SPECIAL VALUES OF KOECHER–MAASS SERIES OF
SIEGEL CUSP FORMS

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A certain finiteness result for special values of character twists of Koecher-Maass series attached to Siegel cusp of genus $g$ is proved.

1. Introduction.

Let $f$ be an elliptic cusp form of even integral weight $k$ on $\Gamma_1 := SL_2(\mathbb{Z})$. Let $\chi$ be a primitive Dirichlet character modulo a positive integer $N$ and denote by $L(f, \chi, s)$ ($s \in \mathbb{C}$) the Hecke $L$-function of $f$ twisted with $\chi$, defined by analytic continuation of the series

$$\sum_{n \geq 1} \chi(n)a(n)n^{-s} \quad (\text{Re}(s) \gg 0; \ a(n) = n\text{-th Fourier coefficient of } f).$$

Let $g(\chi)$ be the Gauss sum attached to $\chi$. As is well-known, there exists a $\mathbb{Z}$-module $M_f \subset \mathbb{C}$ (depending only on $f$) of finite rank such that all the special values

$$i^{s+1}(2\pi)^{-s}g(\chi)L(f, \chi, s)$$

$(s \in \mathbb{N}, 1 \leq s \leq k - 1; \ \chi \text{ a primitive Dirichlet character modulo } N, N \in \mathbb{N})$ lie in $M_f \otimes_{\mathbb{Z}} \mathbb{Z}[\chi]$, where $\mathbb{Z}[\chi]$ is the $\mathbb{Z}$-module obtained from $\mathbb{Z}$ by adjoining the values of $\chi$. In fact, if $f$ is a Hecke eigenform, one has $\text{rk}_{\mathbb{Z}} M_f \leq 2$ [1, 7, 8, 10].

The purpose of this paper is to give a generalization of the above result to the case of a Siegel cusp form $f$, where now $L(f, \chi, s)$ is replaced by an appropriate $\chi$-twist of the Koecher-Maass series attached to $f$.

More precisely, let $f$ be a cusp form of even integral weight $k \geq g + 1$ w.r.t. the Siegel modular group $\Gamma_g := Sp_g(\mathbb{Z})$ of genus $g$ and write $a(T)$ ($T$ a positive definite half-integral matrix of size $g$) for its Fourier coefficients. For $\chi$ as above we set

$$L(f, \chi, s) := \sum_{\{T > 0\}/GL_g, N(\mathbb{Z})} \frac{\chi(\text{tr } T)a(T)}{\epsilon_N(T)(\text{det } T)^s} \quad (\text{Re}(s) \gg 0),$$

(1)
where the summation extends over all positive definite half-integral \((g, g)\)-
matrices \(T\) modulo the action \(T 
\mapsto \cdot T U\) of the group \(GL_{g, N}(\mathbb{Z}) := \{U \in GL_g(\mathbb{Z}) | U \equiv E_g \pmod{N}\}\) and \(\epsilon_N(T) := \#\{U \in GL_{g, N}(\mathbb{Z}) | T[U] = T\}\) is the order of the corresponding unit group of \(T\) (note that \(\epsilon_N(T) = 1\) whenever \(N > 2\) by a classical result of Minkowski). Furthermore, \(\text{tr} T\) denotes the trace of \(T\). Note that \(\chi(\text{tr} T)\) depends only on the \(GL_{g, N}(\mathbb{Z})\)-
class of \(T\).

In §2 (Thm. 1) we shall prove that the series \(L(f, \chi, s)\) have holomorphic
continuations to \(\mathbb{C}\) and satisfy functional equations under \(s \mapsto k - s\). The
proof is fairly standard and follows the same pattern as in [6] for the case
\(N = 1\) (compare also [5]) and [9, §3.6] for \(g = 1\).

