \in \mathcal{B} : B_i e = e, \ 1 \leq i \leq 3 \} \right|.
\]

Proposition 5.4. For \( \Gamma(m_1, m_2, m_3, k_1, k_2, k_3, s, t) \) one has:
\[
\begin{align*}
\dim H^{2p}(\Gamma; Q) &= \sum_{q_1 + q_2 + q_3 = p} \left( \begin{array}{c} n_1 \\ 2q_1 \end{array} \right) \left( \begin{array}{c} n_2 \\ 2q_2 \end{array} \right) \left( \begin{array}{c} n_3 \\ 2q_3 \end{array} \right), \\
\dim H^{2r+1}(\Gamma; Q) &= \sum_{q_1 + q_2 + q_3 = r+1} \left( \begin{array}{c} n_1 \\ 2q_1 + 1 \end{array} \right) \left( \begin{array}{c} n_2 \\ 2q_2 + 1 \end{array} \right) \left( \begin{array}{c} n_3 \\ 2q_3 + 1 \end{array} \right).
\end{align*}
\]

Proof. Let \( e \in \mathcal{B} \). If \( e \in (A^q M)^{Z_2 \oplus Z_2} \), then
\[
a_1 + a_2 \equiv a_1 + a_3 \equiv a_2 + a_3 \equiv 0 \mod 2.
\]

(i) If \( q \) is even, then \( a_1 + a_2 + a_3 \equiv 0 \mod 2 \) and \( a_1 \equiv a_2 \equiv a_3 \equiv 0 \mod 2 \). Conversely, if \( a_i \equiv 0 \mod 2 \), for \( 1 \leq i \leq 3 \), then every \( e \in \mathcal{B} \) of the form
\[
e = f_{i_1} \wedge \cdots \wedge f_{i_{a_1}} \wedge g_{j_1} \wedge \cdots \wedge g_{j_{a_2}} \wedge \cdots \wedge h_{t_{a_3}},
\]
is in \( (A^q M)^{Z_2 \oplus Z_2} \). Therefore we have the first formula.

(ii) If \( q \) is odd, then \( a_1 + a_2 + a_3 \equiv 1 \mod 2 \). As in the previous case we get the second formula.

\[ \square \]

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PRIMITIVE $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-MANIFOLDS

References


[Co] P. Cobb, Manifolds with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and first Betti number zero, J. Differential Geometry, 10 (1975), 221-224.


[RT2] ______, Five-dimensional Bieberbach groups with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, Geom. Dedicata, 77 (1999), 149-172.


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HÖLDER REGULARITY FOR \( \bar{\partial} \) ON THE CONVEX DOMAINS OF FINITE STRICT TYPE

Wei Wang

By using the Cauchy–Fantappiè machinery, the nonhomogeneous Cauchy-Riemann equation on convex domain \( D \) for \((0, q)\) form \( f \) with \( \bar{\partial} f = 0, \bar{\partial} u = f \), has a solution which is a linear combination of integrals on \( bD \) of the following differential forms

\[
\frac{1}{A^{j+1} \beta^{n-j-1}} \partial r \wedge (\bar{\partial} \zeta \partial r)^j \wedge \partial \zeta \beta \\
\wedge \left( \sum_{i=1}^{n} d\zeta_i \wedge d\zeta_i \right)^{n-q-3-j} \wedge \left( \sum_{i=1}^{n} d\zeta_i \wedge dz_i \right)^{q-1} \wedge f,
\]

\( j = 1, \ldots, n-q-3 \), where \( A = \langle \partial r(\zeta), \zeta - z \rangle \), \( \beta = |z - \zeta|^2 \) and \( r \) is the defining function of \( D \). In the case of finite strict type, Bruna et al. estimated \( \langle \partial r(\zeta), \zeta - z \rangle \) by the pseudometric constructed by McNeal. We can estimate the above differential forms and their derivatives. Then, by using a method of estimating integrals essentially due to McNeal and Stein, we prove the following almost sharp Hölder estimate

\[
\|u\|_{C^{1,\kappa}_{0,q-1} (\bar{b}D)} \leq C \|f\|_{L^\infty(bD)}, \quad 1 \leq q \leq n-1
\]

for arbitrary \( \kappa > 0 \). The constant only depends on \( \kappa, D \) and \( q \).

1. Introduction.

Let \( D \) be a convex domain in \( \mathbb{C}^n \) with smooth boundary, \( D = \{ r < 0 \} \), \( bD = \{ r = 0 \} \) and \( dr \neq 0 \) on \( bD \). Suppose the defining function \( r \) is convex near the boundary \( bD \). For \( \zeta \in \bar{D}, T^C_{\zeta} \) denotes the complex-tangential space to \( \{ r = r(\zeta) \} \) at \( \zeta \). By the results in \cite{Mc3}, we say \( p \in bD \) of type \( m \) if the contact order of complex lines \( L \subset T^C_{\zeta} \) with \( bD \) at \( \zeta \) is not greater than \( m \), for all \( \zeta \in bD \).

We say \( D \) is of strict type if there exists \( C = C(D) \) such that for all \( \zeta \in bD \), all directions \( v \in T^C_{\zeta}(bD) \), \( |v| = 1 \), and small \( t \)

\[
\frac{1}{C} r(\zeta + tv) \leq r(\zeta + t\sqrt{-1}v) \leq Cr(\zeta + tv).
\]
The condition implies that the order of contact with \( bD \) at \( \zeta \) of \( \{ \zeta + tv \} \) and \( \{ \zeta + t\sqrt{-1}v \} \) is the same. If \( D \) is both of finite type and of strict type, we say that \( D \) has finite strict type.

Now consider the nonhomogeneous Cauchy-Riemann equation on \( D \)

\[
\overline{\partial}u = f
\]  

with \( f \in L_{0,q}^\infty(\overline{D}) \), where \( L_{0,q}^\infty(\overline{D}) \) is the space of \((0,q)\) forms on \( D \) with coefficients in \( L^\infty(\overline{D}) \). When \( D \) is of type \( m \), i.e., each point \( p \in bD \) is of type less than or equal to \( m \), it is natural to expect that the following Hölder estimate holds: For \( f \in L_{0,q}^\infty(\overline{D}) \) and \( \overline{\partial}f = 0 \), Equation (1.2) has a solution \( u \) satisfying

\[
\|u\|_{C_{0,q-1}^m(\overline{D})} \leq C\|f\|_{L_{0,q}^\infty(\overline{D})}, \quad 1 \leq q \leq n-1,
\]  

where \( C \) is a constant depending on \( D,q \).

One method to study this problem is to establish an integral representation formula using the Cauchy-Fantappiè machinery (see [R2]). The key point of the Cauchy-Fantappiè machinery is to find a barrier form

\[
w(\zeta, z) = \sum_{i=1}^{n} w_i(\zeta, z)d\zeta_i
\]  

satisfying

\[
\sum_{i=1}^{n} w_i(\zeta, z)(\zeta_i - z_i) = 1
\]

for \( \zeta \in bD, z \in D \), where \( w_i(\zeta, z), i = 1, \cdots, n, \) are holomorphic in \( z \). For the strongly pseudoconvex domains, this barrier form, hence the integral representation formula, is constructed and the sharp estimate is obtained (see [R2], for example). For the weakly pseudoconvex domains, little is known, but some important results have been obtained. Range [R1] proved sharp Hölder estimate for some convex domains in \( \mathbb{C}^2 \) and generalized by Bruna and Castillo [BC]. Fornaess [Fo] constructed a barrier form for a kind of pseudoconvex domains of finite type in \( \mathbb{C}^2 \) and proved sup norm estimate. Diederich et al. [DFW] and Chen et al. [CKM] obtained the sharp estimate for ellipsoids. By using Skoda’s estimates, Range [R3] constructed a barrier form for pseudoconvex domain of finite type in \( \mathbb{C}^2 \).

