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COMPOSITION OF POLYNOMIALS OF THE FORM  $z^2 + c_n$

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## GEOMETRIC PROPERTIES OF JULIA SETS OF THE COMPOSITION OF POLYNOMIALS OF THE FORM $z^2 + c_n$

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For a sequence  $(c_n)$  of complex numbers we consider the quadratic polynomials  $f_{c_n}(z) := z^2 + c_n$  and the sequence  $(F_n)$  of iterates  $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$ . The Fatou set  $\mathcal{F}_{(c_n)}$  is by definition the set of all  $z \in \widehat{\mathbb{C}}$  such that  $(F_n)$  is normal in some neighbourhood of  $z$ , while the complement of  $\mathcal{F}_{(c_n)}$  is called the Julia set  $\mathcal{J}_{(c_n)}$ . The aim of this article is to study geometric properties, Lebesgue measure and Hausdorff dimension of the Julia set  $\mathcal{J}_{(c_n)}$  provided that the sequence  $(c_n)$  is bounded.

### 1. Introduction.

For a sequence  $(c_n)$  of complex numbers we consider the quadratic polynomials  $f_{c_n}(z) := z^2 + c_n$  and the sequence  $(F_n)$  of iterates  $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$ . (Note that  $F_n$  depends on  $c_1, \dots, c_n$  which we do not indicate explicitly in the notation.) If  $c_n = c$  for all  $n$ , we write  $f_c^n$  instead of  $F_n$ . The Fatou set  $\mathcal{F}_{(c_n)}$  is by definition the set of all  $z \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  such that  $(F_n)$  is normal (in the sense of Montel) in some neighbourhood of  $z$ , while the complement of  $\mathcal{F}_{(c_n)}$  (in  $\widehat{\mathbb{C}}$ ) is called the Julia set  $\mathcal{J}_{(c_n)}$ . A component of the Fatou set is called a *stable domain*. For iteration theory of a fixed function we refer the reader to the books of Beardon [Be], Carleson and Gamelin [CG], Milnor [M] or Steinmetz [St]. We also mention the survey articles of Blanchard [Bl], Lyubich [L2] or Eremenko and Lyubich [EL].

We always assume that  $|c_n| \leq \delta$  for some  $\delta > 0$ . Then from [Bü2] it is known that to some extent the sequence  $(F_n)$  behaves similar to the sequence  $(f_c^n)$ . There exists a stable domain  $\mathcal{A}_{(c_n)}(\infty)$  which contains the point  $\infty$  and wherein  $F_n \rightarrow \infty$  as  $n \rightarrow \infty$  locally uniformly. This domain need not be *invariant* (i.e.,  $f_{c_k}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$  for all  $k$ ) or *backward invariant* (i.e.,  $f_{c_k}^{-1}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$  for all  $k$ ), but there exists an invariant domain  $M = M_\delta \subset \mathcal{A}_{(c_n)}(\infty)$  which contains the point  $\infty$  and which satisfies  $\mathcal{A}_{(c_n)}(\infty) = \{z \in \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N}\}$ . Therefore, the filled Julia set  $\mathcal{K}_{(c_n)} := \widehat{\mathbb{C}} \setminus \mathcal{A}_{(c_n)}(\infty)$  and the Julia set  $\mathcal{J}_{(c_n)}$  are compact in  $\mathbb{C}$ , and  $\mathcal{K}_{(c_n)}$  is the set of all  $z \in \mathbb{C}$  such that  $(F_k(z))_{k=1}^\infty$  is bounded. Furthermore, we have  $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty) = \partial \mathcal{K}_{(c_n)}$ . Also  $\mathcal{J}_{(c_n)}$  and  $\mathcal{K}_{(c_n)}$  are perfect sets.

Finally,  $\mathcal{J}_{(c_n)}$  and  $\mathcal{F}_{(c_n)}$  are invariant in the sense that  $F_k^{-1}(F_k(\mathcal{J}_{(c_n)})) = \mathcal{J}_{(c_n)}$  and  $F_k^{-1}(F_k(\mathcal{F}_{(c_n)})) = \mathcal{F}_{(c_n)}$  for all  $k \in \mathbb{N}$ . For further results we also refer to [Brü], [BBR], [Bü1] and [FS].

The *Mandelbrot set*  $\mathcal{M}$  is defined as the set of all  $c \in \mathbb{C}$  such that  $(f_c^n(0))_{n=1}^\infty$  is bounded, and  $\mathcal{M}$  is compact in  $\mathbb{C}$ . It plays an important role in iteration of a fixed quadratic polynomial  $f_c$ . We recall that the largest disk with center 0 which is contained in  $\mathcal{M}$  has radius  $\frac{1}{4}$ .

The plan of this article is as follows. After introducing some notations and known auxiliary results (Section 2) we show that the Julia set  $\mathcal{J}_{(c_n)}$  is always uniformly perfect (Section 3).

Our main result (Section 4) states that the Julia set  $\mathcal{J}_{(c_n)}$  is a quasicircle provided that  $|c_n| \leq \delta$  for some  $\delta < \frac{1}{4}$ . This is done by proving that  $\mathcal{F}_{(c_n)}$  consists of two simply connected John domains  $\mathcal{A}_{(c_n)}(0)$  and  $\mathcal{A}_{(c_n)}(\infty)$  which have  $\mathcal{J}_{(c_n)}$  as their common boundaries.

Concerning the two-dimensional Lebesgue measure  $m_2(\mathcal{J}_{(c_n)})$  of Julia sets (Section 5) we show that it is almost surely zero provided that the  $c_n$  are randomly chosen in  $\{z \in \mathbb{C} : |z| \leq \delta\}$  for some  $\delta > \frac{1}{4}$ . For  $\delta < \frac{1}{4}$  we always have  $m_2(\mathcal{J}_{(c_n)}) = 0$ .

Section 6 deals with Hausdorff dimension  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)}$  of Julia sets. We give a lower estimate for  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)}$  depending only on  $\delta$  which implies that  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)}$  is always positive. For that purpose we prove that the Green function of  $\mathcal{A}_{(c_n)}(\infty)$  (which is known to exist) is Hölder continuous. Furthermore, for  $\delta < \frac{1}{4}$  it follows that  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)} < 2$ .

A point  $\zeta \in \mathbb{C}$  is called a repelling fixpoint of the sequence of iterates  $(F_n)$  if  $F_k(\zeta) = \zeta$  for some  $k \in \mathbb{N}$  and  $|F'_k(\zeta)| > 1$ . The set of all those points is denoted by  $\mathcal{R}_{(c_n)}$ . In this general setting it is not necessarily true that  $\mathcal{R}_{(c_n)} \subset \mathcal{J}_{(c_n)}$ . But we prove (Section 7) that if  $|c_n| \leq \delta < \frac{1}{4}$ , then the derived set of  $\mathcal{R}_{(c_n)}$  coincides with  $\mathcal{J}_{(c_n)}$ . In the last section we investigate the asymptotic distribution of certain predecessors.

## 2. Notations and auxiliary results.

We introduce a few further notations and collect some known auxiliary results that are frequently used in the sequel. If  $E \subset \mathbb{C}$ , then  $E'$  denotes the derived set (that is the set of points  $z \in \mathbb{C}$  such that every neighbourhood of  $z$  contains a point  $w \in E \setminus \{z\}$ ),  $\overline{E}$  the closure and  $E^\circ$  the set of interior points of  $E$ . Furthermore, the diameter of  $E$  is defined by  $\text{diam } E := \sup \{|z - w| : z, w \in E\}$ , and the distance of a point  $z \in \mathbb{C}$  from  $E$  by  $\text{dist}(z, E) := \inf \{|z - w| : w \in E\}$ . For  $a \in \mathbb{C}$  and  $r > 0$  we set  $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$ ,  $D_r := D_r(0)$ ,  $\mathbb{D} := D_1$  and  $K_r := \overline{D}_r$ . Finally, for  $R > 0$  let  $\Delta_R := \{z \in \widehat{\mathbb{C}} : |z| > R\}$ .

If  $(c_n) \in K_\delta^\mathbb{N}$ , then the invariant domain  $M \subset \mathcal{A}_{(c_n)}(\infty)$  may be chosen as  $M = \Delta_R$  for any

$$R \geq R_\delta := \frac{1}{2}(1 + \sqrt{1 + 4\delta}) .$$

More precisely, if  $R > R_\delta$ , then  $f_c(\Delta_{R_\delta}) \subset \Delta_{R_\delta}$  and  $f_c(\overline{\Delta}_R) \subset \Delta_R$  for all  $c \in K_\delta$ . This implies that  $\mathcal{K}_{(c_n)} \subset K_{R_\delta}$ . If  $\delta \leq \frac{1}{4}$ , we set

$$r_\delta := \frac{1}{2}(1 + \sqrt{1 - 4\delta}) \in [\frac{1}{2}, 1] , \quad s_\delta := \frac{1}{2}(1 - \sqrt{1 - 4\delta}) \in [0, \frac{1}{2}] .$$

Then we have  $f_c(D_{s_\delta}) \subset D_{s_\delta}$ ,  $f_c(D_{r_\delta}) \subset D_{r_\delta}$  and  $f_c(\overline{D}_r) \subset D_r$  for all  $c \in K_\delta$  and all  $r \in (s_\delta, r_\delta)$ . This implies that there exists a stable domain  $\mathcal{A}_{(c_n)}(0) \supset D_{r_\delta}$ , and there holds  $\mathcal{J}_{(c_n)} \subset K_{R_\delta} \cap \overline{\Delta}_{r_\delta}$ .

From [FS, Theorem 2.1] it follows that  $\mathcal{A}_{(c_n)}(\infty)$  is regular for logarithmic potential theory which means that the Green function of  $\mathcal{A}_{(c_n)}(\infty)$  with pole at infinity exists. More precisely, the function  $g_{(c_n)}$  defined by

$$(2.1) \quad g_{(c_n)}(z) := \lim_{k \rightarrow \infty} \frac{1}{2^k} \log^+ |F_k(z)|$$

is continuous in  $\mathbb{C}$ ,  $g_{(c_n)}(z) = 0$  for  $z \in \mathcal{K}_{(c_n)}$ , and it is the Green function of  $\mathcal{A}_{(c_n)}(\infty)$  with pole at infinity.

Furthermore, we introduce the *critical set* (or set of critical points)

$$\mathcal{C}_{(c_n)} := \{ z \in \mathbb{C} : F_j(z) = 0 \text{ for some } j \in \mathbb{N}_0 \}$$

of  $(F_n)$ , where  $F_0(z) := z$ . This is motivated by the fact that

$$F'_k(z) = 2^k \prod_{j=0}^{k-1} F_j(z)$$

so that  $F'_k(z) = 0$  if and only if  $F_j(z) = 0$  for some  $j \in \{0, 1, \dots, k - 1\}$ . We call a point  $w \in \mathbb{C}$  a *critical value* of  $(F_n)$ , if  $w = F_k(z)$  and  $F'_k(z) = 0$  for some  $k \in \mathbb{N}$  and some  $z \in \mathbb{C}$ . If  $w \in \mathbb{C}$  is not a critical value of  $F_k$ , then in some sufficiently small disk  $D_\varepsilon(w)$  there exist  $2^k$  analytic branches of the inverse function of  $F_k$ .

Finally, we recall a result of Büger [Bü1] that the Julia set  $\mathcal{J}_{(c_n)}$  is self-similar. This means that for any open set  $D$  meeting  $\mathcal{J}_{(c_n)}$  there exists  $k_0 \in \mathbb{N}$  such that  $F_k(\mathcal{J}_{(c_n)} \cap D) = F_k(\mathcal{J}_{(c_n)})$  for all  $k \geq k_0$ .