The main result of the paper (Thm. 2) which will be proved in §3, states
that all the special values
\[
\left(s \in \mathbb{N}, \frac{g + 1}{2} \leq s \leq k - \frac{g + 1}{2} ;
\chi \text{ a primitive Dirichlet character modulo } N, N \in \mathbb{N}\right)
\]
are contained in \(M_f \otimes_{\mathbb{Z}} \mathbb{Z}[\chi]\) where \(M_f \subset \mathbb{C}\) is a finite \(\mathbb{Z}\)-module depending
only on \(f\). Its rank is bounded by the rank of a certain singular relative
homology group of a toroidal compactification of a quotient space of \(H_g \times \mathbb{C}^g\), where \(H_g\) is the
Siegel upper half-space of genus \(g\) and \(w := k - (g + 1)\).
See §3 for details.

For the proof one represents the functions \(L(f, \chi, s)\) (similar as in the case
\(g = 1\)) as finite linear combinations of integrals of certain differential forms
attached to \(f\) along certain \(g(g + 1)/2\)-dimensional real subcycles of \(\Gamma_g\). Our
assertion then can be deduced if we use results of Hatada given in [2, 3].
More precisely, in [2] it is shown that the space of cusp forms of weight
\(k \geq g + 1\) w.r.t. a torsion-free congruence subgroup \(\Gamma \subset \Gamma_g\) is canonically
isomorphic to the space of holomorphic differential forms of highest degree
on a compactification of \(\Gamma \times \mathbb{C}^g\), and in [3] using [2] a certain
finiteness statement for a certain family of integrals of Siegel cusp forms is
derived. (Actually, as we think, some of the assertions of [3] have to be
slightly modified, for complete correctness’ purposes; cf. §3.)

Inspecting the proof of Thm. 2, it is quite suggestive or even more or less
clear that a similar finiteness statement as given there can be proved for
special values of Dirichlet series of a much more general type. In fact, such
a result essentially seems to be true for finite linear combinations of all the
partial series
\[
\sum_{\{T > 0\}/GL_g(Z)} \frac{e^{2\pi i \text{tr}(TS)}a(T)}{\epsilon(S)(T)(\text{det } T)^s} \quad (\text{Re } s > 0),
\]
where $S$ is any rational symmetric matrix of size $g$, $GL_g(Z)$ is the subgroup \{ $U \in GL_g(Z) \mid S[U] \equiv S \pmod{Z}$ \} and $\epsilon(S)(T) := \#\{ U \in GL_g(Z) \mid T[U] = T \}$. However, we do not want to pursue this point further.

We finally remark that in [4] the Koecher-Maass series of a Siegel-Eisenstein series of genus $g$ is explicitly expressed in terms of “elementary” zeta functions. In particular, if $g$ is odd it is shown to be a sum of products of Riemann zeta functions. It would be interesting to investigate if a similar statement as given in Thm. 2 would also hold in this case. In fact, it is suggestive that such an assertion can be derived directly from the explicit formulas given in [4].

One can also ask similar questions in the case of a Klingen-Siegel-Eisenstein series.

**Notations.** If $A$ and $B$ are complex matrices of appropriate sizes, we put $A[B] := B^tAB$. We simply write $E = E_g$ resp. $0 = 0_g$ for the unit resp. zero matrix of size $g$ if there is no confusion.

We often write elements of the group $GSp_g^+(\mathbb{R}) \subset GL_{2g}(\mathbb{R})$ consisting of real symplectic similitudes of size $2g$ with positive scale in the form \[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]
understanding that $A, B, C$ and $D$ are real $(g, g)$-matrices.

If $Y \in \mathbb{R}^{(g,g)}$, we write $Y > 0$ if $Y$ is symmetric and positive definite. The group $GL_g(\mathbb{R})$ operates on $\mathcal{P}_g := \{ Y \in \mathbb{R}^{(g,g)} \mid Y > 0 \}$ in the usual way from the right by $Y \mapsto Y[U]$.

If $f(Z)$ is a complex-valued function on $\mathcal{H}_g$, $k$ a positive integer and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_g^+(\mathbb{R})$, we set
\[
(f|_k\gamma)(Z) := \det (CZ + D)^{-k}f((AZ + B)(CZ + D)^{-1}) \quad (Z \in \mathcal{H}_g).
\]
We often write $f|\gamma$ instead of $f|_k\gamma$ if there is no misunderstanding.