His method was generalized to pseudoconvex domains in \( \mathbb{C}^n \) by Michel [M]. We should also mention the work of Chaumat and Chollet for convex domain which needn’t be pseudoconvex. Chaumat and Chollet’s and Michel’s results, which are far from sharp, are \( C^\infty \) estimates. Bruna et al. [BCD] proved \( L^1 \) estimate for \((0,1)\) forms on convex domains of finite strict type. Hölder estimates can also be obtained by studying the associated \( \partial_b \) and singular integral operators on the boundary (see [FK], [FKM], [Ch]).
H"older Regularity for $\overline{\partial}$

When the domain is convex, there is a natural barrier form

$$w(\zeta, z) = \frac{\partial r(\zeta)}{\langle \partial r(\zeta), \zeta - z \rangle}$$

(1.6)

where $r$ is the defining function of the domain,

$$\partial r(\zeta) = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta)d\zeta_i \quad \text{and} \quad \langle \partial r(\zeta), \zeta - z \rangle = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta)(\zeta_i - z_i).$$

(1.7)

By using the barrier form, we can construct the integral representation formula for $(0,q)$ form by applying Cauchy-Fantappiè machinery. Therefore, to estimate the solution of $\overline{\partial}$, we need to estimate the quantity $\langle \partial r(\zeta), \zeta - z \rangle$ for $\zeta \in bD, z \in D$. Thanks to the work of McNeal [Mc4] and [BCD], there exists a pseudometric on $D$, and we can estimate $\langle \partial r(\zeta), \zeta - z \rangle$ by this pseudometric in the case of finite strict type. Then the integral kernel and its derivatives are estimated. All these results allow us to prove the following theorem.

**Theorem 1.1.** Let $D \subset C^n$ be a bounded convex domain of strict finite type $m$, and let $f \in L_{0,q}^{\infty}(\overline{D})$ be $\overline{\partial}$-closed, $1 \leq q \leq n-1$. Then the nonhomogeneous Cauchy-Riemann Equation (1.2) has a solution $u$ satisfying

$$\|u\|_{C^{\frac{1}{q} - \kappa - 1,q-1}(\overline{D})} \leq C\|f\|_{L_{0,q}^{\infty}(\overline{D})}$$

(1.8)

for arbitrary $\kappa > 0$. The constant only depends on $\kappa, D$ and $q$.

Here is the plan of this paper. In Section 2, we state McNeal’s pseudometric and some propositions for our purpose. In Section 3, we deduce the integral representation formula and some estimates associated to it. In Section 4, we estimate the derivatives of the integral kernel. Then we use a method essentially due to McNeal and Stein [MS] to prove the main Theorem.

In this paper, we will us the following notations. The expression $X \lesssim Y$ and $X \gtrsim Y$ means that there exists some constant $C > 0$, which is independent on the obvious parameters, so that $X \leq CY$ and $Y \geq CY$, respectively. $X \approx Y$ means $X \lesssim Y$ and $Y \lesssim X$ simultaneously.

**2. McNeal’s pseudometric on the convex domain of finite type.**

Let $S^n = \{\zeta \in C^n; |\zeta| = 1\}$. Each element of $S^n$, together with a point $q \in C^n$, determine a complex line in $C^n$. Without loss of generality, then, we may assume that our defining function $r$ has the property that all the set $\{z; r(z) < \eta\}$ are convex for $-\eta_0 < \eta < \eta_0$, for some $\eta_0 > 0$. If $\gamma \in S^n$, $-\eta_0 < \eta < \eta_0$, we denote the distance from $q$ to the level set $\{z; r(z) = \eta\}$ along the complex line $\gamma$ by $\delta_\eta(q, \gamma)$. 
Now recall the definition of $\varepsilon$-extremal basis of McNeal. Let $bD_{q, \varepsilon} = \{ z \in U; r(z) = r(q) + \varepsilon \}$. Let $p \in bD$. If $U$ is sufficiently small a neighbourhood of $p$, the distance from $q \in U$ to $bD_{q, \varepsilon}$ is well defined. Let $n$ be the real normal line of $\{ z, r(z) = r(q) \}$ at $q$ and $p_1$ be the intersection of $n$ with $bD_{q, \varepsilon}$. Set $\tau_1(q, \varepsilon) = |q - p_1|$. Choose a parametrization of the complex line from $q$ to $p_1$, by $z_1$, with $z_1(0) = q$ and $p_1$ lying on the positive $\text{Re} z$ axis. Now consider the orthogonal complement of the span of the coordinate $z_1$, $OC_1$, which is the complex subspace of the tangent space to $\{ z; r(z) = r(q) \}$ at $q$. For any $\gamma \in OC_1 \cap S^n$, compute $\delta_{r(q)+\varepsilon}(q, \gamma)$. Let $\tau_2(q, \varepsilon)$ be the largest such distance and $p_2 \in bD_{q, \varepsilon}$ be any point achieving this distance. The coordinate $z_2$ is defined by parametrization the complex line from $q$ to $p_2$ on such a way that $z_2(0) = q$ and $p_2$ lying on the positive $\text{Re} z_2$ axis. Continuing this process, we obtain the $n$ coordinate functions $z_1, \ldots, z_n$, $n$ quantities $\tau_1(q, \varepsilon), \ldots, \tau_n(q, \varepsilon)$ and $n$ points $p_1, \ldots, p_n$. Let $z_j = x_j + \sqrt{-1}r_{n+j}$ for $1 \leq j \leq n$. Following [BCD], we call coordinates $\{ z_1, \ldots, z_n \}$ $\varepsilon$-extremal coordinates centered at $q$.

If we define the polydisc

\[(2.1^*) \quad P_\varepsilon(q) = \{ z \in U; |z_1| < \tau_1(q, \varepsilon), \ldots, |z_n| < \tau_n(q, \varepsilon) \}, \]

then there exist a constant $C > 0$, independent on $q \in U \cap D$, such that $CP_\varepsilon(q) \subset \{ z \in U; r(z) < r(q) + \varepsilon \}$.

It is obvious that $\tau_1(q, \varepsilon) \approx \varepsilon$. Let

\[(2.1) \quad A_k^i(q) = |a_k^i(q)|, \quad a_k^i(q) = \frac{\partial^k}{\partial x_i^k} r(z(0^0, 0, x_i, 0, \ldots, 0)), \quad 2 \leq i \leq n \]

and set

\[(2.2) \quad \sigma_i(q, \varepsilon) = \min \left\{ \left( \frac{\varepsilon}{A_k^i(q)} \right)^{\frac{1}{k}}, 2 \leq k \leq m \right\}, \]

then

\[(2.3) \quad \sigma_i(q, \varepsilon) \approx \tau_i(q, \varepsilon) \]

[Mc4]. It follows directly from (2.1)-(2.3) that for each $2 \leq i \leq n$,

\[(2.4) \quad \left| \frac{\partial^k}{\partial x_i^k} r(q) \right| \lesssim \varepsilon \tau_i(q, \varepsilon)^{-k}, \quad k = 1, \ldots, m. \]

We have the following three propositions. See [Mc4] for their proofs.

**Proposition 2.1.** For $q \in U \cap D$, let $\gamma_1, \ldots, \gamma_n$ be the orthogonal unit vector determined by $\varepsilon$-extremal coordinate centered at $q$. Suppose $\lambda \in S^n$
can be written as \( \lambda = \sum_{i=1}^{n} a_i \gamma_i \), \( a_i \geq 0 \), \( \sum_{i=1}^{n} a_i = 1 \). For small \( \varepsilon > 0 \), set \( \eta = r(q) + \varepsilon \), then

\[
(2.5) \quad \left( \sum_{i=1}^{n} \frac{a_i}{\tau_i(q, \varepsilon)} \right)^{-1} \approx \delta_{\eta}(q, \varepsilon).
\]

**Proposition 2.2.** There is a constant \( C \) independent of \( q, q^1, q^2 \in D \cap U, \varepsilon > 0 \), so that if \( P_{\varepsilon}(q^1) \cap P_{\varepsilon}(q^2) \neq \emptyset \), then

\[
(2.6) \quad P_{\varepsilon}(q^1) \subset CP_{\varepsilon}(q^2) \text{ and } P_{\varepsilon}(q^2) \subset CP_{\varepsilon}(q^1)
\]

and

\[
(2.7) \quad P_{r\varepsilon}(q) \subset CP_{\varepsilon}(q), \quad 0 \leq r \leq 2.
\]

When \( r = 2 \), (2.7) is Proposition 2.5 in [Mc4]. His proof works in the case of \( 0 \leq r \leq 2 \) (because \( \tau_i(q, r\varepsilon) \approx \sigma_i(q, r\varepsilon) \approx \min\{ \left( \frac{r\varepsilon}{A_{\varepsilon}(q)} \right)^k, 2 \leq k \leq m \} \approx \sigma_i(q, \varepsilon) \}). Note \( r\varepsilon \)-extremal coordinates centered at \( q \) may be different from \( \varepsilon \)-coordinates centered at \( q \) (\( r \leq 1 \)). \( P_{r\varepsilon}(q) \subset P_{\varepsilon}(q) \) may not hold for \( 0 \leq r \leq 1 \).