### 3. Uniform perfectness of Julia sets.

An open set  $A \subset \widehat{\mathbb{C}}$  is called a *conformal annulus*, if it can be mapped conformally onto an annulus  $\{ z \in \mathbb{C} : 1 < |z| < \varrho \}$  for some  $\varrho > 1$ . Then the number  $\varrho$  is uniquely determined and  $\text{mod } A := \frac{1}{2\pi} \log \varrho$  is called the *modulus* of  $A$ . Now, let  $E \subset \widehat{\mathbb{C}}$  be a compact set. A conformal annulus  $A$  *separates*  $E$ , if both components of  $\widehat{\mathbb{C}} \setminus A$  meet  $E$ . The set  $E$  is called *uniformly perfect*, if it is not a single point and if there is a constant  $\alpha > 0$  such that for any

conformal annulus  $A$  which separates  $E$  there holds  $\text{mod } A \leq \alpha$ . Obviously, a uniformly perfect set is also perfect (that is  $E' = E$ ), and every connected compact set with at least two points is uniformly perfect. Uniformly perfect sets were introduced by Beardon and Pommerenke [BeP] (see also [P1]). It is known that the Julia set of a fixed rational function is always uniformly perfect [MR] (see also [CG, p. 64]). We show that this result extends to our situation.

**Theorem 3.1.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then the Julia set  $\mathcal{J}_{(c_n)}$  is uniformly perfect.*

*Proof.* We assume that  $\mathcal{J}_{(c_n)}$  is not uniformly perfect. Then there exists a sequence of conformal annuli  $A_k \subset \mathcal{F}_{(c_n)}$  which separate  $\mathcal{J}_{(c_n)}$  and  $\text{mod } A_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Let  $E_k$  be the component of  $\widehat{\mathbb{C}} \setminus A_k$  with the smaller chordal diameter (which we denote by  $\text{diam}_\chi E_k$ ). Then we have  $\text{diam}_\chi E_k \rightarrow 0$  as  $k \rightarrow \infty$ . If  $\lambda_k: \mathbb{D} \rightarrow A_k \cup E_k$  is a conformal map of  $\mathbb{D}$  onto  $A_k \cup E_k$  with  $\lambda_k(0) \in E_k$ , and if  $M_k := \lambda_k^{-1}(E_k) \subset \mathbb{D}$ , then  $M_k$  is compact and connected,  $0 \in M_k$  and  $\text{diam}_\chi M_k \rightarrow 0$  as  $k \rightarrow \infty$ .

It is elementary to see that  $(f_{c_n})$  satisfies a uniform Lipschitz condition with respect to the chordal metric  $\chi$ , that means that there exists a constant  $L > 0$  (which depends only on  $\delta$  but not on  $n$ ) such that  $\chi(f_{c_n}(z), f_{c_n}(w)) \leq L\chi(z, w)$  for all  $z, w \in \widehat{\mathbb{C}}$  and all  $n \in \mathbb{N}$ . From Lemma 4.1 in [BBR] we know that  $\text{diam } F_k(\mathcal{J}_{(c_n)}) \geq 1$  for all  $k \in \mathbb{N}_0$  so that  $\text{diam}_\chi F_k(\mathcal{J}_{(c_n)}) \geq C := 2(1 + R_\delta^2)^{-1}$ .

We choose  $\varepsilon > 0$  with  $\varepsilon < C$  and

$$(3.1) \quad \frac{C}{3} > L\varepsilon.$$

Let  $k_0 \in \mathbb{N}$  such that  $\text{diam}_\chi E_k < \varepsilon$  for all  $k \geq k_0$ . Since  $(A_k \cup E_k) \cap \mathcal{J}_{(c_n)} \neq \emptyset$  and since  $\mathcal{J}_{(c_n)}$  is self-similar (cf. [Bü1]), for every  $k \geq k_0$  there exists a smallest index  $m(k) \in \mathbb{N}$  such that  $\text{diam}_\chi F_{m(k)}(E_k) > \varepsilon$ . Setting  $G_k := F_{m(k)} \circ \lambda_k$  we obtain

$$(3.2) \quad \text{diam}_\chi G_k(M_k) > \varepsilon$$

for all  $k \geq k_0$ . By the choice of  $m(k)$  we have  $\text{diam}_\chi F_{m(k)-1}(E_k) \leq \varepsilon$  and thus  $\text{diam } F_{m(k)}(E_k) = \text{diam}_\chi f_{c_{m(k)}}(F_{m(k)-1}(E_k)) \leq L\varepsilon$  for all  $k \geq k_0$ .

Because of (3.1) there exist at least three different points  $a_{1,k}, a_{2,k}, a_{3,k} \in F_{m(k)}(\mathcal{J}_{(c_n)})$  whose chordal distance is greater than  $L\varepsilon$ . We have  $G_k(\mathbb{D} \setminus M_k) = F_{m(k)}(\lambda_k(\mathbb{D} \setminus M_k)) = F_{m(k)}(A_k) \subset F_{m(k)}(\mathcal{F}_{(c_n)})$  and  $\text{diam}_\chi G_k(M_k) = \text{diam}_\chi F_{m(k)}(E_k) \leq L\varepsilon$  for all  $k \geq k_0$ . This implies that  $G_k$  omits at least two of the values  $a_{1,k}, a_{2,k}, a_{3,k}$  in  $\mathbb{D}$  and hence  $(G_k)$  is normal in  $\mathbb{D}$  by a generalized version of Montel's theorem (cf. [Be, p. 57]). Since  $\text{diam}_\chi M_k \rightarrow 0$  as  $k \rightarrow \infty$  and  $0 \in M_k$  we get  $\text{diam}_\chi G_k(M_k) \rightarrow 0$  as  $k \rightarrow \infty$  which contradicts (3.2).  $\square$

4. Julia sets and quasicircles.

From iteration theory of a fixed function it is known that  $\mathcal{J}(f_c)$  is a quasicircle if  $c$  is in the interior of the main cardioid of the Mandelbrot set (cf. Yakobson [Y], see also [CG, p. 103]). The goal of this section is to show that this result remains valid in our general situation provided that  $\delta < \frac{1}{4}$ . We do this in several steps, and we first recall some facts on quasicircles and John domains.

A *quasicircle*  $\Gamma \subset \mathbb{C}$  is the image of the unit circle  $\partial\mathbb{D}$  under a quasiconformal homeomorphism of  $\mathbb{C}$  onto itself. An equivalent geometric definition is the three-point property, i.e., there exists a constant  $a > 0$  such that if  $z_1, z_2, z_3 \in \Gamma$  and  $z_2$  is on the arc between  $z_1$  and  $z_3$  with the smaller diameter, then  $|z_1 - z_2| + |z_2 - z_3| \leq a|z_1 - z_3|$ . A quasicircle may be non-rectifiable but it has no cusps. For details we refer, for example, to the books of Ahlfors [A] or Lehto and Virtanen [LV].

A domain  $G \subset \widehat{\mathbb{C}}$  with  $\partial G \subset \mathbb{C}$  is called a *John domain*, if there exists a constant  $b > 0$  and a point  $w_0 \in G$  such that for any  $z_0 \in G$ , there is an arc  $\gamma = \gamma(z_0) \subset G$  joining  $z_0$  and  $w_0$  and satisfying  $\text{dist}(z, \partial G) \geq b|z - z_0|$  for all  $z \in \gamma$ . A simply connected John domain  $G$  has locally connected boundary  $\partial G$  so that by Carathéodory’s theorem (cf. [P2, p. 20]) the Riemann map from  $\mathbb{D}$  onto  $G$  extends continuously to  $\overline{\mathbb{D}}$ . The image of a John domain under a quasiconformal homeomorphism of  $\widehat{\mathbb{C}}$  onto itself is again a John domain. Thus, the two complementary domains of a quasicircle are John domains. Conversely, if the two complementary components of a Jordan curve (a homeomorphic image of the unit circle)  $\Gamma$  are John domains, then  $\Gamma$  is a quasicircle. For this and further background material we refer to [NV].

For  $\delta \leq \frac{1}{4}$  we know that  $\mathcal{J}_{(c_n)}$  is connected [BBR], and since  $\mathcal{J}_{(c_n)} = \partial\mathcal{A}_{(c_n)}(\infty)$ , the stable domain  $\mathcal{A}_{(c_n)}(\infty)$  is simply connected. Furthermore, there exists a stable domain  $\mathcal{A}_{(c_n)}(0)$  containing  $D_{r_\delta}$ . We now show:

**Theorem 4.1.** *Let  $\delta \leq \frac{1}{4}$ ,  $(c_n) \in K_\delta^\mathbb{N}$  and  $s_\delta \leq r \leq r_\delta$ . Then there holds  $\mathcal{A}_{(c_n)}(0) = \bigcup_{k=0}^\infty F_k^{-1}(D_r)$  and  $\partial\mathcal{A}_{(c_n)}(0) = \mathcal{J}_{(c_n)}$ . In particular,  $\mathcal{A}_{(c_n)}(0)$  is simply connected and  $\mathcal{F}_{(c_n)} = \mathcal{A}_{(c_n)}(0) \cup \mathcal{A}_{(c_n)}(\infty)$ .*

*Proof.* We set  $A := \bigcup_{k=0}^\infty U_k$  with  $U_k := F_k^{-1}(D_r)$ . It is elementary to see that each  $U_k$  is a domain containing  $D_r$ , and since  $D_r$  is invariant, we get  $U_k \subset \mathcal{F}_{(c_n)}$ . Thus,  $A$  is a domain with  $D_r \subset A \subset \mathcal{F}_{(c_n)}$  which gives  $A \subset \mathcal{A}_{(c_n)}(0)$ .

We show that  $\mathcal{J}_{(c_n)} \subset \partial A$ . For that purpose, let  $z_0 \in \mathcal{J}_{(c_n)}$  and  $D := D_\varepsilon(z_0)$  for  $\varepsilon > 0$ . By Montel’s theorem the set  $\widehat{\mathbb{C}} \setminus \bigcup_{k=0}^\infty F_k(D)$  contains at most two points so that there exists  $w \in D_r$  such that  $w \in F_m(D)$  for some  $m \in \mathbb{N}_0$ . Therefore,  $D_r \cap F_m(D)$  is a non-empty open set, and this

implies that there exists  $\zeta \in D \setminus \mathcal{J}_{(c_n)}$  with  $F_m(\zeta) \in D_r$ . That means  $\zeta \in A$ , and since  $\varepsilon > 0$  was arbitrary we arrive at  $z_0 \in \partial A$ . Summarizing, we have  $A \subset \mathcal{A}_{(c_n)}(0)$  and  $\partial \mathcal{A}_{(c_n)}(0) \subset \mathcal{J}_{(c_n)} \subset \partial A$  which gives the assertion.  $\square$

For  $\delta < \frac{1}{4}$  and  $\frac{1}{2} < r < r_\delta$  we set  $V := \Delta_r \supset \mathcal{J}_{(c_n)}$ . Then  $V$  is backward invariant, and  $V$  does not contain any critical value of  $(F_n)$  so that in every disk  $D \subset V$  there exist  $2^n$  analytic branches  $F_n^{-1}$  of the inverse function of  $F_n$ . We prove:

**Lemma 4.2.** *Let  $\delta < \frac{1}{4}$ ,  $(c_n) \in K_\delta^\mathbb{N}$  and  $\frac{1}{2} < r < r_\delta$ . Furthermore, let  $\gamma: [0, 1] \rightarrow V$  be a rectifiable curve in  $V := \Delta_r$ ,  $z := \gamma(0)$ ,  $w := \gamma(1)$  and let  $F_n^{-1}$  be an analytic branch of the inverse function of  $F_n$  in some disk  $D \subset V$  with center  $z$ . Finally, we denote the analytic continuation of  $F_n^{-1}$  along  $\gamma$  also by  $F_n^{-1}$ . Then there holds*

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \leq 1 + \alpha \ell(\gamma) e^{\alpha \ell(\gamma)},$$

where  $\alpha := 4r(2r - 1)^{-1}$  and  $\ell(\gamma)$  denotes the length of  $\gamma$ . In particular, for any disk  $D \subset V$  and any analytic branch  $F_n^{-1}$  in  $D$  there holds

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \leq 1 + \alpha e^{\alpha d} |z - w|$$

for all  $z, w \in D$  and  $n \in \mathbb{N}$ , where  $d := \text{diam } D$ .