If $k$ is a positive integer, $\Gamma$ is a subgroup of $\Gamma_g$ and $\chi$ is a character of $\Gamma$ of finite order, we denote by $S_k(\Gamma, \chi)$ the space of Siegel cusp forms of weight $k$ and character $\chi$ w.r.t. $\Gamma$. If $\chi = 1$ we simply write $S_k(\Gamma)$. 
2. Character twists of Koecher-Maass series.

For $N$ a natural number we define
\[
\Gamma^*_{g,0}(N^2) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid C \equiv 0 \pmod{N^2}, \quad D \equiv \lambda E \pmod{N} \text{ for some } \lambda \in \mathbb{Z} \right\}
\]
(note that $\lambda$ must necessarily satisfy $(\lambda, N) = 1$).

It is easy to see that $\Gamma^*_{g,0}(N^2)$ is a subgroup of $\Gamma_g$. If $\chi$ is a Dirichlet character modulo $N$, we extend $\chi$ to a character of $\Gamma^*_{g,0}(N^2)$ by putting $\chi(\gamma) := \chi(\lambda)$ if $\gamma \equiv \begin{pmatrix} * & * \\ 0 & \lambda E \end{pmatrix} \pmod{N}$.

**Lemma 1.** Let $f \in S_k(\Gamma_g)$ with Fourier coefficients $a(T)$ ($T > 0$ half-integral). Let $\chi$ be a primitive Dirichlet character modulo $N$. Then the function
\[
f_\chi(Z) := \sum_{T > 0} \chi(\text{tr} T)a(T)e^{2\pi i \text{tr}(TZ)} \quad (Z \in \mathcal{H}_g)
\]
belongs to $S_k(\Gamma^*_{g,0}(N^2), \chi^2)$.

**Proof.** Let
\[
g(\chi) := \sum_{\nu \pmod{N}} \chi(\nu)e^{2\pi i \nu/N}
\]
be the Gauss sum attached to $\chi$. Since
\[
\sum_{\nu \pmod{N}} \chi(\nu)e^{2\pi i \text{tr}(T)\frac{\nu}{N}} = \chi(\text{tr} T)g(\chi),
\]
we obtain
\[
(2) \quad f_\chi = \frac{1}{g(\chi)} \sum_{\nu \pmod{N}} \chi(\nu) f|_{\alpha_\nu},
\]
where
\[
\alpha_\nu := \begin{pmatrix} E & \nu E \\ 0 & E \end{pmatrix} \quad (\nu \in \mathbb{Z}).
\]

Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^*_{g,0}(N^2)$ and put
\[
A' := A + \frac{\nu}{N} C,
\]
\[
B' := B + \frac{\nu}{N}(E - AD^t)D - \frac{\nu^2}{N^2} CD^tD,
\]
\[
D' := D - \frac{\nu}{N} CD^tD.
\]
Then $A', B'$ and $D'$ are integral matrices, one has $D' \equiv D \pmod{N}$ and
\[
\alpha_{\nu} \gamma = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} E & \frac{\gamma}{E} D D' \\ 0 & E \end{pmatrix};
\]
in particular \( \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma^*_{g,0}(N^2) \), and it follows that
\[
f_{\chi}|\gamma = \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) f \begin{pmatrix} E & \frac{\gamma}{E} D D' \\ 0 & E \end{pmatrix} \]
\[
= \chi(\lambda^2) \cdot \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) f_{\alpha_{\nu}} (D \equiv \lambda E \pmod{N})
\]
\[
= \chi^2(\gamma)f.
\]
This proves the claim.