Suppose \( q^1, q^2 \in U \cap D \), define

\[
(2.8) \quad d(q^1, q^2) = \inf\{ \varepsilon; q^2 \in P_{\varepsilon}(q^1) \},
\]

where \( P_{\varepsilon}(q^1) \) defined by (2.1).

**Proposition 2.3.** \( d(\cdot, \cdot) \) defines a local pseudometric on \( U \cap D \), i.e., for \( q^1, q^2, q^3 \in U \cap D \),

1. \( d(q^1, q^2) = 0 \) iff \( q^1 = q^2 \);
2. \( d(q^1, q^2) \approx d(q^2, q^1) \);
3. \( d(q^1, q^3) \lesssim d(q^1, q^2) + d(q^2, q^3) \).

**Corollary 2.4.** Let \( \varepsilon > 0, q, q' \in U \cap D, \varepsilon \leq d(q, q') \leq 2\varepsilon \), then, in the \( \varepsilon \)-extremal coordinates centered in \( q, q' = (q_1', \cdots, q_n') \),

\[
(2.9) \quad d(q, q') \approx |q_1'| + \sum_{i=2}^{n} \sum_{l=2}^{m} A_i^l(q)|q_i'^l|.
\]

**Proof.** Since \( q' \) lies in the boundary of polydisc \( P_{d(q, q')}(q) \), and \( \frac{1}{C} P_{\varepsilon}(q) \subset P_{d(q, q')}(q) \subset CP_{\varepsilon}(q) \) for some constant \( C > 0 \), by Proposition 2.2, we find

\[
(2.10) \quad |q_i'| \leq C\tau_i(q, \varepsilon), \quad i = 1, \cdots, n,
\]

and there exists \( i_0 \) such that \( |q_{i_0}'| \geq \frac{1}{C} \tau_{i_0}(q, \varepsilon) \). Thus

\[
(2.11) \quad A_i^{i_0}(q)|q_{i_0}'|^{l_i} \gtrsim \varepsilon
\]

for some \( l \) by (2.1)-(2.3), the right side of (2.9) \( \gtrsim \varepsilon \). The right side of (2.9) \( \lesssim \varepsilon \) by (2.10) and (2.1)-(2.3).
Note (2.9) may not hold for each \( q' \in U \cap D \) because \( \varepsilon \)-extremal coordinates centered at \( q \) may change abruptly as \( \varepsilon \). We can also prove:

**Lemma 2.5.** If \( d(z, \zeta) \leq \varepsilon \), then, in the \( \varepsilon \)-extremal coordinates \( \{w_1, \ldots, w_n\} \) centered at \( z, \zeta = \{\zeta_1, \ldots, \zeta_n\} \)

\[
D^\beta r(\zeta) \lesssim \frac{\varepsilon}{\tau(z, \varepsilon)^\beta} \tag{2.12}
\]

for all multiindices \( \beta = (\beta_1, \bar{\beta}_1, \ldots, \beta_n) \), where

\[
D^\beta_r = \frac{\partial^{\beta_1 + \beta_1 + \cdots + \beta_n} r}{\partial \zeta_1^{\beta_1} \partial \bar{\zeta}_1^{\beta_1} \cdots \partial \zeta_n^{\beta_n}}, \quad \tau^\beta(z, \varepsilon) = \tau_1^{\beta_1 + \bar{\beta}_1}(z, \varepsilon) \cdots \tau_n^{\beta_n + \bar{\beta}_n}(z, \varepsilon).
\]

**Proof.** Note (2.12) is obvious by \(|\tau(z, \varepsilon)^\beta| \lesssim \varepsilon\) if \(|\beta| = \sum_{i=1}^n \beta_i + \bar{\beta}_i \geq m\). When \( \zeta = z \),

\[
|D^\beta r(z)| \lesssim \frac{\varepsilon}{\tau_i(z, \varepsilon)^\beta} \tag{2.13}
\]

is proved for \( \beta \) with \( \beta_i + \bar{\beta}_i \neq 0 \) only for two \( i \) in \([BCD, p. 398-399]\), their proof works in the general case. Since \( P_{d(z, \zeta)}(z) \subset CP(z) \) for constant \( C \) by Proposition 2.2, \(|\zeta_i| \lesssim \tau_i(z, \varepsilon), i = 1, \ldots, n\). Then

\[
\left| \frac{\partial r}{\partial w_i}(\zeta) - \frac{\partial r}{\partial w_i}(z) \right| \lesssim \sum_{|\beta| = 1}^m D^\beta r(z) \tau^\beta(z, \varepsilon) + o(|\tau(z, \varepsilon)|^m), \tag{2.14}
\]

where \(|\tau(z, \varepsilon)| = \max_{1 \leq k \leq m} \tau_k(z, \varepsilon)|\). It

\[
\lesssim \sum_{|\beta| = 1}^m \varepsilon^{-|\beta|}(z, \varepsilon) \tau_i^{-1}(z, \varepsilon) \tau^\beta(z, \varepsilon) + o(|\tau(z, \varepsilon)|^m) \lesssim \frac{\varepsilon}{\tau_i(z, \varepsilon)}
\]

by (2.13) and \(|\tau(z, \varepsilon)| \lesssim \varepsilon^{-\frac{1}{m}}\). Thus

\[
\left| \frac{\partial r}{\partial w_i}(\zeta) \right| \lesssim \frac{\varepsilon}{\tau_i(z, \varepsilon)}. \tag{2.15}
\]

For general \( \beta \), (2.12) can be proved similarly. This completes the proof of Lemma 2.5.

**3. The integral representation formula and some estimates.**

It is well known that for convex domains with smooth boundaries, we have the following explicit integral representation for \( \partial \) problem.

**Proposition 3.1 ([R2, p. 176]).** Let \( D \subset \subset C^n \) be convex with smooth boundary and let \( r \in C^2 \) be a defining function for \( D \). Let

\[
C^{(r)}(\zeta, z) = \frac{\partial r(\zeta)}{\langle \partial r(\zeta), \zeta - z \rangle}. \tag{3.1}
\]
where $\langle \partial r(\zeta), \zeta - z \rangle = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta_i - z_i)$ and

$$\hat{C}^{(r)}(\zeta, z) = \lambda C^{(r)}(\zeta, z) + (1 - \lambda)B(\zeta, z),$$

where $0 < \lambda < 1$,

$$B(\zeta, z) = \frac{\partial \beta}{\beta}, \quad \beta = |z - \zeta|^2. \quad (3.2)$$

Define the Cauchy-Fantappie kernel associated to $\hat{C}^{(r)}$ by

$$\Omega_q(\hat{C}^{(r)}) = (-1)^{\frac{q(q-1)}{2}} \binom{n-1}{q} \hat{C}^{(r)} \wedge (\overline{\partial}_\zeta \lambda \hat{C}^{(r)})^{n-q-1} \wedge (\overline{\partial}_z \hat{C}^{(r)})^q. \quad (3.3)$$

for $0 \leq q \leq n-1$, $\overline{\partial}_{\zeta,\lambda} = \overline{\partial}_\zeta + d_\lambda$, and the Bochner-Martinelli-Koppelman kernel

$$K_q = (-1)^{\frac{q(q-1)}{2}} \binom{n-1}{q} B \wedge (\overline{\partial}_\zeta B)^{n-q-1} \wedge (\overline{\partial}_z B)^q. \quad (3.4)$$

Then the operator $T_q^{(r)} : C_{0,q}(D) \rightarrow C_{0,q-1}(D)$ defined by

$$T_q^{(r)} f = \int_{bD \times I} f \wedge \Omega_q(\hat{C}^{(r)}) - \int_D f \wedge K_{q-1}, \quad I = [0, 1] \quad (3.6)$$

satisfies

$$\overline{\partial} T_q^{(r)} f = f$$

on $D$ if $f \in C_{0,q}(D)$ and $\overline{\partial} f = 0$.