*Proof.* For  $n \in \mathbb{N}$  and  $k = 0, 1, \dots, n - 1$  we set  $F_{n,k} := f_{c_n} \circ \dots \circ f_{c_{k+1}}$ . Since

$$(F_n^{-1})'(z) = \frac{1}{F_n'(F_n^{-1}(z))} = \frac{1}{2^n \prod_{j=0}^{n-1} F_j'(F_n^{-1}(z))} = \frac{1}{2^n \prod_{j=0}^{n-1} F_{n,j}^{-1}(z)}$$

and  $V$  is backward invariant we have

$$|(F_n^{-1})'(z)| \leq q^n \quad (z \in V),$$

or

$$|(F_{n,k}^{-1})'(z)| \leq q^{n-k} \quad (z \in V),$$

where  $q := \frac{1}{2r} < 1$ . This implies

$$(4.1) \quad |F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)| \leq \left| \int_z^w |(F_{n,k}^{-1})'(\zeta)| |d\zeta| \right| \leq q^{n-k} \ell(\gamma),$$

where we integrate over the curve  $\gamma$ . Furthermore, we have

$$\frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} = \prod_{k=0}^{n-1} \frac{F_k'(F_n^{-1}(w))}{F_k'(F_n^{-1}(z))} = \prod_{k=0}^{n-1} \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)}.$$

Writing

$$\frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)} = 1 + \frac{F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)}{F_{n,k}^{-1}(z)},$$

we obtain from (4.1)

$$\left| \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)} \right| \leq 1 + 2q^{n-k+1}\ell(\gamma).$$

This implies

$$\begin{aligned} \left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| &\leq \prod_{k=0}^{n-1} (1 + 2q^{n-k+1}\ell(\gamma)) = \prod_{k=2}^{n+1} (1 + 2q^k\ell(\gamma)) \\ &\leq \prod_{k=0}^{\infty} (1 + 2q^k\ell(\gamma)) = \exp\left(\sum_{k=0}^{\infty} \log(1 + 2q^k\ell(\gamma))\right) \\ &\leq \exp\left(\sum_{k=0}^{\infty} 2q^k\ell(\gamma)\right) = e^{\alpha\ell(\gamma)}, \end{aligned}$$

where  $\alpha := 2(1 - q)^{-1}$ . Finally, this gives the assertion since  $e^x \leq 1 + xe^x$  for  $x \geq 0$ . □

**Theorem 4.3.** *Let  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^{\mathbb{N}}$ . Then  $\mathcal{A}_{(c_n)}(\infty)$  is a John domain.*

*Proof.* We first introduce a few notations. For  $z_1, z_2 \in \mathbb{C}$  let  $[z_1, z_2]$  denote the line segment joining  $z_1$  and  $z_2$ . If  $\zeta \in \mathbb{C}$ ,  $\zeta \neq 0$ , and if  $\Gamma$  is the ray from 0 to  $\infty$  passing through  $\zeta$ , then let  $\Gamma_\zeta$  denote that part of  $\Gamma$  from  $\zeta$  to  $\infty$ .

Let  $R > R_\delta$  such that  $R^2 + \delta - R \leq \frac{1}{2}$ ,  $\varepsilon := R - R_\delta \leq 1$  and  $U_k := F_k^{-1}(\Delta_R)$  for  $k \in \mathbb{N}$ . Then we have  $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(\infty)$  and  $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{k=1}^{\infty} U_k$ . Furthermore,  $U_k$  is a simply connected domain (in  $\widehat{\mathbb{C}}$ ) bounded by an analytic Jordan curve. For  $z \in \mathcal{A}_{(c_n)}(\infty)$  let  $d(z) := \text{dist}(z, \mathcal{J}_{(c_n)})$ . We prove a lower estimate for  $d(z)$ , if  $z \in U_k$  for some  $k \in \mathbb{N}$ . We set  $w := F_k(z)$ . If  $U$  denotes the component of  $F_k^{-1}(D_\varepsilon(w))$  containing  $z$ , there holds  $U \subset \mathcal{A}_{(c_n)}(\infty)$ . Let  $\varrho > 0$  such that  $D_\varrho(z) \subset U$ . If  $z' \in D_\varrho(z)$  and  $w' := F_k(z')$ , then

$$\begin{aligned} w' - w &= F_k(z') - F_k(z) = \int_z^{z'} F'_k(\zeta) d\zeta = F'_k(F_k^{-1}(w)) \int_z^{z'} \frac{F'_k(\zeta)}{F'_k(F_k^{-1}(w))} d\zeta \\ &= F'_k(z) \int_z^{z'} \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))} d\zeta, \end{aligned}$$

where we integrate over the line segment  $[z, z']$ . By Lemma 4.2 we obtain

$$\left| \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))} \right| \leq 1 + \alpha e^{\alpha\varepsilon} |w - F_k(\zeta)| \leq 1 + \alpha e^{\alpha\varepsilon} \varepsilon \leq 1 + \alpha e^\alpha$$

and thus

$$|w' - w| \leq |F'_k(z)| |z' - z| (1 + \alpha e^\alpha) \leq |F'_k(z)| \varrho (1 + \alpha e^\alpha).$$

Setting

$$\varrho := \frac{\varepsilon}{|F'_k(z)|(1 + \alpha e^\alpha)}$$

we obtain  $D_\varrho(z) \subset U$  and thus

$$(4.2) \quad d(z) \geq \frac{\varepsilon}{|F'_k(z)|(1 + \alpha e^\alpha)} = \frac{\alpha_1}{|F'_k(z)|} \quad (z \in U_k).$$

In order to prove the John property, let  $w_0 := \infty$  and  $z_0 \in \mathcal{A}_{(c_n)}(\infty)$ . We may assume that  $z_0 \in U_k \setminus U_{k-1}$  for some  $k \in \mathbb{N}$ . Then  $R < |F_k(z_0)| \leq R^2 + \delta$ . We construct an arc in  $U_k$  joining  $z_0$  and  $w_0$  as follows. First, we join  $z_0$  with  $\partial U_{k-1}$  by an arc  $\gamma_k \subset U_k \setminus U_{k-1}$  such that  $F_k(\gamma_k) \subset \Gamma_{F_k(z_0)}$ , and we denote the endpoint of  $\gamma_k$  on  $\partial U_{k-1}$  by  $\zeta_{k-1}$ . Then we join  $\zeta_{k-1}$  with  $\partial U_{k-2}$  by an arc  $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$  such that  $F_{k-1}(\gamma_{k-1}) \subset \Gamma_{F_{k-1}(\zeta_{k-1})}$ , and we denote the endpoint of  $\gamma_{k-1}$  on  $\partial U_{k-2}$  by  $\zeta_{k-2}$ . Proceeding in this way we get an arc in  $U_k \cap \overline{D}_R$  with endpoint  $\zeta_0$  on  $\partial D_R$ . Finally, we set  $\gamma = \gamma(z_0) := \gamma_k \cup \dots \cup \gamma_1 \cup \Gamma_{\zeta_0}$ . We note that the line segments  $F_j(\gamma_j)$  ( $j = 1, \dots, k$ ) all lie in  $\overline{\Delta}_R \cap \overline{D}_{R^2+\delta}$  and thus have lengths at most  $\frac{1}{2}$ .

We now show that the arc  $\gamma$  has the John property. For that purpose, let  $z \in \gamma$ . We may assume that  $z \in D_R$ . First, let  $z \in U_k \setminus U_{k-1}$ . We deduce an upper estimate for  $|z - z_0|$ . There holds

$$\begin{aligned} z - z_0 &= F_k^{-1}(F_k(z)) - F_k^{-1}(F_k(z_0)) = \int_{F_k(z_0)}^{F_k(z)} (F_k^{-1})'(\zeta) d\zeta \\ &= (F_k^{-1})'(F_k(z)) \int_{F_k(z_0)}^{F_k(z)} \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))} d\zeta, \end{aligned}$$

where we integrate over the line segment  $[F_k(z_0), F_k(z)]$ . By Lemma 4.2 we obtain

$$\left| \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))} \right| \leq 1 + \alpha e^\alpha |F_k(z) - \zeta| \leq 1 + \alpha e^\alpha |F_k(z) - F_k(z_0)| \leq 1 + \alpha e^\alpha$$

and thus

$$(4.3) \quad |z - z_0| \leq |(F_k^{-1})'(F_k(z))|(1 + \alpha e^\alpha) |F_k(z) - F_k(z_0)|$$

$$\leq \frac{1 + \alpha e^\alpha}{|F'_k(z)|} = \frac{\alpha_2}{|F'_k(z)|} \quad (z \in \gamma \setminus U_{k-1}).$$

Putting (4.2) and (4.3) together we arrive at

$$d(z) \geq \frac{\alpha_1}{\alpha_2} |z - z_0| = \alpha_3 |z - z_0| \quad (z \in \gamma \setminus U_{k-1}).$$

Now, let  $z \in U_{k-m} \setminus U_{k-m-1}$  for some  $m \in \{1, \dots, k-1\}$ . By (4.2) we have

$$d(z) \geq \frac{\alpha_1}{|F'_{k-m}(z)|}.$$

From the construction of  $\gamma$  and (4.3) we obtain

$$\begin{aligned} |z - z_0| &\leq |z_0 - \zeta_{k-1}| + |\zeta_{k-1} - \zeta_{k-2}| + \dots + |\zeta_{k-m+1} - \zeta_{k-m}| + |\zeta_{k-m} - z| \\ &\leq \alpha_2 \left( \frac{1}{|F'_k(\zeta_{k-1})|} + \frac{1}{|F'_{k-1}(\zeta_{k-2})|} + \dots \right. \\ &\quad \left. + \frac{1}{|F'_{k-m+1}(\zeta_{k-m})|} + \frac{1}{|F'_{k-m}(z)|} \right) \end{aligned}$$

and thus

$$(4.4) \quad \frac{d(z)}{|z - z_0|} \geq \frac{\alpha_3}{1 + \sum_{j=1}^m \left| \frac{F'_{k-m}(z)}{F'_{k-m+j}(\zeta_{k-m+j-1})} \right|}.$$

In order to estimate the denominator of the right hand side we consider a single term

$$\begin{aligned} \frac{F'_{k-m}(z)}{F'_{k-m+j}(\zeta_{k-m+j-1})} &= \frac{1}{2^j F_{k-m+j-1}(\zeta_{k-m+j-1}) \cdots F_{k-m}(\zeta_{k-m+j-1})} \\ &\quad \times \frac{F'_{k-m}(z)}{F'_{k-m}(\zeta_{k-m+j-1})}. \end{aligned}$$

Because of  $|F_{k-m+j-1}(\zeta_{k-m+j-1})| = R$  and the invariance of  $D_r$  we obtain

$$(4.5) \quad \left| \frac{F'_{k-m}(z)}{F'_{k-m+j}(\zeta_{k-m+j-1})} \right| \leq q^j \left| \frac{F'_{k-m}(z)}{F'_{k-m}(\zeta_{k-m+j-1})} \right|,$$

where  $q := \frac{1}{2r} < 1$ .

