SPECIAL VALUES OF KOECHER–MAASS SERIES OF
SIEGEL CUSP FORMS

YOUNGJU CHOE AND WINFRIED KOHNEN
SPECIAL VALUES OF KOECHER–MAASS SERIES OF SIEGEL CUSP FORMS

YOUNGJU CHOE AND WINFRIED KOHNEN

A certain finiteness result for special values of character twists of Koecher-Maass series attached to Siegel cusp of genus $g$ is proved.

1. Introduction.

Let $f$ be an elliptic cusp form of even integral weight $k$ on $\Gamma_1 := SL_2(\mathbb{Z})$. Let $\chi$ be a primitive Dirichlet character modulo a positive integer $N$ and denote by $L(f, \chi, s)$ ($s \in \mathbb{C}$) the Hecke $L$-function of $f$ twisted with $\chi$, defined by analytic continuation of the series

$$
\sum_{n \geq 1} \chi(n)a(n)n^{-s} \quad (\text{Re}(s) \gg 0; \ a(n) = n\text{-th Fourier coefficient of } f).
$$

Let $g(\chi)$ be the Gauss sum attached to $\chi$. As is well-known, there exists a $\mathbb{Z}$-module $M_f \subset \mathbb{C}$ (depending only on $f$) of finite rank such that all the special values

$$
i^{s+1}(2\pi)^{-s}g(\chi)L(f, \chi, s)
$$

($s \in \mathbb{N}$, $1 \leq s \leq k - 1$; $\chi$ a primitive Dirichlet character modulo $N$, $N \in \mathbb{N}$) lie in $M_f \otimes_{\mathbb{Z}} \mathbb{Z}[\chi]$, where $\mathbb{Z}[\chi]$ is the $\mathbb{Z}$-module obtained from $\mathbb{Z}$ by adjoining the values of $\chi$. In fact, if $f$ is a Hecke eigenform, one has $\text{rk}_{\mathbb{Z}} M_f \leq 2$ [1, 7, 8, 10].

The purpose of this paper is to give a generalization of the above result to the case of a Siegel cusp form $f$, where now $L(f, \chi, s)$ is replaced by an appropriate $\chi$-twist of the Koecher-Maass series attached to $f$.

More precisely, let $f$ be a cusp form of even integral weight $k \geq g + 1$ w.r.t. the Siegel modular group $\Gamma_g := Sp_g(\mathbb{Z})$ of genus $g$ and write $a(T)$ ($T$ a positive definite half-integral matrix of size $g$) for its Fourier coefficients. For $\chi$ as above we set

$$
L(f, \chi, s) := \sum_{\{T > 0\}/GL_g, N(\mathbb{Z})} \frac{\chi(\text{tr} T)a(T)}{\epsilon_N(T)(\text{det } T)^s} \quad (\text{Re}(s) \gg 0),
$$

where $\epsilon_N(T)$ is the determinant of $N(T)$ and $N$ is the norm function.

373
where the summation extends over all positive definite half-integral \((g,g)\)-matrices \(T\) modulo the action 
\[ T 
\rightarrow \n T[U] := U^tTU \]

of the group 
\[ GL_{g,N}(\mathbb{Z}) := \{ U \in GL_g(\mathbb{Z}) \mid U \equiv E_g \pmod{N} \} \]

and 
\[ \epsilon_N(T) := \#\{U \in GL_{g,N}(\mathbb{Z}) \mid T[U] = T \} \]

is the order of the corresponding unit group of \(T\) (note that \(\epsilon_N(T) = 1\) whenever \(N > 2\) by a classical result of Minkowski). Furthermore, 
\[ \text{tr } T \]

denotes the trace of \(T\). Note that \(\chi(\text{tr } T)\) depends only on the \(GL_{g,N}(\mathbb{Z})\)-class of \(T\).