We decompose

$$\Omega_q(\hat{C}^{(r)}) = \Omega_q^{(1)} \wedge d\lambda + \Omega_q^{(0)}, \quad (3.7)$$

where $\Omega_q^{(0)}, \Omega_q^{(1)}$ is of degree 0 in $\lambda$. By simple calculation, we can prove the following:

**Lemma 3.2 ([R2, p. 206])**. For $0 \leq q \leq n-2$ and $f \in C_{0,q+1}(bD)$, one has

$$\int_{bD \times I} f \wedge \Omega_q(\hat{C}^{(r)}) = \int_{bD} f \wedge A_q(C^{(r)}; B) \quad (3.8)$$

where

$$A_q(C^{(r)}; B) = \sum_{j=0}^{n-q-2} \sum_{k=0}^{q} a_q^{j,k} A_q^{j,k}(C^{(r)}; B) \quad (3.9)$$

with universal constants $a_q^{j,k}$ and

$$A_q^{j,k}(C^{(r)}; B) = C^{(r)} \wedge B \wedge (\overline{\partial}_\zeta C^{(r)})^j \wedge (\overline{\partial}_\zeta B)^{n-q-2-j} \wedge (\overline{\partial}_z C^{(r)})^k \wedge (\overline{\partial}_z B)^{q-k}. \quad (3.10)$$
A straightforward computation gives

\[ A_j^k(C^{(r)}; B) \]

\[ = \frac{\partial \zeta r(\zeta) \wedge \partial \zeta \beta \wedge (\overline{\partial} \zeta \partial \zeta r)^j \wedge (\overline{\partial} \zeta \partial \zeta \beta)^{n-q-2-j} \wedge (\overline{\partial} \zeta \partial \zeta r)^k \wedge (\overline{\partial} \zeta \partial \zeta \beta)^{q-k}}{\langle \partial r(\zeta), \zeta - z \rangle^{j+k+1} \beta^{n-j-k}} \]

(see [R2, p. 206] for a general formula) and

\[ A_j^k = 0, \quad \text{if } k \geq 1 \]

by \( \overline{\partial} \zeta r(\zeta) = 0 \).

Because the Bochner-Martinelli-Koppelman kernel \( K_{q-1} \) is a kind of Caldéron-Zygmund kernel, we have the following regularity result. See [R2, p. 156], for example.

**Proposition 3.3.** For \( 0 < \alpha < 1 \), we have

\[ \left\| \int_D f \wedge K_{q-1} \right\|_{C^\alpha_{0,q-1}(D)} \lesssim \| f \|_{L^\infty_0(D)}. \]

In the kernels in (3.11), there is a factor \( \langle \partial r(\zeta), \zeta - z \rangle (\zeta \in bD) \) in the dominators. We should estimate this quantity.

**Proposition 3.4 ([BCD, Lemma 4.2]).** If \( D \) is of finite strict type, then

\[ d(\zeta, z) \approx |\langle \partial r(\zeta), \zeta - z \rangle| \]

for \( \zeta \in bD, z \in \overline{D} \).

In order to prove the Hölder estimate in the main Theorem 1.1, we will use the following elementary real variable fact.

**Lemma 3.5 ([R2, p. 204]).** Let \( D \subset \subset R^n \) be a bounded domain with \( C^1 \) boundary. Suppose \( g \) differentiable on \( D \) and that for some \( 0 < \alpha < 1 \), there is a constant \( C \) such that

\[ |dg(x)| \lesssim C\delta_D(x)^{\alpha-1}, \quad x \in D, \]

where \( \delta_D(x) \) is the distance from \( x \) to the boundary \( bD \). Then \( g \in C^\alpha(\overline{D}) \) and there exists a compact subset \( K \) of \( D \) such that

\[ \|g\|_{C^\alpha(\overline{D})} \lesssim C + \|g\|_{L^\infty(K)}. \]

Therefore, if we can prove the following proposition, the Hölder estimate of \( \overline{\partial} \)-problem is proved by Lemma 3.5.

**Proposition 3.6.** Using the above notation, we have

\[ \left| \int_{bD} d_z A^j_0(z, \zeta) \wedge f \right| \lesssim \delta_D(z)^{-1+\frac{1}{m}-\kappa} \| f \|_{L^\infty_0} \]

for \( 0 < \kappa < 1 - \frac{1}{m} \), where the constant depends only on \( D, j, q, \kappa \).
H"older Regularity for $\bar{\partial}$

Theorem 1.1 for $\kappa \geq 1 - \frac{1}{m}$ obviously follows from $\kappa < 1 - \frac{1}{m}$. We will prove (3.16) in Section 4.

We will need the following estimates.

**Lemma 3.7.** If $a, b, a', \alpha, \alpha' > 0, k \geq 1$, then
\[
\int_{\mathbb{C}} \frac{b^{\frac{x}{k}}}{(a + b|\zeta|^k)^{\alpha + \frac{1}{k}}} \cdot \frac{1}{(a' + |\zeta|)^{\alpha' + \frac{1}{k}}} dV(\zeta) \lesssim \frac{1}{a^\alpha} \frac{1}{a'^{\alpha'}},
\]
where $dV(\zeta)$ is the volume element of $\mathbb{C}$.

**Proof.** Denote $\zeta = x + iy$, then the integral
\[
\leq \int_{\mathbb{R}} \frac{b^{\frac{x}{k}}}{(a + b|x|^k)^{\alpha + \frac{1}{k}}} dx \cdot \int_{\mathbb{R}} \frac{dy}{(a' + |y|)^{\alpha' + 1}}
\]
\[
= \int_{\mathbb{R}} \frac{dx}{(a + |x|^k)^{\alpha + \frac{1}{k}}} \cdot \int_{\mathbb{R}} \frac{dy}{(a' + |y|)^{\alpha' + 1}}
\]
\[
\lesssim \frac{1}{a^\alpha a'^{\alpha'}}.
\]

**Lemma 3.8.** If $a, b, \alpha > 0, k \geq 1$, then
\[
\int_{\mathbb{C}} \frac{b^{\frac{x}{k}}}{(a + b|\zeta|^k)^{\alpha + \frac{2}{k}}} dV(\zeta) \lesssim \frac{1}{a^\alpha},
\]
where $dV(\zeta)$ is the volume element of $\mathbb{C}$.

**Proof.** Define
\[
D_1 = \{\zeta; b|\zeta|^k \geq a\}
\]
and
\[
D_2 = \{\zeta; b|\zeta|^k < a\}.
\]
It follows that on the region $D_1$, we have
\[
\int_{D_1} \frac{b^{\frac{x}{k}}}{(a + b|\zeta|^k)^{\alpha + \frac{2}{k}}} dV(\zeta) \lesssim \int_{D_1} \frac{b^{\frac{x}{k}}}{(b|\zeta|^k)^{\alpha + \frac{2}{k}}} dV(\zeta)
\]
\[
\lesssim \int_{L} b^{-\alpha} (p^k)^{-\frac{2}{k} - \alpha} p dp \lesssim a^{-\alpha},
\]
where $L = (\frac{a}{b})^k$. On the region $D_2$, we obtain the same upper bound
\[
\int_{D_2} \frac{b^{\frac{x}{k}}}{(a + b|\zeta|^k)^{\alpha + \frac{2}{k}}} dV(\zeta) \lesssim \frac{b^{\frac{x}{k}}}{a^{\frac{2}{k} + \alpha}} \text{Vol}(D_2) \lesssim a^{-\alpha}.
\]
This completes the proof of Lemma 3.8.

Lemmas 3.7 and 3.8 were used in [MS] implicitly to estimate the Bergman projection operator in the convex domain of finite type.
4. The estimate of the integral.

The purpose of this section is to prove Proposition 3.6.