Now, we deduce an estimate of the right hand side of (4.5). For abbreviation we set  $p := k - m$  and write

$$\frac{F'_p(z)}{F'_p(\zeta_{p+j-1})} = \frac{(F_p^{-1})'(F_p(\zeta_{p+j-1}))}{(F_p^{-1})'(F_p(z))}.$$

From Lemma 4.2 we get

$$(4.6) \quad \left| \frac{F'_p(z)}{F'_p(\zeta_{p+j-1})} \right| \leq 1 + \alpha \ell(\sigma) e^{\alpha \ell(\sigma)},$$

where  $\sigma = \sigma_{p,j}$  is the curve  $F_p(\gamma'_p \cup \gamma_{p+1} \cup \dots \cup \gamma_{p+j-1})$ , and where  $\gamma'_p$  is that part of  $\gamma_p$  joining  $\zeta_p$  with  $z$ . Hence, there holds  $\ell(\sigma) \leq \ell(F_p(\gamma_p)) + \dots + \ell(F_p(\gamma_{p+j-1}))$ . We have  $\ell(F_p(\gamma_p)) \leq \frac{1}{2}$  and  $F_p(\gamma_{p+\nu}) = F_{p+\nu,p}^{-1}(s_{p,\nu})$ , where  $s_{p,\nu} := F_{p+\nu}(\gamma_{p+\nu})$  is a line segment on  $\Gamma_{F_{p+\nu}(\zeta_{p+\nu})}$  of length at most  $\frac{1}{2}$  for  $\nu \geq 1$ . Furthermore, we know that  $F_p(\gamma_{p+\nu}) \subset \Delta_r$ . Therefore, we obtain

$$\ell(F_p(\gamma_{p+1})) = \int_{s_{p,1}} \frac{|dw|}{2\sqrt{|w - c_{p+1}|}} \leq \frac{\ell(s_{p,1})}{2r} \leq \frac{1}{4r}.$$

By induction we get  $\ell(F_p(\gamma_{p+\nu})) \leq \frac{1}{2(2r)^\nu} = \frac{1}{2}q^\nu$  and thus

$$\ell(\sigma) \leq \frac{1}{2}(1 + q + \dots + q^{j-1}) \leq \frac{1}{2(1 - q)} = \alpha_4.$$

Setting  $\alpha_5 := 1 + \alpha \alpha_4 e^{\alpha \alpha_4}$  we obtain together with (4.4), (4.5) and (4.6)

$$\frac{d(z)}{|z - z_0|} \geq \frac{\alpha_3}{1 + \alpha_5 \sum_{j=1}^m q^j} \geq \frac{\alpha_3(1 - q)}{\alpha_5}$$

which finally shows that  $\gamma$  has the John property. □

**Theorem 4.4.** *Let  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then  $\mathcal{A}_{(c_n)}(0)$  is a John domain.*

*Proof.* The proof is very similar to the proof of Theorem 4.3. The only difficulty that arises is that  $\mathcal{A}_{(c_n)}(0)$  contains critical values which all lie in  $D_{s_\delta}$ . Therefore, we only give a sketch and omit the details.

Let  $\frac{1}{2} < r < r' < r_\delta$ ,  $\varepsilon := r' - r \leq 1$  and  $U_k := F_k^{-1}(D_{r'})$  for  $k \in \mathbb{N}$ . Then we have  $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(0)$  and  $\mathcal{A}_{(c_n)}(0) = \bigcup_{k=1}^\infty U_k$ . For  $z \in \mathcal{A}_{(c_n)}(0)$  let  $d(z) := \text{dist}(z, \mathcal{J}_{(c_n)})$ . If  $z \in U_k \setminus U_{k-1}$  for some  $k \in \mathbb{N}$ ,  $k \geq 2$  and  $w := F_{k-1}(z)$ , then  $|w| \geq r'$  and thus  $D_\varepsilon(w) \cap D_r = \emptyset$ . Therefore, we obtain

$$(4.2a) \quad d(z) \geq \frac{\alpha_1}{|F'_{k-1}(z)|} \quad (z \in U_k \setminus U_{k-1}).$$

In order to prove the John property, let  $w_0 := 0$  and  $z_0 \in \mathcal{A}_{(c_n)}(0)$ . We may assume that  $z_0 \in U_k \setminus U_{k-1}$  for some  $k \in \mathbb{N}$ ,  $k \geq 2$ . Then  $|F_{k-1}(z_0)| \geq r'$ . We construct an arc in  $U_k$  joining  $z_0$  and  $w_0$  as follows. First, we join  $z_0$  with  $\partial U_{k-1}$  by an arc  $\gamma_k \subset U_k \setminus U_{k-1}$  such that  $F_{k-1}(\gamma_k) \subset [0, F_{k-1}(z_0)]$ , and we denote the endpoint of  $\gamma_k$  on  $\partial U_{k-1}$  by  $\zeta_{k-1}$ . Then we join  $\zeta_{k-1}$  with  $\partial U_{k-2}$  by an arc  $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$  such that  $F_{k-2}(\gamma_{k-1}) \subset [0, F_{k-2}(\zeta_{k-1})]$ , and we denote the endpoint of  $\gamma_{k-1}$  on  $\partial U_{k-2}$  by  $\zeta_{k-2}$ . Proceeding in this way we get an arc in  $U_k \cap (\mathbb{C} \setminus U_1)$  with endpoint  $\zeta_1$  on  $\partial U_1$ . Finally, we set  $\gamma = \gamma(z_0) := \gamma_k \cup \dots \cup \gamma_2 \cup [0, \zeta_1]$ . We mention that  $[0, \zeta_1] \subset \bar{U}_1$ , since  $U_1$

is a starlike domain with respect to 0 bounded by an analytic Jordan curve. Furthermore, we note that the line segments  $F_{j-1}(\gamma_j)$  ( $j = 2, \dots, k$ ) all lie in  $\overline{\Delta}_{r'} \cap \overline{D}_{R_\delta}$  and thus have lengths at most one.

We now show that the arc  $\gamma$  has the John property. For that purpose, let  $z \in \gamma$ . We may assume that  $z \notin U_1$ . First, let  $z \in U_k \setminus U_{k-1}$ . Then we obtain the upper estimate for  $|z - z_0|$

$$(4.3a) \quad |z - z_0| \leq \frac{\alpha_2}{|F'_{k-1}(z)|} \quad (z \in \gamma \setminus U_{k-1}).$$

Putting (4.2a) and (4.3a) together we arrive at

$$d(z) \geq \alpha_3 |z - z_0| \quad (z \in \gamma \setminus U_{k-1}).$$

Finally, the case that  $z \in U_{k-m} \setminus U_{k-m-1}$  for some  $m \in \{1, \dots, k - 2\}$  is handled as in the proof of Theorem 4.3.  $\square$

**Corollary 4.5.** *Let  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then  $\mathcal{J}_{(c_n)}$  is a quasicircle.*

*Proof.* From Theorem 4.1 we know that  $\mathcal{F}_{(c_n)} = \mathcal{A}_{(c_n)}(\infty) \cup \mathcal{A}_{(c_n)}(0)$ . Then the assertion follows from Theorems 4.3 and 4.4 and the known results mentioned at the beginning of this section.  $\square$

If  $(c_n) \in K_{1/4}^\mathbb{N}$ , then  $\mathcal{J}_{(c_n)}$  need not be a quasicircle. For example, if  $c_n = \frac{1}{4}$  for all  $n$ , then it is known that  $\mathcal{J}_{(f_{1/4})}$  is still a Jordan curve (see for example [CG, p. 97] or [St, p. 124]) but it has cusps. Furthermore, Corollary 4.5 does not hold true in general when all  $c_n$  are contained in the interior of the main cardioid of the Mandelbrot set. This can be seen by the simple example  $c_1 = -\frac{1}{2} - \eta$  and  $c_n = \frac{1}{4} - \varepsilon$  for  $n \geq 2$  with  $0 < \eta < \frac{1}{4}$  and  $0 < \varepsilon < \eta^2$ . In this case we have  $F_n(0) \rightarrow \infty$  as  $n \rightarrow \infty$  so that by Theorem 1.1 in [BBR] the Julia set  $\mathcal{J}_{(c_n)}$  is even disconnected. It would be of interest whether  $\mathcal{J}_{(c_n)}$  is also a Jordan curve in our more general setting provided that  $(c_n) \in K_{1/4}^\mathbb{N}$  or what holds when  $(c_n) \in D_{1/4}^\mathbb{N}$ .

Furthermore, we consider the dynamics of  $(F_n)$  in the stable domain  $\mathcal{A}_{(c_n)}(0)$  provided that  $(c_n) \in K_{1/4}^\mathbb{N}$ . We will show that  $\mathcal{A}_{(c_n)}(0)$  is a *contracting domain*, that is a stable domain  $U$  such that all limit functions of  $(F_n)$  in  $U$  are constant. This property is equivalent to  $\text{diam } F_n(K) \rightarrow 0$  as  $n \rightarrow \infty$  for every compact set  $K \subset U$ .

**Theorem 4.6.** *Let  $(c_n) \in K_{1/4}^\mathbb{N}$ . Then  $\mathcal{A}_{(c_n)}(0)$  is a contracting domain.*

*Proof.* Let  $K \subset \mathcal{A}_{(c_n)}(0)$  be a compact set. We first assume that  $(c_n) \in K_\delta^\mathbb{N}$  for some  $\delta < \frac{1}{4}$ , and we choose  $r \in (s_\delta, \frac{1}{2})$ . Then by Theorem 4.1 there exists  $N \in \mathbb{N}$  such that  $F_N(K) \subset D_r$ . If  $z_1, z_2 \in K$ , then  $w_1 := F_N(z_1)$ ,  $w_2 := F_N(z_2) \in D_r$  and thus  $|f_{c_k}(w_1) - f_{c_k}(w_2)| = |w_1 + w_2||w_1 - w_2| \leq 2r|w_1 - w_2|$  which implies  $|F_{N+k}(z_1) - F_{N+k}(z_2)| \leq (2r)^k|w_1 - w_2|$ . Therefore,

we obtain  $\text{diam } F_{N+k}(K) \leq (2r)^k \text{diam } F_N(K) \rightarrow 0$  as  $k \rightarrow \infty$ , and the assertion follows.

Now, let  $|c_n| \leq \frac{1}{4}$  for all  $n \in \mathbb{N}$ . Again, by Theorem 4.1 there exists  $N \in \mathbb{N}$  such that  $F_N(K) \subset K_{1/2}$ , and we obtain as above  $\text{diam } F_{N+k}(K) \leq \text{diam } F_{N+k-1}(K)$  so that the sequence  $(\text{diam } F_{N+k}(K))$  is monotonically decreasing and thus convergent. In order to deduce  $\text{diam } F_{N+k}(K) \rightarrow 0$  as  $k \rightarrow \infty$  we need a better estimate. If  $w_1, w_2 \in F_N(K)$ , we obtain

$$|f_{c_k}(w_1) - f_{c_k}(w_2)| \leq 2 \left| \int_{w_1}^{w_2} |z| |dz| \right|.$$

For the estimate of the right hand side we consider the worst case which can happen, that is  $|w_1| = |w_2| = \frac{1}{2}$ . For simplicity, we may assume that  $w_2 = \bar{w}_1$ , and we set  $\varrho := \text{Re } w_1 = \text{Re } w_2 \in [0, \frac{1}{2})$ . Then with  $d := \frac{1}{2}|w_1 - w_2|$  we get  $\varrho^2 + d^2 = \frac{1}{4}$  and thus

$$\begin{aligned} 2 \left| \int_{w_1}^{w_2} |z| |dz| \right| &\leq 4 \int_0^d |\varrho + it| dt = 4 \int_0^d \sqrt{\varrho^2 + t^2} dt \\ &= d + 2\varrho^2 \log \frac{2d + 1}{2\varrho} \\ &= \frac{1}{2}|w_1 - w_2| + \frac{1}{4}(1 - |w_1 - w_2|^2) \log \frac{1 + |w_1 - w_2|}{1 - |w_1 - w_2|}. \end{aligned}$$

This implies with  $d_n := \text{diam } F_n(K)$

$$d_{N+k} \leq \frac{1}{2}d_{N+k-1} + \frac{1}{4}(1 - d_{N+k-1}^2) \log \frac{1 + d_{N+k-1}}{1 - d_{N+k-1}}.$$

Setting  $\alpha := \lim_{k \rightarrow \infty} \text{diam } F_{N+k}(K)$  we see that

$$\alpha \leq \frac{1}{2}\alpha + \frac{1}{4}(1 - \alpha^2) \log \frac{1 + \alpha}{1 - \alpha},$$

and an elementary argument shows that this is possible only for  $\alpha = 0$  which gives the assertion. □

If  $(c_n) \in K_\delta^\mathbb{N}$  for some  $\delta \leq \frac{1}{4}$ , we denote by  $L_{(c_n)}$  the set of (constant) limit functions of  $(F_n)$  in  $\mathcal{A}_{(c_n)}(0)$ , that is the set of all  $\zeta \in \mathbb{C}$  such that for some subsequence  $(F_{n_k})$  of  $(F_n)$  there holds  $F_{n_k} \rightarrow \zeta$  as  $k \rightarrow \infty$  locally uniformly in  $\mathcal{A}_{(c_n)}(0)$ . It is easy to see that  $L_{(c_n)}$  is a compact set, and from the proof of Theorem 4.6 it follows that  $L_{(c_n)} \subset K_{s_\delta} \subset K_{1/2}$ . From Theorem 1.6 in [BBR] we know that the case  $L_{(c_n)} = K_{s_\delta}$  may occur. Moreover, this phenomenon happens almost surely, that means that the product measure (cf. Section 5) of the set of these sequences  $(c_n)$  in  $K_\delta^\mathbb{N}$  is one. In a similar way it is possible to construct sequences  $(c_n) \in K_\delta^\mathbb{N}$  such that  $L_{(c_n)} = \partial K_{s_\delta}$ .