**Lemma 2.** Let the notations be as in Lemma 1 and put
\[
W_{N^2} := \begin{pmatrix} 0 & -E \\ N^2 E & 0 \end{pmatrix}.
\]
Then
\[
f_{\chi}|W_{N^2} = g(\overline{\chi})^2 N^{-g-1} f_{\overline{\chi}}.
\]

**Proof.** For $(\nu, N) = 1$ determine $\lambda, \mu \in \mathbb{Z}$ with $\lambda N - \mu \nu = 1$. Then
\[
\alpha_{\nu} W_{N^2} = N \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \begin{pmatrix} N E & -\mu E \\ -\nu E & \lambda E \end{pmatrix} \alpha_{\mu}.
\]
Hence
\[
g(\overline{\chi}) \cdot f_{\chi}|W_{N^2} = N^{-g-1} \sum_{\nu \pmod{N},(\nu,N)=1} \overline{\chi}(\nu) f_{\alpha_{\mu}}
\]
\[
= \chi(-1) N^{-g-1} \sum_{\mu \pmod{N},(\mu,N)=1} \chi(\mu) f_{\alpha_{\mu}}
\]
\[
= \chi(-1) g(\chi) N^{-g-1} f_{\overline{\chi}}.
\]
Since $g(\chi) g(\overline{\chi}) = \chi(-1) N$, we obtain our claim.

**Theorem 1.** Let $k$ be even and let $f \in S_k(\Gamma_g)$. Let $\chi$ be a primitive Dirichlet character modulo $N$ and define $L(f, \chi, s)$ (Re $s \gg 0$) by (1). Let
\[
\gamma_g(s) := (2\pi)^{-gs} \prod_{\nu=1}^{g} \pi(\nu^{-1})/2 \Gamma \left( s - \frac{\nu - 1}{2} \right) \quad (s \in \mathbb{C})
\]
and set
\[
L^*(f, \chi, s) := N^{gs} \gamma_g(s) L(f, \chi, s) \quad (\text{Re } s \gg 0).
\]
Then \( L^*(f, \chi, s) \) extends to a holomorphic function on \( \mathbb{C} \), and the functional equation

\[
L^*(f, \chi, k - s) = (-1)^{\frac{ak}{2}} g(\chi)^2 \frac{1}{N} L^*(f, \overline{\chi}, s)
\]

holds, where \( g(\chi) \) is the Gauss sum attached to \( \chi \).

Proof. Since

\[
\left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in \text{GL}_{g,N}(\mathbb{Z}) \right\} \subset \Gamma_{g,0}(N^2)
\]

and \( k \) is even, the function \( f_\chi(iY) (Y > 0) \) is invariant under \( Y \mapsto Y[U] \) \((U \in \text{GL}_{g,N}(\mathbb{Z}))\). Hence it follows in the usual way that

\[
L^*(f, \chi, s) = \frac{1}{2} N^{gs} \int_{\mathcal{F}_{g,N}} f_\chi(iY)(\det Y)^s d\nu \quad (\text{Re } (s) \gg 0),
\]

where \( \mathcal{F}_{g,N} \) is any fundamental domain for the action of \( \text{GL}_{g,N}(\mathbb{Z}) \) on \( \mathcal{P}_g \) and \( d\nu = (\det Y)^{-(g+1)/2} dY \) is the \( \text{GL}_g(\mathbb{R}) \)-invariant volume element on \( \mathcal{P}_g \).

We fix a set of representatives \( U_1, \ldots, U_r \) for \( \text{GL}_g(\mathbb{Z})/\text{GL}_{g,N}(\mathbb{Z}) \) and now take

\[
\mathcal{F}_{g,N} = \bigcup_{\nu=1}^r \mathcal{R}_g[U_\nu],
\]

where \( \mathcal{R}_g \) is Minkowski’s fundamental domain for the action of \( \text{GL}_g(\mathbb{Z}) \).

Since \( \text{GL}_{g,N}(\mathbb{Z}) \) is closed under transposition, also \( \mathcal{F}_{g,N}^{-1} \) is a fundamental domain for \( \text{GL}_{g,N}(\mathbb{Z}) \).