Let $\beta$ be a multiindex, $\beta = (\beta_1, \bar{\beta}_1, \ldots, \beta_n)$. Define

$$D^\beta = \frac{\partial^{\beta_1}}{\partial z_1^{\beta_1}} \frac{\partial^{\bar{\beta}_1}}{\partial \bar{z}_1^{\bar{\beta}_1}} \cdots \frac{\partial^{\beta_n}}{\partial z_n^{\beta_n}} \frac{\partial^{\bar{\beta}_n}}{\partial \bar{z}_n^{\bar{\beta}_n}}, \quad |\beta| = \sum_{i=1}^n \beta_i + \bar{\beta}_i$$

and

$$\tau^\beta(w, \varepsilon) = \tau^{\beta_1+\bar{\beta}_1}(w, \varepsilon) \cdots \tau^{\beta_n+\bar{\beta}_n}(w, \varepsilon)$$

for $w \in D \cap U, \varepsilon > 0$. If $0 \leq \beta_i, \bar{\beta}_i \leq 1$, define

$$d\zeta^\beta = d\zeta_1^{\beta_1} \wedge d\zeta_1^{\bar{\beta}_1} \wedge \cdots \wedge d\zeta_n^{\beta_n},$$

where $d\zeta_i$ or $d\bar{\zeta}_i$ absents if $\beta_i = 0$ or $\bar{\beta}_i = 0$, respectively.

Suppose $bD$ is covered by $U_1, \ldots, U_N$, where $U_i$ are balls centered at $z_i \in bD$ with radius $r_i$. Furthermore, all results of Proposition 2.1-2.5 hold for $U = U'_i, i = 1, \ldots, N$, where $U'_i = B(z_i, 2r_i)$. Let $V = \cup_{i=1}^N U'_i$ and set

$$A_{j,0}^{q-1} = \sum_J A_J^j d\zeta^J$$

where $A_J^j$ are $(n, n-q-1)$ forms in $\zeta$ and $\bar{\zeta}$, and $J$ takes over all multiindices with $|J| = q-1, 1 \leq J_i, \bar{J}_i \leq 1$. The estimate for $d_z A_J^j$ is as follows. We will use the following notation. For $(p, p')$ differential form $A(\zeta)$, $\|A(\zeta)\|_{bD}$ denote the norm of $A(\zeta)$ acting on $(\bigotimes T^\zeta(bD))^{p+p'}$.

**Proposition 4.1.**

1. If $z \in U_i$ for some $i$, then

$$\|d_z A_J^j(z, \zeta)\|_{bD} \lesssim \sum_{\beta} \frac{1}{\tau^\beta(z, \varepsilon)} |z - \zeta|^{2n-2j-3}$$

for $\zeta \in bD \cap U'_i$, where $\varepsilon = d(z, \zeta)$, and $\beta$ takes over all multiindices satisfying the following condition C:

   (C1) $\sum_{i=1}^n \beta_i + \bar{\beta}_i = 2j + 2$;

   (C2) There exists at most one $i_0 > 1$ such that $\beta_{i_0} + \bar{\beta}_{i_0} = 3$ and $\beta_l + \bar{\beta}_l \leq 2$ for all $l \neq i_0$. If such $i_0$ exists, we must have $\beta_1 + \bar{\beta}_1 = 1$.

2. If $\zeta \notin U'_i$, then

$$\|d_z A_J^j\|_{bD} \lesssim 1.$$
Note
\begin{align}
(4.7) \quad \left| \int_{\partial D \cap U'_i} d_z A^j_f(z, \zeta) \wedge f \right| & \lesssim \int_{\partial D \cap U'_i} \| d_z A^j_f(z, \zeta) \wedge f \|_{bD} dV(\zeta) \\
& \lesssim \| f \|_{L^\infty_{\infty,q}} \int_{\partial D \cap U'_i} \| d_z A^j_f(z, \zeta) \|_{bD} dV(\zeta),
\end{align}

where $dV(\zeta)$ is the volume element of $bD$, and
\begin{align}
(4.8) \quad \int_{\partial D \cap U'_i} \| d_z A^j_f \|_{bD} & \lesssim 1
\end{align}
by (4.6) for $z \in U_i$, and $|r(z)| \approx \delta_D(z)$, the proof of Proposition 3.6 is reduced to the following estimate.

Lemma 4.2. For $\beta$ satisfying condition $C$ and $z \in U_i$ for some $i$, we have
\begin{align}
(4.9) \quad I_\beta(z) = \int_{\partial D \cap U'_i} \frac{1}{\tau^\beta(z, \epsilon)|z - \zeta|^{2n-2j-3}} dV(\zeta) \lesssim |r(z)|^{-1-\kappa+\frac{1}{m}}, \quad \epsilon = d(z, \zeta)
\end{align}
for $0 < \kappa < 1 - \frac{1}{m}$, where $dV(\zeta)$ is the volume element of $bD$.

Before we begin to prove Proposition 4.1, we give a lemma. Since
\begin{align}
(4.10) \quad \bar{\partial}_\zeta \partial_\zeta \beta = \sum_{i=1}^n d\zeta_i \wedge d\zeta_i,
\end{align}
\begin{align}
(4.11) \quad A^j_{q-1} = \frac{1}{A^{j+1}_q \|\tau^{j+1}_L(z, \epsilon)\|} \partial_\zeta r \wedge (\bar{\partial}_\zeta \partial_\zeta r)^j \wedge \partial_\zeta \beta \\
& \quad \wedge \left( \sum_{i=1}^n d\zeta_i \wedge d\zeta_i \right)^{n-q-3-j} \wedge \left( \sum_{i=1}^n d\zeta_i \wedge d\zeta_i \right)^{q-1},
\end{align}
we get
\begin{align}
(4.12) \quad C = \partial_\zeta r \wedge (\bar{\partial}_\zeta \partial_\zeta r)^j.
\end{align}

Lemma 4.3. For $z \in U_i, \zeta \in bD \cap U'_i$ for some $i$, we have
\begin{align}
(4.13) \quad \| C \|_{bD} \lesssim \sum_L \frac{\epsilon^{j+1}}{\tau^L(z, \epsilon)}, \quad d_z C = 0, \quad \epsilon = d(z, \zeta),
\end{align}
where the sum takes over all multiindices satisfying

\[0 \leq L_i, T_i \leq 1, \quad \sum_{i=1}^{n} L_i + T_i = 2j + 1, \quad L_1 = 0.\]

Proof of Lemma 4.3. \(d_z C = 0\) is obvious since \(C\) does not depend on \(z\). Now fix \(z \in U_i\). Note formula (3.11) for \(A_{q,0}\) is stated in the standard coordinates \(\zeta_1, \cdots, \zeta_n\) in \(\mathbb{C}^n\). Denote the \(d(z, \zeta)\)-extremal coordinates centered at \(z\) by \(w_1, \cdots, w_n\). Then there exists an unitary matrix \(U_z\), which is only depending on \(z\), and the translation \(T_z\) from the origin to \(z\), such that \(U_z \circ T_z\) transforms coordinates \(\zeta_1, \cdots, \zeta_n\) to coordinates \(w_1, \cdots, w_n\). It follows from the invariance of differential forms under a linear transform that we can write \(\partial_{\zeta} r, \overline{\partial}_{\zeta} \partial_{\zeta} r\) in coordinates \(w_1, \cdots, w_n\) as

\[
\partial_{\zeta} r = \partial_{w} r, \quad \overline{\partial}_{\zeta} \partial_{\zeta} r = \overline{\partial}_{w} \partial_{w} r.
\]

Thus

\[
C = \partial_{w} r \wedge (\overline{\partial}_{w} \partial_{w} r)^j
= \sum_{l_1, \cdots, l_j} \frac{\partial r}{\partial w_{l_1}} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial \overline{w}_{l_i} \partial w_{k_i}} \cdot dw_{l_1} \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_j} \wedge dw_{k_j},
\]

where \(l_1 \cdots, l_j\) are different, and \(t, k_1, \cdots, k_j\) are different. Notice \(dr = 0\) when restricted to the space tangential to \(bD\), we find that

\[
\frac{\partial r}{\partial w_1} dw_1 = - \frac{\partial r}{\partial w_2} dw_2 - \cdots - \frac{\partial r}{\partial w_n} dw_n
- \frac{\partial r}{\partial \overline{w}_1} d\overline{w}_1 - \frac{\partial r}{\partial \overline{w}_2} d\overline{w}_2 - \cdots - \frac{\partial r}{\partial \overline{w}_n} d\overline{w}_n
\]

holds on tangential space \(T(bD)\), \(dw_1\) disappeared in the differential forms in the right side of (4.15) if we substitute (4.16) into (4.15) (see [CKM,
p. 133] for the same fact). Note if $k_s = 1$, and substitute (4.16) into (4.15),

\begin{equation}
\frac{\partial r}{\partial w_t} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial w_i \partial w_k_i} \cdot dw_t \wedge d\bar{w}_i \wedge dw_{k_1} \wedge \cdots \wedge d\bar{w}_{l_j} \wedge dw_{k_j}
\end{equation}

\[ = - \sum_{v \neq t, k_1, \ldots, k_j} \frac{\partial r}{\partial w_t} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial w_i \partial w_k_i} \cdot \frac{\partial r}{\partial w_v} \cdot dw_t \wedge d\bar{w}_i \wedge dw_{k_1} \wedge \cdots \wedge d\bar{w}_{l_s} \wedge dw_v \wedge \cdots \wedge dw_{k_j}.
\]