On the other hand, if  $L_{(c_n)}$  consists of a single point  $\zeta$ , then  $F_n \rightarrow \zeta$  as  $n \rightarrow \infty$  locally uniformly in  $\mathcal{A}_{(c_n)}(0)$ , and since  $F_{n+1}(z) = (F_n(z))^2 + c_n$  we obtain  $c_n \rightarrow c \in K_\delta$  as  $n \rightarrow \infty$ , where  $c = \zeta - \zeta^2$ . Therefore, the set  $C_\delta$  of all these points  $\zeta$  is the component of the preimage of  $K_\delta$  under the map  $z \mapsto z - z^2$  which is contained in  $K_{s_\delta}$ . Therefore,  $C_\delta$  is a proper subset of  $K_{s_\delta}$  and  $C_\delta \cap \partial K_{s_\delta} = \{s_\delta\}$ . It would be of interest to characterize those compact sets  $K \subset K_{s_\delta}$  such that  $K = L_{(c_n)}$  for some sequence  $(c_n) \in K_\delta^\mathbb{N}$ .

The stable domain  $\mathcal{A}_{(c_n)}(\infty)$  may be viewed as a Böttcher domain. If it is simply connected, then there exists a conformal map  $\phi$  of  $\mathcal{A}_{(c_n)}(\infty)$  onto  $\Delta_1$  normalized at infinity by

$$(4.7) \quad \phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

Note that the capacity of  $\mathcal{K}_{(c_n)}$  (cf. Section 8) is equal to one. Like in the iteration of a fixed polynomial we show that  $\phi$  may be described dynamically.

**Theorem 4.7.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^\mathbb{N}$  such that  $\mathcal{A}_{(c_n)}(\infty)$  is simply connected. Then the conformal map  $\phi$  of  $\mathcal{A}_{(c_n)}(\infty)$  onto  $\Delta_1$  with the normalization (4.7) is given by*

$$\phi(z) = \lim_{k \rightarrow \infty} \sqrt[2^k]{F_k(z)} = z \lim_{k \rightarrow \infty} \sqrt[2^k]{\frac{F_k(z)}{z^{2^k}}}$$

with locally uniform convergence in  $\mathcal{A}_{(c_n)}(\infty)$ , and where the branch of the root is determined by  $\sqrt[2^k]{1} = 1$ .

*Proof.* Let  $R > R_\delta$  such that  $R^2 \geq 2\delta$  and  $U_m := F_m^{-1}(\Delta_R)$  for  $m \in \mathbb{N}$ . Then we have  $U_m \subset U_{m+1} \subset \mathcal{A}_{(c_n)}(\infty)$  and  $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{m=1}^\infty U_m$ . For  $k \in \mathbb{N}$  we define

$$\phi_k(z) := \sqrt[2^k]{F_k(z)} = z \sqrt[2^k]{\frac{F_k(z)}{z^{2^k}}}$$

Then  $\phi_k$  maps  $U_k$  conformally onto  $\Delta_{R_k}$ , where  $R_k := \sqrt[2^k]{R}$ . For  $z \in U_m$  and  $k \geq m$  we have

$$\left| \frac{c_k}{(F_k(z))^2} \right| \leq \frac{\delta}{R^2} \leq \frac{1}{2},$$

and the elementary inequality

$$|\sqrt[p]{1+u} - 1| \leq \frac{1}{p} \quad (u \in K_{1/2})$$

yields

$$\left| \frac{\phi_{k+1}(z)}{\phi_k(z)} - 1 \right| = \left| \sqrt[2^{k+1}]{\frac{F_{k+1}(z)}{(F_k(z))^2}} - 1 \right| = \left| \sqrt[2^{k+1}]{1 + \frac{c_k}{(F_k(z))^2}} - 1 \right| \leq \frac{1}{2^{k+1}}.$$

Therefore, the limit

$$\phi(z) := \lim_{k \rightarrow \infty} \phi_k(z) = z \prod_{k=0}^{\infty} \frac{\phi_{k+1}(z)}{\phi_k(z)}$$

exists uniformly in  $U_m$ , and  $\phi$  is the desired conformal map. □

### 5. Lebesgue measure of Julia sets.

From a result of Lyubich [L2] (see also [CG, p. 90] or [St, p. 144]) it follows that the Julia set of a hyperbolic rational function has two-dimensional Lebesgue measure (which we denote by  $m_2$ ) zero. In particular, this is true for  $\mathcal{J}(f_c)$  provided that  $c$  is contained in a hyperbolic component of the interior of the Mandelbrot set  $\mathcal{M}$  or  $c \notin \mathcal{M}$ . In this section we show that this is true to a certain extent in our situation.

We begin with  $\delta < \frac{1}{4}$ . Then by Section 4 we know that if  $(c_n) \in K_\delta^{\mathbb{N}}$ , then  $\mathcal{J}_{(c_n)}$  is a quasicircle, and from the differentiability properties of quasiconformal maps it follows that quasicircles always have two-dimensional Lebesgue measure zero (see for example [LV, p. 165]).

**Corollary 5.1.** *Let  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^{\mathbb{N}}$ . Then  $m_2(\mathcal{J}_{(c_n)}) = 0$ .*

Now, we will show that  $m_2(\mathcal{J}_{(c_n)})$  is almost surely zero provided that the  $c_n$  are randomly chosen in  $K_\delta$  for some  $\delta > \frac{1}{4}$ . To be more precise, let  $\lambda_\delta$  denote the two-dimensional Lebesgue measure on  $K_\delta$  normalized by  $\lambda_\delta(K_\delta) = 1$ . Then the product space  $K_\delta^{\mathbb{N}}$  carries the usual product measure  $\tilde{\lambda}_\delta := \bigotimes_{k=1}^{\infty} \lambda_\delta$ . We set

$$(5.1) \quad \mathfrak{N}_\delta := \{ (c_n) \in K_\delta^{\mathbb{N}} : m_2(\mathcal{J}_{(c_n)}) = 0 \}.$$

Then the goal is to show that  $\tilde{\lambda}_\delta(\mathfrak{N}_\delta) = 1$ . In order to do this we recall:

**Theorem 5.2** ([BBR]). *Let  $\delta > \frac{1}{4}$  and  $R > 0$ . Then for every  $z \in \widehat{\mathbb{C}}$  there exists an open set  $\mathfrak{U}_z \subset K_\delta^{\mathbb{N}}$  with the following properties:*

- (a)  $\tilde{\lambda}_\delta(\mathfrak{U}_z) = 1$ ,
- (b) for every  $(c_n) \in \mathfrak{U}_z$  there holds  $|F_k(z)| > R$  for all sufficiently large  $k$ .

**Theorem 5.3.** *Let  $\delta > \frac{1}{4}$ , and let  $\mathfrak{N}_\delta \subset K_\delta^{\mathbb{N}}$  be defined by (5.1). Then  $\tilde{\lambda}_\delta(\mathfrak{N}_\delta) = 1$ .*

*Proof.* Let  $M = \Delta_R$  be an invariant domain and

$$\tilde{\mathfrak{E}} := \{ ((c_n), z) \in K_\delta^{\mathbb{N}} \times \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \}.$$

By Theorem 5.2 we have  $\tilde{\lambda}_\delta(\tilde{\mathfrak{E}}_z) = 1$  for  $z \in \widehat{\mathbb{C}}$ , where

$$\tilde{\mathfrak{E}}_z := \{ (c_n) \in K_\delta^{\mathbb{N}} : ((c_n), z) \in \tilde{\mathfrak{E}} \}.$$

If  $\sigma$  denotes the two-dimensional Lebesgue measure on  $\widehat{\mathbb{C}}$  normalized by  $\sigma(\widehat{\mathbb{C}}) = 1$ , it follows

$$(\tilde{\lambda}_\delta \otimes \sigma)(\tilde{\mathfrak{E}}) = \int_{\widehat{\mathbb{C}}} \tilde{\lambda}_\delta(\tilde{\mathfrak{E}}_z) d\sigma(z) = 1.$$

Now, let

$$\mathfrak{E} := \{ (c_n) \in K_\delta^\mathbb{N} : \sigma(\tilde{\mathfrak{E}}_{(c_n)}) = 1 \},$$

where

$$\tilde{\mathfrak{E}}_{(c_n)} := \{ z \in \widehat{\mathbb{C}} : ((c_n), z) \in \tilde{\mathfrak{E}} \}.$$

Since

$$1 = (\tilde{\lambda}_\delta \otimes \sigma)(\tilde{\mathfrak{E}}) = \int_{K_\delta^\mathbb{N}} \sigma(\tilde{\mathfrak{E}}_{(c_n)}) d\tilde{\lambda}_\delta((c_n))$$

we obtain  $\tilde{\lambda}_\delta(\mathfrak{E}) = 1$ . If  $(c_n) \in \mathfrak{E}$ , then

$$\sigma(\mathcal{A}_{(c_n)}(\infty)) = \sigma(\{ z \in \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \}) = \sigma(\tilde{\mathfrak{E}}_{(c_n)}) = 1$$

which implies  $\sigma(\mathcal{J}_{(c_n)}) = 0$ . □

It would be of interest whether Theorem 5.3 remains valid for  $\delta = \frac{1}{4}$ . Concerning the question whether there exists a sequence  $(c_n) \in K_\delta^\mathbb{N}$  for some  $\delta \geq \frac{1}{4}$  such that  $m_2(\mathcal{J}_{(c_n)}) > 0$ , the referee mentioned that, recently, a group of mathematicians around P.W. Jones at Yale University have constructed such an example. More precisely, there exists a sequence  $(c_n)$  with  $c_n \in \{0, \pm\frac{1}{4}, \frac{1}{2}\}$  such that  $m_2(\mathcal{J}_{(c_n)}) > 0$ . This result was communicated to the author by P.W. Jones. The author is grateful to both for bringing this information to his attention.