In §2 (Thm. 1) we shall prove that the series 
\[ L(f, \chi, s) \]

have holomorphic continuations to \(\mathbb{C}\) and satisfy functional equations under 
\[ s \rightarrow k - s \]

The main result of the paper (Thm. 2) which will be proved in §3, states that all the special values
\[ L(f, \chi, s) \]

are contained in 
\[ M_f \otimes_{\mathbb{Z}} \mathbb{Z}[\chi] \]

where \(M_f \subset \mathbb{C}\) is a finite \(\mathbb{Z}\)-module depending only on \(f\). Its rank is bounded by the rank of a certain singular relative homology group of a toroidal compactification of a quotient space of \(H_g \times C^{gw}\), where \(H_g\) is the Siegel upper half-space of genus \(g\) and 
\[ w := k - (g + 1). \]

For the proof one represents the functions 
\[ L(f, \chi, s) \]

as finite linear combinations of integrals of certain differential forms attached to \(f\) along certain \(g(g+1)/2\)-dimensional real subcycles of \(\Gamma_g \backslash H_g\). Our assertion then can be deduced if we use results of Hatada given in [2, 3]. More precisely, in [2] it is shown that the space of cusp forms of weight \(k \geq g + 1\) w.r.t. a torsion-free congruence subgroup \(\Gamma \subset \Gamma_g\) is canonically isomorphic to the space of holomorphic differential forms of highest degree on a compactification of \(\Gamma \times H_g \backslash H_g \times C^{gw}\), and in [3] using [2] a certain finiteness statement for a certain family of integrals of Siegel cusp forms is derived. (Actually, as we think, some of the assertions of [3] have to be slightly modified, for complete correctness’ purposes; cf. §3.)

Inspecting the proof of Thm. 2, it is quite suggestive or even more or less clear that a similar finiteness statement as given there can be proved for special values of Dirichlet series of a much more general type. In fact, such a result essentially seems to be true for finite linear combinations of all the
partial series
\[
\sum_{\{T>0\}/GL_g(Z)} \frac{e^{2\pi i \text{tr}(TS)}a(T)}{\epsilon(S)(T)(\det T)^s} \quad (\Re(s) \gg 0),
\]
where \(S\) is any rational symmetric matrix of size \(g\), \(GL_g(Z)\) is the subgroup \(\{U \in GL_g(Z) \mid S[U] \equiv S \pmod{Z}\}\) and \(\epsilon(S)(T) := \#\{U \in GL_g(Z) \mid T[U] = T\}\). However, we do not want to pursue this point further.

We finally remark that in \([4]\) the Koecher-Maass series of a Siegel-Eisenstein series of genus \(g\) is explicitly expressed in terms of “elementary” zeta functions. In particular, if \(g\) is odd it is shown to be a sum of products of Riemann zeta functions. It would be interesting to investigate if a similar statement as given in Thm. 2 would also hold in this case. In fact, it is suggestive that such an assertion can be derived directly from the explicit formulas given in \([4]\).

One can also ask similar questions in the case of a Klingen-Siegel-Eisenstein series.

**Notations.** If \(A\) and \(B\) are complex matrices of appropriate sizes, we put \(A[B] := B^tAB\). We simply write \(E = E_g\) resp. \(0 = 0_g\) for the unit resp. zero matrix of size \(g\) if there is no confusion.

We often write elements of the group \(GSp^+_g(\mathbb{R}) \subset GL_{2g}(\mathbb{R})\) consisting of real symplectic similitudes of size \(2g\) with positive scale in the form \(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\), understanding that \(A, B, C\) and \(D\) are real \((g, g)\)-matrices.

If \(Y \in \mathbb{R}^{(g, g)}\), we write \(Y > 0\) if \(Y\) is symmetric and positive definite. The group \(GL_g(\mathbb{R})\) operates on \(\mathcal{P}_g := \{Y \in \mathbb{R}^{(g, g)} \mid Y > 0\}\) in the usual way from the right by \(Y \mapsto Y[U]\).

If \(f(Z)\) is a complex-valued function on \(\mathcal{H}_g\), \(k\) a positive integer and \(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp^+_g(\mathbb{R})\), we set
\[
(f|k\gamma)(Z) := \det((CZ + D)^{-1})^k f((AZ + B)(CZ + D)^{-1}) \quad (Z \in \mathcal{H}_g).
\]
We often write \(f|\gamma\) instead of \(f|k\gamma\) if there is no misunderstanding.