We let

\[
\mathcal{P}_{g,+} := \{ Y \in \mathcal{P}_g \mid \det Y \geq N^{-g} \}, \quad \mathcal{P}_{g,-} := \{ Y \in \mathcal{P}_g \mid \det Y \leq N^{-g} \},
\]

write

\[
\mathcal{F}_{g,N} = (\mathcal{F}_{g,N} \cap \mathcal{P}_{g,+}) \cup (\mathcal{F}_{g,N} \cap \mathcal{P}_{g,-})
\]

and observe that \( \mathcal{F}_{g,N} \cap \mathcal{P}_{g,-} \) under the map \( Y \mapsto (N^2Y)^{-1} \) is transformed bijectively onto \( \mathcal{F}_{g,N}^{-1} \cap \mathcal{P}_{g,+} \). We also observe that both \( \mathcal{F}_{g,N} \cap \mathcal{P}_{g,+} \) and \( \mathcal{F}_{g,N}^{-1} \cap \mathcal{P}_{g,+} \) are fundamental domains for the induced action of \( \text{GL}_{g,N}(\mathbb{Z}) \) on \( \mathcal{P}_{g,+} \), the integral in (3) is absolutely convergent and the integrand is invariant under \( \text{GL}_{g,N}(\mathbb{Z}) \).

Therefore, since by Lemma 2

\[
f_\chi(i(N^2Y)^{-1}) = (-1)^{\frac{ak}{2}} g(\chi)^2 N^{gk-1}(\det Y)^k f_\overline{\chi}(iY),
\]
we conclude that
\[ L^s(f, \chi, s) = \frac{1}{2} \int_{Fg,N \cap Fg,+} \left( f_\chi(iY)(N^g \det Y)^s \right. \\
+ \left. (-1)^{\frac{g^2}{2}} g(\chi)^2 N^{-1} f_\chi(iY)(N^g \det Y)^{k-s} \right) dv. \]

Standard arguments and estimates taking into account (4) and properties of \( R_g \) (compare e.g., [5, Chap. VI]) now show that the integral on the right of (5) is (absolutely) convergent for all \( s \in \mathbb{C} \) and represents a holomorphic function of \( s \).

Since
\[ g(\chi)g(\overline{\chi}) = \chi(-1)N, \]
we also easily see the claimed functional equation. This concludes the proof of the Theorem.

3. Special values.

In this section we shall prove:

**Theorem 2.** Let \( k \) be even, \( k \geq g+1 \) and let \( f \in S_k(\Gamma_g) \). If \( \chi \) is a primitive Dirichlet character modulo \( N \), define \( L(f, \chi, s) \) (\( s \in \mathbb{C} \)) by holomorphic continuation of the series (1) (Theorem 1). Let \( g(\chi) \) be the Gauss sum attached to \( \chi \) and let \( Z[\chi] \) be the \( \mathbb{Z} \)-module obtained from \( \mathbb{Z} \) by adjoining the values of \( \chi \).

Then there exists a \( \mathbb{Z} \)-module \( M_f \subset \mathbb{C} \) depending only on \( f \) of finite rank such that all the special values
\[ s \in \mathbb{N}, \quad \frac{g^2+1}{2} \leq s \leq k - \frac{g^2+1}{2} \]
and \( \chi \) runs over all primitive Dirichlet characters modulo all positive integers \( N \), are contained in \( M_f \otimes \mathbb{Z} Z[\chi] \).

**Proof.** From (2) and (3) and the proof of Theorem 1 we find that
\[ g(\overline{\chi})\gamma_g(s)L(f, \chi, s) = \frac{1}{2} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) \int_{Fg,N} f(iY + \frac{\nu}{N} E)(\det Y)^s\gamma \frac{g^2+1}{2} dY \]
for all \( s \in \mathbb{C} \).

Note that the individual integrands on the right of (6) are \( GL_{g,N}(\mathbb{Z}) \)-invariant since \( f(Z) \) is invariant under \{ \( \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in GL_{g,N}(\mathbb{Z}) \) \} and under translations. Let \( w \in \mathbb{Z}, w \geq 0 \) and \( Sp_g(\mathbb{R}) \rtimes \mathbb{R}^{2gw} \) be the semi-direct product of \( Sp_g(\mathbb{R}) \) and \( \mathbb{R}^{2gw} \cong (\mathbb{R}^{2g})^w \) with multiplication given by
\[ (\gamma, \lambda)(\gamma', \lambda') = (\gamma\gamma', \lambda\gamma'^t + \lambda') \]
where by $\gamma \mapsto \gamma^\dagger$ we denote the diagonal embedding of $Sp_g(\mathbb{R})$ into $GL_{2gw}(\mathbb{R})$.