Without loss of generality, we can assume $|\frac{\partial r}{\partial w_1}(w)| \approx 1$ for $w \in U'_1$. Since

\[
\left| \frac{\partial^2 r}{\partial w_i \partial w_k_i} \right| \lesssim \frac{\varepsilon}{\tau_1(z, \varepsilon) \tau_{k_i}(z, \varepsilon)},
\]

\[
\left| \frac{\partial^2 r}{\partial w_i \partial w_1} \right| \left| \frac{\partial r}{\partial w_v} \right| \lesssim \frac{\varepsilon}{\tau_1(z, \varepsilon) \tau_{k_i}(z, \varepsilon)} \cdot \frac{\varepsilon}{\tau_v(z, \varepsilon) \tau_{l_s}(z, \varepsilon)}
\]

by Lemma 2.5, where $\varepsilon = d(z, \zeta)$, we see that the absolute value of the coefficient of $dw_t \wedge d\bar{w}_i \wedge dw_{k_1} \wedge \cdots \wedge d\bar{w}_{l_s} \wedge dw_v \wedge \cdots \wedge dw_{k_j}$ in the right side of (4.17)

\[
(4.18) \lesssim \frac{\varepsilon}{\tau_1(z, \varepsilon)} \cdot \prod_{i \neq s} \frac{\varepsilon}{\tau_{k_i}(z, \varepsilon) \tau_{l_i}(z, \varepsilon)} \cdot \frac{\varepsilon}{\tau_v(z, \varepsilon) \tau_{l_s}(z, \varepsilon)}
\]

the coefficient of $dw_t \wedge d\bar{w}_i \wedge dw_{k_1} \wedge \cdots \wedge d\bar{w}_{l_s} \wedge dw_v \wedge \cdots \wedge dw_{k_j}$ in the right side of (4.17) has the same bound (4.18). If $t = 1$ in the right side of (4.15), after substituting (4.16) into (4.15), we have the similar results. Now, we find that, as differential form acting on $(\bigotimes T(bD))^{2j+1}$,

\[
C = \sum_L a_L dw^L, \quad |a_L| \lesssim \frac{\varepsilon^{j+1}}{\tau^L(z, \varepsilon)}
\]

where multiindices $L$ satisfy $0 \leq L_i, T_i \leq 1, L_1 = 0$ and $\sum_{i=1}^n L_i + \bar{T}_i = 2j+1$.

By using the inverse of transformation $U_z \circ T_z$, we can write $dw^L$ as a linear combination of differential forms $d\zeta^I$, \n
\[
dw^L = \sum_I a_I^L d\zeta^I, \quad |a_I^L| \lesssim 1
\]

by each entry of the matrix $U_z^{-1}$ has absolute value $\leq 1$, where multiindices $I$ satisfy $0 \leq I, \bar{I} \leq 1, |I| = 2j + 1$. Thus, $\|C\|_{bD} \lesssim \sum_L |a_L|$, where the
summation takes over all multiindices \( L \) satisfying \( 0 \leq L_i, T_i \leq 1, L_1 = 0 \) and \( \sum_{i=1}^{n} L_i + T_i = 2j + 1 \). This completes the proof of Lemma 4.3.

Now we can prove Proposition 4.1.

**Proof of Proposition 4.1.** (1) Note \( \partial \frac{C}{\partial z_i} = 0, \ i = 1, \ldots, n, \)

\[
(4.20) \quad \frac{\partial}{\partial z_i} \left( \frac{Cd\zeta^j}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^{n} (\xi_i - \zeta_i) d\zeta_i \right) = \frac{(j + 1)Cd\zeta^j \frac{\partial \tau^r}{\partial \zeta}}{(\partial \tau^r(\zeta), \zeta - z)^{j+2}|z - \zeta|^{2n-2j-2}} \wedge \sum_{i=1}^{n} (\xi_i - \zeta_i) d\zeta_i + \frac{(n - j - 1)Cd\zeta^j (\xi_i - \zeta_i)}{(\partial \tau^r(\zeta), \zeta - z)^{j+1}|z - \zeta|^{2n-2j}} \wedge \sum_{i=1}^{n} (\xi_i - \zeta_i) d\zeta_i
\]

by \( \frac{\partial}{\partial z_i} (\partial \tau^r(\zeta), \zeta - z) = -\frac{\partial \tau^r}{\partial \zeta} \). Note \( |\frac{\partial \tau^r}{\partial \zeta}| \lesssim |\sum_{j=1}^{n} \frac{1}{\tau^l_i(z, \varepsilon)}| \lesssim \sum_{j=1}^{n} \frac{1}{\tau^l_i(z, \varepsilon)} \)

by each entry of matrix \( U_z \) having absolute value not bigger than 1 and \( d(z, \zeta) \approx d(\zeta, z) \approx |(\partial \tau^r(\zeta), \zeta - z)| \) for \( \zeta \in bD \cap U_1 \) by Proposition 2.3 and Proposition 3.4. It follows that

\[
(4.21) \quad \left\| \frac{\partial}{\partial z_i} \left( \frac{Cd\zeta^j}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^{n} (\xi_i - \zeta_i) d\zeta_i \right) \right\|_{bD} \lesssim \sum_{I} \frac{1}{d(z, \zeta)^{j+2}|z - \zeta|^{2n-2j-3}} \frac{\varepsilon^{j+1}}{\tau^l(z, \varepsilon)} \sum_{i=1}^{n} \frac{1}{\tau^l_i(z, \varepsilon)} + \frac{1}{d(z, \zeta)^{j+1}|z - \zeta|^{2n-2j-2}} \frac{\varepsilon^{j+1}}{\tau^l(z, \varepsilon)}
\]

by Lemma 4.3 and \( \varepsilon = d(z, \zeta) \), where \( I \) takes over all multiindices satisfying \( 0 \leq I_i, T_i \leq 1, I_1 = 0 \) and \( \sum_{i=1}^{n} I_i + T_i = 2j + 1 \). For such \( I, 1 \leq i \leq n, \)

\( \tau^l(z, \varepsilon) \tau_i(z, \varepsilon) = \tau^\beta(z, \varepsilon) \) for some multiindex \( \beta \) satisfying condition C, i.e.,

\( C1 \) \( \sum_{i=1}^{n} \beta_i + \beta_i = 2j + 2; \ C2 \) There exists at most one \( i_0 > 1 \) such that \( \beta_{i_0} + \beta_{i_0} = 3 \) and \( \beta_l + \beta_l \leq 2 \) for all \( l \neq i_0 \). If such \( i_0 \) exists, we must have \( \beta_1 + \beta_1 = 1 \). Notice

\[
(4.22) \quad \tau_i(z, \zeta) \approx \varepsilon = d(z, \zeta) \lesssim |z - \zeta|
\]

we get

\[
(4.23) \quad \left\| \frac{\partial}{\partial z_i} \left( \frac{Cd\zeta^j}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^{n} (\xi_i - \zeta_i) d\zeta_i \right) \right\|_{bD} \lesssim \sum_{\beta} \frac{1}{\tau^\beta(z, \varepsilon)|z - \zeta|^{2n-2j-3}},
\]
similarly, we can prove

$$\left\| \frac{\partial}{\partial z_i} \left( \frac{Cd\zeta^J}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^{n} (z_i - \zeta_i)d\zeta_i \right) \right\|_{bD} \lesssim \sum_{\beta} \tau^{\beta}(z,\varepsilon) |z - \zeta|^{2n-2j-3},$$

where $\beta$ takes over all multiindices satisfying condition $C$. This completes the proof of (1).