Finally, we prove:

**Theorem 5.4.** *Let  $\delta > 2$  (which is equivalent to  $\delta > R_\delta$ ), and let  $\varepsilon > 0$  such that  $R_\delta + \varepsilon \leq |c_n| \leq \delta$  for all  $n \in \mathbb{N}$ . Then  $m_2(\mathcal{J}_{(c_n)}) = 0$ .*

*Proof.* We choose  $R$  such that  $R_\delta < R < R_\delta + \varepsilon$  and  $\eta := R_\delta + \varepsilon - R > 0$ . Then we have  $\mathcal{J}_{(c_n)} \subset \overline{D}_{R_\delta} \subset D := D_R$ , and  $D$  is backward invariant, that is  $f_{c_n}^{-1}(D) \subset D$  for all  $n \in \mathbb{N}$ . Furthermore, there holds  $|f_{c_n}(0)| = |c_n| \geq R_\delta + \varepsilon = R + \eta$  and thus  $|(f_{c_m} \circ \dots \circ f_{c_{k+1}})(0)| \geq R + \eta$  for  $k = 0, 1, \dots, m - 1$  and all  $m \in \mathbb{N}$ . Therefore,  $D$  does not contain any critical value of  $(F_n)$  so that in  $D$  there exist  $2^k$  analytic branches of the inverse function of  $F_k$  which we denote by  $G_{j,k}$  for  $j = 1, \dots, 2^k$  and  $k \in \mathbb{N}$ . Furthermore, we set  $D_{j,k} := G_{j,k}(D) \subset D$  and  $D_k := \bigcup_{j=1}^{2^k} D_{j,k}$ . Then  $D_{1,k}, \dots, D_{2^k,k}$  are mutually disjoint simply connected domains, and  $\mathcal{J}_{(c_n)} \subset D_{k+1} \subset D_k$ . Finally, we set  $U_k := D_k \setminus \overline{D}_{k+1}$  and  $U_{j,k} := D_{j,k} \setminus \overline{D}_{k+1}$  so that  $U_{1,k}, \dots, U_{2^k,k}$  are mutually disjoint multiply connected domains, and  $U_k = \bigcup_{j=1}^{2^k} U_{j,k}$ .

Now, we prove that there exists a constant  $q > 0$  such that

$$(5.2) \quad \frac{m_2(U_k)}{m_2(D_k)} \geq q$$

for all  $k \in \mathbb{N}$ . For that purpose it is enough to show that

$$(5.3) \quad \frac{m_2(U_{j,k})}{m_2(D_{j,k})} \geq q$$

for  $j = 1, \dots, 2^k$  and all  $k \in \mathbb{N}$ .

Let  $V_{1,k}$  and  $V_{2,k}$  denote the two components of  $f_{c_{k+1}}^{-1}(D)$ , and let  $W_k := D \setminus (\overline{V}_{1,k} \cup \overline{V}_{2,k})$ . Then  $U_{j,k} = G_{j,k}(W_k)$ , and we obtain

$$\frac{m_2(U_{j,k})}{m_2(D_{j,k})} = \frac{\int_{W_k} |G'_{j,k}(z)|^2 dm_2(z)}{\int_D |G'_{j,k}(z)|^2 dm_2(z)} \geq \frac{|G'_{j,k}(z_{j,k})|^2 m_2(W_k)}{|G'_{j,k}(\zeta_{j,k})|^2 m_2(D)},$$

where  $z_{j,k} \in W_k \subset D$  and  $\zeta_{j,k} \in D$  such that  $|G'_{j,k}(z_{j,k})| = \min_{z \in W_k} |G'_{j,k}(z)|$  and  $|G'_{j,k}(\zeta_{j,k})| = \max_{z \in D} |G'_{j,k}(z)|$ . By the Koebe distortion theorem (see for example [P2, p. 9]) applied to the disk  $D_{R+\eta}$  there holds

$$\left| \frac{G'_{j,k}(z)}{G'_{j,k}(\zeta)} \right| \geq \left( \frac{\eta}{\eta + 2R} \right)^4$$

for all  $z, \zeta \in D$ . Therefore, it remains to show that there exists a constant  $\gamma > 0$  such that

$$\frac{m_2(W_k)}{m_2(D)} \geq \gamma$$

for all  $k \in \mathbb{N}$ .

For simplicity we write  $c = c_{k+1}$ , and let  $V \in \{V_{1,k}, V_{2,k}\}$ . Then

$$m_2(V) = \frac{1}{4} \int_D \frac{dm_2(z)}{|z - c|} = \frac{1}{4} \int_0^R \int_0^{2\pi} \frac{\varrho}{|\varrho e^{it} - c|} dt d\varrho.$$

By the Cauchy-Schwarz inequality we get

$$\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|} \leq \sqrt{2\pi} \left( \int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|^2} \right)^{1/2},$$

and the Poisson integral formula yields

$$\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|^2} = \frac{2\pi}{|c|^2 - \varrho^2}.$$

Therefore, we arrive at

$$m_2(V) \leq \frac{\pi}{2} \int_0^R \frac{\varrho}{\sqrt{|c|^2 - \varrho^2}} d\varrho = \frac{\pi}{2} (|c| - \sqrt{|c|^2 - R^2}) \leq \frac{1}{2} \pi R.$$

This implies  $m_2(V_{1,k} \cup V_{2,k}) \leq \pi R$  and thus

$$\frac{m_2(W_k)}{m_2(D)} \geq 1 - \frac{1}{R} \geq \frac{1}{2}$$

which proves (5.3).

Finally, (5.2) gives  $m_2(\mathcal{J}_{(c_n)}) \leq m_2(D_{k+1}) = m_2(D_k) - m_2(U_k) \leq (1 - q)m_2(D_k)$  so that  $m_2(\mathcal{J}_{(c_n)}) \leq (1 - q)^k m_2(D) \rightarrow 0$  as  $k \rightarrow \infty$  which completes the proof.  $\square$

### 6. Hausdorff dimension of Julia sets.

We first recall the notion of Hausdorff dimension. Let  $E \subset \mathbb{C}$  be a non-empty compact set, and denote by  $(D_j)_\varepsilon$  any covering of  $E$  by finitely many open sets  $D_j$  with  $\text{diam } D_j < \varepsilon$ . Then for  $t \in (0, 2]$

$$m_t(E) := \sup_{\varepsilon > 0} \inf_{(D_j)_\varepsilon} \sum_j (\text{diam } D_j)^t$$

is called the *t-dimensional Hausdorff measure* of  $E$ . Obviously,  $m_t(E) < \infty$  implies  $m_s(E) = 0$  for  $s > t$ , and conversely,  $m_t(E) > 0$  implies  $m_s(E) = \infty$  for  $s < t$ . Hence, there exists a unique  $\tau \in [0, 2]$  such that  $m_s(E) = 0$  and  $m_t(E) = \infty$  for  $0 < t < \tau < s \leq 2$ . This number  $\tau$  is called the *Hausdorff dimension* of  $E$  and is denoted by  $\dim_{\mathbb{H}} E$ .

It is well-known (cf. [G], see also [Be, p. 251] or [St, p. 169]) that the Hausdorff dimension of the Julia set of any fixed rational function  $f$  is positive. More precisely, if  $\infty \notin \mathcal{J}(f)$  and if  $d$  denotes the degree of  $f$ , then

$$\dim_{\mathbb{H}} \mathcal{J}(f) \geq \frac{\log d}{\log \max_{z \in \mathcal{J}(f)} |f'(z)|}.$$

We show that this estimate holds true in a certain sense in our situation.

**Theorem 6.1.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^{\mathbb{N}}$ . Then  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)} > 0$ . More precisely, there holds*

$$\dim_{\mathbb{H}} \mathcal{J}_{(c_n)} \geq \frac{\log 2}{\log (2R_\delta)} = \frac{\log 2}{\log (1 + \sqrt{1 + 4\delta})}.$$

*Proof.* We show that the Green function  $g$  of  $\mathcal{A}_{(c_n)}(\infty)$  is Hölder continuous with exponent

$$\alpha = \frac{\log 2}{\log 2 + \log (2R - R_\delta)}$$

for any  $R > R_\delta$ . Then a result of Carleson [C] gives  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)} \geq \alpha$ . For that purpose, it suffices to show that there exists a constant  $\gamma > 0$  such that  $g(z) \leq \gamma(d(z))^\alpha$  for all  $z \in \mathcal{A}_{(c_n)}(\infty)$ , where  $d(z) := \text{dist}(z, \mathcal{J}_{(c_n)})$ . Of course, we may assume that  $d(z)$  is small.

Let  $R > R_\delta$  and  $U_k := F_k^{-1}(\Delta_R)$  for  $k \in \mathbb{N}$ . Then we have  $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(\infty)$  and  $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{k=1}^\infty U_k$ . The Green function  $g_k$  of  $U_k$  with pole at infinity is given by

$$g_k(z) = \frac{1}{2^k} \log \frac{|F_k(z)|}{R} \quad (z \in U_k).$$

There holds  $g_k(z) \leq g_{k+1}(z) \leq g(z)$  for  $z \in U_k$  and  $g_k \rightarrow g$  as  $k \rightarrow \infty$  locally uniformly in  $\mathcal{A}_{(c_n)}(\infty)$ .

We will show that there exists some constant  $C > 0$  such that  $g(z) \leq g_k(z) + \frac{C}{2^k}$  for  $z \in U_k$ . There holds  $|F_{k+1}(z)| = |(F_k(z))^2 + c_{k+1}| \leq |F_k(z)|^2 + \delta$  and this gives

$$g_{k+1}(z) \leq \frac{1}{2^{k+1}} \log \frac{|F_k(z)|^2 + \delta}{R}.$$

If  $a, b > 0$ , then  $\log^+(a + b) \leq \log^+ a + \log^+ b + \log 2$ , and thus

$$\begin{aligned} g_{k+1}(z) &\leq \frac{1}{2^{k+1}} \left( \log \frac{|F_k(z)|^2}{R} + \log^+ \frac{\delta}{R} + \log 2 \right) \\ &= \frac{1}{2^{k+1}} \left( 2 \log \frac{|F_k(z)|}{R} + \log^+ \frac{\delta}{R} + \log(2R) \right) \\ &= g_k(z) + \frac{C}{2^{k+1}}, \end{aligned}$$

where  $C := \log^+ \frac{\delta}{R} + \log(2R)$ . From this we obtain by induction

$$g_{k+m}(z) \leq g_k(z) + C \left( \frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+m}} \right) \leq g_k(z) + \frac{C}{2^k}$$

for all  $m \in \mathbb{N}$ . Letting  $m \rightarrow \infty$  we get

$$g(z) \leq g_k(z) + \frac{C}{2^k} \quad (z \in U_k).$$

Now, let  $z \in U_k \setminus U_{k-1}$  for some  $k \in \mathbb{N}$ . Then  $|F_{k-1}(z)| \leq R$  which implies  $|F_k(z)| \leq R^2 + \delta$ . Hence, we have  $g_k(z) \leq \frac{1}{2^k} \log(R + \frac{\delta}{R})$  and thus

$$(6.1) \quad g(z) \leq \frac{\Gamma}{2^k} \quad (z \in U_k \setminus U_{k-1}),$$

where  $\Gamma := C + \log(R + \frac{\delta}{R})$ .

Finally, we prove a lower estimate for  $d(z)$ , if  $z \in U_k$  for some  $k \in \mathbb{N}$ . We set  $w := F_k(z)$  and  $\eta := |w| - R_\delta$ . If  $U$  denotes the component of  $F_k^{-1}(D_\eta(w))$  containing  $z$ , there holds  $U \subset \mathcal{A}_{(c_n)}(\infty)$ . Let  $\varrho > 0$  such that  $D_\varrho(z) \subset U$ . Then  $F_k(D_\varrho(z)) \subset D_\eta(w) \subset D_{|w|+\eta}$  which implies  $F_j(D_\varrho(z)) \subset D_{|w|+\eta}$  for

$j = 0, 1, \dots, k$  and thus  $|F'_k(t)| \leq 2^k(|w| + \eta)^k$  for all  $t \in D_\varrho(z)$ . If  $z' \in D_\varrho(z)$  and  $w' := F_k(z')$ , then

$$w' - w = \int_z^{z'} F'_k(t) dt,$$

where we integrate over the line segment joining  $z$  and  $z'$ . This yields

$$|w' - w| \leq 2^k(|w| + \eta)^k |z' - z| < 2^k(|w| + \eta)^k \varrho.$$

Setting

$$\varrho := \frac{\eta}{2^k(|w| + \eta)}$$

we obtain  $D_\varrho(z) \subset U$  and thus

$$d(z) \geq \frac{\eta}{2^k(|w| + \eta)} = \frac{|w| - R_\delta}{2^k(2|w| - R_\delta)} \geq \frac{R - R_\delta}{2^k(2R - R_\delta)} \quad (z \in U_k).$$

We choose  $q := R - R_\delta$  and

$$\alpha := \frac{\log 2}{\log 2 + \log(R + q)}$$

and arrive at

$$(6.2) \quad (d(z))^\alpha \geq \frac{q^\alpha}{2^k} \quad (z \in U_k).$$

Finally, putting (6.1) and (6.2) together we get

$$g(z) \leq \frac{\Gamma}{q^\alpha} (d(z))^\alpha \quad (z \in U_k \setminus U_{k-1})$$

which completes the proof. □

Gehring and Väisälä [GV] have shown that quasicircles always have Hausdorff dimension less than two and thus by Corollary 4.5 we obtain:

**Corollary 6.2.** *Let  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^{\mathbb{N}}$ . Then  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)} < 2$ .*

If  $0 < \delta \leq \frac{1}{4}$  and  $(c_n) \in K_\delta^{\mathbb{N}}$ , then the Julia set  $\mathcal{J}_{(c_n)}$  is connected so that its Hausdorff dimension is at least one. Moreover, Sullivan [Su] has shown, that if  $c \neq 0$  is in the interior of the main cardioid of the Mandelbrot set, then  $\dim_{\mathbb{H}} \mathcal{J}(f_c) > 1$ . Furthermore, it follows by a result of Shishikura [Sh] that  $\dim_{\mathbb{H}} \mathcal{J}(f_{1/4}) = 2$ . It would be of interest whether  $\dim_{\mathbb{H}} \mathcal{J}_{(c_n)}$  is almost surely (in the sense of Section 5) greater than one if  $(c_n) \in K_\delta^{\mathbb{N}}$  for some  $\delta < \frac{1}{4}$ . In our general setting, it is clear that we can only expect such an almost surely statement.