The group $Sp_g(\mathbb{R}) \ltimes \mathbb{R}^{2gw}$ acts on $\mathcal{H}_g \times \mathbb{C}^{gw}$ (with $\mathbb{C}^{gw} \cong (\mathbb{C}^g)^w$) from the left by

$$(\gamma, \lambda) \circ (Z, (\zeta_1, \ldots, \zeta_w)) = \left((AZ + B)(CZ + D)^{-1}, \left(\begin{array}{c} \zeta_1 + (\mu_1, \nu_1) \\ \vdots \\ \zeta_w + (\mu_w, \nu_w) \end{array}\right) \left(CZ + D\right)^{-1}\right)$$

where $\gamma = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)$ and $\lambda = ((\mu_1, \nu_1), \ldots, (\mu_w, \nu_w))$ with $\mu_j, \nu_j \in \mathbb{R}^g$ for all $j$. The discrete subgroup $\Gamma_g \propto \mathbb{Z}^{2gw}$ acts properly discontinuously.

Let $\Gamma \subset \Gamma_g$ be any congruence subgroup acting without fixed points on $\mathcal{H}_g$ (e.g., the principal congruence subgroup $\Gamma_g(\ell)$ with $\ell \geq 3$) and view $f$ as an element of $S_k(\Gamma)$.

Put $w := k - (g + 1)$. It was shown in [2] that the map

$$h(Z) \mapsto h(Z)dzd\zeta$$

gives an isomorphism between $S_k(\Gamma)$ and the space of holomorphic differential forms of degree $\frac{g(g+1)}{2} + gw$ of (any) non-singular compactification of the quotient space $\Gamma \times \mathbb{Z}^{2gw}\backslash \mathcal{H}_g \times \mathbb{C}^{gw}$.

Using toroidal compactifications, in [3] from this a certain finiteness statement for certain cycle integrals attached to $h$ was derived which we now want to describe in the special case we need.

Let $S$ be a given rational symmetric matrix of size $g$ and let $n$ be an integer with $0 \leq n \leq w$. Define

$$T_g(S; n) := \bigcup_{Y \in \mathcal{P}} \{S + iY\}$$

$$\times \left(\mathbb{R}^{g}^{w-n} \times \{(\mu_1iY, \ldots, \mu_niY) \mid \mu_1, \ldots, \mu_n \in \mathbb{R}^{g}\}\right)$$

$$\subset \mathcal{H}_g \times \mathbb{C}^{gw}.$$

Then $T_g(S; n)$ is a real submanifold of $\mathcal{H}_g \times \mathbb{C}^{gw}$ of dimension $\frac{g(g+1)}{2} + gw$.

(In the notation of [3, §6] we have taken $a_1 = a_2 = \ldots = a_{w-n} \in \{g + 1, \ldots, 2g\}$ and $a_{w-n+1} = \ldots a_w \in \{1, \ldots, g\}$. Also note that in the definition of $T_g(a_1, \ldots, a_w; X)$ in [3, p. 401] we have replaced the “$Z$” in $W(a_1, \ldots, a_w)[Z]$ by “$iY$”. We think that this is the correct definition, since otherwise the corresponding integrals in [3, Lemma 6.2 and Thm. 5] in general would not be convergent.)
Put
\[ U_g := \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} | U \in GL_g(\mathbb{R}) \right\} \subset Sp_g(\mathbb{R}), \]

\[ V_{g,n} := \{(\lambda_1, \ldots, \lambda_{w-n}, (\mu_1, 0), \ldots, (\mu_n, 0)) | \lambda_1, \ldots, \lambda_{w-n} \in \mathbb{R}^g, \mu_1, \ldots, \mu_n \in \mathbb{R}^g \} \]

and
\[ H_{g,n} := U_g \otimes V_{g,n} \subset Sp_g(\mathbb{R}) \otimes \mathbb{R}^{2gw}. \]

Let
\[ \alpha^{(S)} := \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}. \]

Then one easily checks that the conjugate subgroup
\[ H_{g,n}^{(S)} := (\alpha^{(S)}, 0) \cdot H_{g,n} \cdot (\alpha^{(S)}, 0)^{-1} \]
leaves \( T_g(S; n) \) stable.