2) For $z \in U_i$, $\zeta \in bD$ and $\zeta \notin U'_i$, $\langle \partial \xi r(\zeta), \zeta - z \rangle \neq 0$ and $|z - \zeta| \neq 0$. Note $U_1, \cdots, U_N$ covering $bD$. It follows

$$\langle \partial \xi r(\zeta), \zeta - z \rangle \gtrsim 1, \quad |z - \zeta| \gtrsim 1$$

by compactness. It follows that the coefficients of differential forms $dz^J A_{q}^{j,0}$ are bounded. The Proposition 4.1 is proved.

Now we are ready to prove Lemma 4.2.

**Proof of Lemma 4.2.** Define

$$S_0 = \{ i; \beta_i + \overline{\beta_i} = 0 \}$$

$$S_1 = \{ i; \beta_i + \overline{\beta_i} = 1 \}$$

$$S_2 = \{ i; \beta_i + \overline{\beta_i} = 2 \}$$

$$S_3 = \{ i; \beta_i + \overline{\beta_i} = 3 \}$$

for the multiindex $\beta$ satisfying condition $C$. Denote the cardinal of $S_i$ by $n_i, i = 0, 1, 2, 3$. We know that $n_3 \leq 1$ from condition $C$. We consider three cases: Case A, $n_3 = 0$ and $1 \in S_2$; Case B, $n_3 = 0$ and $1 \notin S_2$; Case C, $n_3 = 1$.

Note

$$\frac{1}{\tau_1(z,\varepsilon)} \lesssim \frac{1}{\varepsilon} \approx \frac{1}{\tau_1(z,\varepsilon)}.$$ 

If we replace $\tau_i$ by $\tau_1$ in (4.9) for some $l \in S_2$, $\beta_1 + \overline{\beta_1}$ will increase 1. Case B is reduced to Case A. In Case C, $\beta_1 + \overline{\beta_1} = 1$. If we replace $\tau_1$ by $\tau_1$ for $l \in S_3$, $\beta_1 + \overline{\beta_1}$ will increase to 2. Case C is reduced to Case A. Thus we only need to consider Case A.
For such $\beta$: $\beta_i + \overline{\beta}_j \leq 2$, $i = 1, \ldots, n$ and $\beta_1 + \overline{\beta}_1 = 2$, we will calculate $I_\beta$ as in [MS]. Recall the definition of $\sigma_i$ and $r_i \approx \sigma_i$, we get

\[
I_\beta \lesssim \int_{bD \cap U_\ell^I} e^{-\beta_1 - \overline{\beta}_1} \left( \sum_{i_2} [A^2_{i_2}(z)]^{\beta_2 + \overline{\beta}_2} \frac{\partial_2 + \overline{\partial}_2}{\partial_2} \right) \cdots \\
\cdot \left( \sum_{i_n} [A^n_{i_n}(z)]^{\beta_n + \overline{\partial}_n} \frac{\partial_n + \overline{\partial}_n}{\partial_n} \right) \cdot \frac{dV(\zeta)}{|z - \zeta|^{2n-2\ell-3}}
\]

by (4.5) and $e^{-\kappa} \gtrsim 1$, $|z - \zeta|^{-\kappa} \gtrsim 1$, where the summation takes over all $(i_2, \ldots, i_n)$ with $2 \leq i_2, \ldots, i_n \leq m$, $\varepsilon = d(z, \zeta)$. Let

\[
D_0 = \{ \zeta \in bD | d(z, \zeta) \leq |r(z)| \}, \\
D_q = \{ \zeta \in bD | 2^{q-1}|r(z)| \leq d(z, \zeta) \leq 2^q|r(z)| \}, \quad q = 1, 2, \ldots.
\]

Note for $\zeta \in D_q$, on the $2^q-1|r(z)|$-extremal coordinates centered at $z$, $\zeta = (\zeta_1, \ldots, \zeta_n)$

\[
d(z, \zeta) \approx 2^q|r(z)| + |\zeta_1| + \sum_{l=2}^{m} \sum_{l=2}^{n} A_l(z) |\zeta_l|^l
\]

by Corollary 2.4. Note $|z - \zeta| \gtrsim d(z, \zeta) \gtrsim 2^q|r(z)|$. (4.27) is less than

\[
\lesssim \sum_{q=0}^{\infty} \sum_{(i_2 \cdots i_n)} [A^2_{i_2}(z)]^{\beta_2 + \overline{\beta}_2} \cdots [A^n_{i_n}(z)]^{\beta_n + \overline{\partial}_n} \int_{bD \cap U_\ell^I \cap D_q} 2^q|r(z)| + |\zeta_1| \\
+ \sum_{k=1}^{n} \sum_{l=2}^{m} A_l^k(z) |\zeta_k|^l \frac{dV(\zeta)}{(2^q|r(z)| + |\zeta|)^{2n-2\ell-3+\kappa}}
\]

\[
= \sum_{q=0}^{\infty} \sum_{(i_2 \cdots i_n)} I_{\beta, i_2 \cdots i_n}^q.
\]

We will prove

\[
I_{\beta, i_2 \cdots i_n}^q \lesssim |r(z)|^{-1 + \frac{1}{m} - \kappa} 2^q(-1 + \frac{1}{m} - \kappa)
\]

for $0 < \kappa < 1 - \frac{1}{m}$, $2 \leq i_2, i_3, \cdots, i_n \leq m$, $\beta$ satisfying condition $C$. Hence, $\sum_{q=0}^{\infty} I_{\beta, i_2 \cdots i_n}^q < \infty$. This gives Lemma 4.2.
For $l \in S_2$, in the $2^{q-1}|r(z)|$-extremal coordinates centered at $z$,

\begin{equation}
I_{\beta,i_2\cdots i_n}^q \lesssim \Pi_{j \neq l} [A_{i_j}^j (z)]^{-\frac{\beta_j + \bar{\beta}_j}{i_j}} \int_{\mathbb{R}^{2n-3}} \int_{C} [A_{i_l}^l (z)]^\frac{2}{\eta_l} \left( 2^q |r(z)| + \sum_{k \neq l} |\zeta_k| \right)^{-(2n-2j-3+\kappa)}
\end{equation}

\begin{align*}
&\cdot \left( 2^q |r(z)| + |\zeta_1| + \sum_{k \neq l} \sum_{t=2}^m A_k^k(z) |\zeta_k|^t \right)^{-\frac{2-\sum_{j \neq l,j \geq 2} \beta_j + \bar{\beta}_j}{\eta_l} - \frac{\kappa}{\eta_l}} \\
&\quad \cdot dx_2 \cdots dx_{2n}
\end{align*}

by the volume element $dV(\zeta)$ on $bD \approx dx_2 \cdots dx_n$. Now apply Lemma 3.8 to (4.31) with

\begin{align*}
b &= A_{i_l}^l(z), \quad k = i_t, \quad \alpha = 2 + \sum_{j \neq l,j \geq 2} \frac{\beta_j + \bar{\beta}_j}{i_j} + \kappa \\
a &= 2^q |r(z)| + |\zeta_1| + \sum_{k \neq l} \sum_{t=2}^m A_k^k(z) |\zeta_k|^t
\end{align*}

to get

\begin{equation}
I_{\beta,i_2\cdots i_n}^q \lesssim \Pi_{j \neq l} [A_{i_j}^j (z)]^{-\frac{\beta_j + \bar{\beta}_j}{i_j}} \int_{\mathbb{R}^{2n-3}} \int_{C} [A_{i_l}^l (z)]^\frac{2}{\eta_l} \left( 2^q |r(z)| + \sum_{k \neq l} |\zeta_k| \right)^{-(2n-2j-3+\kappa)}
\end{equation}

\begin{align*}
&\cdot \left( 2^q |r(z)| + |\zeta_1| + \sum_{k \neq l} \sum_{t=2}^m A_k^k(z) |\zeta_k|^t \right)^{-\frac{2-\sum_{j \neq l,j \geq 2} \beta_j + \bar{\beta}_j}{\eta_l} - \frac{\kappa}{\eta_l}} \\
&\quad \cdot dV(\zeta)
\end{align*}
where $dV(\zeta)$ denote the volume element of $R^{2n-3}$. Repeating this procedure, we can integrate all variables $\zeta_i$ with $i \in S_2 \setminus \{1\}$. Then

\begin{equation}
I^q_{\beta,j_2\cdots i_n} \lesssim \Pi_{j \notin S_2 \setminus \{1\}} \left[ A_{i_j}^j (z) \right]^{\beta_j + \bar{\beta}_j \over \bar{\tau}_j - \tau_j} \left( 2^q |r(z)| + |\zeta_1| + \sum_{k \notin S_2} |\zeta_k| \right)^{-(2n-2j-3+\kappa)} \\
\cdot \left( 2^q |r(z)| + |\zeta_1| + \sum_{k \notin S_2} \sum_{t=2}^m A_k^t (z) |\zeta_k|^t \right)^{-(2+\sum_{j \notin S_2} \beta_j + \bar{\beta}_j + \kappa)} dV(\zeta).
\end{equation}