If \(k\) is a positive integer, \(\Gamma\) is a subgroup of \(\Gamma_g\) and \(\chi\) is a character of \(\Gamma\) of finite order, we denote by \(S_k(\Gamma, \chi)\) the space of Siegel cusp forms of weight \(k\) and character \(\chi\) w.r.t. \(\Gamma\). If \(\chi = 1\) we simply write \(S_k(\Gamma)\).
2. Character twists of Koecher-Maass series.

For \( N \) a natural number we define
\[
\Gamma^*_g(N^2) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid C \equiv 0 \pmod{N^2}, \quad D \equiv \lambda E \pmod{N} \text{ for some } \lambda \in \mathbb{Z} \right\}
\]
(note that \( \lambda \) must necessarily satisfy \( (\lambda, N) = 1 \)).

It is easy to see that \( \Gamma^*_g(N^2) \) is a subgroup of \( \Gamma_g \). If \( \chi \) is a Dirichlet character modulo \( N \), we extend \( \chi \) to a character of \( \Gamma^*_g(N^2) \) by putting
\[
\chi(\gamma) := \chi(\lambda) \text{ if } \gamma \equiv \begin{pmatrix} * & * \\ 0 & \lambda E \end{pmatrix} \pmod{N}.
\]

**Lemma 1.** Let \( f \in S_k(\Gamma_g) \) with Fourier coefficients \( a(T) \) \( (T > 0 \text{ half-integral}) \). Let \( \chi \) be a primitive Dirichlet character modulo \( N \). Then the function
\[
f_\chi(Z) := \sum_{T>0} \chi(\text{tr}(T))a(T)e^{2\pi i \text{tr}(TZ)} \quad (Z \in \mathcal{H}_g)
\]
belongs to \( S_k(\Gamma^*_g(N^2), \chi^2) \).

**Proof.** Let
\[
g(\chi) := \sum_{\nu \pmod{N}} \overline{\chi}(\nu)e^{2\pi i \nu/N}
\]
be the Gauss sum attached to \( \chi \). Since
\[
\sum_{\nu \pmod{N}} \overline{\chi}(\nu)e^{2\pi i \text{tr}(T)\nu/N} = \chi(\text{tr}T)g(\chi),
\]
we obtain
\[
f_\chi = \frac{1}{g(\chi)} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) f|\alpha_{\nu},
\]
where
\[
\alpha_{\nu} := \begin{pmatrix} E & \nu E^t \\ 0 & N \end{pmatrix} \quad (\nu \in \mathbb{Z}).
\]

Let \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^*_g(N^2) \) and put
\[
A' := A + \frac{\nu}{N} C,
\]
\[
B' := B + \frac{\nu}{N} (E - AD^t)D - \frac{\nu^2}{N^2} CD^t D,
\]
\[
D' := D - \frac{\nu}{N} CD^t D.
\]
Then $A', B'$ and $D'$ are integral matrices, one has $D' \equiv D \pmod{N}$ and

$$\alpha_\nu \gamma = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} E & \frac{\sqrt{N} D D}{E} \\ 0 & E \end{pmatrix};$$

in particular $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma^*_g(N^2)$, and it follows that

$$f_{\chi}|\gamma = \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) f\begin{pmatrix} E & \frac{\sqrt{N} D D}{E} \\ 0 & E \end{pmatrix}$$

$$= \chi(\lambda^2) \cdot \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) f|\alpha_\nu \quad (D \equiv \lambda E \pmod{N})$$

$$= \chi^2(\gamma)f.$$

This proves the claim.