### 7. Density of repelling fixpoints.

From iteration theory of a fixed rational function it is well-known that the repelling periodic points are dense in the Julia set (cf. [Be, p. 148], [CG, p. 63] or [St, p. 35]). In our setting we consider the set  $\mathcal{R}_{(c_n)}$  of *repelling fixpoints* of the sequence of iterates  $(F_n)$ , i.e.,

$$\mathcal{R}_{(c_n)} := \{ \zeta \in \mathbb{C} : F_k(\zeta) = \zeta \text{ for some } k \in \mathbb{N} \text{ and } |F'_k(\zeta)| > 1 \}.$$

It is not necessarily true that  $\mathcal{R}_{(c_n)} \subset \mathcal{J}_{(c_n)}$ . But from a result of Fornæss and Sibony [FS, Theorem 2.3] it follows that if  $\delta > 0$  is sufficiently small and  $(c_n) \in K_\delta^{\mathbb{N}}$ , then  $(\mathcal{R}_{(c_n)})' = \mathcal{J}_{(c_n)}$ . More precisely, we show:

**Theorem 7.1.** *Let  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^{\mathbb{N}}$ . Then  $(\mathcal{R}_{(c_n)})' = \mathcal{J}_{(c_n)}$ .*

*Proof.* Since  $\delta < \frac{1}{4}$  we have  $f_c(\overline{D}_r) \subset D_r$  for all  $c \in K_\delta$  and  $s_\delta < r < r_\delta$ . This implies that  $F_k(z) \neq z$  for all  $k \in \mathbb{N}$  and  $s_\delta < |z| < r_\delta$ . Since  $F'_k(z) = 2^k \prod_{j=0}^{k-1} F'_j(z)$  and  $f_c(K_{1/2}) \subset K_{1/2}$ , we have  $\mathcal{R}_{(c_n)} \cap K_{1/2} = \emptyset$ . Setting  $K := K_r$  for some  $r \in (\frac{1}{2}, r_\delta)$ , we also have  $\mathcal{R}_{(c_n)} \cap K = \emptyset$ . We set  $U := \mathbb{C} \setminus K$ . If  $z \in U$  and  $F_j(z) \in U$  for all  $j = 1, \dots, k-1$ , then  $|F'_k(z)| \geq q^k$  with  $q := 2r > 1$ .

We first show that  $(\mathcal{R}_{(c_n)})' \subset \mathcal{J}_{(c_n)}$ . For that purpose let  $F_{k_\ell}(z_\ell) = z_\ell$ ,  $|F'_{k_\ell}(z_\ell)| > 1$  and  $z_\ell \rightarrow \zeta$  as  $\ell \rightarrow \infty$ . If  $\zeta \in \mathbb{C} \setminus \mathcal{K}_{(c_n)}$ , then  $F_{k_\ell} \rightarrow \infty$  as  $\ell \rightarrow \infty$  uniformly in some neighbourhood of  $\zeta$ . This gives  $F_{k_\ell}(z_\ell) \rightarrow \infty$  as  $\ell \rightarrow \infty$  which is a contradiction. Now, assume that  $\zeta \in (\mathcal{K}_{(c_n)})^\circ$ . If  $F_j(\zeta) \in U$  for all  $j \in \mathbb{N}_0$ , then  $|F'_k(\zeta)| \geq q^k \rightarrow \infty$  as  $k \rightarrow \infty$ . But this is impossible since  $(F_k)$  is normal and bounded in  $(\mathcal{K}_{(c_n)})^\circ$ . Therefore, we have  $F_{k_0}(\zeta) \in K$  for some  $k_0 \in \mathbb{N}_0$ , and thus  $F_k(\zeta) \in K$  for all  $k \geq k_0$ . By passing to a subsequence we may assume that  $F_{k_\ell} \rightarrow \phi$  as  $\ell \rightarrow \infty$  uniformly in some neighbourhood  $U_\zeta$  of  $\zeta$ , where  $\phi$  is holomorphic in  $U_\zeta$ . This implies  $z_\ell = F_{k_\ell}(z_\ell) \rightarrow \phi(\zeta)$  as  $\ell \rightarrow \infty$  and thus  $z_\ell \in K$  for all  $\ell$  large enough which is again a contradiction.

Now, we show that  $\mathcal{J}_{(c_n)} \subset (\mathcal{R}_{(c_n)})'$ . Suppose that there exists  $\zeta \in \mathcal{J}_{(c_n)}$  and a neighbourhood  $V$  of  $\zeta$  such that  $F_k(z) \neq z$  for all  $z \in V$  and  $k \geq k_0 = k_0(V)$ . We set

$$h_k(z) := \frac{1}{2^k} \log |F_k(z) - z|.$$

Then  $h_k$  is harmonic and uniformly bounded above in  $V$ . By Eq. (2.1) we have  $h_k \rightarrow g_{(c_n)}$  as  $k \rightarrow \infty$  in  $V \setminus \mathcal{K}_{(c_n)}$ , and thus  $h_k \rightarrow h$  as  $k \rightarrow \infty$  for some harmonic function  $h$  in  $V$ . Furthermore, there holds  $h_k \rightarrow 0$  as  $k \rightarrow \infty$  in  $V \cap \mathcal{K}_{(c_n)}$  so that  $h = 0$  in  $V \cap \mathcal{K}_{(c_n)}$ . But this is a contradiction to the minimum principle for harmonic functions.

Therefore, for every  $\zeta \in \mathcal{J}_{(c_n)}$  there exists a strictly increasing sequence  $(k_\ell)$  in  $\mathbb{N}$  and  $z_\ell \in U$  such that  $z_\ell \rightarrow \zeta$  as  $\ell \rightarrow \infty$  and  $F_{k_\ell}(z_\ell) = z_\ell$ . Then we

have  $F_j(z_\ell) \in U$  for  $j = 1, \dots, k_\ell - 1$  which gives  $|F'_{k_\ell}(z_\ell)| \geq q^{k_\ell} > 1$  so that  $z_\ell \in \mathcal{R}_{(c_n)}$ . □

It would be of interest whether Theorem 7.1 holds for all  $\delta > 0$ . However, the proof shows that we always have  $(\mathcal{R}_{(c_n)})' \subset \mathcal{K}_{(c_n)}$ .

**8. Asymptotic distribution of predecessors.**

If  $(c_n) \in K_\delta^\mathbb{N}$  and if  $a \in \Delta_{R_\delta}$ , then the predecessors  $F_k^{-1}(a)$  of  $a$  are all contained in  $\mathcal{A}_{(c_n)}(\infty)$ , and they only accumulate on the Julia set  $\mathcal{J}_{(c_n)}$ . In fact, this follows from the invariance of  $\Delta_R$  for  $R > R_\delta$  and  $F_k \rightarrow \infty$  as  $k \rightarrow \infty$  locally uniformly in  $\mathcal{A}_{(c_n)}(\infty)$ . We want to study the asymptotic distribution of  $F_k^{-1}(a)$  as  $k \rightarrow \infty$ . For iteration of a fixed polynomial this was done by Brolin [Bro].

We first recall some facts from potential theory which are needed in the sequel and which can be found, for example, in the book of Tsuji [T]. Let  $E \subset \mathbb{C}$  be an infinite compact set, and let  $D$  be its outer domain, that is the component of  $\widehat{\mathbb{C}} \setminus E$  containing the point  $\infty$ . Furthermore, we denote by  $\text{cap } E \geq 0$  the logarithmic capacity (or transfinite diameter) of  $E$ . (We do not recall the definition of  $\text{cap } E$  because it will not be needed.) We suppose that the Green function  $g_D$  of  $D$  with pole at infinity exists. Then

$$g_D(z) = \log |z| + V + o(1) \quad \text{as } z \rightarrow \infty$$

and  $\text{cap } E = e^{-V} > 0$ . Note that by Eq. (2.1) this is true for  $E = \mathcal{J}_{(c_n)}$  with  $\text{cap } E = 1$ . Now, let  $\mu$  be any probability measure on  $E$ . Then the energy integral

$$I[\mu] := \iint_{E \times E} \log \frac{1}{|\zeta - \omega|} d\mu(\zeta) d\mu(\omega)$$

is finite, and the logarithmic potential

$$p_\mu(z) := \int_E \log \frac{1}{|z - \zeta|} d\mu(\zeta)$$

is harmonic in  $D$ . Furthermore, there exists a unique probability measure  $\mu^*$  on  $E$  which minimizes the energy integral  $I[\mu]$ , and there holds

$$g_D(z) - V = -p_{\mu^*}(z) \quad (z \in D).$$

This measure  $\mu^*$  is called the equilibrium measure on  $E$ . In the following  $\mu^*$  always denotes the equilibrium measure on the Julia set  $\mathcal{J}_{(c_n)}$ , and  $\text{supp } \mu^*$  denotes its support, that is the set of points  $z \in \mathcal{J}_{(c_n)}$  such that  $\mu^*(D_\varepsilon(z) \cap \mathcal{J}_{(c_n)}) > 0$  for every  $\varepsilon > 0$ . Note that  $\text{supp } \mu^*$  is a closed set.

In order to study the asymptotic distribution of  $F_k^{-1}(a)$  for  $a \in \Delta_{R_\delta}$  as  $k \rightarrow \infty$  we consider the following sequence  $(\mu_k^a)$  of probability measures.

If  $\delta_z$  denotes the *Dirac measure* concentrated at the point  $z \in \mathbb{C}$  (that is  $\delta_z(E) = 1$  if  $z \in E$  and  $\delta_z(E) = 0$  if  $z \notin E$ ), then let

$$(8.1) \quad \mu_k^a := \frac{1}{2^k} \sum_{F_k(z)=a} \delta_z.$$

We will show that  $(\mu_k^a)$  is weakly convergent to  $\mu^*$ , that is  $\mu_k^a(E) \rightarrow \mu^*(E)$  as  $k \rightarrow \infty$  for every Borel set  $E \subset \mathbb{C}$  with  $\mu^*(E^\circ) = \mu^*(\overline{E})$ . For that purpose we first collect some auxiliary results.