Note that \( H_{g,n}^{(S)} \) consists of all pairs
\[ \left( \begin{pmatrix} U & S(U^t)^{-1} - US \\ 0 & (U^t)^{-1} \end{pmatrix}, (\lambda_1, \ldots, \lambda_{w-n}, (\mu_1, -\mu_1 S), \ldots, (\mu_n, -\mu_n S)) \right) \]
with \( \lambda_1, \ldots, \lambda_{w-n} \in \mathbb{R}^g \) and \( \mu_1, \ldots, \mu_n \in \mathbb{R}^g \).

Let
\[ H_{g,n,\Gamma}^{(S)} := H_{g,n}^{(S)} \cap \Gamma \otimes \mathbb{Z}^{2gw}. \]

Write \( M := \Gamma \otimes \mathbb{Z}^{2gw} \backslash \mathcal{H}_g \times C^{gw} \) and denote by \( \overline{M} \) a fixed toroidal compactification of \( M \). Let \( \partial M = \overline{M} \setminus M \). Then according to [3, Lemma 6.1] the closure of the image of \( H_{g,n,\Gamma}^{(S)} \setminus T_g(S; n) \) in \( \overline{M} \) w.r.t. the usual complex topology is the support of a singular relative \( \frac{g(g+1)}{2} + gw \)-cycle with integral coefficients w.r.t. \( (\overline{M}, \partial M) \).

Since \( H_{g,n+1}^{(S)} + gw(\overline{M}, \partial M, \mathbb{Z}) \) is of finite rank, one concludes that for any given \( h \in S_k(\Gamma) \) all the numbers
\[ \int_{H_{g,n,\Gamma}^{(S)} \setminus T_g(S; n)} h(Z) dZ d\zeta \quad (S \in \mathbb{Q}^{(g,g)}, S = S^t) \]
are contained in a finite \( \mathbb{Z} \)-module (depending only on \( h \)) whose rank is bounded by the rank of the above cohomology group ([3, Thm. 5], compare our above remark).
On the other hand (compare [3, Lemma 6.2]) one has the equality

\[
\int_{H^{(S)} \backslash \mathcal{T}_g(S; n)} h(Z) dZ d\zeta = \sum_{\alpha(S) \notin \U_g(\alpha(S))^{-1} \cap \Gamma \{ S + iY \mid Y \in \mathcal{P}_g \}} h(Z) \det (Z - S)^n dZ.
\]

In particular, now take \( \Gamma = \Gamma_g(\ell) \) with some fixed \( \ell \geq 3 \). Then the integral on the right of (7) is equal to

\[
\int_{\mathcal{P}_g/GL^{(S)}_{g,\ell}(Z)} h(S + iY) (\det Y)^n dY,
\]

where

\[
GL^{(S)}_{g,\ell}(Z) := \{ U \in GL_{g,\ell}(Z) \mid S[U^t] \equiv S \pmod{\ell(Z)} \}.
\]

Let \( S = \frac{\nu}{N} E \) with \( \nu \in \mathbb{Z} \) (so \( \alpha^{(S)} = \alpha_\nu \) in the notation of §2). Then we see that \( GL_{g,\ell N}(Z) \) is contained in \( GL^{(S)}_{g,\ell}(Z) \). Since the index of \( GL_{g,\ell N}(Z) \) in \( GL_{g,N}(Z) \) is bounded by a number depending only on \( \ell \), the assertion of Thm. 2 now follows taking into account (6) and the fact that \( \Gamma(\frac{1}{2} + \nu) \in \mathbb{Q}\sqrt{\pi} \) for \( \nu = 0, 1, 2 \ldots \).

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