**Lemma 2.** Let the notations be as in Lemma 1 and put

$$W_{N^2} := \begin{pmatrix} 0 & -E \\ N^2E & 0 \end{pmatrix}.$$ 

Then

$$f_{\chi}|W_{N^2} = g(\chi)^2 N^{-gk-1} f_{\overline{\chi}}.$$ 

**Proof.** For $(\nu, N) = 1$ determine $\lambda, \mu \in \mathbb{Z}$ with $\lambda N - \mu \nu = 1$. Then

$$\alpha_\nu W_{N^2} = N \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \begin{pmatrix} NE & -\mu E \\ -\nu E & \lambda E \end{pmatrix} \alpha_\mu.$$ 

Hence

$$g(\overline{\chi}) \cdot f_{\chi}|W_{N^2} = N^{-gk} \sum_{\nu \pmod{N}, (\nu, N) = 1} \overline{\chi}(\nu) f|\alpha_\mu$$

$$= \chi(-1) N^{-gk} \sum_{\mu \pmod{N}, (\mu, N) = 1} \chi(\mu) f|\alpha_\mu$$

$$= \chi(-1) g(\chi) N^{-gk} f_{\overline{\chi}}.$$ 

Since $g(\chi)g(\overline{\chi}) = \chi(-1)N$, we obtain our claim.

**Theorem 1.** Let $k$ be even and let $f \in S_k(\Gamma_g)$. Let $\chi$ be a primitive Dirichlet character modulo $N$ and define $L(f, \chi, s)$ $(\text{Re}(s) \gg 0)$ by (1). Let

$$\gamma_g(s) := (2\pi)^{-gs} \prod_{\nu=1}^{g} \pi^{(\nu - 1)/2} \Gamma \left(s - \frac{\nu - 1}{2}\right) \quad (s \in \mathbb{C})$$

and set

$$L^*(f, \chi, s) := N^{gs} \gamma_g(s)L(f, \chi, s) \quad (\text{Re}(s) \gg 0).$$
Then $L^*(f, \chi, s)$ extends to a holomorphic function on $\mathbb{C}$, and the functional equation

$$L^*(f, \chi, k - s) = (-1)^{\frac{ak}{2}} g(\chi)^2 \frac{1}{N} L^*(f, \overline{\chi}, s)$$

holds, where $g(\chi)$ is the Gauss sum attached to $\chi$.

Proof. Since

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} | U \in GL_{g,N}(\mathbb{Z}) \right\} \subset \Gamma_{g,0}(N^2)$$

and $k$ is even, the function $f_{\chi}(iY) (Y > 0)$ is invariant under $Y \mapsto Y[U]$ ($U \in GL_{g,N}(\mathbb{Z})$). Hence it follows in the usual way that

$$L^*(f, \chi, s) = \frac{1}{2} N^{gs} \int_{F_{g,N}} f_{\chi}(iY)(\det Y)^s dv \quad (\text{Re } s \gg 0),$$

where $F_{g,N}$ is any fundamental domain for the action of $GL_{g,N}(\mathbb{Z})$ on $P_g$ and $dv = (\det Y)^{-(g+1)/2} dY$ is the $GL_g(\mathbb{R})$-invariant volume element on $P_g$.

We fix a set of representatives $U_1, \ldots, U_r$ for $GL_g(\mathbb{Z})/GL_{g,N}(\mathbb{Z})$ and now take

$$F_{g,N} = \bigcup_{\nu=1}^r R_g[U_\nu],$$

where $R_g$ is Minkowski’s fundamental domain for the action of $GL_g(\mathbb{Z})$.

Since $GL_{g,N}(\mathbb{Z})$ is closed under transposition, also $F_{g,N}^{-1}$ is a fundamental domain for $GL_{g,N}(\mathbb{Z})$.

We let

$$P_{g,+} := \{ Y \in P_g | \det Y \geq N^{-g} \}, \quad P_{g,-} := \{ Y \in P_g | \det Y \leq N^{-g} \},$$

write

$$F_{g,N} = (F_{g,N} \cap P_{g,+}) \cup (F_{g,N} \cap P_{g,-})$$

and observe that $F_{g,N} \cap P_{g,-}$ under the map $Y \mapsto (N^2Y)^{-1}$ is transformed bijectively onto $F_{g,N}^{-1} \cap P_{g,+}$. We also observe that both $F_{g,N} \cap P_{g,+}$ and $F_{g,N}^{-1} \cap P_{g,+}$ are fundamental domains for the induced action of $GL_{g,N}(\mathbb{Z})$ on $P_{g,+}$, the integral in (3) is absolutely convergent and the integrand is invariant under $GL_{g,N}(\mathbb{Z})$.