**Lemma 8.1** ([Bro, Lemma 15.4]). *Let  $E \subset \mathbb{C}$  be a compact set, and let  $f$  be a function defined on  $E$  such that for some constant  $L$  there holds  $|f(z_1) - f(z_2)| \leq L|z_1 - z_2|$  for all  $z_1, z_2 \in E$ . If  $\text{cap } E = 0$ , then  $\text{cap } f(E) = 0$ .*

**Lemma 8.2.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then  $\text{cap}(\mathcal{J}_{(c_n)} \setminus \text{supp } \mu^*) = 0$ .*

*Proof.* Since  $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty)$  and  $\text{cap } \mathcal{J}_{(c_n)} > 0$ , the assertion immediately follows from Theorem III.31 in [T, p. 79]. □

**Lemma 8.3.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then  $\text{supp } \mu^* = \mathcal{J}_{(c_n)}$ .*

*Proof.* We assume that  $\mathcal{J}^* := \mathcal{J}_{(c_n)} \setminus \text{supp } \mu^* \neq \emptyset$ . By Lemma 8.2 we have  $\text{cap } \mathcal{J}^* = 0$ . Since  $\mathcal{J}^*$  is an open set in  $\mathcal{J}_{(c_n)}$  we may choose  $z_0 \in \mathcal{J}^*$  and  $\varepsilon > 0$  such that  $\mathcal{J}_\varepsilon := \mathcal{J}^* \cap D_\varepsilon(z_0) \subset \mathcal{J}^*$ . We also have  $\text{cap } \mathcal{J}_\varepsilon = 0$ . But by the self-similarity of  $\mathcal{J}_{(c_n)}$  (cf. [Bü1]) there exists  $m \in \mathbb{N}$  such that  $F_m(\mathcal{J}_\varepsilon) = F_m(\mathcal{J}_{(c_n)})$ . Since  $|f_{c_k}(z_1) - f_{c_k}(z_2)| = |z_1 + z_2||z_1 - z_2| \leq 2R_\delta|z_1 - z_2|$  for all  $k \in \mathbb{N}$  and  $z_1, z_2 \in \mathcal{J}_{(c_n)}$ , we obtain  $\text{cap } F_m(\mathcal{J}_\varepsilon) = 0$  by Lemma 8.1. On the other hand there holds  $F_m(\mathcal{J}_{(c_n)}) = \mathcal{J}_{(c_n+m)}$  and thus  $\text{cap } F_m(\mathcal{J}_{(c_n)}) = 1$  which gives a contradiction. □

**Lemma 8.4** ([Bro, Lemma 15.5]). *Let  $E, H \subset \mathbb{C}$  be compact sets with  $E \subset H$  and  $\text{cap } E = e^{-V} > 0$ . Furthermore, let  $(\mu_n)$  be a sequence of probability measures on  $H$  which converges weakly to a probability measure  $\mu$  on  $E$ . If  $u_n$  denotes the logarithmic potential with respect to  $\mu_n$  and  $\mu^*$  denotes the equilibrium measure on  $E$ , then suppose  $\liminf_{n \rightarrow \infty} u_n(z) \geq V$  for  $z \in E$  and  $\text{supp } \mu^* = E$ . Then there holds  $\mu = \mu^*$ .*

**Theorem 8.5.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then for any  $a \in \Delta_{R_\delta}$  the sequence  $(\mu_k^a)$  of probability measures defined by (8.1) converges weakly to the equilibrium measure  $\mu^*$  on  $\mathcal{J}_{(c_n)}$ .*

*Proof.* For  $k \in \mathbb{N}$  let  $z_{1,k}, \dots, z_{2^k,k}$  be the solutions of the equation  $F_k(z) = a$ . Then we have  $z_{j,k} \in \mathcal{A}_{(c_n)}(\infty)$  and  $z_{j,k} \in H := K_{|a|}$  for  $j = 1, \dots, 2^k$  so that  $\text{supp } \mu_k^a \subset H$ . Since  $|F_k(z)| \leq R_\delta$  for  $z \in \mathcal{J}_{(c_n)}$  and

$$|F_k(z) - a| = \prod_{j=1}^{2^k} |z - z_{j,k}|,$$

we obtain for  $z \in \mathcal{J}_{(c_n)}$

$$\sum_{j=1}^{2^k} \log |z - z_{j,k}| = \log |F_k(z) - a| \leq \log (R_\delta + |a|) = C$$

and thus

$$u_k(z) := \frac{1}{2^k} \sum_{j=1}^{2^k} \log \frac{1}{|z - z_{j,k}|} \geq -\frac{C}{2^k}.$$

This can be written as

$$u_k(z) = \int_H \log \frac{1}{|z - \zeta|} d\mu_k^a(\zeta) \geq -\frac{C}{2^k}$$

so that

$$(8.2) \quad \liminf_{k \rightarrow \infty} u_k(z) \geq 0 = \log \operatorname{cap} \mathcal{J}_{(c_n)} \quad (z \in \mathcal{J}_{(c_n)}).$$

By the Selection Theorem (cf. [T, p. 34]) every sequence of probability measures on  $H$  contains a weakly convergent subsequence. Therefore, we only have to show that for every subsequence of  $(\mu_k^a)$  which converges weakly to some probability measure  $\nu$  there holds  $\nu = \mu^*$ . In fact, since the predecessors  $F_k^{-1}(a)$  of  $a$  do not accumulate in  $\mathcal{A}_{(c_n)}(\infty)$  we obtain  $\operatorname{supp} \nu \subset \mathcal{J}_{(c_n)}$ , and because of (8.2) the assertion follows from Lemma 8.3 and 8.4.  $\square$

**Remark 8.6.** If  $\delta < \frac{1}{4}$  and  $(c_n) \in K_\delta^\mathbb{N}$ , then the assertion of Theorem 8.5 also holds for any  $a \in D_{r_\delta}$ . This requires only a few simple modifications in the proof.

Like in the iteration of a fixed function there holds that for any  $a \in \mathcal{J}_{(c_n)}$  the set  $\bigcup_{k=1}^\infty F_k^{-1}(F_k(a))$  is dense in  $\mathcal{J}_{(c_n)}$  (cf. [Bü1]). We also want to study the asymptotic distribution of  $F_k^{-1}(F_k(a))$  as  $k \rightarrow \infty$ . For that purpose, we consider the following sequence  $(\nu_k^a)$  of probability measures defined by

$$(8.3) \quad \nu_k^a := \frac{1}{2^k} \sum_{F_k(z)=F_k(a)} \delta_z.$$

Then  $\operatorname{supp} \nu_k^a \subset \mathcal{J}_{(c_n)}$ , and from iteration theory of a fixed polynomial  $f_c$  it is known (cf. [Bro], see also [St, p. 148]) that  $(\nu_k^a)$  converges weakly to the equilibrium measure  $\mu^*$  on  $\mathcal{J}(f_c)$ . We show that this holds true in our situation.

**Theorem 8.7.** *Let  $\delta > 0$  and  $(c_n) \in K_\delta^\mathbb{N}$ . Then for any  $a \in \mathcal{J}_{(c_n)}$  the sequence  $(\nu_k^a)$  of probability measures defined by (8.3) converges weakly to the equilibrium measure  $\mu^*$  on  $\mathcal{J}_{(c_n)}$ .*

*Proof.* For  $k \in \mathbb{N}$  let  $z_{1,k}, \dots, z_{2^k,k}$  be the solutions of the equation  $F_k(z) = F_k(a)$ . Then we have for  $z \in \mathcal{A}_{(c_n)}(\infty)$

$$\frac{1}{2^k} \log |F_k(z) - F_k(a)| = \frac{1}{2^k} \sum_{j=1}^{2^k} \log |z - z_{j,k}| = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| d\nu_k^a(\zeta).$$

Again, we only have to show that every weakly convergent subsequence  $(\lambda_\ell)$  of  $(\nu_k^a)$  has the limit  $\mu^*$ . If  $\lambda_\ell \rightarrow \lambda$  as  $\ell \rightarrow \infty$  weakly, then for  $z \in \mathcal{A}_{(c_n)}(\infty)$

$$\lim_{\ell \rightarrow \infty} \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| d\lambda_\ell(\zeta) = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| d\lambda(\zeta).$$

On the other hand we have

$$\begin{aligned} \frac{1}{2^k} \log |F_k(z) - F_k(a)| &= \frac{1}{2^k} \log \left| \frac{F_k(z) - F_k(a)}{F_k(z)} \right| \\ &+ \frac{1}{2^k} \log |F_k(z)| \rightarrow g_{(c_n)}(z) \quad \text{as } k \rightarrow \infty. \end{aligned}$$

This implies

$$g_{(c_n)}(z) = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| d\lambda(\zeta) \quad (z \in \mathcal{A}_{(c_n)}(\infty)),$$

and since  $\mu^*$  is unique the assertion follows.  $\square$

## References

- [A] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, van Nostrand, Princeton, 1966.
- [Be] A.F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, New York, 1991.
- [BeP] A.F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2), **41** (1979), 475-483.
- [Bl] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. (N.S.), **11** (1984), 85-141.
- [Bro] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat., **6** (1965), 103-144.
- [Brü] R. Brück, *Connectedness and stability of Julia sets of the composition of polynomials of the form  $z^2 + c_n$* , J. London Math. Soc. (2), **61** (2000), 462-470.
- [BBR] R. Brück, M. Büger and S. Reitz, *Random iterations of polynomials of the form  $z^2 + c_n$ : Connectedness of Julia sets*, Ergodic Theory Dynam. Systems, **19** (1999), 1221-1231.

- [Bü1] M. Büger, *Self-similarity of Julia sets of the composition of polynomials*, Ergodic Theory Dynam. Systems, **17** (1997), 1289-1297.
- [Bü2] ———, *On the composition of polynomials of the form  $z^2 + c_n$* , Math. Ann., **310** (1998), 661-683.
- [C] L. Carleson, *Removable singularities of continuous harmonic functions in  $\mathbb{R}^m$* , Math. Scand., **12** (1963), 15-18.
- [CG] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York, 1993.
- [EL] A.E. Eremenko and M.Yu. Lyubich, *The dynamics of analytic transformations*, Algebra i Analiz, **1** (1989), 1-70; English transl. in Leningrad Math. J., **1** (1990), 563-634.
- [FS] J.E. Fornæss and N. Sibony, *Random iterations of rational functions*, Ergodic Theory Dynam. Systems, **11** (1991), 687-708.
- [G] V. Garber, *On the iteration of rational functions*, Math. Proc. Cambridge Philos. Soc., **84** (1978), 497-505.
- [GV] F.W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2), **6** (1973), 504-512.
- [LV] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, Berlin, 1973.
- [L1] M.Yu. Lyubich, *On typical behavior of the trajectories of a rational mapping of the sphere*, Dokl. Akad. Nauk SSSR, **268** (1983), 29-32; English transl. in Soviet Math. Dokl., **27** (1983), 22-25.
- [L2] ———, *The dynamics of rational transforms: The topological picture*, Uspekhi Mat. Nauk, **41** (1986), 35-95; English transl. in Russian Math. Surveys, **41** (1986), 43-117.
- [MR] R. Mañé and L.F. da Rocha, *Julia sets are uniformly perfect*, Proc. Amer. Math. Soc., **116** (1992), 251-257.
- [M] J. Milnor, *Dynamics in One Complex Variable*, Vieweg Verlag, Braunschweig, 1999.
- [NV] R. Näkki and J. Väisälä, *John disks*, Exposition. Math., **9** (1991), 3-43.
- [P1] Ch. Pommerenke, *Uniformly perfect sets and the Poincaré metric*, Arch. Math. (Basel), **32** (1979), 192-199.
- [P2] ———, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.
- [Sh] M. Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. (2), **147** (1998), 225-267.
- [St] N. Steinmetz, *Rational Iteration (Complex Analytic Dynamical Systems)*, Walter de Gruyter, Berlin, 1993.
- [Su] D. Sullivan, *Conformal dynamical systems*, in 'Geometric Dynamics' (J. Palis, editor), Lecture Notes in Mathematics, **1007**, Springer-Verlag, New York, 1983, 725-752.
- [T] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

- [Y] M.V. Yakobson, *The boundaries of certain normality domains for rational maps*, Uspekhi Mat. Nauk, **39** (1984), 211-212; English transl. in Russian Math. Surveys, **39** (1984), 229-230.

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