Now integrate all variables $\zeta_i$ with $i \in S_0$ by

\begin{equation}
\int_C \frac{d\zeta d\bar{\zeta}}{(|\zeta| + C)^k} \lesssim \frac{1}{C^{k-2}}
\end{equation}

for $k > 2$, we get

\begin{equation}
I^q_{\beta,j_2\cdots i_n} \lesssim \Pi_{j \in S_1} \left[ A_{i_j}^j (z) \right]^{\beta_j + \bar{\beta}_j \over \bar{\tau}_j - \tau_j} \left( 2^q |r(z)| + \sum_{k \in S_1} |\zeta_k| + |\zeta_1| \right)^{-(2n-2j-2n_0-3+\kappa)} \\
\cdot \left( 2^q |r(z)| + |\zeta_1| + \sum_{k \in S_1} \sum_{t=2}^m A_k^t (z) |\zeta_k|^t \right)^{-(2+\sum_{j \in S_1} \beta_j + \bar{\beta}_j + \kappa)} dV(\zeta).
\end{equation}

By condition $C$, $S_3 = \emptyset$ and $1 \in S_2$, we see that

\begin{equation}
2n_0 + 2n_1 + 2n_2 = 2n, \\
2n_2 + n_1 = 2j + 2.
\end{equation}

Therefore

\begin{equation}
2n - 2j - 3 - 2n_0 = n_1 - 1, \\
2n - 2n_2 - 2n_0 + 1 = 2n_1 + 1.
\end{equation}
Now if $n_1 \geq 2$, $l \in S_1$, then

\begin{equation}
I^q_{|\beta;i_2...i_n} \lesssim \Pi_{j \in S_1 \setminus \{l\}} \left[ A^j_{ij}(z) \right]^{\frac{1}{ij}}
\end{equation}

\begin{equation}
\cdot \int_{\mathbb{R}^{2n_1+1}} \left[ A^j_{ij}(z) \right]^{\frac{1}{ij}} \left( 2^{|r(z)|} + \sum_{k \in S_1 \setminus \{l\}} |\zeta_k| + |\zeta_1| \right)^{-n_1+1-\kappa}
\end{equation}

\begin{equation}
\cdot \left( 2^{|r(z)|} + |\zeta_1| + \sum_{k \in S_1 \setminus \{l\}} \sum_{t=2}^m A^k_t(z) |\zeta_k|^t 
\right)^{-2-\sum_{j \in S_1, j \neq l} \frac{1}{ij} - \frac{1-\kappa}{ij}}
\end{equation}

\begin{equation}
\cdot \left( 2^{|r(z)|} + |\zeta_1| + \sum_{k \in S_1 \setminus \{l\}} \sum_{t=2}^m A^k_t(z) |\zeta_k|^t 
\right)^{-2-\sum_{j \in S_1, j \neq l} \frac{1}{ij} - \frac{1-\kappa}{ij}}
\end{equation}

\begin{equation}
dV(\zeta).
\end{equation}

Now apply Lemma 3.7 to (4.38) with

\begin{equation}
a = 2^{|r(z)|} + |\zeta_1| + \sum_{k \in S_1 \setminus \{l\}} \sum_{t=2}^m A^k_t(z) |\zeta_k|^t
\end{equation}

\begin{equation}
a' = 2^{|r(z)|} + \sum_{k \in S_1, k \neq l} |\zeta_k| + |\zeta_1|
\end{equation}

\begin{equation}
k = i_t, \alpha = 2 + \sum_{j \in S_1, j \neq l} \frac{1}{ij} + \kappa, \alpha' = n_1 - 2 + \kappa > 0
\end{equation}

to get

\begin{equation}
I^q_{|\beta;i_2...i_n} \lesssim \Pi_{j \in S_1 \setminus \{l\}} \left[ A^j_{ij}(z) \right]^{\frac{1}{ij}}
\end{equation}

\begin{equation}
\cdot \int_{\mathbb{R}^{2n_1-1}} \left( 2^{|r(z)|} + \sum_{k \in S_1 \setminus \{l\}} |\zeta_k| + |\zeta_1| \right)^{-n_1+2-\kappa}
\end{equation}

\begin{equation}
\cdot \left( 2^{|r(z)|} + |\zeta_1| + \sum_{k \in S_1 \setminus \{l\}} \sum_{t=2}^m A^k_t(z) |\zeta_k|^t 
\right)^{-2-\sum_{j \in S_1, j \neq l} \frac{1}{ij} - \frac{1-\kappa}{ij}}
\end{equation}

\begin{equation}
dV(\zeta).
\end{equation}
Repeating this procedure, we can integrate out \((n_1 - 1)\) variables \(\zeta_i\) with \(i \in S_1\). Let \(\zeta_s\) be the remaining variable. We get

\[
I_{\beta,i_2\ldots i_n}^q \lesssim \int_{\mathbb{R}^3} [A_{i_s}^s(z)]^{\frac{1}{n_2}} \left(2^q |r(z)|\right)^{-\kappa} \left\{2^q |r(z)| + \left|\zeta_1\right| + \sum_{t=2}^m A_t^s(z)|\zeta_s|^t\right\}^{-2-\frac{1}{r_s}-\kappa} dV(\zeta),
\]

where \(dV(\zeta) = dx_2dx_sdx_{n+s}\). Now integrate out variable \(x_2\) to get

\[
I_{\beta,i_2\ldots i_n}^q \lesssim \int_{\mathbb{R}^2} [A_{i_s}^s(z)]^{\frac{1}{n_2}} \left(2^q |r(z)|\right)^{-\kappa} \cdot \left(2^q |r(z)| + \sum_{t=2}^m A_t^s(z)|\zeta_s|^t\right)^{-1-\frac{1}{r_s}-\frac{1}{2}} dV(\zeta)
\]

\[
\lesssim \int_{\mathbb{R}^2} [A_{i_s}^s(z)]^{\frac{1}{n_2}} \left(2^q |r(z)|\right)^{-\kappa} \left(2^q |r(z)| + A_k^s(z)|\zeta_s|^{k_0}\right)^{-\frac{1}{r_s}-\frac{1}{2}} dV(\zeta)
\]

\[
\lesssim \int_{\mathbb{R}^2} \left(2^q |r(z)|\right)^{-\kappa} \left[\frac{1}{A_k^s(z)}\right]^{\frac{1}{r_s}} \left(2^q |r(z)|\right)^{\frac{1}{r_s}+\frac{\kappa}{2}} \cdot \left(2^q |r(z)|\right)^{\frac{1}{r_s}}
\]

\[
= \sigma_s(z, 2^q |r(z)|) \lesssim |r(z)|^{\frac{1}{m}-1-2\kappa}(\frac{1}{m}-1-2\kappa)^q,
\]

by \(\sigma_s(z, 2^q |r(z)|) \lesssim (2^q |r(z)|)^{\frac{1}{r}}\). This completes the proof of (4.30) (take \(\kappa\) to be \(\frac{\kappa}{2}\)), therefore Lemma 4.2.

**Note added:** This paper is the revised form of a paper titled *Hölder estimate for \(\overline{\partial}\) on the convex domains of finite type* written in 1995, where Lemma 4.2 in [BCD] was incorrectly stated for all convex domains of finite type. The referee informed the author that Diederich and Forness announced similar results at the Hayama symposium in December, 1998.
References


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