Therefore, since by Lemma 2

$$f_{\chi}(i(N^2Y)^{-1}) = (-1)^{\frac{ak}{2}} g(\chi)^2 N^{gk-1}(\det Y)^k f_{\overline{\chi}}(iY),$$
we conclude that
\begin{equation}
L^s(f, \chi, s) = \frac{1}{2} \int_{F_{g,N} \cap F_{g,+}} \left( f_\chi(iY)(N^g \det Y)^s \right. \\
+ \left. (-1)^{\frac{g}{2}} g(\chi)^2 N^{-1} f_\chi(iY)(N^g \det Y)^{k-s} \right) dv.
\end{equation}

Standard arguments and estimates taking into account (4) and properties of \(R_g\) (compare e.g., [5, Chap. VI]) now show that the integral on the right of (5) is (absolutely) convergent for all \(s \in \mathbb{C}\) and represents a holomorphic function of \(s\).

Since
\[ g(\chi)g(\overline{\chi}) = \chi(-1)N, \]
we also easily see the claimed functional equation. This concludes the proof of the Theorem.

3. Special values.

In this section we shall prove:

**Theorem 2.** Let \(k\) be even, \(k \geq g+1\) and let \(f \in S_k(\Gamma_g)\). If \(\chi\) is a primitive Dirichlet character modulo \(N\), define \(L(f, \chi, s)\) \((s \in \mathbb{C})\) by holomorphic continuation of the series (1) (Theorem 1). Let \(g(\chi)\) be the Gauss sum attached to \(\chi\) and let \(Z[\chi]\) be the \(\mathbb{Z}\)-module obtained from \(\mathbb{Z}\) by adjoining the values of \(\chi\).

Then there exists a \(\mathbb{Z}\)-module \(M_f \subset \mathbb{C}\) depending only on \(f\) of finite rank such that all the special values
\[ i^{gs+\frac{g(s+1)}{2}} \pi^{-\frac{gs}{4} + \frac{s}{2}} (2\pi)^{-gs} g(\chi) L(f, \chi, s) \]
where \(s \in \mathbb{N}, \frac{g+1}{2} \leq s \leq k - \frac{g+1}{2}\) and \(\chi\) runs over all primitive Dirichlet characters modulo all positive integers \(N\), are contained in \(M_f \otimes \mathbb{Z} Z[\chi]\).

**Proof.** From (2) and (3) and the proof of Theorem 1 we find that
\begin{equation}
\label{eq:6}
g(\overline{\chi})\gamma_g(s)L(f, \chi, s) = \frac{1}{2} \sum_{\nu \mod N} \overline{\chi}(\nu) \int_{F_{g,N}} f(iY + \frac{\nu}{N}E)(\det Y)^{s-\frac{g+1}{2}}dY
\end{equation}
for all \(s \in \mathbb{C}\).

Note that the individual integrands on the right of (6) are \(GL_{g,N}(\mathbb{Z})\)-invariant since \(f(Z)\) is invariant under
\[ \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in GL_{g,N}(\mathbb{Z}) \right\} \]
and under translations. Let \(w \in \mathbb{Z}, w \geq 0\) and \(Sp_g(\mathbb{R}) \rtimes \mathbb{R}^{2gw}\) be the semi-direct product of \(Sp_g(\mathbb{R})\) and \(\mathbb{R}^{2gw}\) with multiplication given by
\[ (\gamma, \lambda)(\gamma', \lambda') = (\gamma\gamma', \lambda\gamma'^t + \lambda') \]
where by $\gamma \mapsto \gamma^\dagger$ we denote the diagonal embedding of $Sp_g(\mathbb{R})$ into $GL_{2gw}(\mathbb{R})$.

The group $Sp_g(\mathbb{R}) \propto \mathbb{R}^{2gw}$ acts on $\mathcal{H}_g \times C^{gw}$ (with $C^{gw} \cong (C^g)^w$) from the left by

$$(\gamma, \lambda) \circ (Z, (\zeta_1, \ldots, \zeta_w)) = \left( (AZ + B)(CZ + D)^{-1}, \left( \begin{array}{c} \zeta_1 + (\mu_1, \nu_1) \\ \vdots \\ \zeta_w + (\mu_w, \nu_w) \end{array} \right) \left( \begin{array}{c} \frac{Z}{E_g} \\ \vdots \\ \frac{Z}{E_g} \end{array} \right) (CZ + D)^{-1} \right)$$

where $\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$ and $\lambda = ((\mu_1, \nu_1), \ldots, (\mu_w, \nu_w))$ with $\mu_j, \nu_j \in \mathbb{R}^g$ for all $j$. The discrete subgroup $\Gamma_g \propto \mathbb{Z}^{2gw}$ acts properly discontinuously.

Let $\Gamma \subset \Gamma_g$ be any congruence subgroup acting without fixed points on $\mathcal{H}_g$ (e.g., the principal congruence subgroup $\Gamma_g(\ell)$ with $\ell \geq 3$) and view $f$ as an element of $S_k(\Gamma)$.

Put $w := k - (g + 1)$. It was shown in [2] that the map $h(Z) \mapsto h(Z)dZd\zeta$ gives an isomorphism between $S_k(\Gamma)$ and the space of holomorphic differential forms of degree $g(g+1)/2 + gw$ of (any) non-singular compactification of the quotient space $\Gamma \propto \mathbb{Z}^{2gw}\backslash \mathcal{H}_g \times C^{gw}$.

Using toroidal compactifications, in [3] from this a certain finiteness statement for certain cycle integrals attached to $h$ was derived which we now want to describe in the special case we need.

Let $S$ be a given rational symmetric matrix of size $g$ and let $n$ be an integer with $0 \leq n \leq w$. Define

$$T_g(S; n) := \bigcup_{Y \in P_g} \{ S + iY \} \times \left( (\mathbb{R}^g)^{w-n} \times \{ (\mu_1, \ldots, \mu_n) | \mu_1, \ldots, \mu_n \in \mathbb{R}^g \} \right) \subset \mathcal{H}_g \times C^{gw}.$$ 

Then $T_g(S; n)$ is a real submanifold of $\mathcal{H}_g \times C^{gw}$ of dimension $\frac{g(g+1)}{2} + gw$.

(In the notation of [3, §6] we have taken $a_1 = a_2 = \ldots = a_{w-n} \in \{ g + 1, \ldots, 2g \}$ and $a_{w-n+1} = \ldots a_w \in \{ 1, \ldots, g \}$. Also note that in the definition of $T_g(a_1, \ldots, a_w; X)$ in [3, p. 401] we have replaced the “$Z$” in $W(a_1, \ldots, a_w)[Z]$ by “$iY$”. We think that this is the correct definition, since otherwise the corresponding integrals in [3, Lemma 6.2 and Thm. 5] in general would not be convergent.)
Put
\[ \mathcal{U}_g := \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \Bigm| U \in GL_g(\mathbb{R}) \right\} \subset Sp_g(\mathbb{R}), \]

\[ V_{g,n} := \{ (\lambda_1, \ldots, \lambda_{w-n}, (\mu_1, 0), \ldots, (\mu_n, 0)) \mid \lambda_1, \ldots, \lambda_{w-n} \in \mathbb{R}^{2g}, \mu_1, \ldots, \mu_n \in \mathbb{R}^g \} \]

and
\[ H_{g,n} := \mathcal{U}_g \propto V_{g,n} \subset Sp_g(\mathbb{R}) \propto \mathbb{R}^{2gw}. \]

Let
\[ \alpha^{(S)} := \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}. \]

Then one easily checks that the conjugate subgroup
\[ H_{g,n}^{(S)} := (\alpha^{(S)}, 0) \cdot H_{g,n} \cdot (\alpha^{(S)}, 0)^{-1} \]

leaves \( T_g(S; n) \) stable.

Note that \( H_{g,n}^{(S)} \) consists of all pairs
\[ \left( \begin{pmatrix} U & S(U^t)^{-1} - US \\ 0 & (U^t)^{-1} \end{pmatrix}, (\lambda_1, \ldots, \lambda_{w-n}, (\mu_1, -\mu_1 S), \ldots, (\mu_n, -\mu_n S)) \right) \]

with \( \lambda_1, \ldots, \lambda_{w-n} \in \mathbb{R}^{2g} \) and \( \mu_1, \ldots, \mu_n \in \mathbb{R}^g \).

Let
\[ H_{g,n}^{(S), \Gamma} := H_{g,n}^{(S)} \cap \Gamma \propto \mathbb{Z}^{2gw}. \]

Write \( M := \Gamma \propto \mathbb{Z}^{2gw} \setminus \mathcal{H}_g \times C^{gw} \) and denote by \( \overline{M} \) a fixed toroidal compactification of \( M \). Let \( \partial M = \overline{M} \setminus M \). Then according to [3, Lemma 6.1] the closure of the image of \( H_{g,n}^{(S), \Gamma} \setminus T_g(S; n) \) in \( \overline{M} \) w.r.t. the usual complex topology is the support of a singular relative \( \frac{g(g+1)}{2} + gw \)-cycle with integral coefficients w.r.t. \( (\overline{M}, \partial M) \).

Since \( H_{g,n}^{(S), \Gamma} \setminus gw(\overline{M}, \partial M, \mathbb{Z}) \) is of finite rank, one concludes that for any given \( h \in S_k(\Gamma) \) all the numbers
\[ \int_{H_{g,n}^{(S), \Gamma} \setminus T_g(S; n)} h(Z) dZ d\zeta \quad (S \in \mathbb{Q}^{(g,g)}, S = S^t) \]

are contained in a finite \( \mathbb{Z} \)-module (depending only on \( h \)) whose rank is bounded by the rank of the above cohomology group ([3, Thm. 5], compare our above remark).
On the other hand (compare [3, Lemma 6.2]) one has the equality

\begin{equation}
\int_{H(S)g,n,\Gamma \setminus S \Gamma g(\alpha(S))} h(Z)dZd\zeta = \int_{\alpha(S)\cdot U_g \cdot \alpha^{-1}(S+iY) \cap \{S+iY \mid Y \in P_g\}} h(Z) \det (Z - S)^n dZ.
\end{equation}

In particular, now take $\Gamma = \Gamma_g(\ell)$ with some fixed $\ell \geq 3$. Then the integral on the right of (7) is equal to

$$
\iota^{g(n+\frac{g(g+1)}{2})} \int_{P_g/GL_{g,\ell}(Z)} h(S+iY) (\det Y)^n dY,
$$

where

$$
GL_{g,\ell}(Z) := \{ U \in GL_{g,\ell}(Z) \mid S[U^t] \equiv S \pmod {\ell,Z}\}.
$$

Let $S = \frac{\pi}{N}E$ with $\nu \in Z$ (so $\alpha(S) = \alpha_{\nu}$ in the notation of §2). Then we see that $GL_{g,\ell N}(Z)$ is contained in $GL_{g,\ell}(Z)$. Since the index of $GL_{g,\ell N}(Z)$ in $GL_{g,N}(Z)$ is bounded by a number depending only on $\ell$, the assertion of Thm. 2 now follows taking into account (6) and the fact that $\Gamma(\frac{1}{2}+\nu) \in Q\sqrt{\pi}$ for $\nu = 0, 1, 2 \ldots$.

**Acknowledgements.** The authors would like to thank E. Freitag, T. Oda, R. Scharlau and H. Yoshida for valuable conversations.

**References**


Received June 15, 1999. The first author was partially supported by KOSEF 98-0701-01-01-3 and POSTECH fund in the program year of 1999.

Department of Mathematics
Pohang Institute of Science & Technology
Pohang 790-784
Korea
E-mail address: yjc@yjc.postech.ac.kr

Universität Heidelberg
Mathematisches Institut,
INF 288, D-69120 Heidelberg
Germany
E-mail address: winfried@mathi.uni-heidelberg.de